13. Fresnel's Equations for Reflection and Transmission

Incident, transmitted, and reflected beams

Boundary conditions: tangential fields are continuous

Reflection and transmission coefficients

The "Fresnel Equations"

Brewster's Angle

Total internal reflection

Power reflectance and transmittance

Augustin Fresnel
1788-1827
Posing the problem

What happens when light, propagating in a uniform medium, encounters a smooth interface which is the boundary of another medium (with a different refractive index)?

First we need to define some terminology.
Definitions: Plane of Incidence and plane of the interface

Plane of incidence (in this illustration, the yz plane) is the plane that contains the incident and reflected k-vectors.

Plane of the interface (y=0, the xz plane) is the plane that defines the interface between the two materials.
Definitions: “S” and “P” polarizations

A key question: which way is the E-field pointing? There are two distinct possibilities.

1. “S” polarization is the perpendicular polarization, and it sticks up out of the plane of incidence.

Here, the plane of incidence (z=0) is the plane of the diagram.

2. “P” polarization is the parallel polarization, and it lies parallel to the plane of incidence.

The plane of the interface (y=0) is perpendicular to this page.
Definitions: “S” and “P” polarizations

Note that this is a different use of the word “polarization” from the way we’ve used it earlier in this class.

The amount of reflected (and transmitted) light is different for the two different incident polarizations.
Fresnel Equations—Perpendicular E field

Augustin Fresnel was the first to do this calculation (1820’s).

We treat the case of s-polarization first:

Beam geometry for light with its electric field sticking up out of the plane of incidence (i.e., out of the page).

This red dashed line represents the xz plane (y = 0), which is the plane of the interface.
Boundary Condition for the Electric Field at an Interface: \textit{s} polarization

The Tangential Electric Field is Continuous

In other words, the component of the E-field that lies in the xz plane is continuous as you move across the plane of the interface.

Here, all E-fields are in the z-direction, which is in the plane of the interface.

So: \( E_i(y = 0) + E_r(y = 0) = E_t(y = 0) \)  
(We’re not explicitly writing the x, z, and t dependence, but it is still there.)
Boundary Condition for the Magnetic Field at an Interface: s polarization

The Tangential Magnetic Field* is Continuous

In other words, the total B-field in the plane of the interface is continuous.

Here, all B-fields are in the xy-plane, so we take the x-components:

\[-B_i(y = 0) \cos \theta_i + B_r(y = 0) \cos \theta_r = -B_t(y = 0) \cos \theta_t\]

*It's really the tangential $B/\mu$, but we're assuming $\mu_i = \mu_t = \mu_0$
Reflection and Transmission for Perpendicularly Polarized Light

Ignoring the rapidly varying parts of the light wave and keeping only the complex amplitudes:

\[ E_{0i} + E_{0r} = E_{0t} \]

\[ -B_{0i} \cos(\theta_i) + B_{0r} \cos(\theta_r) = -B_{0t} \cos(\theta_i) \]

But \( B = E/(c_0/n) = nE/c_0 \) and \( \theta_i = \theta_r \).

Substituting into the second equation:

\[ n_i (E_{0r} - E_{0i}) \cos(\theta_i) = -n_t E_{0i} \cos(\theta_i) \]

Substituting for \( E_{0t} \) using \( E_{0i} + E_{0r} = E_{0t} \):

\[ n_i (E_{0r} - E_{0i}) \cos(\theta_i) = -n_t (E_{0r} + E_{0i}) \cos(\theta_i) \]
Reflection & Transmission Coefficients for Perpendicularly Polarized Light

Rearranging \( n_i(E_{0r} - E_{0i})\cos(\theta_i) = -n_t(E_{0r} + E_{0i})\cos(\theta_t) \) yields:
\[
E_{0r} \left[ n_i \cos(\theta_i) + n_t \cos(\theta_t) \right] = E_{0i} \left[ n_i \cos(\theta_i) - n_t \cos(\theta_t) \right]
\]

