12. Reflection and Refraction

Wave fronts

Spherical waves are solutions to Maxwell’s equations

Molecules scatter spherical waves

Spherical waves can add up to plane waves

Reflection and refraction from surfaces

Snell’s Law

Mirages and prisms
Constructive vs. destructive interference; Coherent vs. incoherent interference

Waves that combine in phase add up to a higher irradiance.

Waves that combine 180° out of phase cancel out and yield zero irradiance.

Waves that combine with lots of different phases nearly cancel out and yield low irradiance.

Often, constructive interference will occur in only one direction in space.
Why can’t we see a light beam?

Unless the light beam is propagating right into your eye or is scattered into it, you won’t see it. This is true for any light beam.

This eye sees no light

This eye is blinded…

To photograph light beams, you need something for the beam to scatter off…
Incoherent interference

Incoherent scattering produces a weak, but non-zero intensity. This is because all of the wavefronts have random phases relative to one another.

Because the scattered radiation is relatively weak, the most dramatic effects are often related to the frequency-dependence of the scattering.
Coherent interference

Incident wave

Coherent scattering produces a wave with higher intensity, since it involves constructive interference. All of the scattered wavelets are in phase.

It is often possible to define a surface in space, connecting all the points at which each wavelet reaches a maximum.

This surface is known as a “wave front”.

In this illustration, the wave front is planar.
Wave-fronts

At a given time, a wave's "wave-fronts" are the surfaces where the wave has its maxima: in other words, a **surface of constant phase**

Wave-fronts are helpful for drawing pictures of the shape of a wave.

A wave's wave-fronts are equally spaced, one wavelength apart.

They're perpendicular to the propagation direction, and they propagate at the speed of the wave.
Plane wave: a wave with a planar wave front

By now we are very familiar with this solution to the wave equation:

$$E(\vec{r}, t) = E_0 e^{j(\vec{k} \cdot \vec{r} - \omega t + \theta)}$$

To find out what the wavefronts of this wave look like, we set the phase equal to a constant. For example, a wave traveling in the $z$ direction:

$$kz - \omega t + \theta = \text{constant}$$

At a particular instant of time, $t = \text{constant}$, this translates to: $z = \text{constant}$. That is the definition of a plane perpendicular to the $z$ axis.

$$E_0 e^{j(\vec{k} \cdot \vec{r} - \omega t + \theta)}$$ is a plane wave.

defined by the mathematical condition: $\vec{k} \cdot \vec{r} = \text{constant}$
Wave fronts don’t have to be planar

Consider a single molecule or point particle, emitting radiation at frequency $\omega$ in all directions. This produces a *spherical wave*.

A spherical wave has spherical wave-fronts.

Note: this does not mean that the intensity is the same in every direction. For a spherical wave, the phase is uniform on a sphere, but the amplitude *need not be the same* everywhere on the sphere.

Unlike a plane wave, whose amplitude remains constant as it propagates, a spherical wave weakens. The irradiance in any direction drops in proportion to $1/r^2$, where $r$ is the distance from the emission point.
To deal with spherical waves, we use spherical coordinates.

This point is both \((x, y, z)\) and \((r, \theta, \phi)\)

\[
r = \sqrt{x^2 + y^2 + z^2}
\]

\[
\theta = \arccos\left(\frac{z}{r}\right)
\]

\[
\phi = \arctan\left(\frac{y}{x}\right)
\]
Wave equation in spherical coordinates

This is really just three independent wave equations, one each for the x-, y-, and z-components of E.

1st derivative:
\[
\frac{\partial \vec{E}}{\partial x} = \frac{\partial \vec{E}}{\partial r} \cdot \frac{\partial r}{\partial x} = \frac{\partial \vec{E}}{\partial r} \cdot \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2}
\]

2nd derivative:
\[
\frac{\partial^2 \vec{E}}{\partial x^2} = \frac{\partial \vec{E}}{\partial r} \cdot \frac{1}{r} + \frac{\partial^2 \vec{E}}{\partial r^2} \cdot \frac{x^2}{r^2} - \frac{\partial \vec{E}}{\partial r} \cdot \frac{\partial}{\partial x} \frac{x^2}{r^3} = \frac{\partial^2 \vec{E}}{\partial r^2} \cdot \frac{x^2}{r^2} + \frac{\partial \vec{E}}{\partial r} \left( \frac{r^2 - x^2}{r^3} \right)
\]

and similar expressions for the y and z derivatives.

