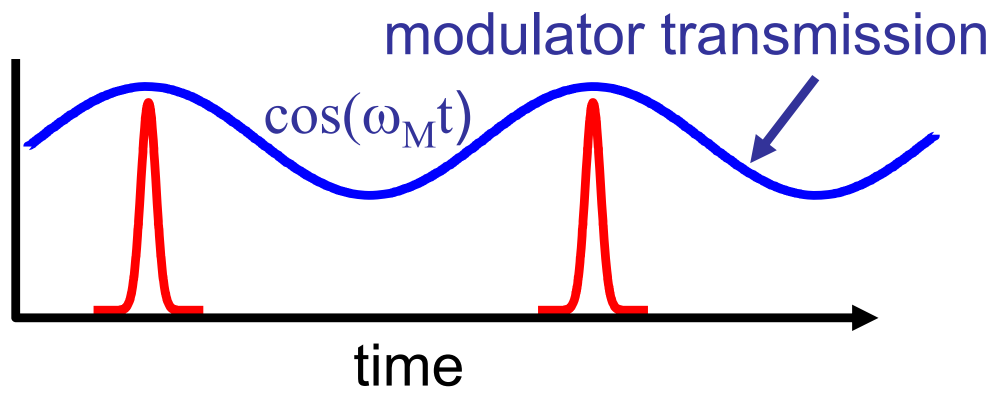
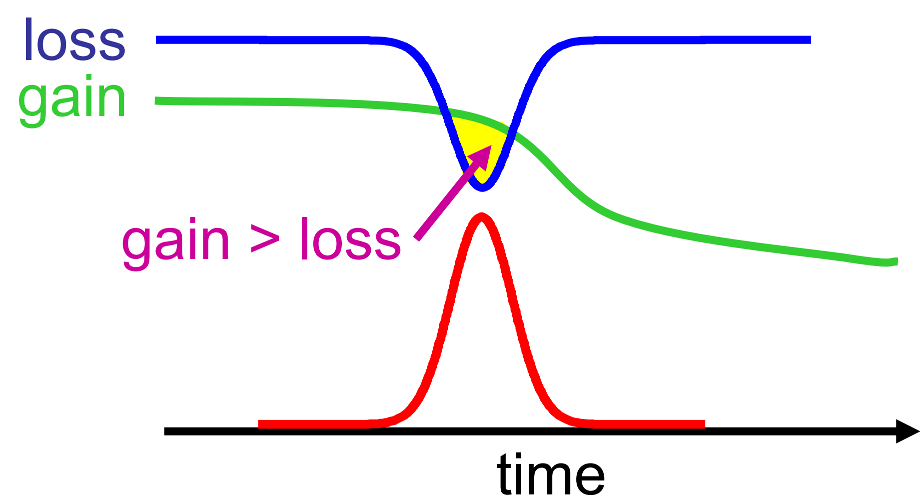
Theory of Ultrashort Laser Pulse Generation

Active mode-locking



Passive mode-locking



Including dispersion and nonlinearity: The Landau-Ginzberg Equation

The Nonlinear Schrodinger Equation

Solitons

Gaussians?
Or hyperbolic secants?

Reference: Hermann Haus, “Short pulse generation,” in *Compact Sources of Ultrashort Pulses*, Irl N. Duling, ed. (Cambridge University Press, 1995).

An actual quotation

"The Laser is numbered among the most miraculous gifts of nature and lends itself to a variety of applications."

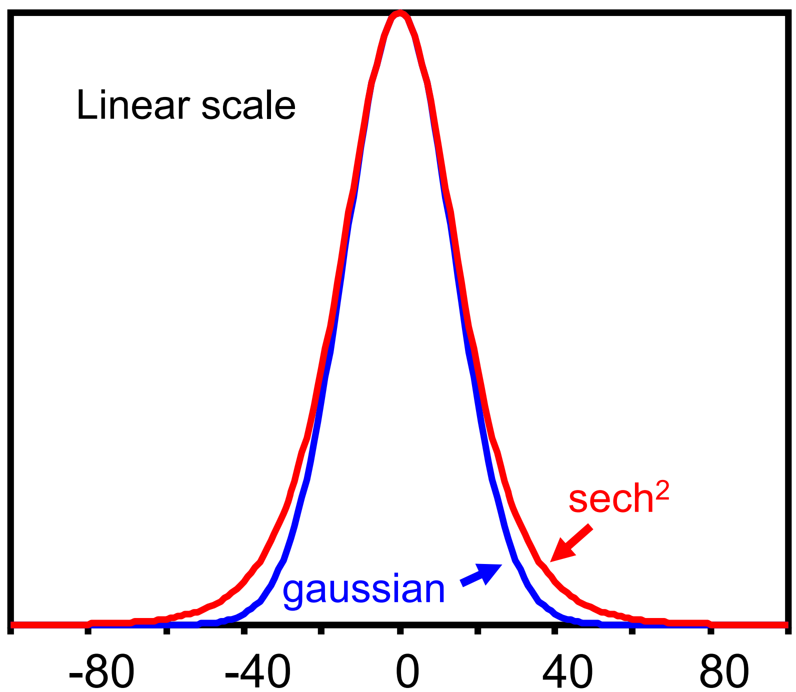
- Pliny (the elder) Natural History XXII, 49 (1st Century AD)

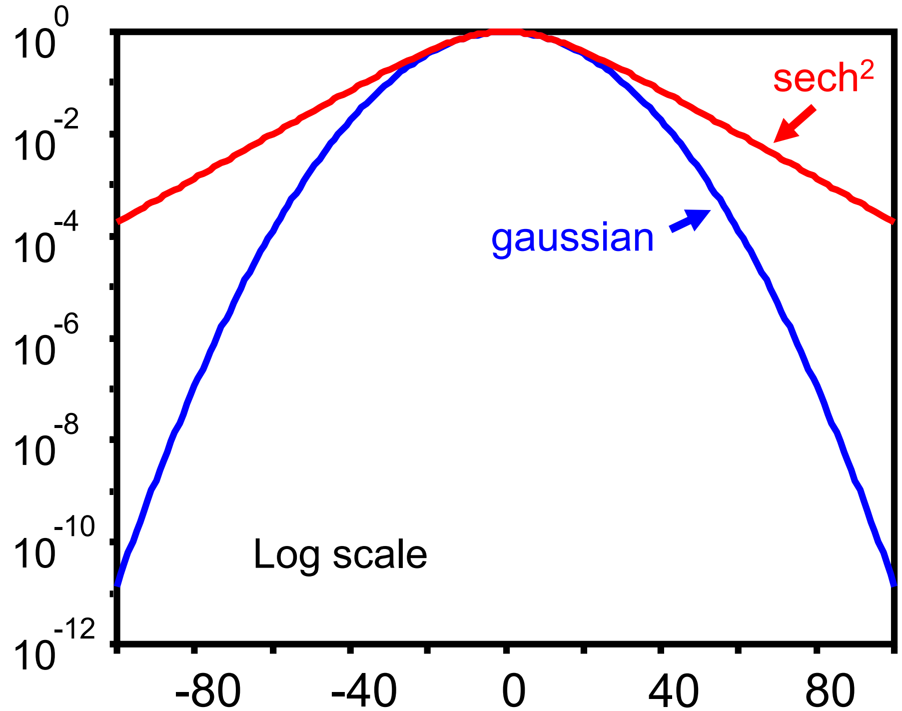
“The number of uses of compounds made with laser is immeasurable," said Pliny. Among them were the following: diuretic, healing ointment for sores, antidote for wounds caused by poison-tipped weapons, snakebites, and scorpion stings, for shrinking corns and carbuncles, healing dog bites, soothing chilblains, alleviating coughing and wheezing, and as a cure for gout, cramps, pleurisy, and tetanus.

silphium laciniatum, a modern relative of laser.

The burning question: gaussian or sech^2?