Solving for \( E_{0r} / E_{0i} \) yields the reflection coefficient:
\[
r_\perp = E_{0r} / E_{0i} = \left[ n_i \cos(\theta_i) - n_t \cos(\theta_t) \right] / \left[ n_i \cos(\theta_i) + n_t \cos(\theta_t) \right]
\]

Analogously, the transmission coefficient, \( E_{0t} / E_{0i} \), is
\[
t_\perp = E_{0t} / E_{0i} = 2n_i \cos(\theta_i) / \left[ n_i \cos(\theta_i) + n_t \cos(\theta_t) \right]
\]

These equations are called the Fresnel Equations for perpendicularly polarized (s-polarized) light.
Fresnel Equations—Parallel electric field

Now, the case of P polarization:

Note that the reflected magnetic field must point into the screen to achieve $\vec{E} \times \vec{B} \propto \vec{k}$ for the reflected wave. The x with a circle around it means “into the screen.”

Note that Hecht uses a different notation for the reflected field, which is confusing! Ours is better!

This leads to a difference in the signs of some equations...
Reflection & Transmission Coefficients for Parallel Polarized Light

For parallel polarized light, \( B_{0i} - B_{0r} = B_{0t} \)

and

\[
E_{0i}\cos(\theta_i) + E_{0r}\cos(\theta_r) = E_{0t}\cos(\theta_t)
\]

Solving for \( E_{0r} / E_{0i} \) yields the reflection coefficient, \( r_{||} \):

\[
r_{||} = E_{0r} / E_{0i} = [n_i \cos(\theta_t) - n_t \cos(\theta_i)] / [n_i \cos(\theta_t) + n_t \cos(\theta_i)]
\]

Analogously, the transmission coefficient, \( t_{||} = E_{0t} / E_{0i} \), is

\[
t_{||} = E_{0t} / E_{0i} = 2n_i \cos(\theta_i) / [n_i \cos(\theta_t) + n_t \cos(\theta_i)]
\]

These equations are called the Fresnel Equations for parallel polarized (p-polarized) light.
To summarize...

s-polarized light:

\[ r_\perp = \frac{n_i \cos(\theta_i) - n_t \cos(\theta_t)}{n_i \cos(\theta_i) + n_t \cos(\theta_t)} \]
\[ t_\perp = \frac{2n_i \cos(\theta_i)}{n_i \cos(\theta_i) + n_t \cos(\theta_t)} \]

p-polarized light:

\[ r_\parallel = \frac{n_i \cos(\theta_i) - n_t \cos(\theta_t)}{n_i \cos(\theta_i) + n_t \cos(\theta_t)} \]
\[ t_\parallel = \frac{2n_i \cos(\theta_i)}{n_i \cos(\theta_i) + n_t \cos(\theta_t)} \]

And, for both polarizations:

\[ n_i \sin(\theta_i) = n_t \sin(\theta_t) \]
Reflection Coefficients for an Air-to-Glass Interface

- The two polarizations are indistinguishable at $\theta = 0^\circ$
- Total reflection at $\theta = 90^\circ$ for both polarizations.
- Zero reflection for parallel polarization at: “Brewster's angle”

The value of this angle depends on the value of the ratio $n_i/n_t$:

$$\theta_{Brewster} = \tan^{-1}\left(\frac{n_t}{n_i}\right)$$

For air to glass ($n_{glass} = 1.5$), this value is $56.3^\circ$.

Sir David Brewster
1781 - 1868
Reflection Coefficients for a Glass-to-Air Interface

\[ n_{\text{glass}} > n_{\text{air}} \]

- Total internal reflection above the "critical angle"

\[ \theta_{\text{crit}} \equiv \sin^{-1}(n_i/n_t) \]

\[ \approx 41.8^\circ \text{ for glass-to-air} \]

(The sine in Snell's Law can't be greater than one!)
The transmission coefficient does not always have a value between zero and one.

