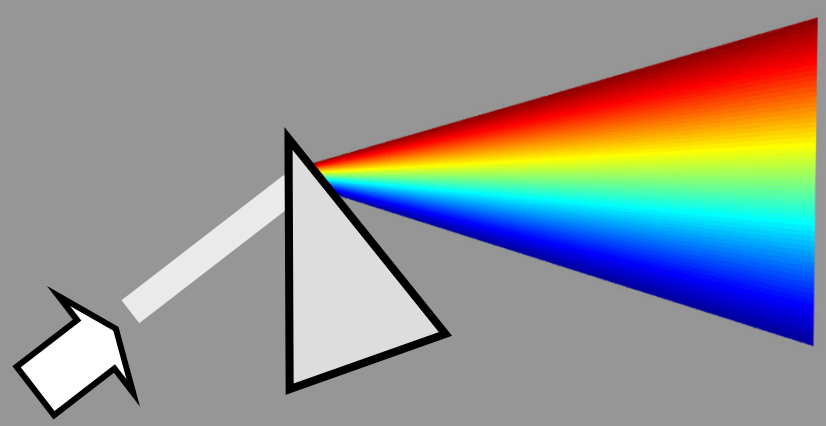
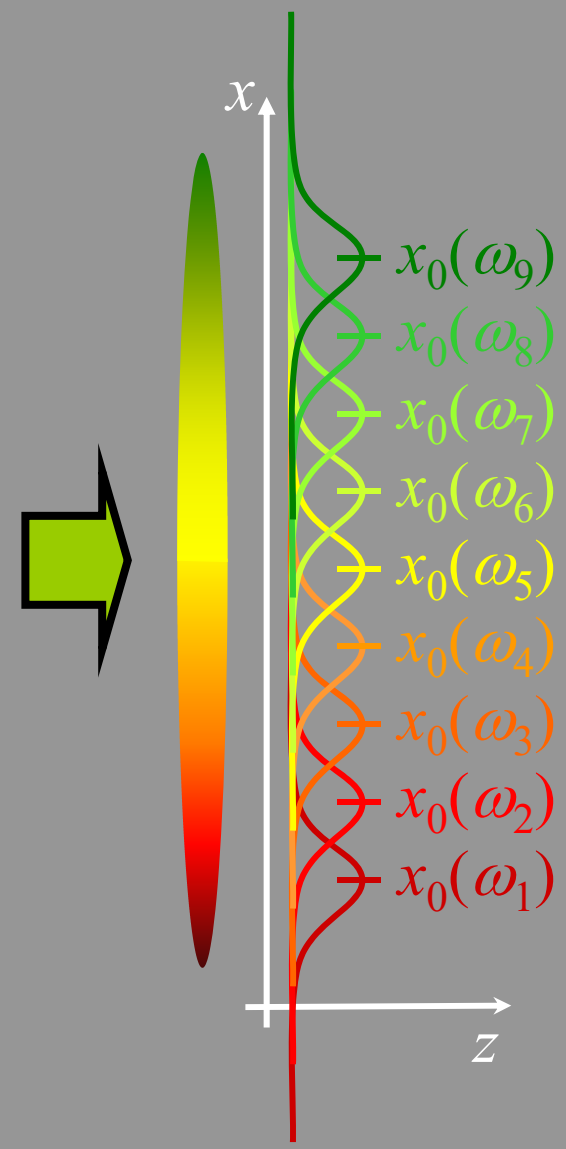
Spatio-Temporal Characteristics of Light Pulses and How to Model Them

Spatio-temporal distortions

- Angular dispersion
- Spatial chirp
- Pulse front tilt

A first step in how to model them:
Ray-pulse matrices





Spatio-Temporal Distortions

Ordinarily, we assume that the pulse-field spatial and temporal factors (or their Fourier-domain equivalents) separate:

Example

$$E(x,y,z,t) = E_{xyz}(x,y,z)\, E_t(t) \longrightarrow \exp(ik_0 z)\exp(i\omega_0 t)$$

$$\tilde{E}(x,y,z,\omega) = E_{xyz}(x,y,z)\, \tilde{E}_t(\omega) \longrightarrow \exp(ik_0 z)\, \delta(\omega-\omega_0)$$

$$\hat{E}(k_x,k_y,k_z,t) = \hat{E}_{xyz}(k_x,k_y,k_z)\, E_t(t) \longrightarrow \delta(k_x)\delta(k_y)\delta(k_z-k_0)\exp(i\omega_0 t)$$

$$\hat{\tilde{E}}(k_x,k_y,k_z,\omega) = \hat{E}_{xyz}(k_x,k_y,k_z)\, \tilde{E}_t(\omega) \longrightarrow \delta(k_x)\delta(k_y)\delta(k_z-k_0)\delta(\omega-\omega_0)$$

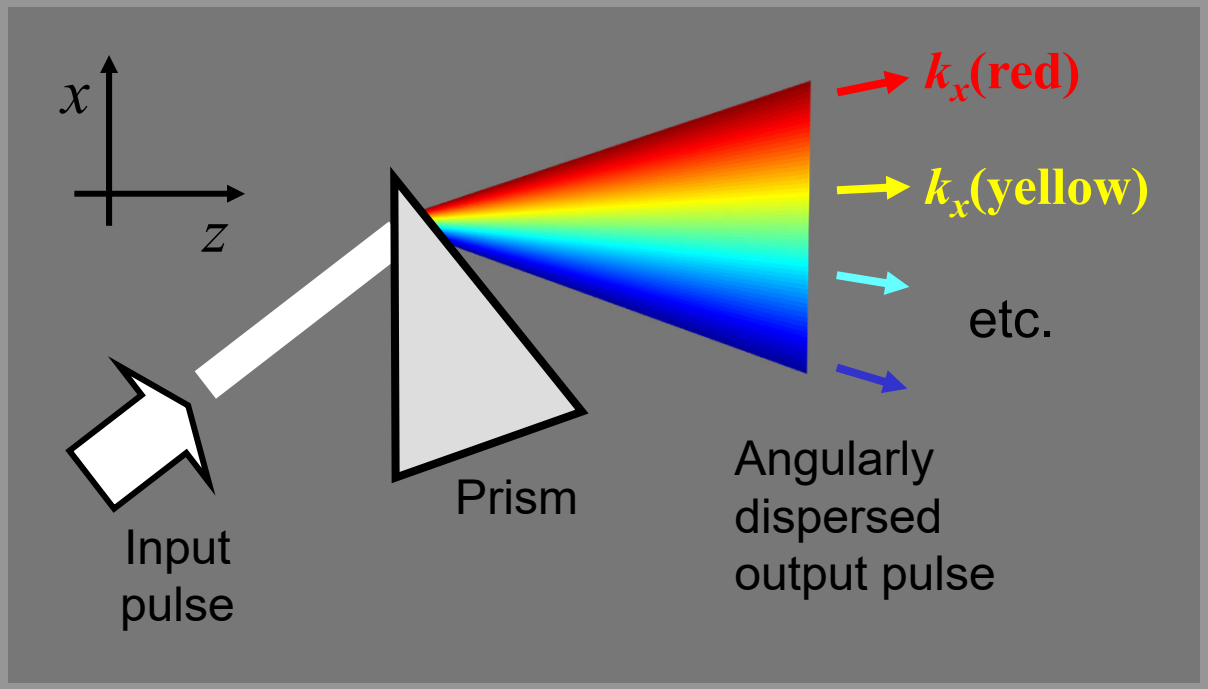
where the tilde and hat mean FTs with respect to t and x, y, z

Sometimes, this separation of variables is not possible.

Angular dispersion is a spatio-temporal distortion. $\frac{dk_x}{d\omega}$

Note: k_x is an input variable, and $k_x(\omega)$ is a function: the mean k_x of light of frequency ω.

In the presence of angular dispersion, the off-axis k-vector component k_x depends on ω:



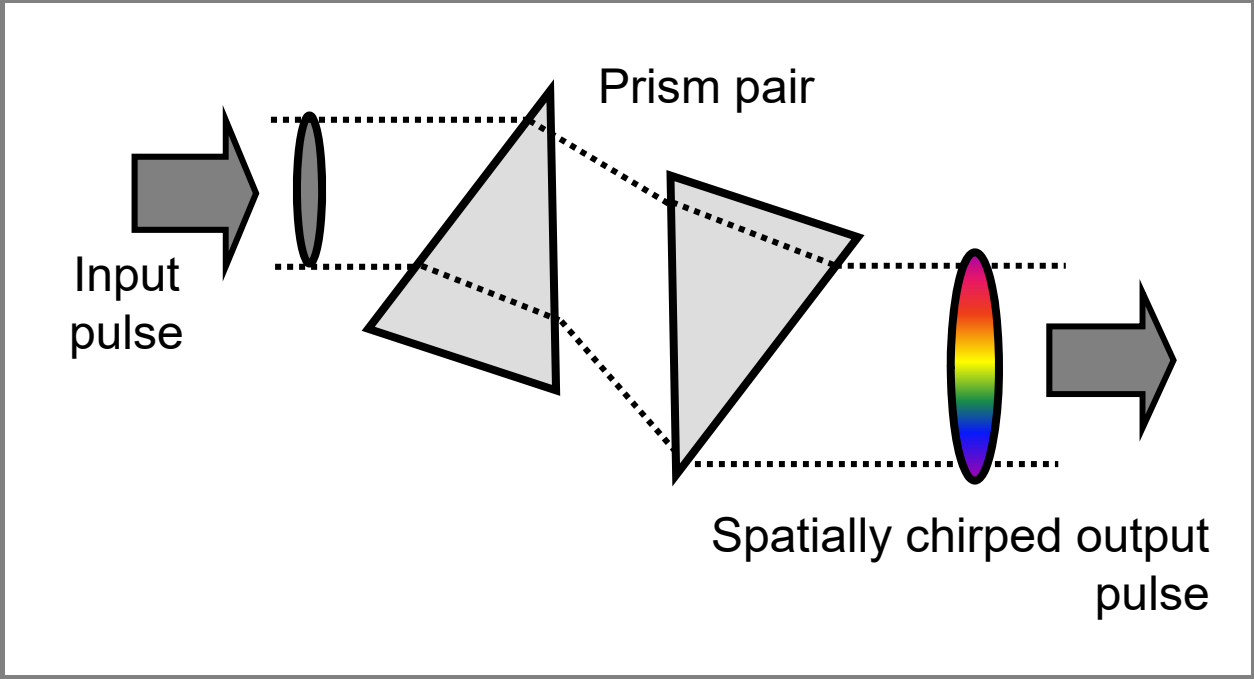
If $\hat{\tilde{E}}(k_x, k_y, k_z, \omega)$ is a function that peaks at $k_x = 0$, then adding the coupling term moves the peak to:

$$k_x = \frac{dk_x}{d\omega}(\omega - \omega_0)$$

$$\hat{\tilde{E}}(k_x, k_y, k_z, \omega) = \hat{\tilde{E}}[k_x - \frac{dk_x}{d\omega}(\omega - \omega_0), k_y, k_z, \omega]$$

Spatial chirp is a spatio-temporal distortion in which the color varies spatially across the beam.