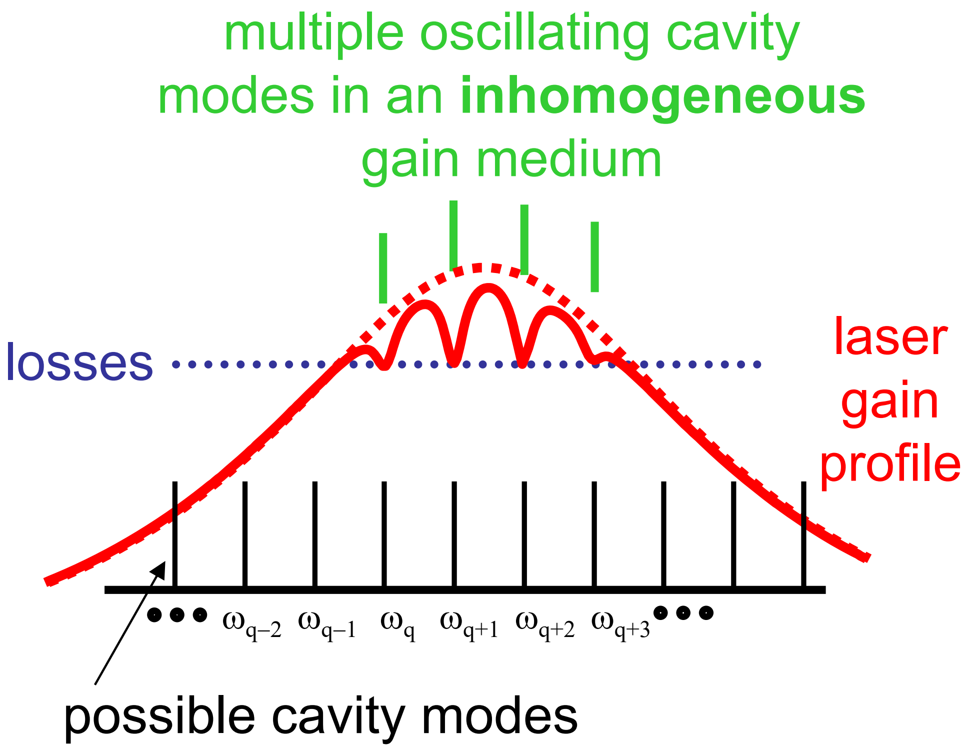
After all this talk about Gaussian pulses… what does Ti:sapphire really produce?





$$e^{-\left(t/\tau\right)^2} \text{ or } \operatorname{sech}^2\left(t/\tau\right)??$$

Mode-locking yields ultrashort pulses

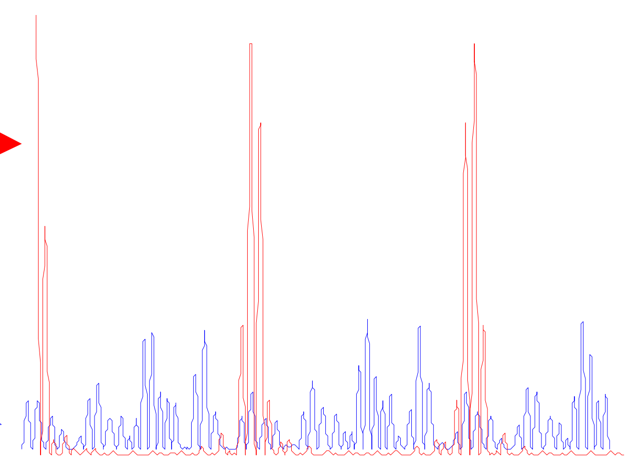


Recall that many frequencies ("modes") oscillate simultaneously in a laser, and when their phases are locked, an ultrashort pulse results.

Locking the modes of a laser requires nonlinear optics. There are numerous strategies.

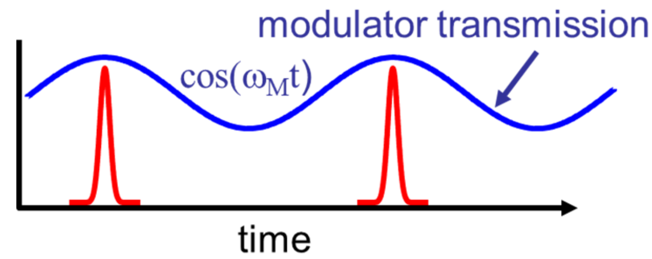
Sum of ten modes with the same relative phase

Sum of ten modes w/ random phase



Two categories of Mode Locking

Active Mode Locking



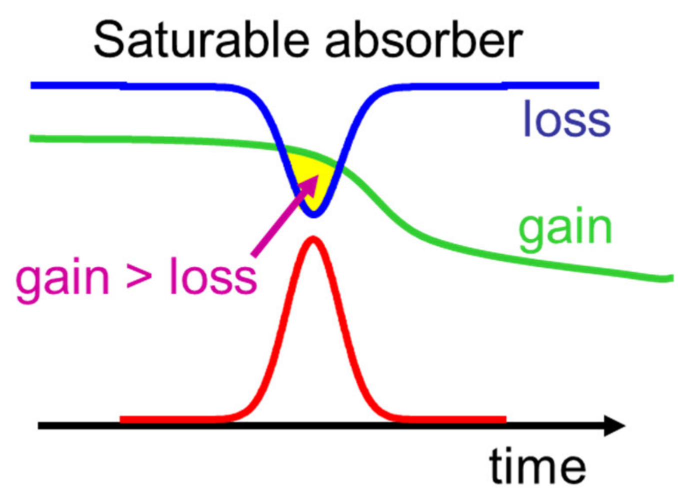
We insert something into the laser cavity that sinusoidally modulates the amplitude of the pulse.

$\Rightarrow$ mode competition couples each mode to modulation sidebands
$\Rightarrow$ eventually, all the modes are coupled and phase-locked

Passive Mode Locking

We insert something into the laser cavity that favors high intensities.

$\Rightarrow$ strong maxima will grow stronger at the expense of weaker ones
$\Rightarrow$ eventually, all of the energy is concentrated in one packet



The Modulation Theorem: The Fourier Transform of E(t)cos(ω_Mt)

$$\mathcal{F}\{E(t)\cos(\omega_M t)\} = \int_{-\infty}^{\infty} E(t)\cos(\omega_M t)\exp(-i\omega t)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} E(t)\left[\exp(i\omega_M t) + \exp(-i\omega_M t)\right]\exp(-i\omega t)\,dt$$

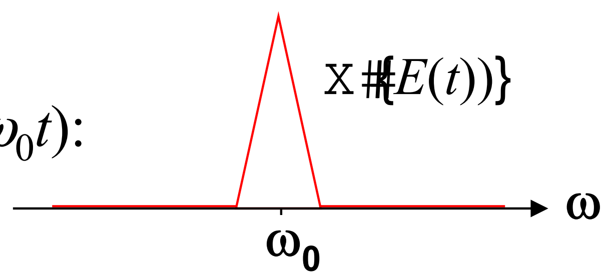
$$= \frac{1}{2}\int_{-\infty}^{\infty} E(t)\exp(-i[\omega-\omega_M]t)\,dt + \frac{1}{2}\int_{-\infty}^{\infty} E(t)\exp(-i[\omega+\omega_M]t)\,dt$$

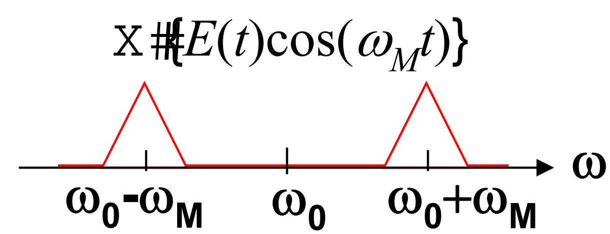
$$\mathcal{F}\{E(t)\cos(\omega_M t)\} = \boxed{\frac{1}{2}\tilde{E}(\omega-\omega_M) + \frac{1}{2}\tilde{E}(\omega+\omega_M)}$$

where $\tilde{E}(\omega)$ is the FT of the original unmodulated function $E(t)$

Multiplication by cos($\omega_M t$) introduces side-bands.

If $E(t) = \mathrm{sinc}^2(t)\exp(\mathrm{i}\omega_0 t)$:

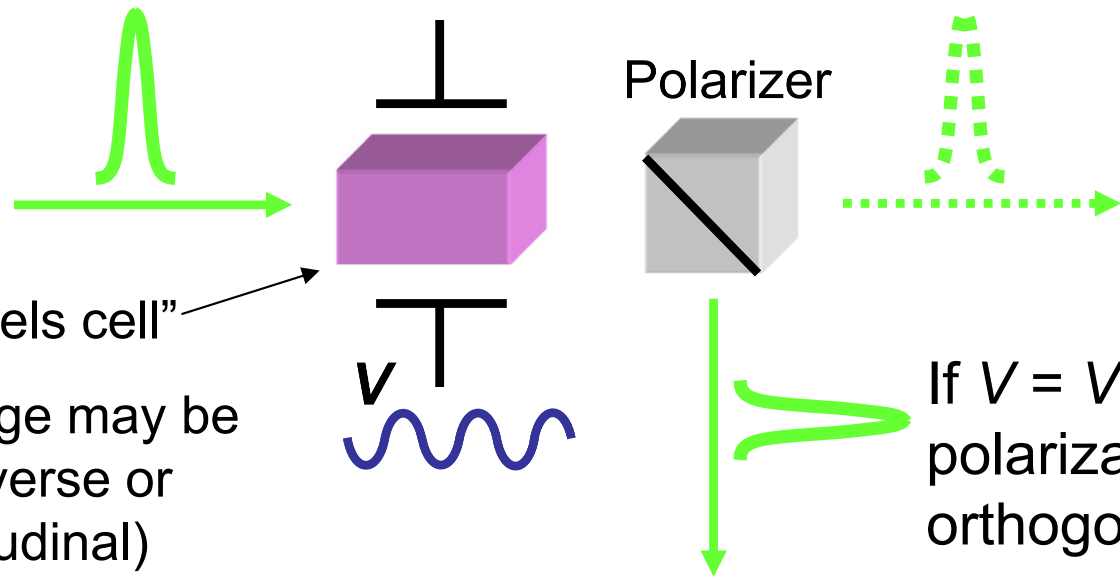




One option for active mode-locking: the electro-optic modulator

Applying a voltage to certain crystalline materials changes the refractive indices and introduces birefringence.

