16. Theory of Ultrashort Laser Pulse Generation

Active mode-locking

Passive mode-locking

Build-up of mode-locking: The Landau-Ginzberg Equation

The Nonlinear Schrodinger Equation

Solitons

The burning question: gaussian or sech^2?

After all this talk about Gaussian pulses… what does Ti:sapphire really produce?

\[ e^{-\left(\frac{t}{\tau}\right)^2} \text{ or } \text{sech}^2\left(\frac{t}{\tau}\right) \]
Recall that many frequencies ("modes") oscillate simultaneously in a laser, and when their phases are locked, an ultrashort pulse results.

Mode-locking yields ultrashort pulses

Possible cavity modes

Losses

Laser gain profile

Multiple oscillating cavity modes

Sum of ten modes with the same relative phase

Sum of ten modes w/ random phase

Recall that many frequencies ("modes") oscillate simultaneously in a laser, and when their phases are locked, an ultrashort pulse results.
Mode Locking

**Active Mode Locking**

Insert something into the laser cavity that sinusoidally modulates the amplitude of the pulse.

⇒ mode competition couples each mode to modulation sidebands
⇒ eventually, all the modes are coupled and phase-locked

**Passive Mode Locking**

Insert something into the laser cavity that favors high intensities.

⇒ strong maxima will grow stronger at the expense of weaker ones
⇒ eventually, all of the energy is concentrated in one packet
Active mode-locking: the electro-optic modulator

Applying a voltage to a crystal changes its refractive indices and introduces birefringence.

A few kV can turn a crystal into a half- or quarter-wave plate.

If \( V = 0 \), the pulse polarization doesn’t change.

If \( V = V_p \), the pulse polarization switches to its orthogonal state.

Applying a sinusoidal voltage yields sinusoidal modulation to the beam. An electro-optic modulator can also be used without a polarizer to simply introduce a phase modulation, which works by sinusoidally shifting the modes into and out of the actual cavity modes.
Active mode-locking: the acousto-optic modulator

An acoustic wave induces sinusoidal density, and hence sinusoidal refractive-index, variations in a medium. This will diffract away some of a light wave’s energy.

Pressure, density, and refractive-index variations due to acoustic wave

Acoustic transducer

Input beam

Quartz

Output beam

Diffraction Beam (Loss)

Such diffraction can be quite strong: ~70%. Sinusoidally modulating the acoustic wave amplitude yields sinusoidal modulation of the transmitted beam.
The Modulation Theorem: The Fourier Transform of \( E(t)\cos(\omega_M t) \)

\[
\mathcal{F}\{E(t)\cos(\omega_M t)\} = \int_{-\infty}^{\infty} E(t)\cos(\omega_M t) \exp(-i\omega t) \, dt \\
= \frac{1}{2} \int_{-\infty}^{\infty} E(t) \left[ \exp(i\omega_M t) + \exp(-i\omega_M t) \right] \exp(-i\omega t) \, dt \\
= \frac{1}{2} \int_{-\infty}^{\infty} E(t) \exp(-i[\omega - \omega_M]t) \, dt + \frac{1}{2} \int_{-\infty}^{\infty} E(t) \exp(-i[\omega + \omega_M]t) \, dt \\
\mathcal{F}\{E(t)\cos(\omega_M t)\} = \frac{1}{2} \tilde{E}(-\omega - \omega_M) + \frac{1}{2} \tilde{E}(\omega + \omega_M)
\]

Multiplication by \( \cos(\omega_M t) \) introduces side-bands.

If \( E(t) = \text{sinc}^2(t)\exp(i\omega_0 t) \):

\[
\mathcal{F}\{E(t)\}\quad \mathcal{F}\{E(t)\cos(\omega_M t)\}
\]
Active mode-locking

In the frequency domain, a modulator introduces side-bands of every mode.

For mode-locking, adjust $\omega_M$ so that $\omega_M = \text{mode spacing}$.

This means that:

$$\omega_M = 2\pi / \text{cavity round-trip time} = 2\pi / (2L/c) = \pi c/L$$

Each mode competes for gain with adjacent modes. Most efficient operation is for phases to lock. Result is global phase locking.