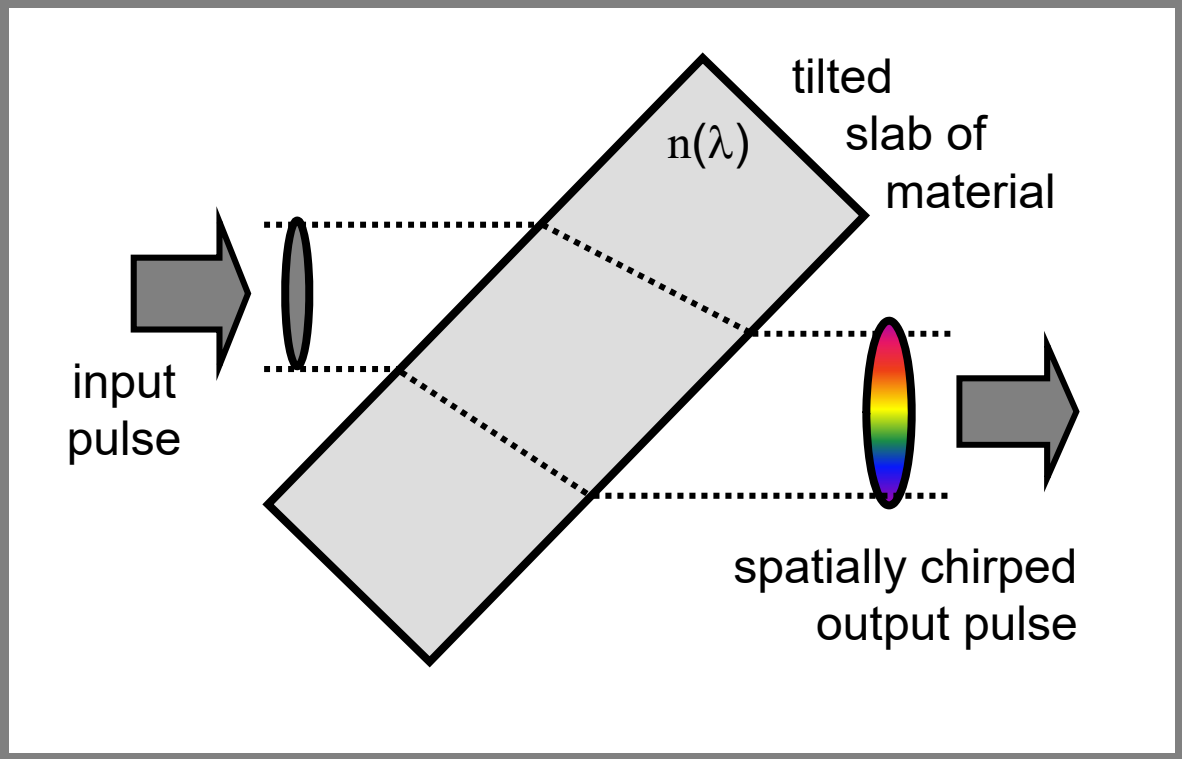
Propagation through a well-aligned prism pair produces a beam with no angular dispersion, but with *spatial* dispersion, often called spatial chirp.



Prism pairs are found inside many ultrafast lasers.

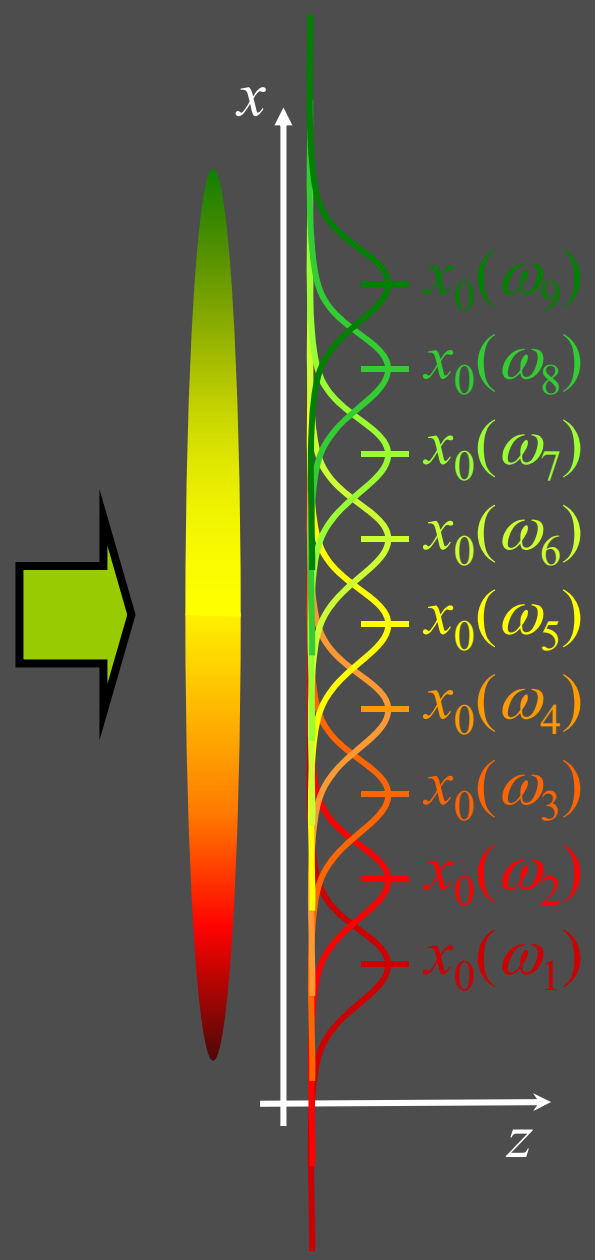
Spatial chirp can be difficult to avoid.

Simply propagating through a tilted window causes spatial chirp!



Because ultrashort pulses are so broadband, this distortion can be very noticeable—and sometimes problematic!

How to think about spatial chirp $\frac{dx_0}{d\omega}$



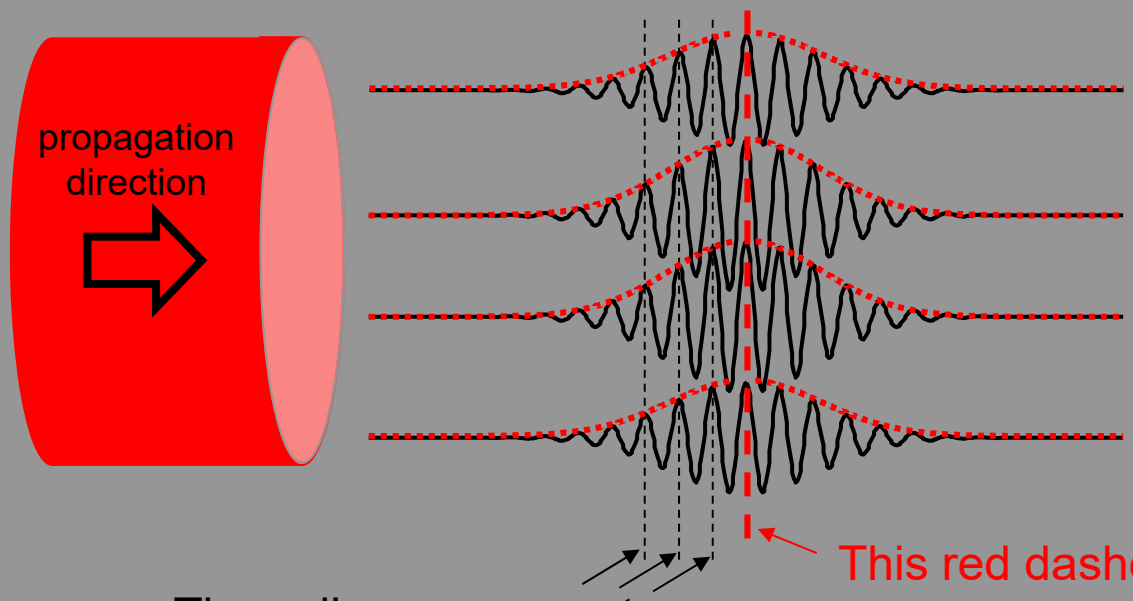
Suppose we send the pulse through a set of monochromatic filters and find the beam center position, x_0, for each frequency, ω.

$$\tilde{E}(x, y, z, \omega) \rightarrow$$

$$\tilde{E}[x - \frac{dx_0}{d\omega}(\omega - \omega_0), y, z, \omega]$$

where x_0 is the center of the beam component of frequency ω.

Pulse fronts vs. phase fronts

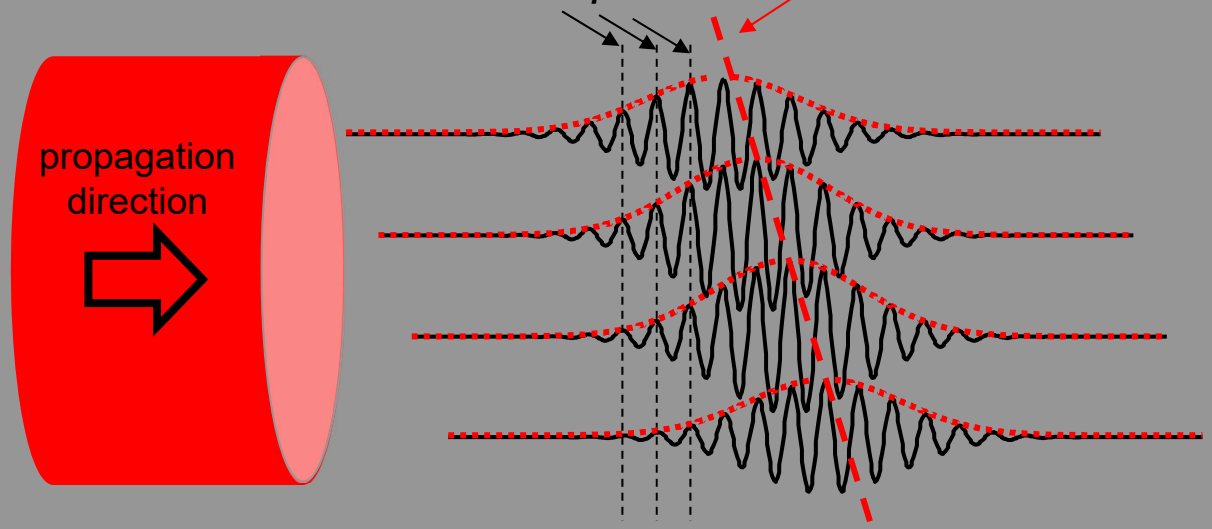


Light always propagates in a direction perpendicular to the **phase fronts**.