A few kV can turn a crystal into a half- or quarter-wave plate.



If $V = 0$, the pulse polarization doesn't change.

"Pockels cell"

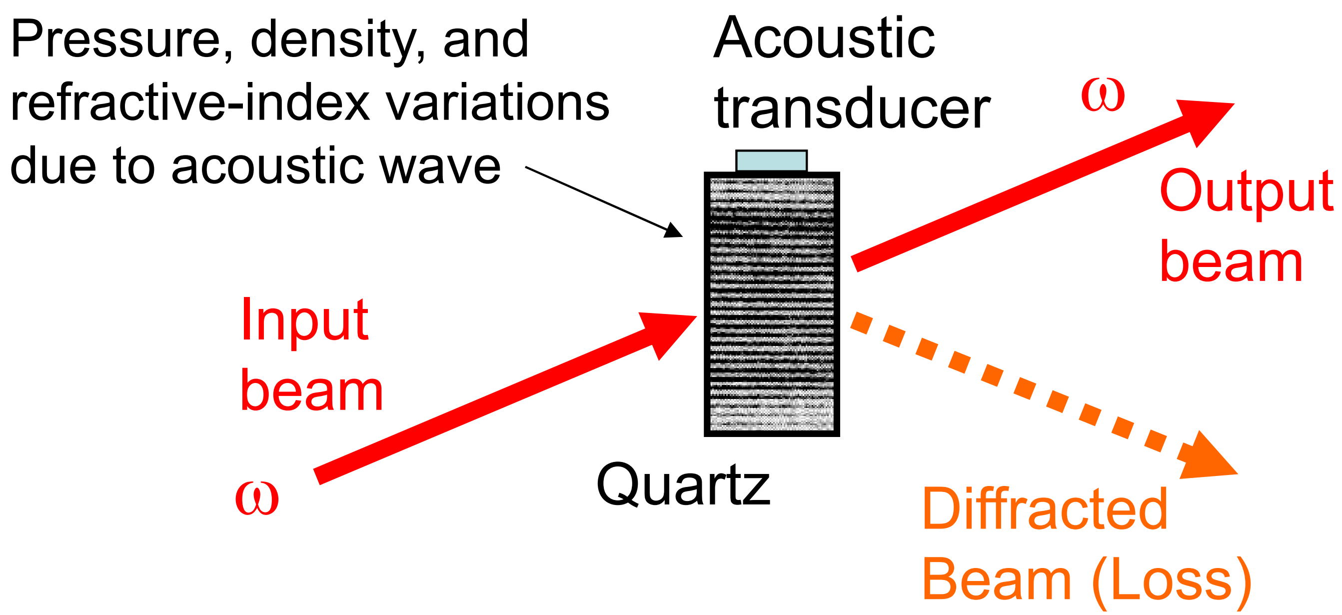
(voltage may be transverse or longitudinal)

If $V = V_p$, the pulse polarization switches to its orthogonal state.

Applying a sinusoidal voltage yields sinusoidal modulation to the beam's amplitude. Or, use it without a polarizer to simply introduce a phase modulation, which sinusoidally shifts the modes into and out of resonance with the actual cavity modes.

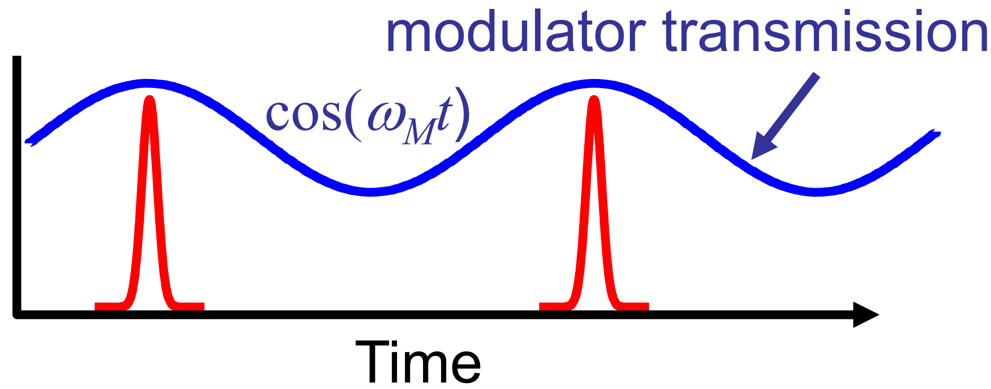
2nd option for active mode-locking: the acousto-optic modulator

Here, an acoustic wave induces sinusoidal density, and hence sinusoidal refractive-index, variations in a medium. This will diffract away some of a light wave's energy.

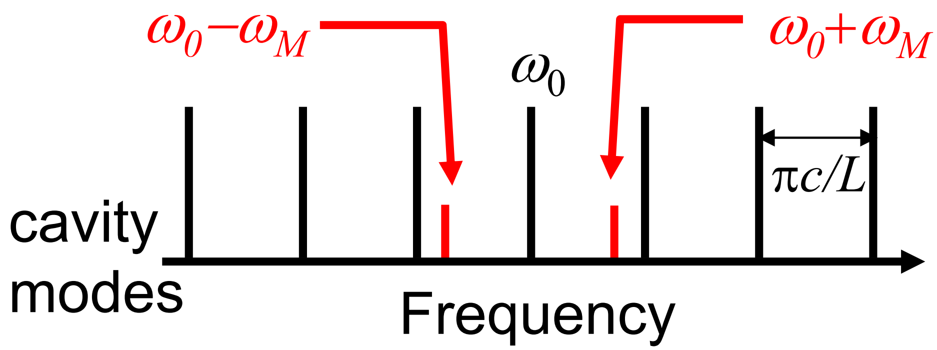


Such diffraction can be quite strong: ~70%. Sinusoidally modulating the acoustic wave amplitude yields sinusoidal modulation of the transmitted beam.

Active mode-locking



In the frequency domain, a modulator introduces side-bands of every laser mode (although, this diagram only shows the side bands for one of them).



For mode-locking, we adjust ω_M so that ω_M = mode spacing (which is not what is shown in this diagram).

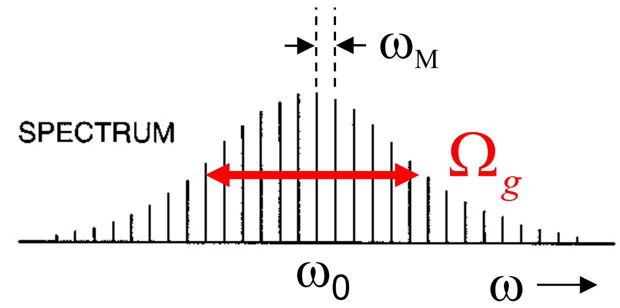
$$\omega_M = 2\pi/\text{cavity round-trip time} = 2\pi/(2L/c) = \pi c/L$$

Under this condition, each mode competes for gain with adjacent modes. The most efficient operation is for phases to lock, resulting in global phase locking.

This is described by a system of N coupled equations: $E_n \Leftrightarrow E_{n+1}, E_{n-1}$

Modeling laser modes and gain

Gain profile and resulting laser modes



Lasers have a mode spacing:

$$\omega_M = \frac{2\pi}{T_R} = \frac{\pi c}{L}$$

Let the zero^th mode be at the center of the gain, ω_0. The n^{th} mode frequency is then:

$$\omega_n = \omega_0 + n\,\omega_M \qquad \text{where } n = \ldots, -1, 0, 1, \ldots$$

Let a_n be the amplitude of the n^{th} mode and assume a Lorentzian gain profile, $G(n)$. Then, the amplitude increases after one round trip according to:

$$a_n^{(k+1)} = G(n)\,a_n^{(k)} = \left[1 + \frac{g}{1 + (n\omega_M)^2 / \Omega_g^2}\right] a_n^{(k)} \quad \text{where } \Omega_g = \text{gain bandwidth}$$

and where the superscript *(k)* indicates the round trip number.

Modeling an amplitude modulator

An amplitude modulator uses the electro-optic or acousto-optic effect to deliberately cause losses at the laser round-trip frequency, ω_M.