N coupled equations: $E_n \Leftrightarrow E_{n+1}, E_{n-1}$
Modeling laser modes and gain

Lasers have a mode spacing:

\[ \omega_M = \frac{2\pi}{T_R} = \frac{\pi c}{L} \]

Let the zero\(^{th}\) mode be at the center of the gain, \(\omega_0\). The \(n^{th}\) mode frequency is then:

\[ \omega_n = \omega_0 + n \omega_M \quad \text{where} \quad n = \ldots, -1, 0, 1, \ldots \]

Let \(a_n\) be the amplitude of the \(n^{th}\) mode and assume a Lorentzian gain profile, \(G(n)\):

\[
G(n) a_n = \left[ 1 + \frac{g}{1 + (n\omega_M)^2 / \Omega_g^2} \right] a_n
\]

\[
\approx \left\{ 1 + g \left[ 1 - \frac{(n\omega_M)^2}{\Omega_g^2} \right] \right\} a_n
\]
Modeling an amplitude modulator

An amplitude modulator uses the electro-optic or acousto-optic effect to deliberately cause losses at the laser round-trip frequency, $\omega_M$.

A modulator multiplies the laser light (i.e., each mode) by $M[1-\cos(\omega_M t)]$

$$M \left[ 1 - \cos(\omega_M t) \right] a_n e^{i(\omega_o+n\omega_M) t} = M \left[ -\frac{1}{2} \exp(-i\omega_M t) + 1 - \frac{1}{2} \exp(i\omega_M t) \right] a_n e^{i(\omega_o+n\omega_M) t}$$

$$= -Ma_n e^{i\omega_o t} \left[ \frac{1}{2} e^{i(n-1)\omega_M t} - e^{i\omega_M t} + \frac{1}{2} e^{i(n+1)\omega_M t} \right]$$

Notice that this spreads the energy from the $n^{th}$ to the $(n+1)^{st}$ and $(n-1)^{st}$ modes. Including the passive loss, $\ell$, we can write this as:

$$a_n^{(k+1)} = a_n^{(k)} + g \left( 1 - \frac{(n\omega_M)^2}{\Omega_g^2} \right) a_n^{(k)} - \ell a_n^{(k)} + \frac{M}{2} \left( a_{n+1}^{(k)} - 2a_n^{(k)} + a_{n-1}^{(k)} \right)$$

where the superscript indicates the $k^{th}$ round trip.
Solve for the steady-state solution

\[
a^{(k+1)}_n = a^{(k)}_n + g \left( 1 - \frac{(n\omega_M)^2}{\Omega_g^2} \right) a^{(k)}_n - \ell a^{(k)}_n + \frac{M}{2} \left( a^{(k)}_{n+1} - 2a^{(k)}_n + a^{(k)}_{n-1} \right)
\]

In steady state, \( a^{(k+1)}_n = a^{(k)}_n \)

Also, the finite difference becomes a second derivative when the modes are many and closely spaced:

\[
a^{(k)}_{n+1} - 2a^{(k)}_n + a^{(k)}_{n-1} \rightarrow \omega_M^2 \frac{d^2}{d\omega^2} a(\omega)
\]

where, in this continuous limit,

\[
a^{(k)}_n \rightarrow a(\omega)
\]

where:

\[
\omega = n \omega_M
\]

Thus we have:

\[
0 = \left[ g \left( 1 - \frac{\omega^2}{\Omega_g^2} \right) - \ell + \frac{M \omega_M^2}{2} \frac{d^2}{d\omega^2} \right] a(\omega)
\]
Solve for the steady-state solution

\[ 0 = \left[ g \left( 1 - \frac{\omega^2}{\Omega_g^2} \right) - \ell + \frac{M \omega_M^2}{2} \frac{d^2}{d\omega^2} \right] a(\omega) \]

This differential equation has the solution:

\[ a(\omega) = H_\nu(\omega \tau) e^{-\omega^2 \tau^2 / 2} \quad \text{(Hermite Gaussians)} \]

with the constraints:

\[ \frac{1}{\tau^4} = \frac{M \omega_M^2 \Omega_g^2}{2g} \quad g - \ell = M \omega_M^2 \tau^2 \left( \nu + \frac{1}{2} \right) \]

In practice, the lowest-order mode occurs:

\[ a(\omega) = Ae^{-\omega^2 \tau^2 / 2} \quad \text{A Gaussian spectrum!} \]
Fourier transforming to the time domain