Pulse fronts: refers to the intensity not the phase

This red dashed line represents a surface of maximum *intensity*.

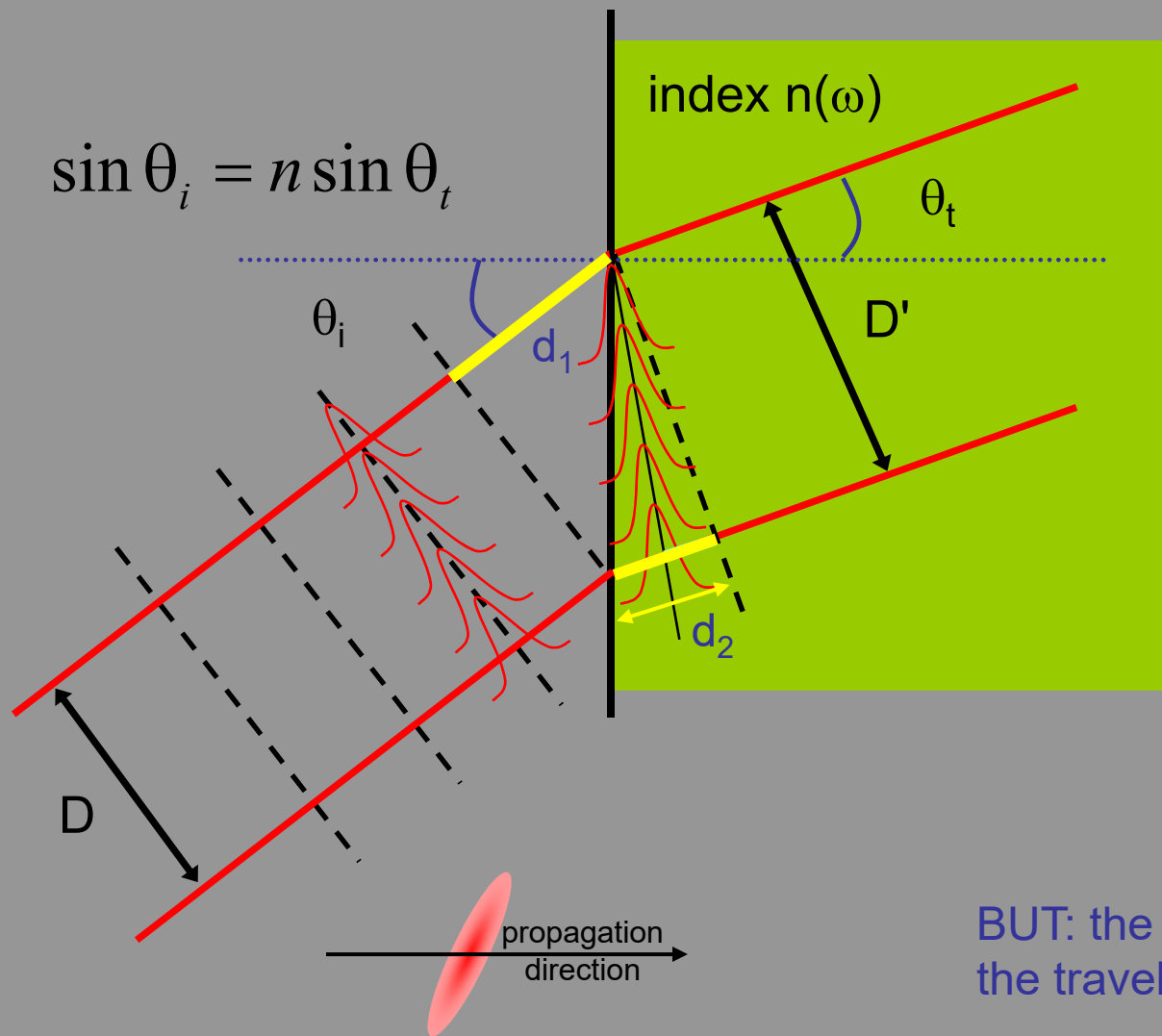
These lines represents surfaces of constant *phase*.



Pulse front tilt: when the surface of max intensity is not perpendicular to the propagation direction

Refraction: a source of pulse front tilt

$$\sin\theta_i = n\sin\theta_t$$



Beam width:

$$D' = D\frac{\cos\theta_t}{\cos\theta_i}$$

The time to propagate the distance d_1:

$$T_{phase} = \frac{d_1}{c} = \frac{D\tan\theta_i}{c}$$

must equal the time to propagate the distance d_2:

$$T_{phase} = d_2\frac{n}{c} = \frac{nD'\tan\theta_t}{c}$$

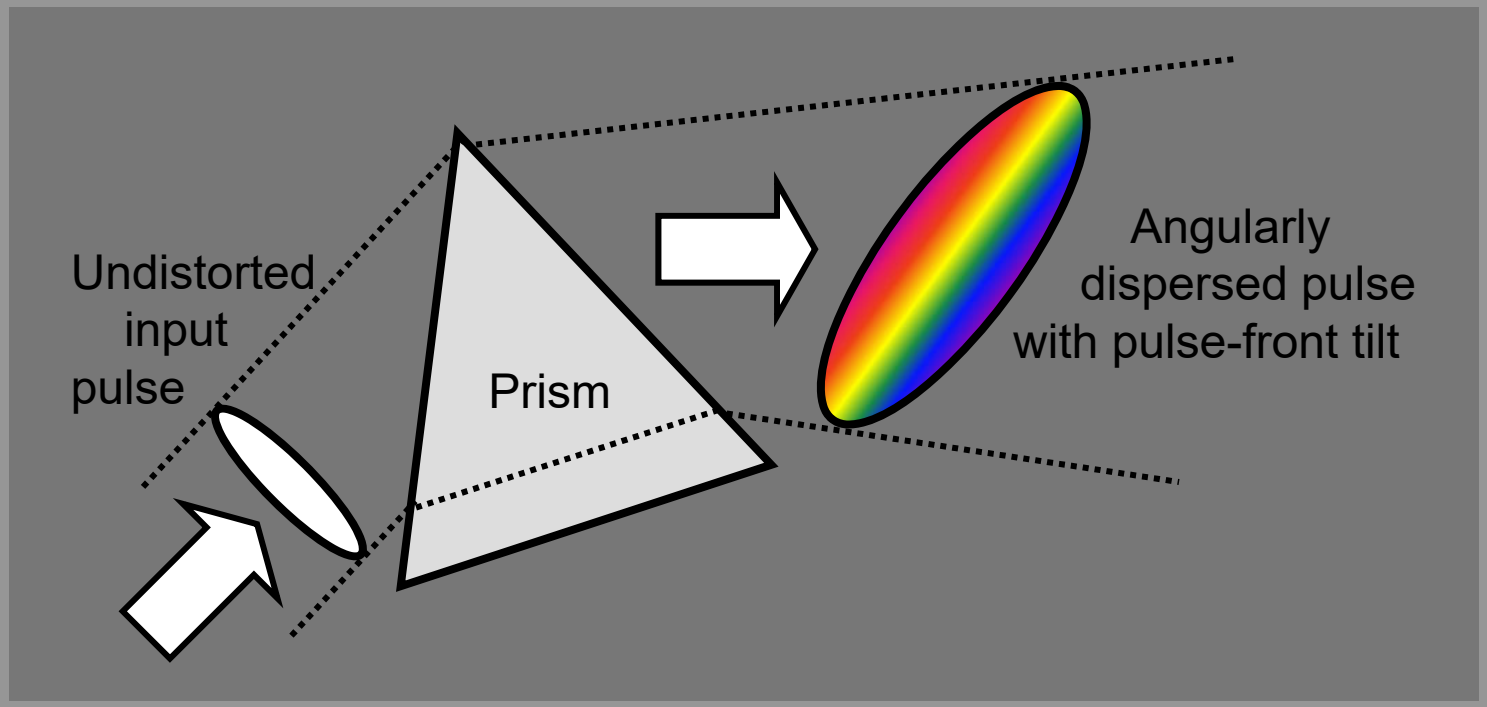
BUT: the pulse front travels at V_g, so the travel distance is less by:

$$\Delta d = \left(V_\phi - V_g\right)\cdot T_{phase}$$

<u>phase fronts:</u> perpendicular to propagation
<u>pulse fronts:</u> not necessarily!

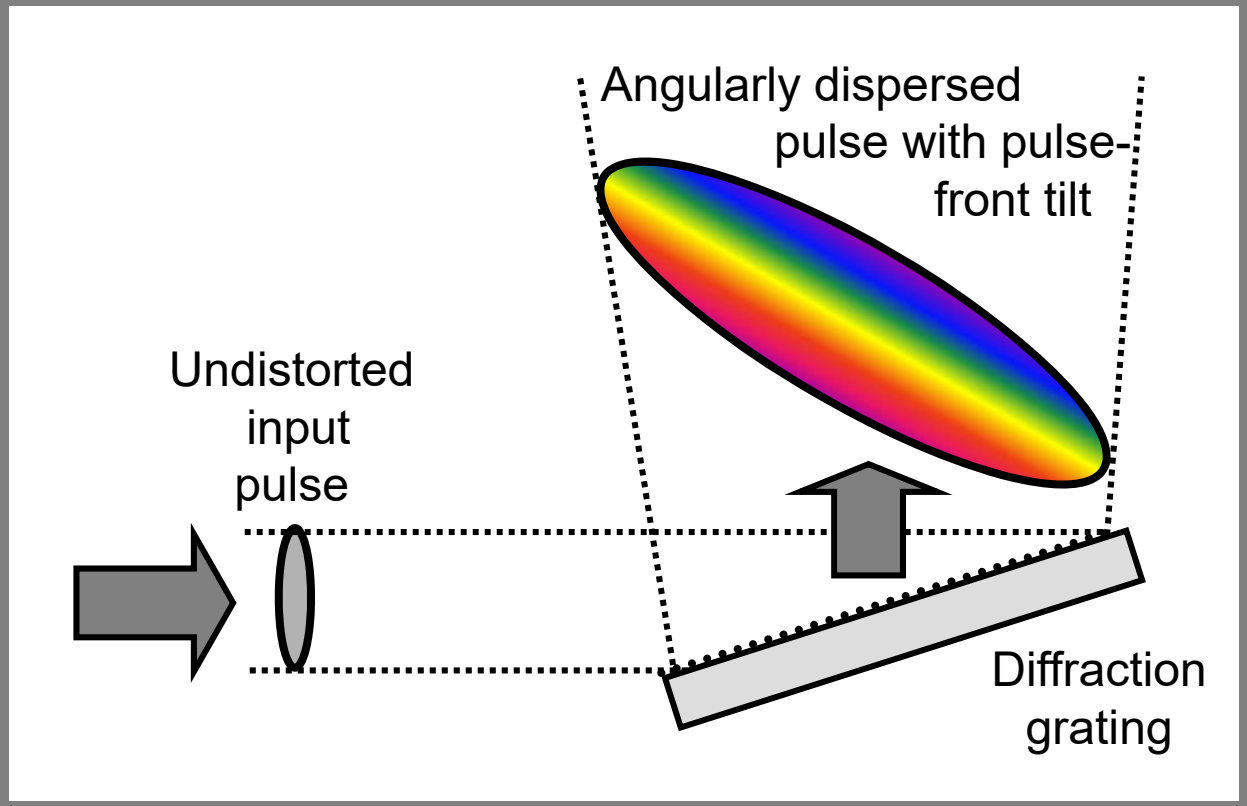
Pulse-front tilt is another common spatio-temporal distortion.