A modulator multiplies the laser light (i.e., each mode) by $M[1-\cos(\omega_M t)]$

$$M\left[1-\cos(\omega_M t)\right]a_n e^{i(\omega_o+n\omega_M)\,t} = M\left[-\frac{1}{2}\exp(-i\omega_M t)+1-\frac{1}{2}\exp(i\omega_M t)\right]a_n e^{i(\omega_o+n\omega_M)\,t}$$

$$= -Ma_n e^{i\omega_o t}\left[\frac{1}{2}e^{i(n-1)\omega_M t} - e^{in\omega_M t} + \frac{1}{2}e^{i(n+1)\omega_M t}\right]$$

Notice that this spreads the energy from the n^{th} to the $(n+1)^{st}$ and $(n\text{-}1)^{st}$ modes. Including the passive loss, ℓ, we can write this as:

$$a_n^{(k+1)} = a_n^{(k)} + g\left(1+\frac{(n\omega_M)^2}{\Omega_g^2}\right)^{-1} a_n^{(k)} - \ell a_n^{(k)} + \frac{M}{2}\left(a_{n+1}^{(k)} - 2a_n^{(k)} + a_{n-1}^{(k)}\right)$$

Reminder: the superscript indicates the k^{th} round trip.
The subscript n is the mode index: $\omega_n = \omega_0 + n\,\omega_M$

Solve for the steady-state solution

Approximate: modes near the center of the Lorentzian gain, so $n\omega_M \ll \Omega_g$

thus: $$\frac{g}{1+\left(n\omega_M\right)^2 / \Omega_g^2} \approx g\left[1-\frac{\left(n\omega_M\right)^2}{\Omega_g^2}\right]$$

which results in:

$$\cancel{a_n^{(k+1)}} = \cancel{a_n^{(k)}} + g\left(1-\frac{\left(n\omega_M\right)^2}{\Omega_g^2}\right)a_n^{(k)} - \ell a_n^{(k)} + \frac{M}{2}\left(a_{n+1}^{(k)} - 2a_n^{(k)} + a_{n-1}^{(k)}\right)$$

In steady state, $a_n^{(k+1)} = a_n^{(k)}$

Also, the finite difference becomes a second derivative when the modes are many and closely spaced:

$$a_{n+1}^{(k)} - 2a_n^{(k)} + a_{n-1}^{(k)} \quad \rightarrow \quad \omega_M^2 \frac{d^2}{d\omega^2} a(\omega)$$

where, in this continuous limit,

$a_n^{(k)} \rightarrow a(\omega)$

where: $\omega = n\,\omega_M$

Solve for the steady-state solution

Thus we have:

$$0 = \left[g\left(1 - \frac{\omega^2}{\Omega_g{}^2}\right) - \ell + \frac{M\omega_M^2}{2}\frac{d^2}{d\omega^2} \right] a(\omega)$$

This differential equation has the solution:

$$a(\omega) = H_\nu(\omega\tau)\, e^{-\omega^2\tau^2/2}$$

(Hermite Gaussians)

with the constraints:

$$\frac{1}{\tau^4} = \frac{M\omega_M^2\Omega_g{}^2}{2g} \qquad g - \ell = M\omega_M^2\tau^2\left(\nu + \frac{1}{2}\right)$$

Focus on the lowest-order (ν = 1) mode:

$$a(\omega) = Ae^{-\omega^2\tau^2/2}$$

Active sinusoidal modulation of the laser modes at the round-trip frequency produces a Gaussian spectrum!

AM mode locking: pulse duration

The solution: $$a(\omega) = H_{\nu}(\omega\tau)\, e^{-\omega^2\tau^2/2}$$

with the constraints: $$\frac{1}{\tau^4} = \frac{M\omega_M^2 \Omega_g{}^2}{2g} \qquad g - \ell = M\omega_M^2 \tau^2 \left(\nu + \frac{1}{2}\right)$$

The parameter τ determines the spectral bandwidth, which in turn determines the shortest possible pulse duration:

$$\tau = \frac{1}{\sqrt{\Omega_g}} \cdot \sqrt[4]{\frac{2g}{M\omega_M^2}}$$

In other words, the spectral bandwidth of the mode-locked pulse varies as the square root of the gain bandwidth.

→ AM mode-locking does not exploit the full bandwidth of an inhomogeneous medium!

Fourier transforming to the time domain

Recalling that multiplication by $-\omega^2$ in the frequency domain is just a second derivative in the time domain (and vice versa).

So this:
$$a_n^{(k+1)} - a_n^{(k)} = \left[g\left(1 - \frac{\omega^2}{{\Omega_g}^2}\right) - \ell + \frac{M\omega_M^2}{2}\frac{d^2}{d\omega^2} \right] a(\omega)$$

becomes this:
$$a^{(k+1)}(t) - a^{(k)}(t) = \left[g\left(1 + \frac{1}{{\Omega_g}^2}\frac{d^2}{dt^2}\right) - \ell - \frac{M}{2}\omega_M^2 t^2 \right] a^{(k)}(t)$$

which (in the continuous steady-state limit) has the solution:
$$a(t) = \frac{(i)^\nu}{\sqrt{2\pi}} H_\nu\left(\frac{t}{\tau}\right) e^{-t^2/2\tau^2}$$

This makes sense because Hermite-Gaussians are their own Fourier transforms.

The time-domain will prove to be better for modeling passive mode-locking.

Other active mode-locking techniques

FM mode-locking

produce a phase shift per round trip
implementation: electro-optic modulator
similar results in terms of steady-state pulse duration

Synchronous pumping

gain medium is pumped with a pulsed laser, at a rate of 1 pulse per round trip
requires an actively mode-locked laser to pump your laser ($$)
requires the two cavity lengths to be accurately matched
useful for converting long AM pulses into short AM pulses
(e.g., 150 psec argon-ion pulses $\Rightarrow$ sub-psec dye laser pulses)

Additive-pulse or coupled-cavity mode-locking

external cavity that feeds pulses back into main cavity synchronously
requires two cavity lengths to be matched
can be used to form sub-100-fsec pulses

Passive mode-locking

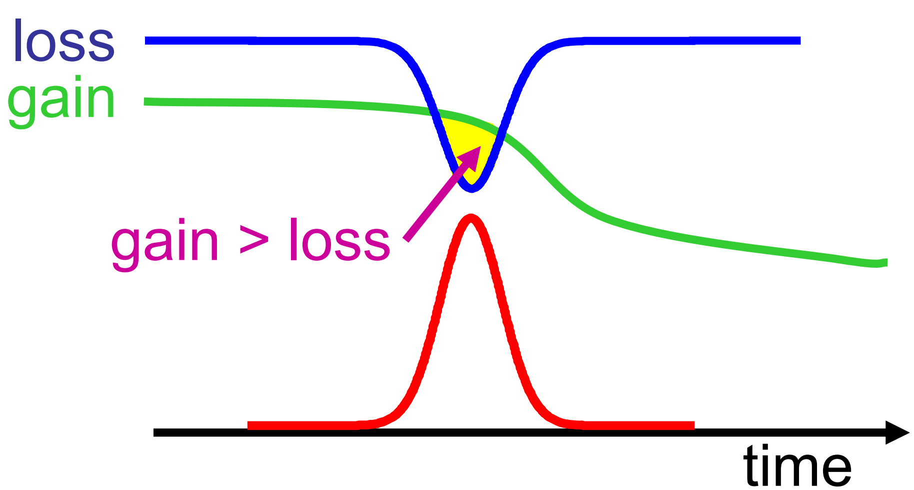
Saturable absorption:

- absorption saturates during the passage of the pulse
- leading edge is selectively eroded

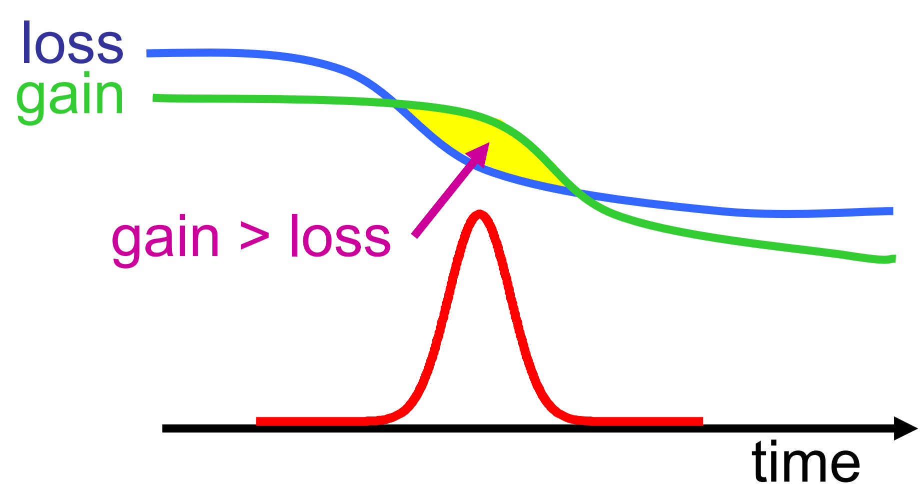
Saturable gain:

- gain saturates during the passage of the pulse
- leading edge is selectively amplified