Recalling that multiplication by \(-\omega^2\) in the frequency domain is just a second derivative in the time domain (and vice versa):

\[
a_n^{(k+1)} - a_n^{(k)} = \left[ g \left( 1 - \frac{\omega^2}{\Omega_g^2} \right) - \ell + \frac{M \omega_M^2}{2} \frac{d^2}{d\omega^2} \right] a(\omega)
\]

becomes:

\[
a^{(k+1)}(t) - a^{(k)}(t) = \left[ g \left( 1 + \frac{1}{\Omega_g^2} \frac{d^2}{dt^2} \right) - \ell - \frac{M}{2} \omega_M^2 t^2 \right] a^{(k)}(t)
\]

which (in the continuous limit) has the solution:

\[
a(t) = \frac{(i)^v}{\sqrt{2\pi}} H_v \left( \frac{t}{\tau} \right) e^{-t^2/2\tau^2}
\]

This makes sense because Hermite-Gaussians are their own Fourier transforms.

The time-domain will prove to be a better domain for modeling passive mode-locking.
AM mode-locking – a round-trip analysis

A simple Gaussian pulse (one round trip) analysis:

\[
\exp \left[ -\frac{(\omega - \omega_0)^2}{4\Gamma'^2} \right] = \exp \left[ -\frac{4gL(\omega - \omega_g)^2}{\Delta\omega_g^2} \right] \times \exp \left[ -\frac{(\omega - \omega_0)^2}{4\Gamma^2} \right]
\]

modified spectral width \( \Gamma' \)

approximate (Gaussian) form for round trip gain. (\( g = \) saturated gain)

Assuming \( \omega_g = \omega_0 \):

\[
\Gamma' - \Gamma \approx -\frac{16gL}{\Delta\omega_g^2} \Gamma^2
\]

approximate (Gaussian) form for modulated loss

\[
\exp \left[ -\Gamma''t^2 \right] = \exp \left[ -\frac{\eta\Omega_m^2 t^2}{2} \right] \times \exp \left[ -\Gamma't^2 \right] \quad \rightarrow \quad \Gamma'' - \Gamma' \approx \frac{\eta\Omega_m^2}{2}
\]
AM mode-locking (continued)

\[ \Gamma'' - \Gamma \approx \frac{-16gL}{\Delta \omega_g} \Gamma^2 + \frac{\eta}{2} \Omega_m^2 = 0 \]

steady-state condition

Steady-state pulse duration (FWHM):

\[ \tau_p = \sqrt{\frac{2\sqrt{2 \ln 2}}{\pi^2}} \times \left( \frac{gL}{\eta} \right)^{1/4} \times \sqrt{\frac{4\pi^2}{\Omega_m \cdot \Delta \omega_g}} \sim \frac{\tau_{RT}}{\sqrt{N}} \]

number of oscillating modes

BUT: with an inhomogeneously broadened gain medium, using the full bandwidth \( \Delta \omega_g \):

\[ \tau_p \approx \frac{1}{\Delta \omega_g} \sim \frac{\tau_{RT}}{N} \]

AM mode-locking does not exploit the full bandwidth of an inhomogeneous medium!
Other active mode-locking techniques

FM mode-locking
produce a phase shift per round trip
implementation: electro-optic modulator
similar results in terms of steady-state pulse duration

Synchronous pumping
 gain medium is pumped with a pulsed laser, at a rate of 1 pulse per round trip
requires an actively mode-locked laser to pump your laser ($$)
requires the two cavity lengths to be accurately matched
useful for converting long AM pulses into short AM pulses
  (e.g., 150 psec argon-ion pulses \(\Rightarrow\) sub-psec dye laser pulses)

Additive-pulse or coupled-cavity mode-locking
 external cavity that feeds pulses back into main cavity synchronously
requires two cavity lengths to be matched
can be used to form sub-100-fsec pulses
Passive mode-locking

Saturable absorption:
absorption saturates during the passage of the pulse
leading edge is selectively eroded

Saturable gain:
gain saturates during the passage of the pulse
leading edge is selectively amplified

Fast absorber

Slow absorber
Kerr lensing is a type of saturable absorber

If a pulse experiences additional focusing due to the Kerr lens nonlinearity, and we align the laser for this extra focusing, then a high-intensity beam will have better overlap with the gain medium.