Because the group velocity is usually less than phase velocity, pulse fronts tilt when light traverses a prism.



Diffraction gratings also yield pulse-front tilt.

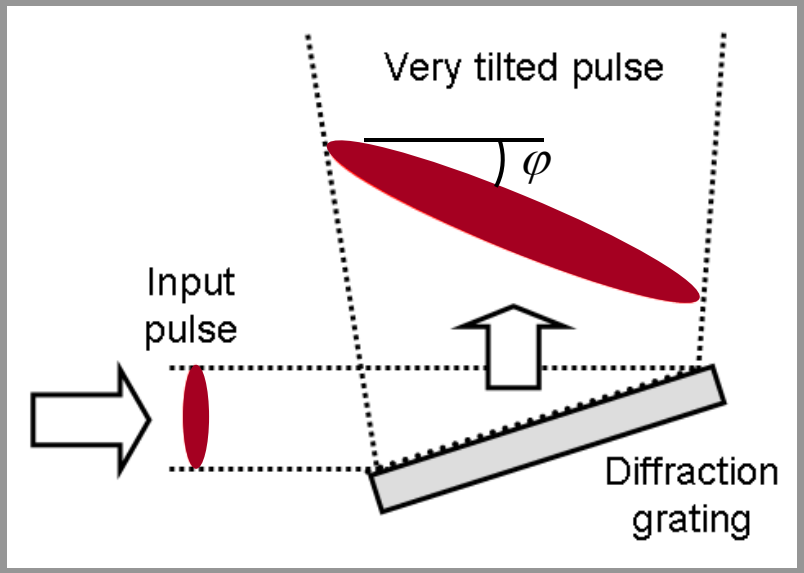
The path is simply shorter for rays that impinge on the near side of the grating.



Of course, angular dispersion and spatial chirp occur, too.

Gratings yield about ten times more pulse-front tilt than prisms do.

Pulse-Front Tilt from a Grating



For a diffraction grating, use a grazing (large) incidence angle (for largest PFT).

In the limit of grazing incidence: The extra distance traveled by the ray that impinges on the back edge of the grating is d, where d is the length of the grating.

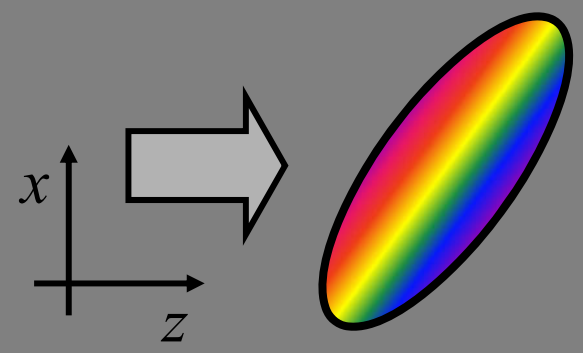
But, in the time it takes for this ray to travel this extra distance, the distance traveled by the ray that impinges on the front edge is also *d*.

So the maximum pulse-front tilt angle achievable using a grating is given by:

$$\tan(\varphi) = d/d, \text{ or } \varphi = \sim 45^{\circ}.$$

Modeling pulse-front tilt

$$\frac{dt_0}{dx}$$



Pulse-front tilt involves coupling between the space and time domains:

$$E(x,y,z,t) \rightarrow E[x,y,z,t - \frac{dt_0}{dx}(x - x_0)]$$

For a given transverse position in the beam, x, the pulse mean time, t_0, varies in the presence of pulse-front tilt.

Angular dispersion always causes pulse-front tilt!

Angular dispersion means that the off-axis k-vector depends on ω:

$$\hat{\tilde{E}}(k_x, k_y, k_z, \omega) = \hat{\tilde{E}}_0[k_x - \gamma(\omega - \omega_0), k_y, k_z, \omega]$$ where $\gamma = dk_{x0}/d\omega$

Inverse Fourier-transforming with respect to k_x, k_y, and k_z yields:

$$\tilde{E}(x, y, z, \omega) = \tilde{E}_0(x, y, z, \omega)\, e^{-i\gamma(\omega-\omega_0)x}$$ using the shift theorem

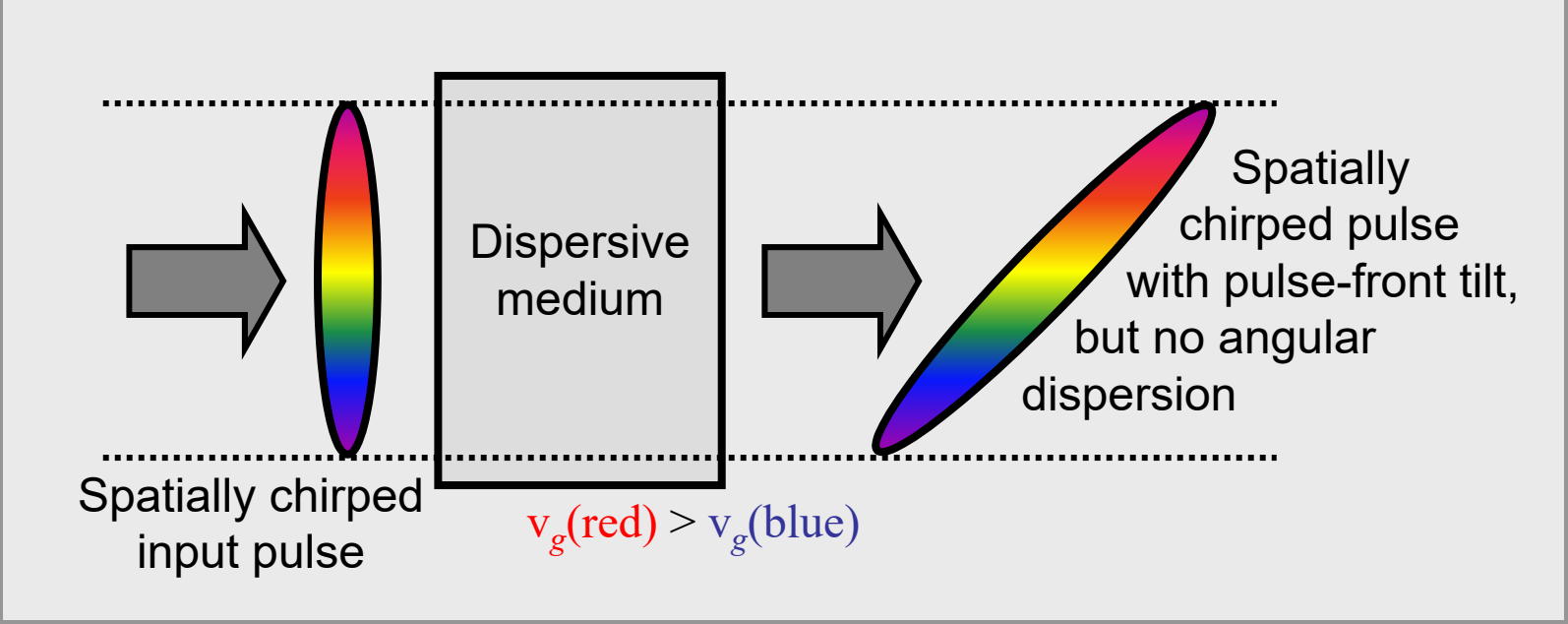
Inverse Fourier-transforming with respect to ω (or $\omega-\omega_0$) yields:

$$\Rightarrow\ E(x, y, z, t) \propto E_0(x, y, z, t - \gamma x)$$ using the shift theorem again

which is just pulse-front tilt!

The combination of spatial and temporal chirp also causes pulse-front tilt.

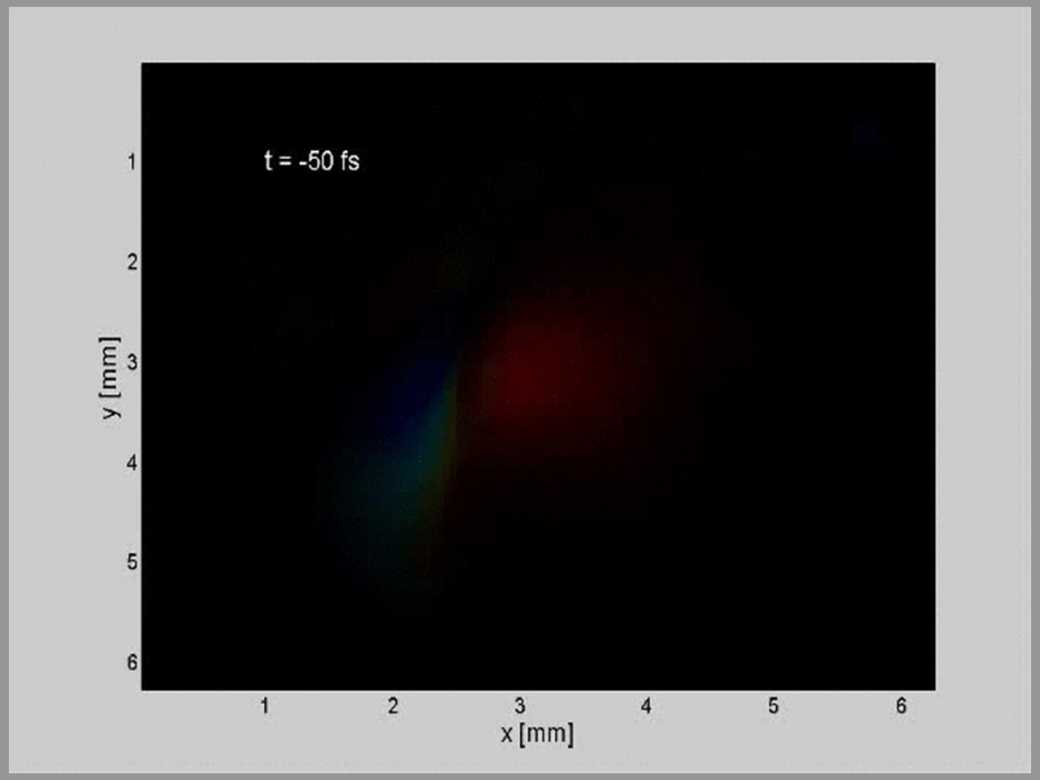
The theorem we just proved assumed no spatial chirp, however. So it neglects another contribution to the pulse-front tilt.