Fast absorber

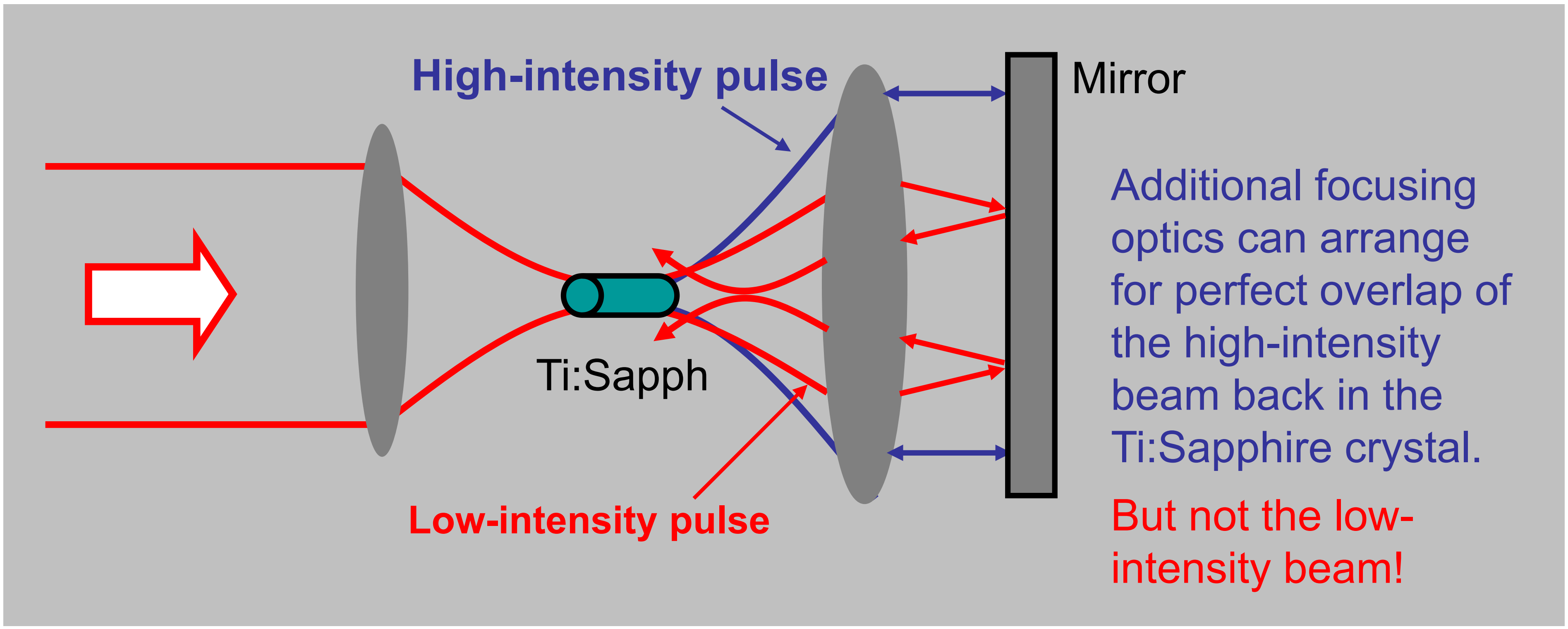


Slow absorber



Kerr lensing is a type of saturable absorber

If a pulse experiences additional focusing due to the Kerr lens nonlinearity, and we align the laser for this extra focusing, then a high-intensity beam will have better overlap with the gain medium.



This is a type of saturable loss, with the same saturation behavior we've seen before:

$$\frac{\Delta I}{I} \propto \frac{1}{1 + I/I_{sat}}$$

Saturable-absorber mode-locking

Neglect gain saturation, and model a fast saturable absorber:

$$\alpha\left(|a(t)|^2\right) = \frac{\alpha_0}{1 + \dfrac{|a(t)|^2}{I_{sat}}}$$

Intensity

Saturation intensity

The transmission through a fast saturable absorber of length L_a:

$$e^{-\alpha L_a} \approx 1 - \alpha L_a \approx 1 - \alpha_0 L_a \left(1 - \frac{|a(t)|^2}{I_{sat}}\right) = 1 - \alpha_0 L_a + \gamma |a|^2$$

where: $\gamma = \dfrac{\alpha_0 L_a}{I_{sat}}$

Including saturable absorption in the mode-amplitude equation (and removing the active mode-locking term proportional to M):

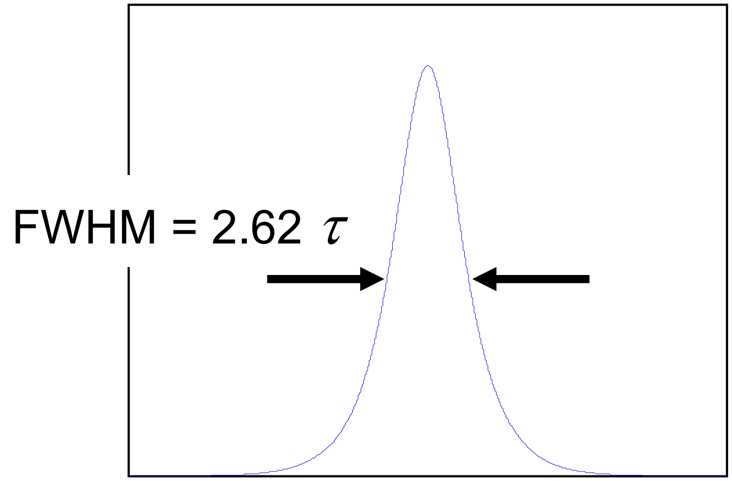
$$a^{(k+1)} - a^{(k)} = \left\{ g\left(1 + \frac{1}{\Omega_g{}^2}\frac{d^2}{dt^2}\right) - \ell + \gamma \left|a^{(k)}\right|^2 \right\} a^{(k)}$$

Here, we lumped the constant loss into ℓ

The sech pulse shape

In steady state, this equation has the solution:

$$a(t) = A_0 \operatorname{sech}\left(\frac{t}{\tau}\right)$$



where the conditions on τ and A_0 are:

$$\gamma |A_0|^2 = \frac{2g}{\Omega_g^2 \tau^2} \qquad g - \ell + \frac{g}{\Omega_g^2 \tau^2} = 0$$

$$\tau = \frac{1}{\Omega_g} \sqrt{\frac{2g}{\gamma |A_0|^2}}$$

Note: the pulse duration is now proportional to the inverse of the gain bandwidth. Passive mode locking produces shorter pulses!

The Master Equation: including GVD

Expand k to second order in ω: $k(\omega) = k(\omega_0) + k'\Delta\omega + \frac{1}{2}k''\Delta\omega^2$

After propagating a distance L_d, the amplitude becomes:

$$a(L_d,\omega) = \exp\left\{-i\left[k(\omega_o) + k'\Delta\omega + \frac{1}{2}k''\Delta\omega^2\right]L_d\right\}a(0,\omega)$$

Ignore the constant phase and v_g, and approximate the 2nd-order phase:

$$\exp\left\{-\frac{1}{2}ik''\Delta\omega^2 L_d\right\}a(0,\omega) \approx \left(1 - \frac{1}{2}ik''\Delta\omega^2 L_d\right)a(0,\omega)$$

Inverse-Fourier-transforming:

$$a(L_d,t) = \left(1 + \frac{1}{2}ik''L_d\frac{d^2}{dt^2}\right)a(0,t) \equiv \left(1 + iD\frac{d^2}{dt^2}\right)a(0,t)$$

where: $D \equiv \frac{1}{2}k''L_d$

The Master Equation (continued): including the Kerr effect

The Kerr Effect: $n = n_0 + n_2 I$ so: $\Phi = \frac{2\pi}{\lambda} n_2 L_k \left| a(t) \right|^2 \equiv \delta \left| a(t) \right|^2$

where $\delta \propto n_2$

The master equation (assuming small effects) becomes:

$$a^{(k+1)} - a^{(k)} = \left\{ g \left(1 + \frac{1}{\Omega_g{}^2} \frac{d^2}{dt^2} \right) + iD \frac{d^2}{dt^2} - \ell + (\gamma - i\delta) \left| a^{(k)} \right|^2 \right\} a^{(k)}$$

In steady state: $a^{(k+1)} - a^{(k)} = i\psi a(t)$ where ψ is the phase shift per round trip.

$$\boxed{\left\{ -i\psi + (g - \ell) + \left(\frac{g}{\Omega_g{}^2} + iD \right) \frac{d^2}{dt^2} + (\gamma - i\delta) \left| a \right|^2 \right\} a(t) = 0}$$

This important equation is called the Landau-Ginzberg Equation.