This is a type of saturable loss, with the same saturation behavior we’ve seen before:

$$\frac{\Delta I}{I} \sim \frac{1}{1 + \frac{I}{I_{sat}}}$$
Saturable-absorber mode-locking

Neglect gain saturation, and model a fast saturable absorber:

\[ \alpha \left( |a(t)|^2 \right) = \alpha_0 \sqrt{1 + \frac{|a(t)|^2}{I_{sat}}} \]

The transmission through a fast saturable absorber:

\[ e^{-\alpha L_a} \approx 1 - \alpha L_a \approx 1 - \alpha_0 L_a \left( 1 - \frac{|a(t)|^2}{I_{sat}} \right) = 1 - \alpha_0 L_a + \gamma |a|^2 \]

where: \[ \gamma = \frac{\alpha_0 L_a}{I_{sat}} \]

Including saturable absorption in the mode-amplitude equation:

\[ a^{(k+1)} - a^{(k)} = \left\{ g \left( 1 + \frac{1}{\Omega g^2} \frac{d^2}{dt^2} \right) - \ell + \gamma |a^{(k)}|^2 \right\} a^{(k)} \]

Lumping the constant loss into \( \ell \)
The sech pulse shape

In steady state, this equation has the solution:

\[ a(t) = A_0 \text{sech}\left(\frac{t}{\tau}\right) \]

where the conditions on \( \tau \) and \( A_0 \) are:

\[ \gamma |A_0|^2 = \frac{2g}{\Omega_g^2 \tau^2} \]

\[ g - \ell + \frac{g}{\Omega_g^2 \tau^2} = 0 \]
The Master Equation: including GVD

Expand $k$ to second order in $\omega$:  

$$k(\omega) = k(\omega_0) + k'\Delta\omega + \frac{1}{2}k''\Delta\omega^2$$

After propagating a distance $L_d$, the amplitude becomes:

$$a(L_d, \omega) = \exp\left\{ -i \left[ k(\omega_0) + k'\Delta\omega + \frac{1}{2}k''\Delta\omega^2 \right] L_d \right\} a(0, \omega)$$

Ignore the constant phase and $v_g$, and approximate the 2nd-order phase:

$$\exp\left\{ -\frac{1}{2}ik''\Delta\omega^2 L_d \right\} a(0, \omega) \approx \left( 1 - \frac{1}{2}ik''\Delta\omega^2 L_d \right) a(0, \omega)$$

Inverse-Fourier-transforming:

$$a(L_d, t) = \left( 1 + \frac{1}{2}ik''L_d \frac{d^2}{dt^2} \right) a(0, t) \equiv \left( 1 + iD \frac{d^2}{dt^2} \right) a(0, t)$$

where:  

$$D \equiv \frac{1}{2}k''L_d$$
The Master Equation (continued): including the Kerr effect

The Kerr Effect: \( n = n_0 + n_2 I \) so: \( \Phi = \frac{2\pi}{\lambda} n_2 L_k |a(t)|^2 \equiv \delta |a(t)|^2 \)

The master equation (assuming small effects) becomes:

\[
a^{(k+1)} - a^{(k)} = \left\{ g \left( 1 + \frac{1}{\Omega g^2} \frac{d^2}{dt^2} \right) + iD \frac{d^2}{dt^2} - \ell + (\gamma - i\delta) |a^{(k)}|^2 \right\} a^{(k)}
\]

In steady state: \( a^{(k+1)} - a^{(k)} = i\psi a(t) \) where \( \psi \) is the phase shift per round trip.

\[
\left\{ -i\psi + (g - \ell) + \left( \frac{g}{\Omega g^2} + iD \right) \frac{d^2}{dt^2} + (\gamma - i\delta) |a|^2 \right\} a(t) = 0
\]

This important equation is called the Landau-Ginzberg Equation.
Solution to the Master Equation

It is:

\[ a(t) = A_0 \left[ \text{sech} \left( \frac{t}{\tau} \right) \right]^{(1+i\beta)} \]

where:

\[-i\psi + g - \ell + \frac{(1+i\beta)}{\tau^2} \left( \frac{g}{\Omega_g^2} + iD \right) = 0\]

\[ \frac{1}{\tau^2} \left( \frac{g}{\Omega_g^2} + iD \right) \left( 2 + 3i\beta - \beta^2 \right) = (\gamma - i\delta) A_0^2 \]

The complex exponent yields chirp.
The pulse length and chirp parameter

\[ D_n = \frac{\Omega_g^2}{g} D \]

\[ \beta \]

\[ \tau_n = \Omega_g \tau \]