The total pulse-front tilt is the sum of that due to angular dispersion and that due to this effect.

A pulse with temporal chirp, spatial chirp, and pulse-front tilt.

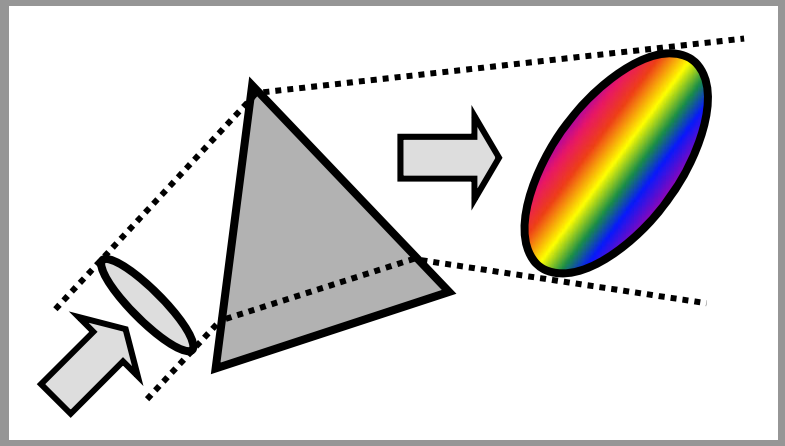
It's best to make a movie of such a pulse (coming at you):



We'll need a nice formalism for calculating these distortions.

We'll also need a nice technique for measuring them. The above pulse was measured using a technique called STRIPED FISH.

Spatio-temporal distortions can be useful or inconvenient.



Good:

They allow pulse compression.

They can be used to measure pulses (tilted pulse fronts).

They allow pulse shaping.

They can increase bandwidth and conversion efficiency in some nonlinear-optical processes (e.g. high-intensity terahertz pulses!)

Bad:

They usually increase the pulse length.

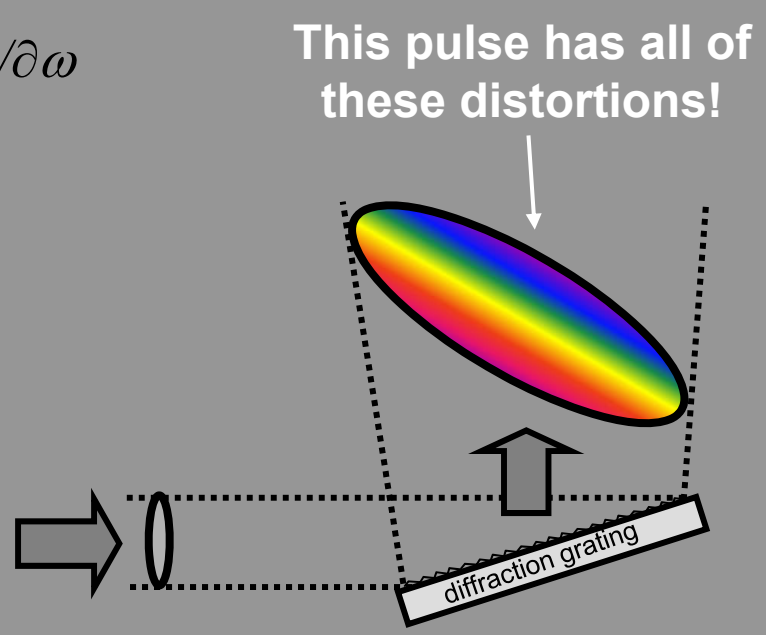
They reduce intensity.

They can be hard to measure.

Modeling the time and frequency distributions of a light pulse

We'd like a matrix formalism to predict such effects as the:

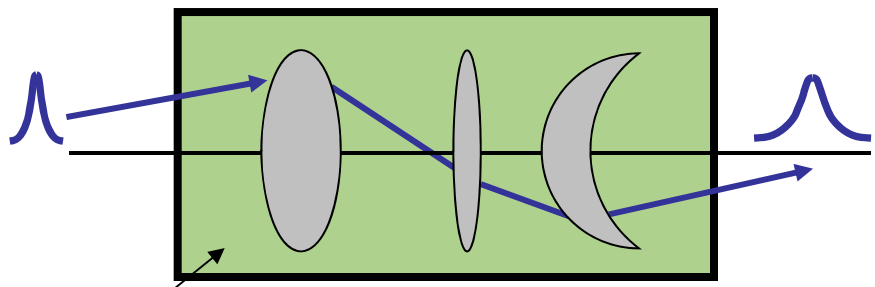
- group-delay dispersion $\partial t/\partial \omega$
- angular dispersion $\partial k_x/\partial \omega$ or $\partial \theta/\partial \omega$
- spatial chirp $\partial x/\partial \omega$
- pulse-front tilt $\partial t/\partial x$
- time vs. angle $\partial t/\partial \theta$.



We'll need to consider, not only the position (x) and slope (θ) of the ray, but also the time (t) and frequency (ω) of the pulse.

Propagation in space and time: Ray-pulse Kostenbauder matrices

Kostenbauder matrices are a generalization of the 2x2 ray matrices that we discussed earlier. They are 4x4 matrices that multiply 4-vectors comprising the position, slope, time (group delay), and frequency.



Optical system ↔ 4x4 Ray-pulse matrix

$$\begin{bmatrix} x_{in} \\ \theta_{in} \\ t_{in} \\ \nu_{in} \end{bmatrix} \longrightarrow \begin{bmatrix} A & B & 0 & E \\ C & D & 0 & F \\ G & H & 1 & I \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} x_{out} \\ \theta_{out} \\ t_{out} \\ \nu_{out} \end{bmatrix}$$

where each vector component corresponds to the deviation from a mean value for the ray or pulse.

A Kostenbauder matrix requires five additional parameters, E, F, G, H, I.

Kostenbauder matrix elements

As with 2x2 ray matrices, consider each element to correspond to a small deviation from its mean value ($x_{in} = x - x_0$). So we can think in terms of partial derivatives.

the usual 2x2 ray matrix

angular dispersion $\frac{\partial \theta_{out}}{\partial \nu_{in}}$

spatial chirp $\frac{\partial x_{out}}{\partial \nu_{in}}$

$$\begin{bmatrix} x_{out} \\ \theta_{out} \\ t_{out} \\ \nu_{out} \end{bmatrix} = \begin{bmatrix} A & B & 0 & E \\ C & D & 0 & F \\ G & H & 1 & I \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \\ t_{in} \\ \nu_{in} \end{bmatrix}$$

pulse-front tilt $\frac{\partial t_{out}}{\partial x_{in}}$

time vs. angle $\frac{\partial t_{out}}{\partial \theta_{in}}$

GDD $\frac{\partial t_{out}}{\partial \nu_{in}}$

A. G. Kostenbauder, IEEE J. Quant. Electron. 26, 1148 (1990)

Some Kostenbauder matrix elements are always zero or one.

Passive optical components don't change with time, so nothing but t_{out} depends on t_{in}. So this column must be zero, except for the 3rd element, which must be 1.

$$\begin{bmatrix} x_{out} \\ \theta_{out} \\ t_{out} \\ \nu_{out} \end{bmatrix} = \begin{bmatrix} A & B & 0 & E \\ C & D & 0 & F \\ G & H & 1 & I \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \\ t_{in} \\ \nu_{in} \end{bmatrix}$$

The output frequency, ν_{out}, is always equal to the input frequency, ν_{in}. So these elements are always zero, and the last element must be 1.

Kostenbauder matrix for propagation through free space or a uniform material

The ABCD elements are always the same as the ray matrix.

Here, the only other interesting element is the GDD: $I = \partial t_{out} / \partial \nu_{in}$

So:

$$\mathcal{K}_{material} = \begin{bmatrix} 1 & L/n & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2\pi L k'' \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The 2π is due to the definition of K-matrices in terms of ν, not ω.

where L is the thickness of the medium, n is its refractive index, and k'' is the GVD:

$$k'' \equiv \left. \frac{d^2 k}{d\omega^2} \right|_{\omega_0} = \frac{\lambda^3}{2\pi c^2} \frac{d^2 n}{d\lambda^2}$$

Example: Using the Kostenbauder matrix for propagation through a uniform medium

Apply the free-space propagation matrix to an input vector:

$$\begin{bmatrix} 1 & L/n & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2\pi L k'' \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \\ t_{in} \\ \nu_{in} \end{bmatrix} = \begin{bmatrix} x_{in} + (L/n)\theta_{in} \\ \theta_{in} \\ t_{in} + 2\pi L k'' \nu_{in} \\ \nu_{in} \end{bmatrix}$$

The position varies in the usual way, and the beam angle remains the same.

The group delay increases by $k'' L \omega_{in}$

The frequency remains the same.

Because the group delay depends on frequency, the pulse broadens.

This approach works in much more complex situations, too.

Kostenbauder Matrix for a Lens

The ABCD elements are always the same as the ray matrix. Everything else is a zero or one.

So:

$$\mathscr{K}_{lens} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/f & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where f is the lens focal length.

As with ray matrices, the same holds for a curved mirror.

To include the GDD of a lens, just multiply by a Kostenbauder material-propagation matrix for the thickness of the lens.

While chromatic aberrations can be modeled using a wavelength-dependent focal length, other lens imperfections cannot be modeled using Kostenbauder matrices.