For example, for the case of glass-to-air (1.5 to 1): if $\theta_i$ is equal to the critical angle, then $\theta_t = 90^\circ$, and we have:

$$t_\perp = \frac{2n_i \cos(\theta_i)}{n_i \cos(\theta_i) + n_t \cos(\theta_t)} = 2$$

$$t_\parallel = \frac{2n_i \cos(\theta_i)}{n_i \cos(\theta_t) + n_t \cos(\theta_i)} = \frac{2n_i}{n_t} = 3$$

This may seem counter-intuitive, but remember that E fields are not conserved. Energy is conserved!
Reflectance \((R)\)

\[ R \equiv \frac{\text{Reflected Power}}{\text{Incident Power}} = \frac{I_rA_r}{I_iA_i} \]

Because the angle of incidence = the angle of reflection, the beam’s area doesn’t change on reflection.

Also, \(n\) is the same for both incident and reflected beams.

So: \[ R = r^2 \]

since \[ \frac{|E_{0r}|^2}{|E_{0i}|^2} = r^2 \]
Transmittance ($T$)

$T \equiv \frac{\text{Transmitted Power}}{\text{Incident Power}} = \frac{I_t A_t}{I_i A_i}$

If the beam has width $w_i$:

$$\frac{A_t}{A_i} = \frac{w_t}{w_i} = \frac{\cos(\theta_t)}{\cos(\theta_i)}$$

The beam expands (or contracts) in one dimension on refraction.

$$T = \frac{I_t A_t}{I_i A_i} = \left(\frac{n_t \frac{\varepsilon_0 c_0}{2}}{n_i \frac{\varepsilon_0 c_0}{2}}\right) \left|\frac{E_{0t}}{E_{0i}}\right|^2 \left[\frac{w_t}{w_i}\right] = \frac{n_t |E_{0t}|^2 w_t}{n_i |E_{0i}|^2 w_i} = \frac{n_t w_t}{n_i w_i} t^2 \quad \text{since} \quad \frac{|E_{0t}|^2}{|E_{0i}|^2} = t^2$$

$$\Rightarrow \quad T = \left[\frac{\left(n_t \cos(\theta_t)\right)}{\left(n_i \cos(\theta_i)\right)}\right] t^2$$
Reflectance and Transmittance for an Air-to-Glass Interface

Note: it is NOT true that: $r + t = 1$.

But: it is ALWAYS true that: $R + T = 1$
Perpendicular polarization

Incidence angle, $\theta_i$

Reflectance and Transmittance for a Glass-to-Air Interface

Parallel polarization

Incidence angle, $\theta_i$

Note that the critical angle is the same for both polarizations.

And still, $R + T = 1$
Reflection at normal incidence, $\theta_i = 0$

When $\theta_i = 0$, the Fresnel equations reduce to:

$$R = \left( \frac{n_t - n_i}{n_t + n_i} \right)^2$$

$$T = \frac{4n_t n_i}{(n_t + n_i)^2}$$

For an air-glass interface ($n_i = 1$ and $n_t = 1.5$),

$$R = 4\% \quad \text{and} \quad T = 96\%$$

The values are the same, whichever direction the light travels, from air to glass or from glass to air.

This 4\% value has big implications for photography.

If it weren’t for “lens flare,” there would be no J. J. Abrams.
Where you’ve seen Fresnel’s Equations in action

Windows look like mirrors at night (when you’re in a brightly lit room).

One-way mirrors (used by police to interrogate bad guys) are just partial reflectors (actually, with a very thin aluminum coating).

Disneyland puts ghouls next to you in the haunted house using partial reflectors (also aluminum-coated one-way mirrors).

Smooth surfaces can produce pretty good mirror-like reflections, even though they are not made of metal.
Optical fibers only work because of total internal reflection.

Many lasers use Brewster’s angle components to avoid reflective losses:

\[ R = 100\% \]

\[ R = 90\% \]