Figure 1.7. The plots of pulsewidth \( \tau \) and chirp parameter \( \beta \) as functions of dispersion, with the SPM coefficient as parameter. \( \beta \), chirp parameter; \( D_n \), normalized GVD parameter; \( \gamma_n \), normalized equivalent fast saturable absorber, \( \gamma_n = \gamma \frac{W}{2g} \Omega_g \); \( \delta_n \), normalized self-phase modulation parameter, \( \delta_n = \delta \frac{W}{2g} \Omega_g \); \( \tau_n \) normalized pulse width \( \tau_n = \Omega_g \tau \).
The spectral width

The spectral width vs. dispersion for various SPM values.

A broader spectrum is possible if some positive chirp is acceptable.
The Nonlinear Schrodinger Equation

Recall the master equation:

\[
a^{(k+1)} - a^{(k)} = \left\{ g \left( 1 + \frac{1}{\Omega_g^2} \frac{d^2}{dt^2} \right) + iD \frac{d^2}{dt^2} - \ell + (\gamma - i\delta) \left| a^{(k)} \right|^2 \right\} a^{(k)}
\]

If you were interested in light pulses *not* propagating inside a laser cavity, but in a medium outside the laser, then you would change this to a continuous differential equation (rather than a difference equation with a discrete round-trip index \( k \)).

You would also ignore gain and loss (both linear loss and saturable loss).

Then you would have the Nonlinear Schrodinger Equation:

\[
\frac{\partial a}{\partial z} = \left( iD \frac{\partial^2}{\partial t^2} - i\delta \left| a \right|^2 \right) a
\]

- \( D \) – GVD parameter
- \( \delta \) – Kerr nonlinearity
The Nonlinear Schrödinger Equation (NLSE)

\[
\frac{\partial a}{\partial z} = \left( iD \frac{\partial^2}{\partial t^2} - i\delta |a|^2 \right) a
\]

The solution to the nonlinear Schrödinger equation is:

\[
a(z, t) = A_0 \text{sech}\left( \frac{t}{\tau} \right) \exp\left[ -i \frac{\delta |A_0|^2}{2} z \right]
\]

where:

\[
\frac{1}{\tau^2} = -\frac{\delta |A_0|^2}{2D}
\]

Note that \(\delta / D < 0\), or no solution exists.

But note that, despite dispersion, the pulse length and shape do not vary with distance: solitons
Collision of two solitons
Evolution of a soliton from a square wave
So then do all mode-locked lasers produce sech(t) pulses?

\[ \frac{\partial a}{\partial z} = \left( iD \frac{\partial^2}{\partial t^2} - i\delta |a|^2 \right) a(z) \]

The Master Equation assumed that the dispersion is uniform throughout the laser cavity, so that the pulse is always experiencing a certain (constant) GVD as it propagates through one full round-trip.

But that is far from being true.

Ti: sapphire crystal: positive GVD

between the prisms: negative GVD
Distributed dispersion within the laser cavity

So the dispersion $D$ should depend on position within the laser cavity, $D = D(z)$. In principle, so should the Kerr nonlinearity, $\delta = \delta(z)$ since this nonlinearity only exists inside the gain medium.
Perturbed nonlinear Schrödinger equation

\[ \frac{\partial a}{\partial z} = \left( iD(z) \frac{\partial^2}{\partial t^2} - i\delta(z)|a|^2 \right) a(z) \]

\( \delta(z) = \) zero except inside the Ti:sapphire crystal
\( D(z) = \) cavity dispersion map (positive inside the crystal, negative between the prisms)

The pulse experiences these perturbations periodically, once per cavity round-trip.

Solution: requires numerical integration except in the asymptotic limit (infinite \(|D|\))

Result: the pulse shape is not invariant! It varies during the round trip. But it is still stable at any particular location in the laser.

“Dispersion-managed solitons”

In real lasers, the pulse shape is complicated

Dispersion management can produce many different pulse shapes.

- small dispersion swing: sech pulses
- moderate dispersion: Gaussian pulses
- large dispersion: the result depends more sensitively on gain filtering

Also:
The perturbed NLSE neglects gain and saturable loss, both of which are required for mode-locked operation.

In principle, both would also need to be included in a distributed way:
\[ g = g(z) \quad \text{and} \quad \gamma = \gamma(z) \]

Bottom line: it’s complicated.