Kostenbauder matrix for a diffraction grating

Gratings introduce magnification, angular dispersion and pulse-front tilt:

So:

spatial magnification

angular magnification

no spatial chirp (yet)

$$\mathscr{K}_{grating} = \begin{bmatrix} -\dfrac{\cos(\beta')}{\cos(\beta)} & 0 & 0 & 0 \\ 0 & -\dfrac{\cos(\beta)}{\cos(\beta')} & 0 & \dfrac{\lambda[\sin(\beta')-\sin(\beta)]}{c\cos(\beta')} \\ \dfrac{\sin(\beta)-\sin(\beta')}{c\cos(\beta)} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

angular dispersion

pulse-front tilt

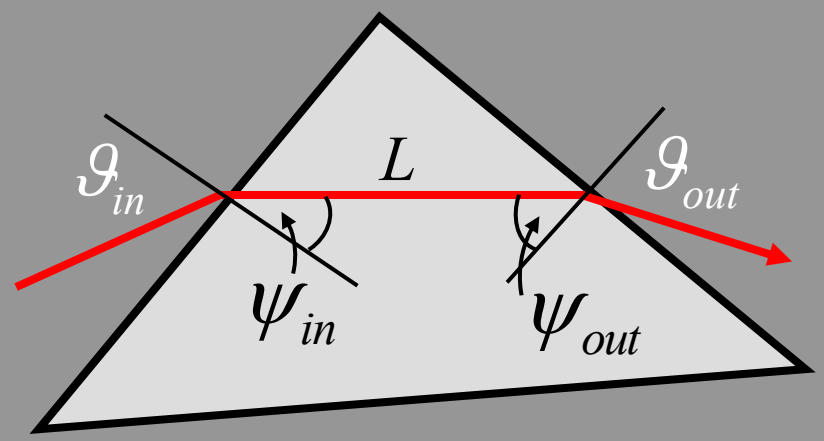
time is independent of angle

no GDD (yet)

where β is the incidence angle, and β' is the diffraction angle (note that Kostenbauder uses different angle definitions in his paper).

The zero elements (E, H, I) become nonzero when propagation follows.

Kostenbauder matrix for a general prism



All new elements are nonzero.

spatial magnification

angular magnification

angular dispersion

spatial chirp

$$\mathscr{K} = \begin{bmatrix} m_{in}m_{out} & Lm_{out}/nm_{in} & 0 & -2\pi\frac{dn}{d\omega}Lm_{out}\tan\psi_{in}/n \\ 0 & 1/m_{in}m_{out} & 0 & -2\pi\frac{dn}{d\omega}(\tan\psi_{in}+\tan\psi_{out})/m_{out} \\ -2\pi\frac{dn}{d\omega}m_{in}(\tan\psi_{in}+\tan\psi_{out})/\lambda_0 & -2\pi\frac{dn}{d\omega}L\tan\psi_{out}/nm_{in}\lambda_0 & 1 & 4\pi^2\left(\frac{dn}{d\omega}\right)^2 L\tan\psi_{in}\tan\psi_{out}/n\lambda_0+2\pi Lk'' \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

pulse-front tilt

time vs. angle

GDD

where $k'' \equiv \left.\frac{d^2k}{d\omega^2}\right|_{\omega_0}$ is the GVD, $m_{in} \equiv \frac{\cos\psi_{in}}{\cos\vartheta_{in}}$ and $m_{out} \equiv \frac{\cos\vartheta_{out}}{\cos\psi_{out}}$

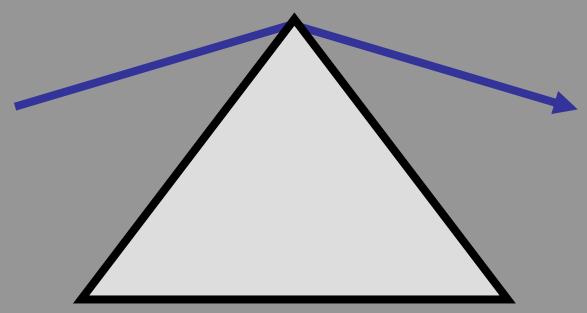
Kostenbauder matrix for a Brewster prism

If the beam passes through the apex of the prism:

$$L \to 0$$

(this simplifies the calculation a lot!)

Brewster angle incidence and exit



$$\psi_{in} = \psi_{out} \equiv \psi$$

$$m_{in} = 1/m_{out}$$

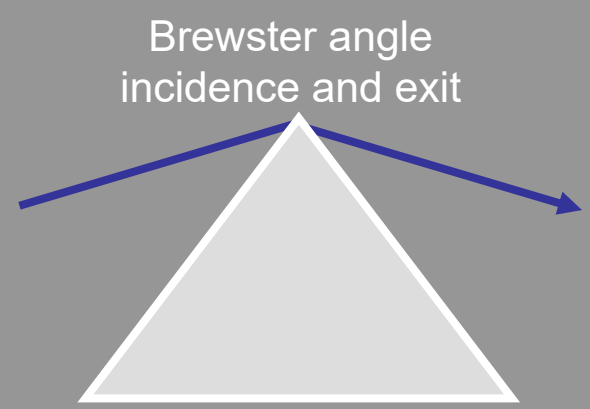
$$\mathscr{K}_{prism} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \pm\mathscr{D} \\ \pm\mathscr{D}/\lambda_0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Use + if the prism is oriented as above; use – if it's inverted.

Just angular dispersion and pulse-front tilt. No GDD etc.

where $\mathscr{D} = -4\pi(dn/d\omega)\tan\psi / m_{out} = -4\pi m_{in}(dn/d\omega)\tan\psi$

Using the Kostenbauder matrix for a Brewster prism



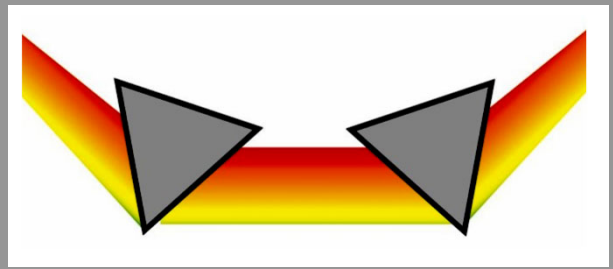
This matrix takes into account all that we need to know for pulse compression.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \pm\mathscr{D} \\ \pm\mathscr{D}/\lambda_0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \\ t_{in} \\ \nu_{in} \end{bmatrix} = \begin{bmatrix} x_{in} \\ \theta_{in} \pm \mathscr{D}\nu_{in} \\ t_{in} \pm x_{in}\mathscr{D}/\lambda_0 \\ \nu_{in} \end{bmatrix}$$

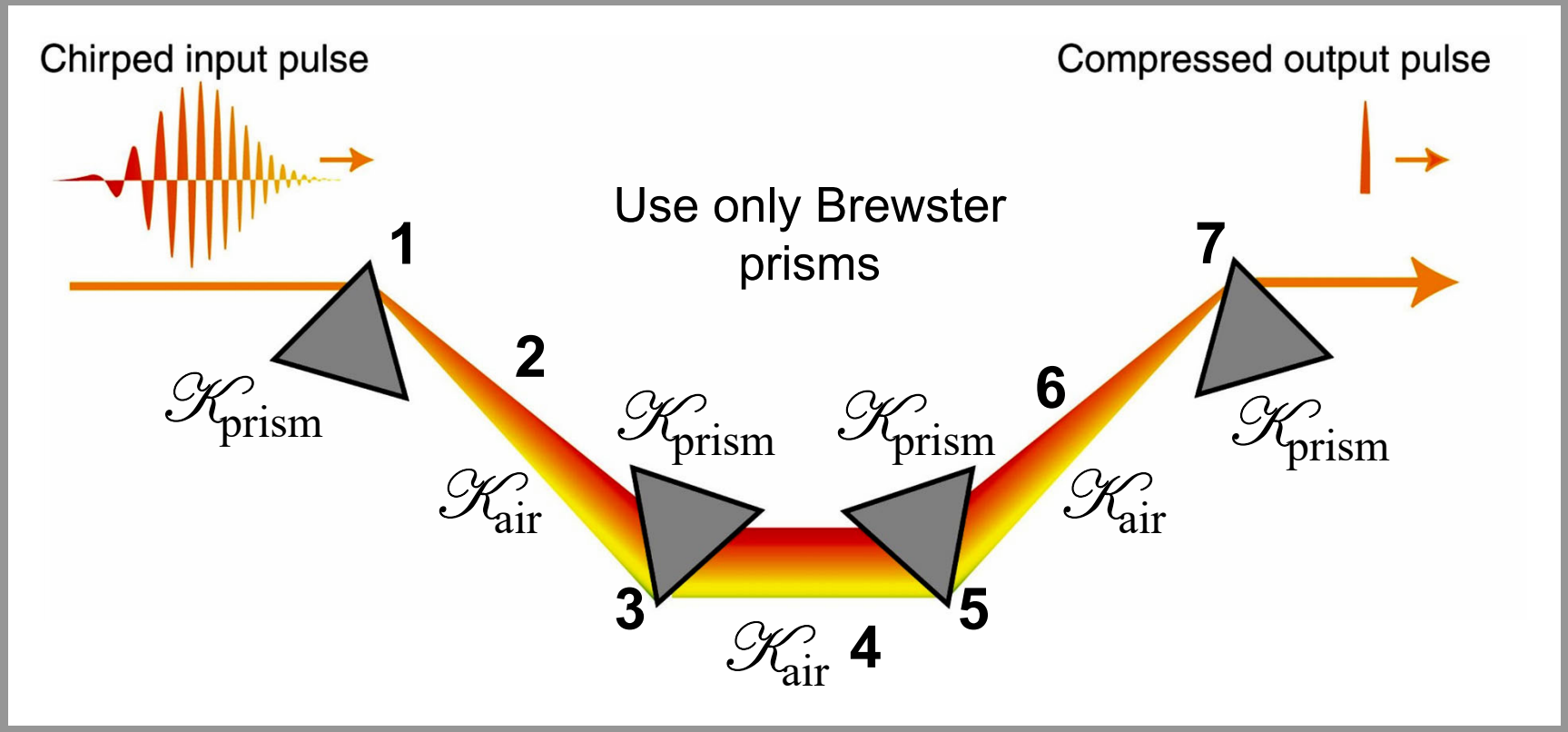
Dispersion changes the beam angle.

Pulse-front tilt yields GDD.

When the pulse reaches the two inverted prisms, this effect becomes very important, yielding longer group delay for longer wavelengths ($\mathscr{D} < 0$; and use the minus sign for inverted prisms).