Solution to the Master Equation

It has an analytic solution:

$$\boxed{a(t) = A_0 \left[\operatorname{sech}\left(\frac{t}{\tau} \right) \right]^{(1+i\beta)}}$$

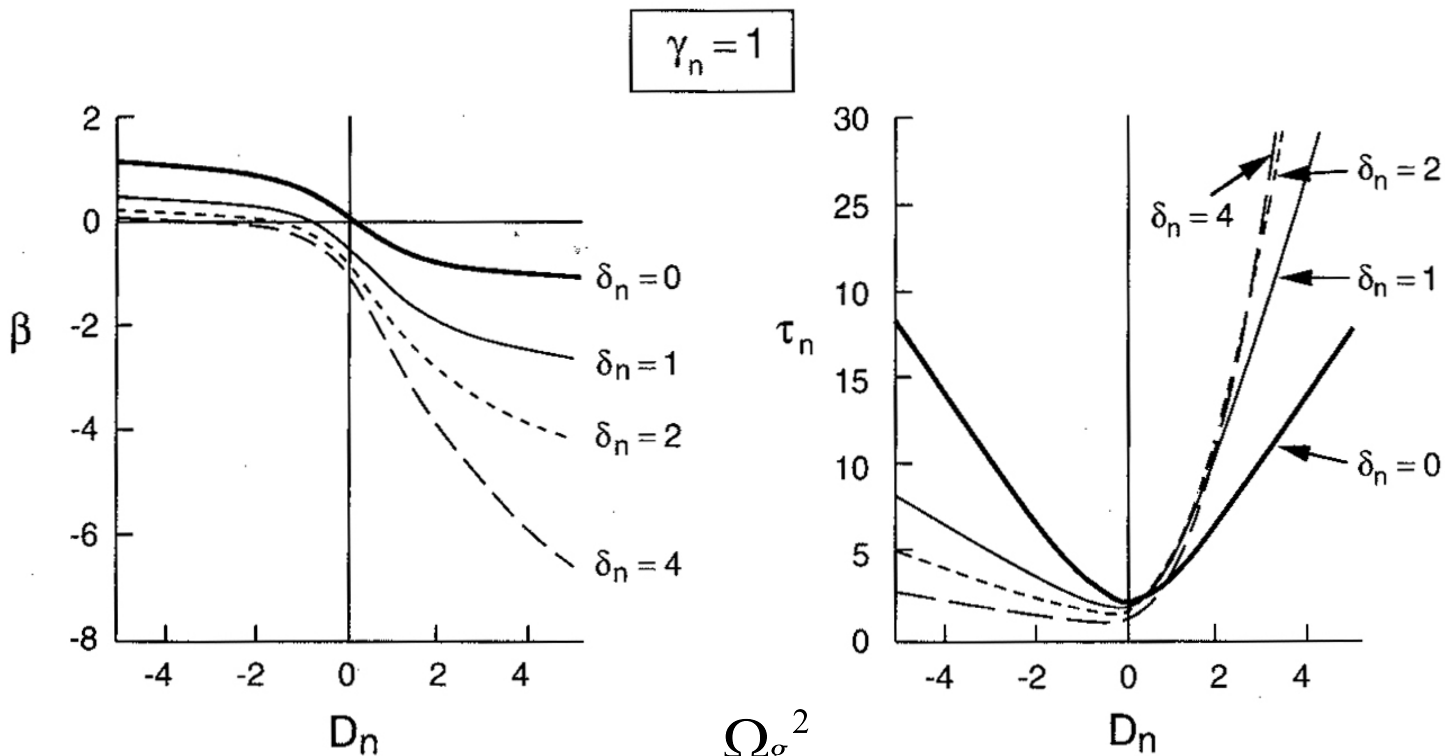
where:

$$-i\psi + g - \ell + \frac{(1+i\beta)}{\tau^2}\left(\frac{g}{{\Omega_g}^2} + iD \right) = 0$$

$$\frac{1}{\tau^2}\left(\frac{g}{{\Omega_g}^2} + iD \right)\left(2 + 3i\beta - \beta^2\right) = (\gamma - i\delta)\, A_0^2$$

The complex exponent yields chirp.

The pulse length and chirp parameter

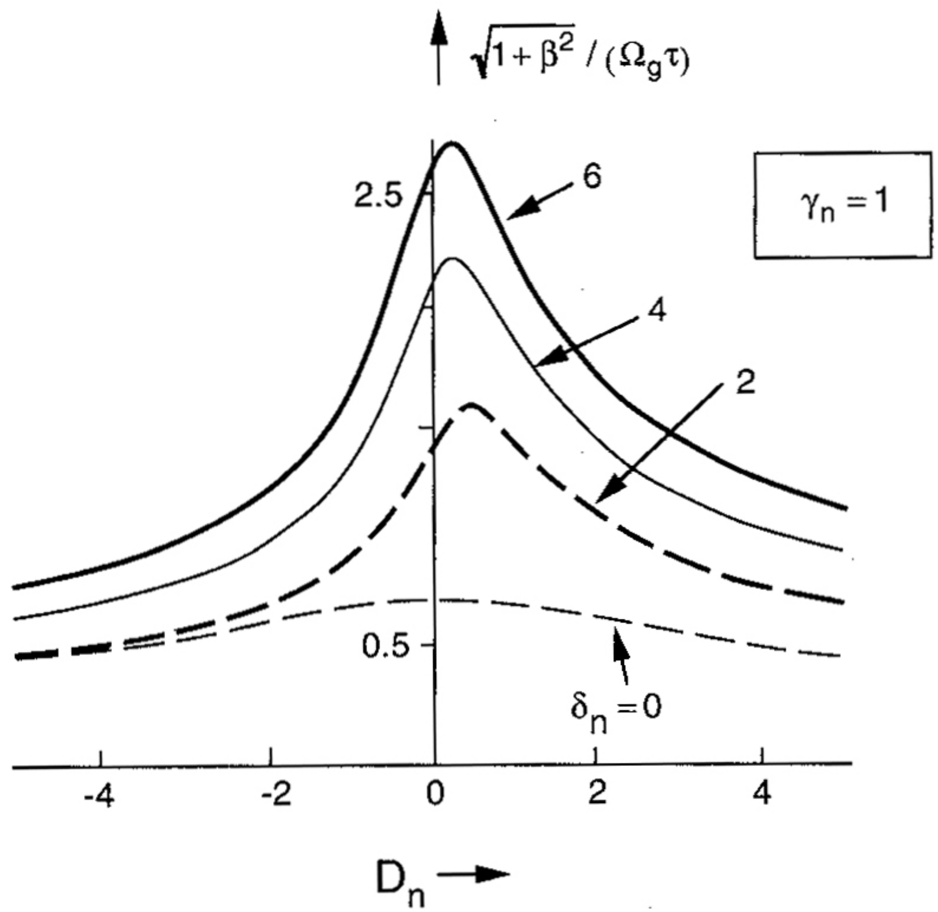


$$D_n = \frac{\Omega_g{}^2}{g} D$$

Figure 1.7. The plots of pulsewidth τ and chirp parameter β as functions of dispersion, with the SPM coefficient as parameter. β, chirp parameter; D_n, normalized GVD parameter; γ_n, normalized equivalent fast saturable absorber, $\gamma_n = \gamma \frac{W}{2g} \Omega_g$; δ_n, normalized self-phase modulation parameter, $\delta_n = \delta \frac{W}{2g} \Omega_g$; τ_n normalized pulse width $\tau_n = \Omega_g \tau$.

The spectral width

The spectral width vs. dispersion for various SPM values.



A broader spectrum is possible if some positive chirp is acceptable.

The Nonlinear Schrodinger Equation

Recall the master equation:

$$a^{(k+1)} - a^{(k)} = \left\{ g\left(1 + \frac{1}{\Omega_g{}^2}\frac{d^2}{dt^2}\right) + iD\frac{d^2}{dt^2} - \ell + (\gamma - i\delta)\left|a^{(k)}\right|^2 \right\} a^{(k)}$$

If you were interested in light pulses *not* propagating inside a laser cavity, but in a medium outside the laser, then you would change this to a continuous differential equation (rather than a difference equation with a discrete round-trip index k).

You would also ignore gain and loss (both linear loss and saturable loss).

Then you would have the Nonlinear Schrodinger Equation:

$$\boxed{\frac{\partial a}{\partial z} = \left(iD\frac{\partial^2}{\partial t^2} - i\delta\left|a\right|^2 \right) a}$$

D – GVD parameter

δ – Kerr nonlinearity

The Nonlinear Schrodinger Equation (NLSE)

$$\frac{\partial a}{\partial z} = \left(iD \frac{\partial^2}{\partial t^2} - i\delta \left| a \right|^2 \right) a$$

The solution to the nonlinear Shrodinger equation is:

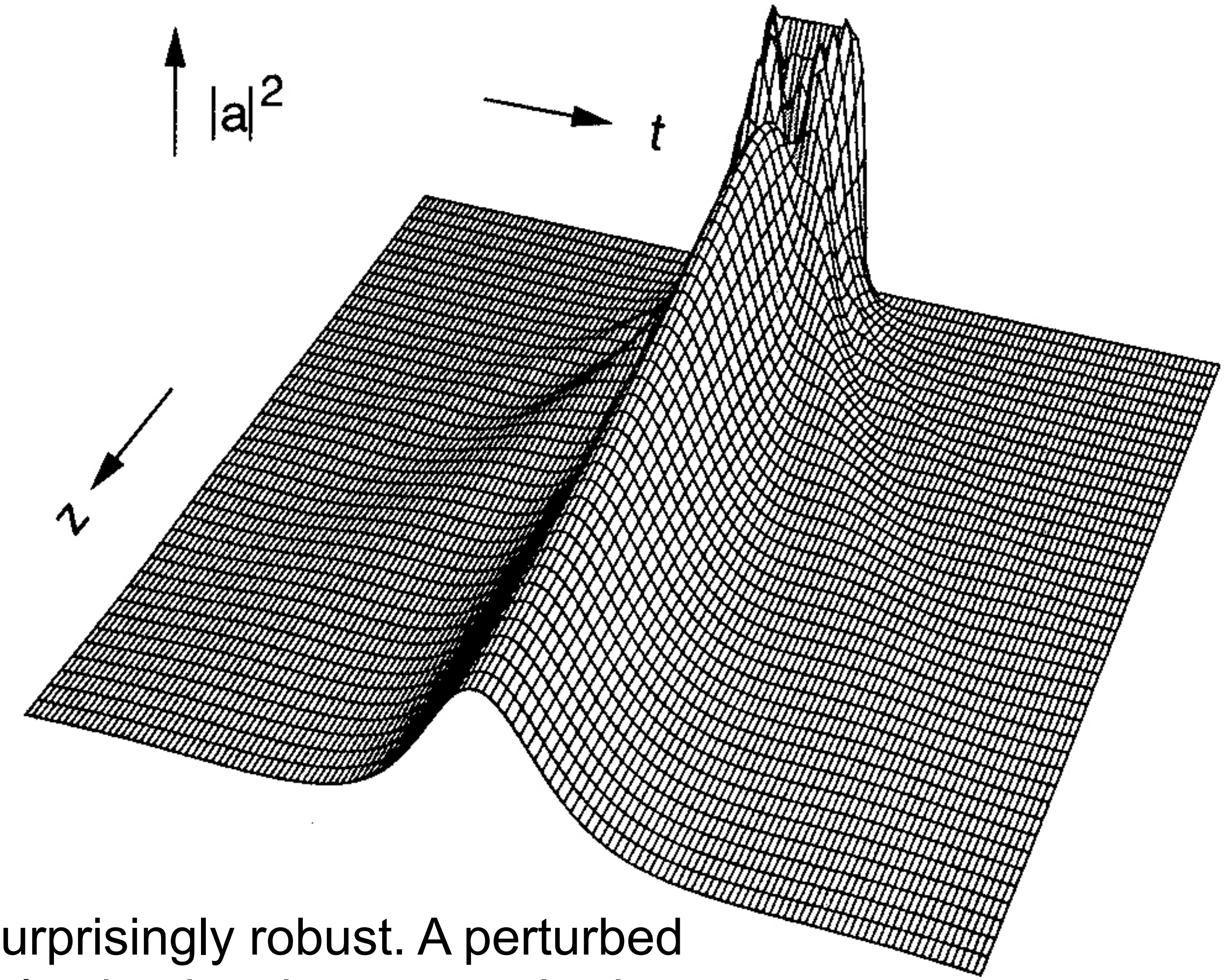
$$a(z,t) = A_0 \operatorname{sech}\left(\frac{t}{\tau}\right) \exp\left[-i \frac{\delta \left| A_0 \right|^2}{2} z \right]$$

where: $$\frac{1}{\tau^2} = -\frac{\delta \left| A_0 \right|^2}{2D}$$

Note that δ / D, must be negative, or no solution exists.

But note that, despite dispersion, the pulse length and shape do not vary with distance: **this predicts solitons**

Evolution of a soliton from a square wave



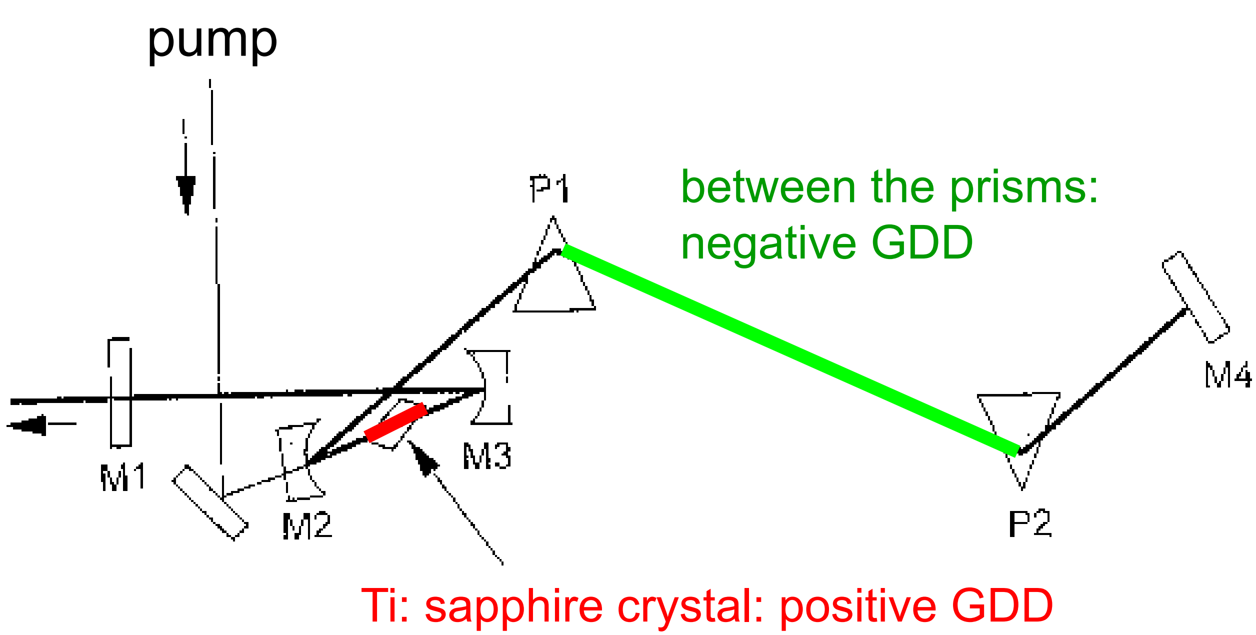
Solitons are surprisingly robust. A perturbed soliton will evolve back to its unperturbed state as long as the perturbation is not too large.

So then do all mode-locked lasers produce sech(t) pulses?

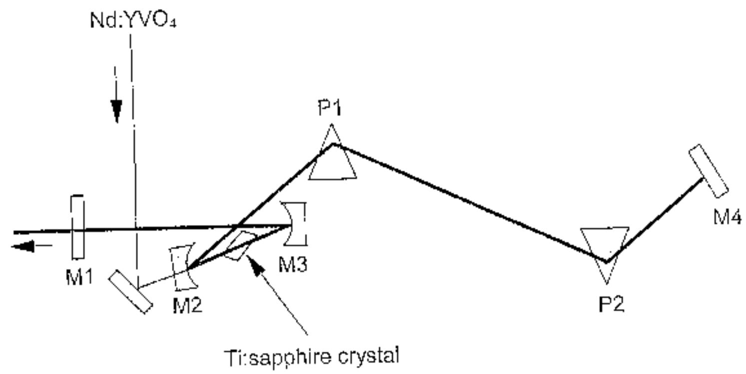
$$\frac{\partial a}{\partial z} = \left(iD\frac{\partial^2}{\partial t^2} - i\delta |a|^2 \right) a(z)$$

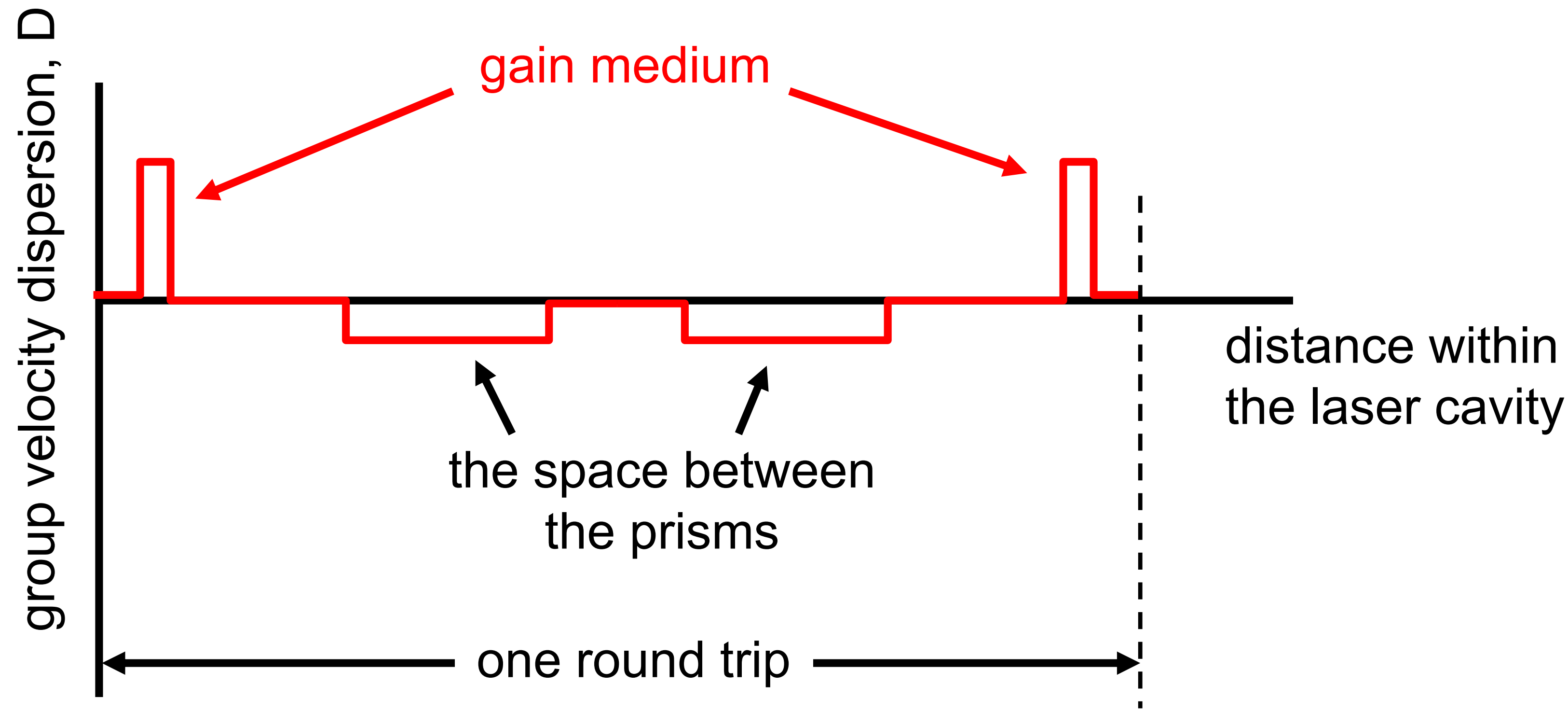
The Master Equation assumed that the dispersion is uniform throughout the laser cavity, so that the pulse is always experiencing a certain (constant) GVD as it propagates through one full round-trip.