Put them all together to find the wave equation in spherical coordinates:
\[
\frac{\partial^2 \vec{E}}{\partial r^2} + \frac{2}{r} \frac{\partial \vec{E}}{\partial r} = \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}
\]
Spherical wave solutions to the wave equation

This wave equation can be recast in the following form:

$$\mu \varepsilon \frac{\partial^2}{\partial t^2} \left( r\vec{E} \right) = \frac{\partial^2}{\partial r^2} \left( r\vec{E} \right)$$

which is exactly the same form as the 1D wave equation with which we are very familiar. Thus, if the product $rE$ has the same form as the known solutions to the 1D equation:

$$r\vec{E} (r, t) = \vec{E}_0 e^{j(kr - \omega t + \theta)}$$

then we can immediately determine the solution to the 3D equation:

$$\vec{E} (r, t) = \frac{\vec{E}_0}{r} e^{j(kr - \omega t + \theta)}$$

Irradiance: \[ I \propto \left| \vec{E} \right|^2 \propto \frac{1}{r^2} \]

Surfaces of constant phase are spheres!

Decreases with increasing distance from the emission point.
A plane wave impinging on a molecule or particle scatters into a spherical wave.

Huygens’ Principle says that waves propagate as if each point on a wave-front emits a spherical wave (whether or not there’s a molecule or particle involved).

Scattering from an individual molecule or particle is weak, but many such scatterings can add up—especially if one can arrange it so that interference is constructive.
Most wave fronts are complicated

Plane waves and spherical waves are important examples. But most wavefronts have much more complicated shapes.

- A source of spherical waves (upper left corner)
- A region of lower refractive index (blue area across the middle)

Light reflected from a non-smooth object has a curved and distorted wavefront in the shape of the object.

Any arbitrarily complicated wavefront can be written as a sum of plane waves or of spherical waves.
Spherical waves can combine to form plane waves.

A plane wave impinging on a surface will produce a reflected plane wave because all the spherical wavelets interfere constructively along a flat surface.
A note on the way we define angles

When considering the problem of a light beam approaching a surface, we always define the angles relative to the surface normal, *not* to the plane of the surface.
What happens to light when it encounters a surface?

It is scattered by the surface molecules. But a beam can remain a beam if there is a direction for which constructive interference occurs.

Constructive interference occurs for a reflected beam if the angle of incidence = the angle of reflection.

For a transmitted beam, the result is a bit more complicated. The angle of the beam changes. This is known as ‘refraction’.
Snell's Law

Snell’s Law was first described by Ibn Sahl in 984 AD.

Unaware of Sahl, Snell derived it in 1621. Descartes worked it out at about the same time. In France, it is known as the Snell-Decartes Law.

One interesting description comes from Feynman…

shortest distance ≠ shortest time!
One proof of Snell's Law

The time it takes for a wave to propagate a certain number of wavelengths is independent of the medium, because both the speed and the wavelength change by the same factor.

\[ AD = BD/\sin(\theta_i) \quad \text{and} \quad AD = AE/\sin(\theta_t) \]

So: \( BD/\sin(\theta_i) = AE/\sin(\theta_t) \)

But: \( BD = v_i \Delta t = (c_0/n_i) \Delta t \)

& \( AE = v_t \Delta t = (c_0/n_t) \Delta t \)

So: \( (c_0/n_i) \Delta t / \sin(\theta_i) = (c_0/n_t) \Delta t / \sin(\theta_t) \)

Or: \( n_i \sin(\theta_i) = n_t \sin(\theta_t) \)

Snell’s Law java applet: http://www.ps.missouri.edu/rickspage/refract/refraction.html
Snell's Law explains many everyday effects
Snell's Law explains why things look bent when they stick out of a pool of water
Refraction makes lenses possible

Because refraction bends rays toward the thickest part of a lens, parallel rays incident on a convex lens converge.

Technically we need to check that the interference is constructive, too.

Parallel input rays

The distance at which parallel rays converge is the lens focal length, $f$.

Dispersion means that the focal length of a lens can depend on wavelength, $f(\lambda)$. This is called ‘chromatic aberration’. Lens designers go to great lengths to compensate for dispersion.
Snell's Law explains mirages

Snell’s law helps us to understand the origin of mirages...

The refractive index varies with density (and hence temperature)

A mirage appears off the shore of Penglai City in eastern China's Shandong Province on Sunday, May 7, 2005.
Snell's Law explains why the sun flattens when it sets
Snell’s Law explains why stars twinkle

The atmosphere has non-uniform temperature and hence non-uniform refractive index.

And these regions move about in time.

As the air masses move about, the amount of light reaching our eyes from the star varies.
Prisms disperse light.

Because the refractive index depends on wavelength, the refraction angle also depends on wavelength.

Because $n$ generally decreases with wavelength ($dn/d\lambda < 0$), smaller wavelengths experience greater refraction angles.

Differentiating implicitly with respect to $\lambda$:

$$\cos(\theta_t) \frac{d\theta_t}{d\lambda} = \frac{dn}{d\lambda} \sin(\theta_i)$$

We obtain the “prism dispersion:”

$$\frac{d\theta_t}{d\lambda} = \frac{dn}{d\lambda} \frac{\sin(\theta_i)}{\cos(\theta_t)}$$

Assuming $n_{air} = 1$:

$$\sin(\theta_t) = n(\lambda) \sin(\theta_i)$$

If the prism were made from a material for which $dn/d\lambda = 0$, then that prism would make no rainbow!