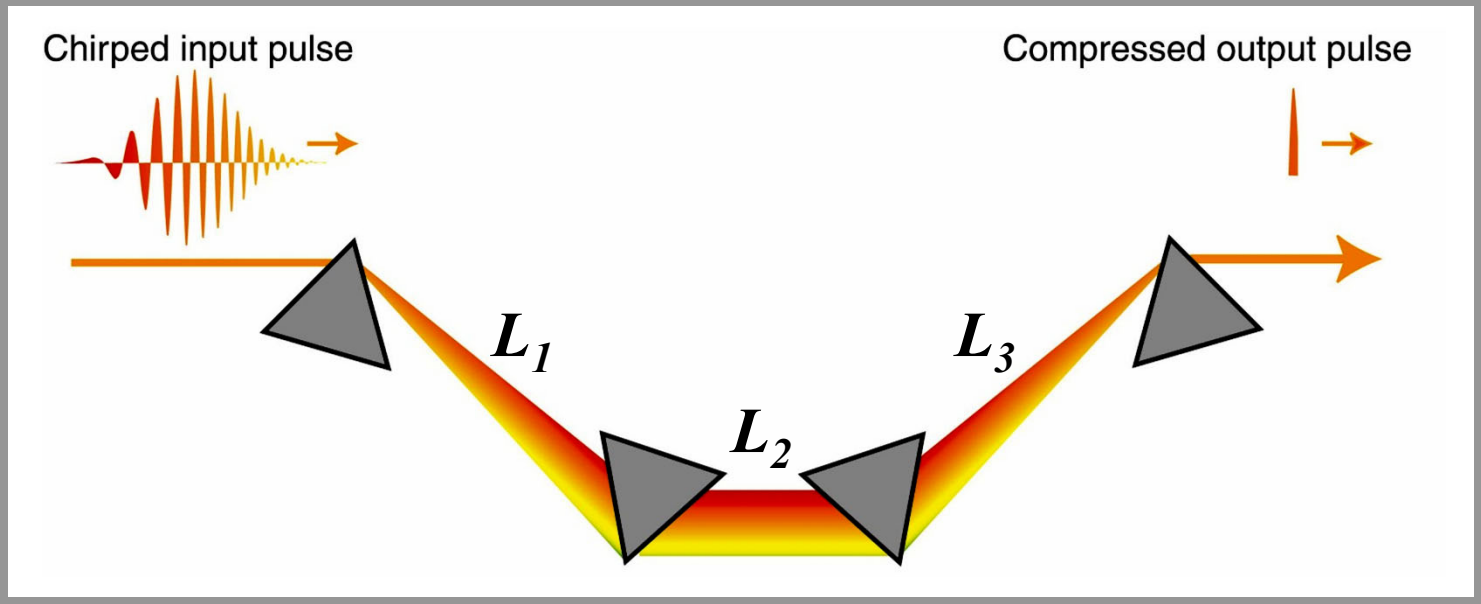
Modeling a prism pulse compressor using Kostenbauder matrices



$$\mathcal{K} = \mathcal{K}_7\,\mathcal{K}_6\,\mathcal{K}_5\,\mathcal{K}_4\,\mathcal{K}_3\,\mathcal{K}_2\,\mathcal{K}_1$$

Free space propagation in a pulse compressor

There are three distances in this problem.



$$\mathcal{K}_{free-space} = \begin{bmatrix} 1 & L_i & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

n = 1 in free space

K-matrix for a prism pulse compressor

$$\mathscr{K} = \mathscr{K}_7 \mathscr{K}_6 \mathscr{K}_5 \mathscr{K}_4 \mathscr{K}_3 \mathscr{K}_2 \mathscr{K}_1$$

Spatial chirp unless $L_1 = L_3$.

$$\mathscr{K} = \begin{bmatrix} 1 & L_1 + L_2 + L_3 & 0 & \mathscr{D}(L_1 - L_3) \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{\mathscr{D}}{\lambda_0}(L_1 - L_3) & 1 & \boxed{-\frac{\mathscr{D}^2}{\lambda_0}(L_1 + L_3)} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

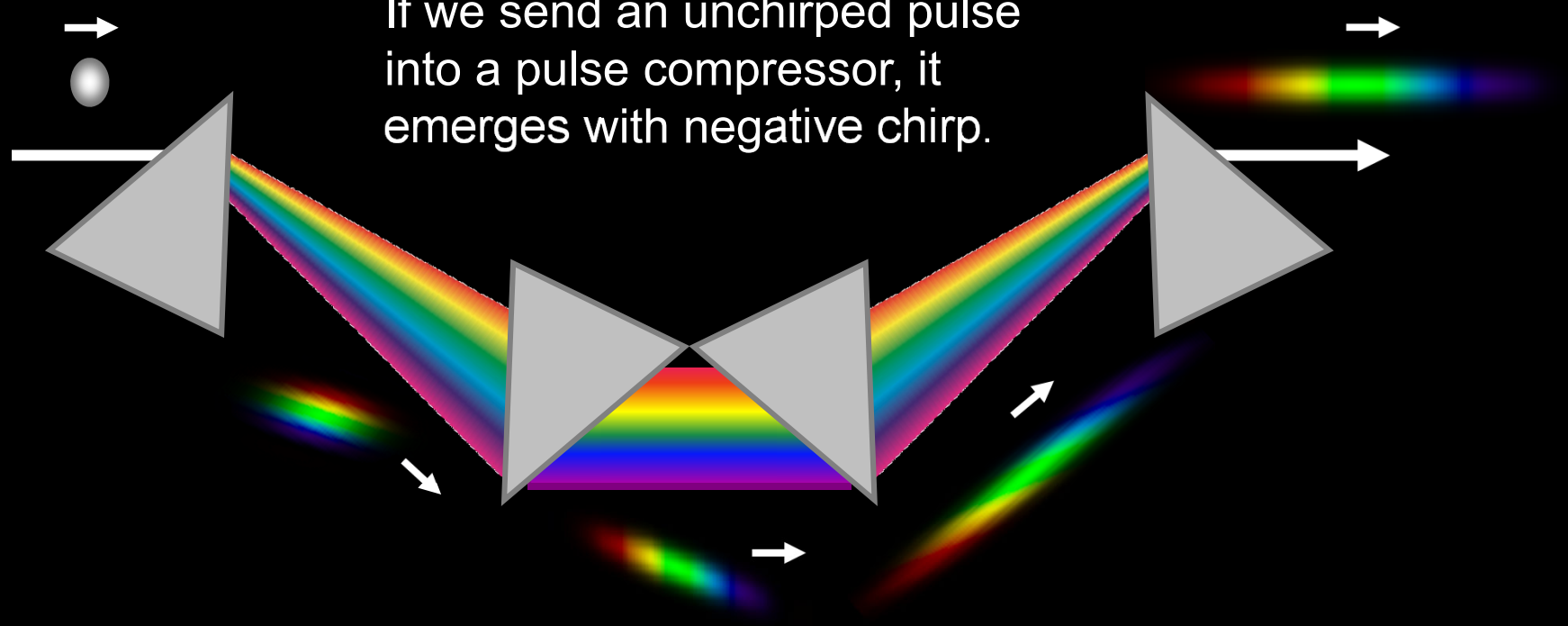
Time vs. angle unless $L_1 = L_3$.

Negative GDD!

The GDD is negative and can be tuned by changing the amount of extra glass in the beam (which we haven't included yet, but which is easy).

What does the pulse look like inside a pulse compressor?

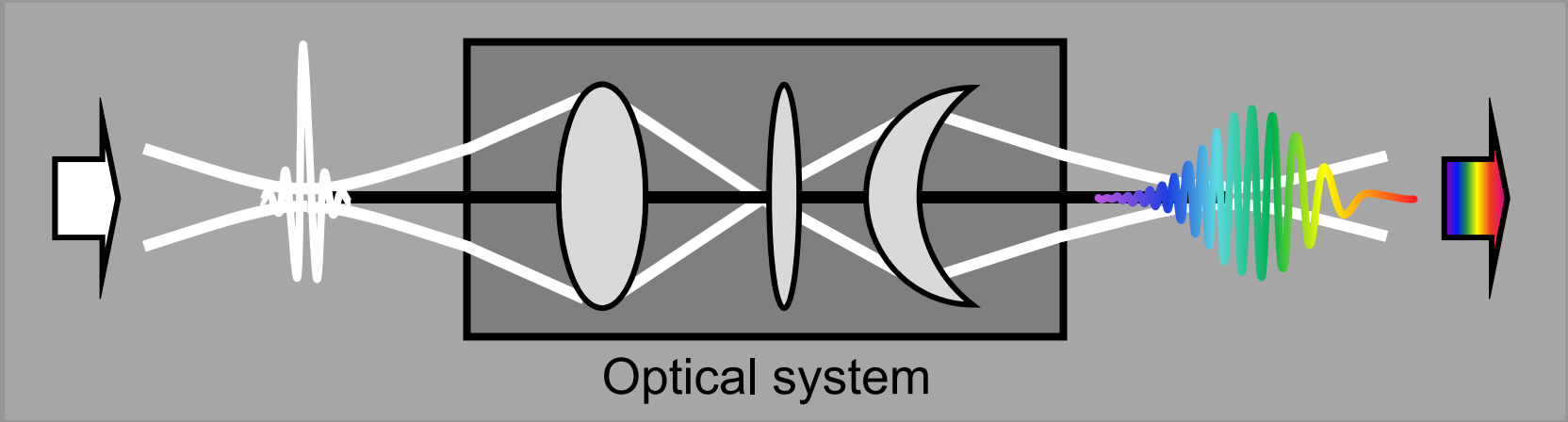
If we send an unchirped pulse into a pulse compressor, it emerges with negative chirp.



Note all the spatio-temporal distortions.

Coupling spatial and temporal propagation

To follow beams that are Gaussian in both **space** and **time**:



We can propagate Gaussian beams in space because they're quadratic in x and y:

$$E(x,y) \propto \exp[-(1/w^2 + i\pi/\lambda R)(x^2 + y^2)]$$

A Gaussian pulse is quadratic in time. And the real and imaginary parts also have important meanings (pulse length and chirp):

$$E(t) \propto \exp[-(1/\tau_G^2 + i\beta)t^2]$$

The complex-Q matrix

We define the complex Q-matrix analogously to the complex-q parameter, so the space *and* time dependence of the pulse can be written:

$$E(x,t) \propto \exp\left\{-i\frac{\pi}{\lambda}[x,-t]\,Q^{-1}\begin{bmatrix} x \\ t \end{bmatrix}\right\} = \exp\left\{-i\frac{\pi}{\lambda}[x,-t]\begin{bmatrix} Q_{11}^{-1} & Q_{12}^{-1} \\ Q_{21}^{-1} & Q_{22}^{-1} \end{bmatrix}\begin{bmatrix} x \\ t \end{bmatrix}\right\}$$

$$= \exp\left\{-i\frac{\pi}{\lambda}[x,-t]\begin{bmatrix} Q_{11}^{-1}x + Q_{12}^{-1}t \\ Q_{21}^{-1}x + Q_{22}^{-1}t \end{bmatrix}\right\}$$

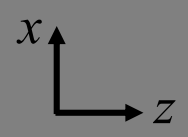
Note: $Q_{21}^{-1} = -Q_{12}^{-1}$

$$= \exp\left\{-i\frac{\pi}{\lambda}[Q_{11}^{-1}x^2 + 2Q_{12}^{-1}xt - Q_{22}^{-1}t^2]\right\}$$

These complex matrix elements contain all the parameters of beams/pulses that are Gaussian in space and time. And they can be propagated using the K-matrices.