But that is far from being true.



Distributed dispersion within the laser cavity





So the dispersion D should depend on position within the laser cavity, $D = D(z)$. In principle, so should the Kerr nonlinearity, $\delta = \delta(z)$ since this nonlinearity only exists inside the gain medium.

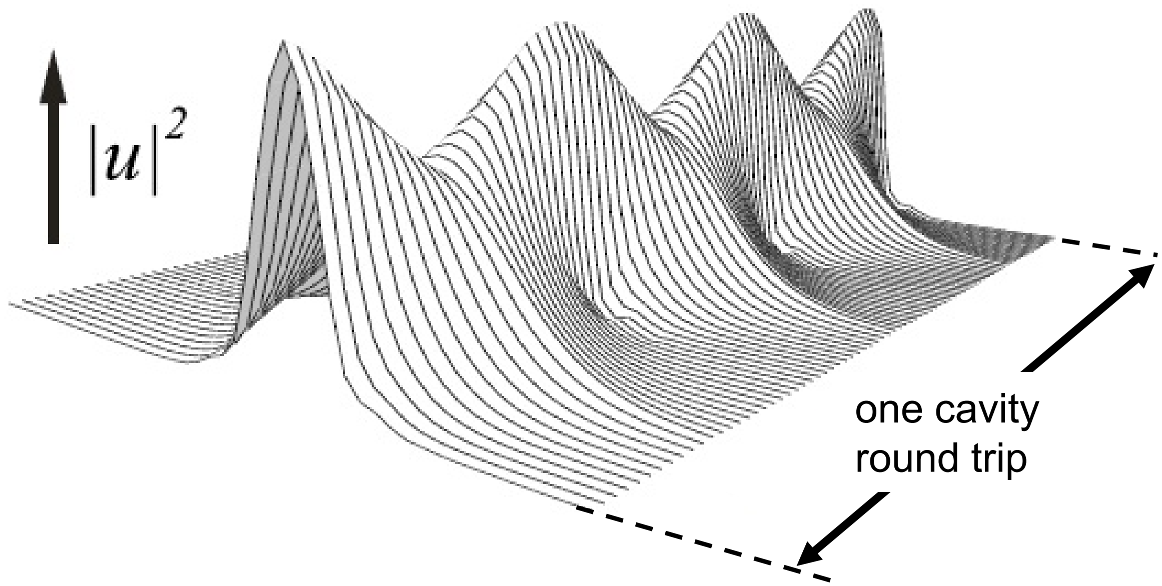
Perturbed nonlinear Schrodinger equation

$$\frac{\partial a}{\partial z} = \left(iD(z)\frac{\partial^2}{\partial t^2} - i\delta(z)|a|^2 \right) a(z)$$

$\delta(z)$ = zero except inside the Ti:sapphire crystal
$D(z)$ = cavity dispersion map (positive inside the crystal, negative between the prisms)

The pulse experiences these perturbations periodically, once per cavity round-trip.

Solution: requires numerical integration except in the asymptotic limit (infinite $|D|$)



Result: the pulse shape is not invariant! It varies during the round trip. But it is still stable at any particular location in the laser.

→ "Dispersion-managed solitons"

Q. Quraishi *et al.*, *Phys. Rev. Lett.* **94**, 243904 (2005)

In real lasers, the pulse shape is complicated

Dispersion management can produce many different pulse shapes.

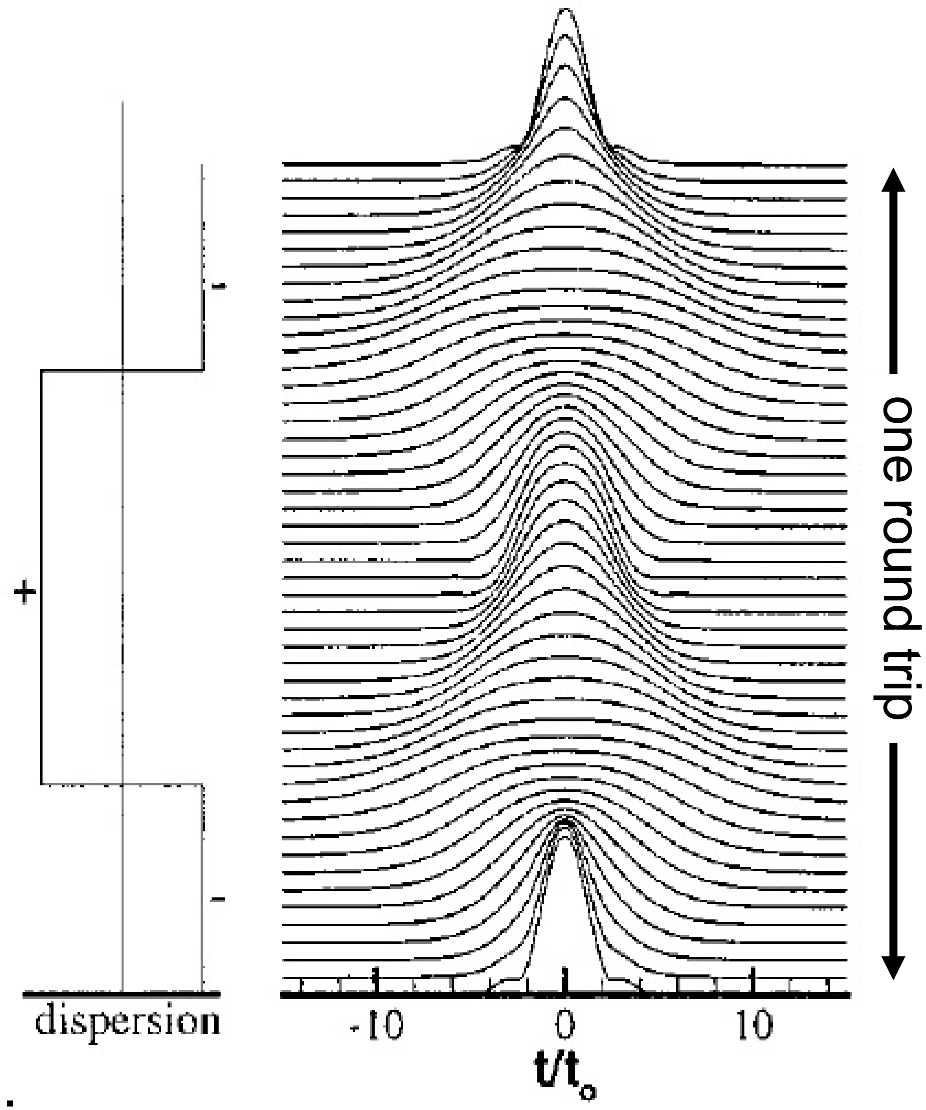
- small dispersion swing: sech pulses
- moderate dispersion: Gaussian pulses
- large dispersion: neither; the result depends sensitively on gain filtering

Also:

The perturbed NLSE neglects gain and saturable loss, both of which are required for mode-locked operation.

In principle, both would also need to be included in a distributed way:

$$g = g(z) \quad \text{and} \quad \gamma = \gamma(z)$$



Bottom line: **it's complicated**.

Y. Chen *et al.*, *J. Opt. Soc Am. B* **16**, 1999 (1999)