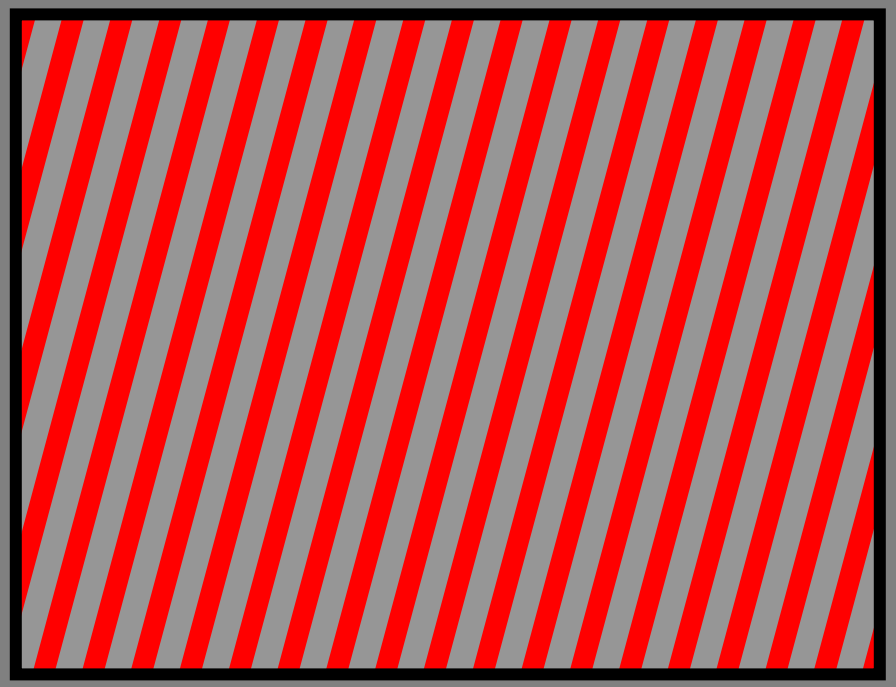
An example: spatio-temporal phase distortions

The imaginary parts of the spatio-temporal distortions are not well-known, but they are interesting. Consider the imaginary part of $\tilde{Q}_{xt}$:



The electric field vs. x and z.

Red = +
Gray = -



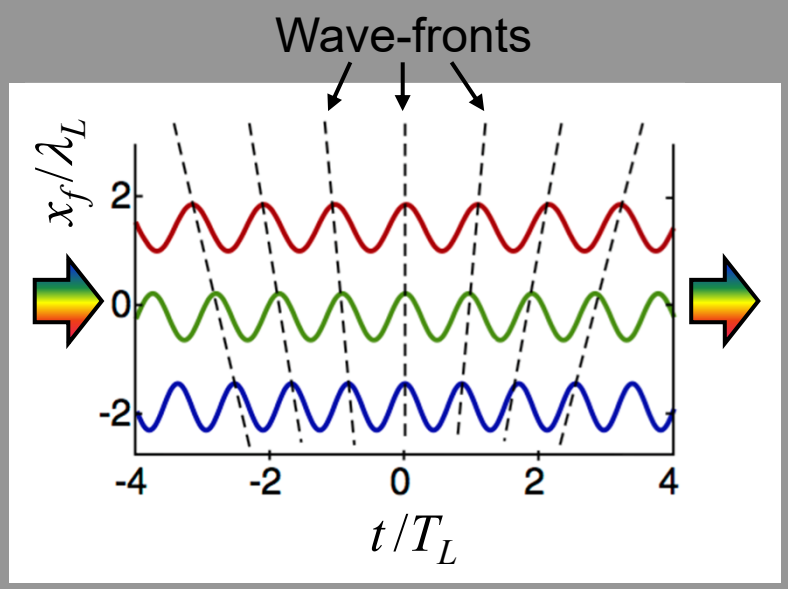
Average wave-propagation direction

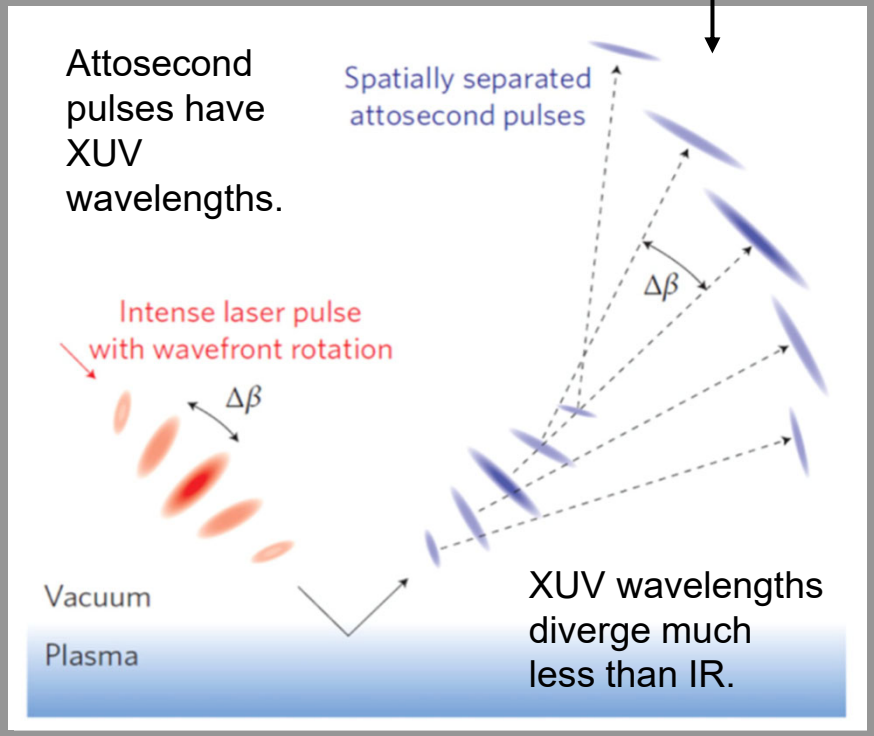
This effect is called **Wave-Front Rotation**.

Wave-front Rotation and the Attosecond Lighthouse

Spatial chirp causes wave-front rotation.

Wave-front rotation leads to the lighthouse effect.





The ultrafast lighthouse effect, generated in this manner, is now used to separate out one atto-second pulse from a train of them.

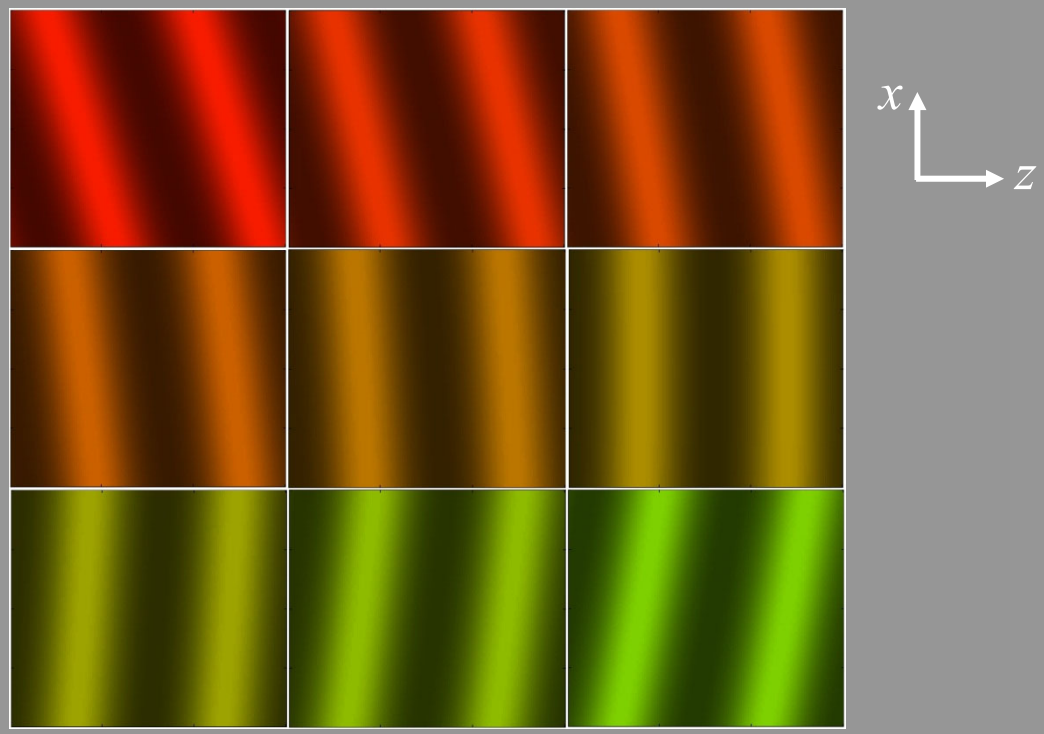
Because attosecond pulses have smaller divergence angles, all that's then required is an aperture.

Quéré and coworkers

Dispersion of spatio-temporal phase distortions

The dynamic rotation of the wave front need not be the same for all frequencies.

Plots of the electric field vs. x and z for different colors.



This distortion is called **Wave-Front-Tilt Dispersion**.

Summary of First-Order Spatio-Temporal Distortions

In all, there are eight first-order spatio-temporal distortions, four in amplitude (intensity) and four in phase.

Table 1. Intensity and phase couplings in different domains.

Domain	Intensity coupling	Phase coupling
(x, t)	Pulse-front tilt	Wavefront rotation
(x, ω)	Spatial chirp	Wavefront tilt dispersion
(k, ω)	Angular dispersion	Angular spectral chirp
(k, t)	Time versus angle	Angular temporal chirp

S. Akturk, X. Gu, P. Bowlan, and R. Trebino. "Spatio-temporal couplings in ultrashort laser pulses." J. Optics **12: 093001** (2010).