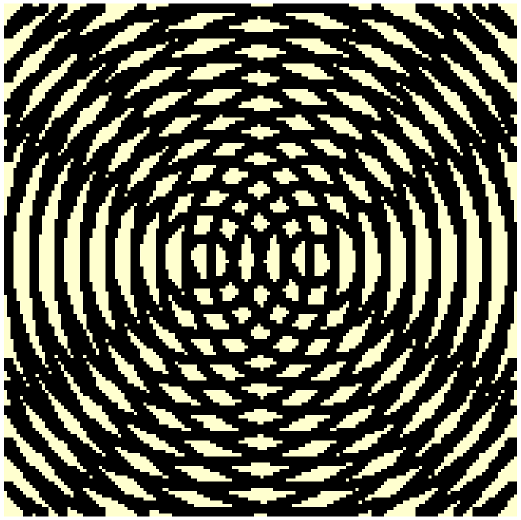
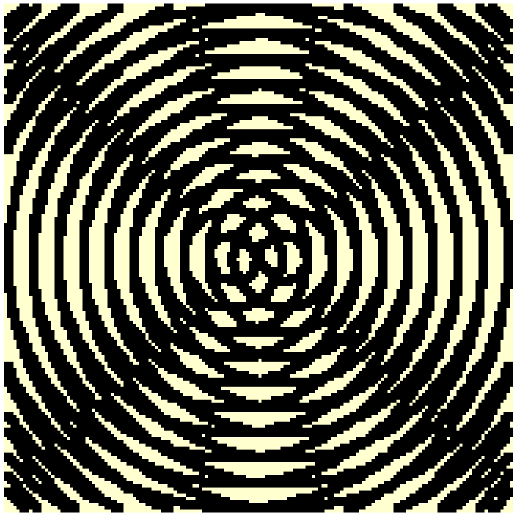
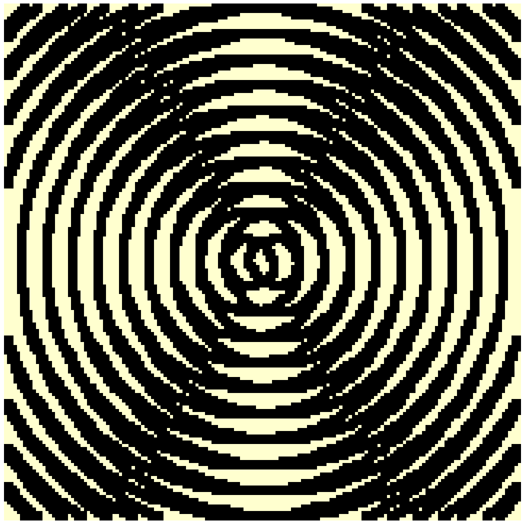
18. Standing Waves, Beats, and Group Velocity



Superposition again

Standing waves: the sum of two oppositely traveling waves





Beats: the sum of two different frequencies

Group velocity: the speed of information

Going faster than light... (but not really)

Sometimes, the refractive index is less than one.

Example: a bunch of free electrons, above the plasma frequency

$$n(\omega) \approx \sqrt{1 - \frac{\omega_P^{\,2}}{\omega^2}}$$

In the x-ray range (where $\omega >> \omega_P$), the refractive index of most materials is slightly less than unity:

$$n(\omega) \approx 1 - \delta(\omega)$$

δ is small and positive

So, how fast do waves propagate in these situations? What is the speed of light?

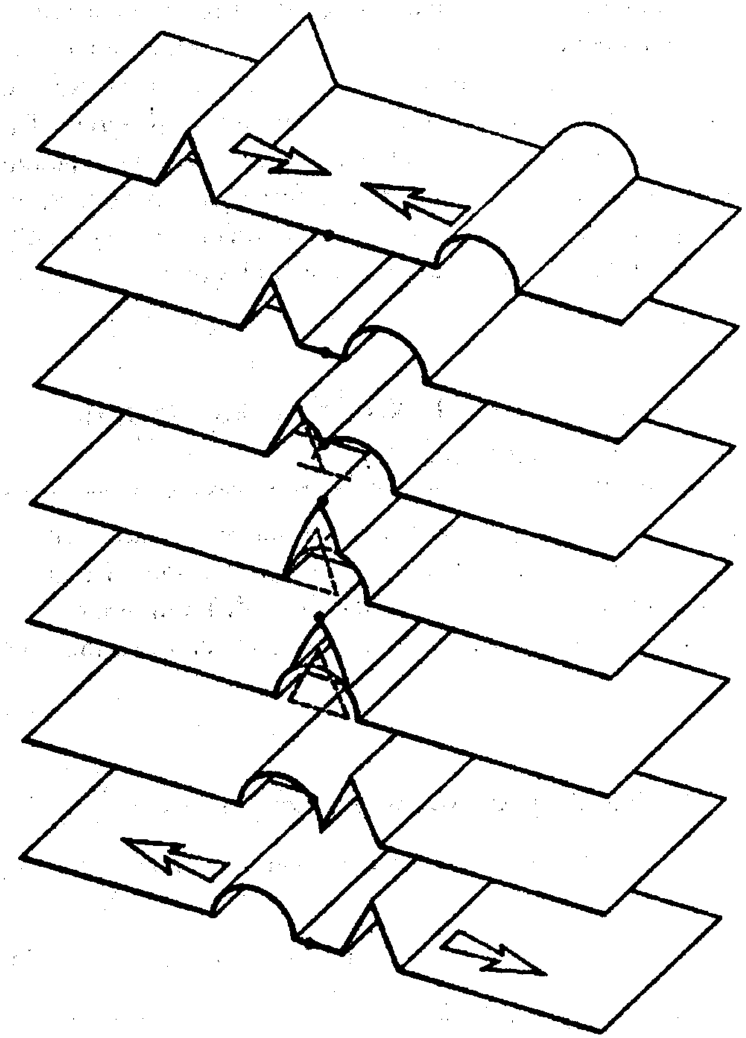
To answer this question, we need to think more carefully about what we mean by “speed”.

Superposition allows waves to pass through each other.

Recall:

If $E_1(x,t)$ and $E_2(x,t)$ are both solutions to the wave equation, then so is their sum.

Otherwise they'd get screwed up while overlapping, and wouldn't come out the same as they went in.

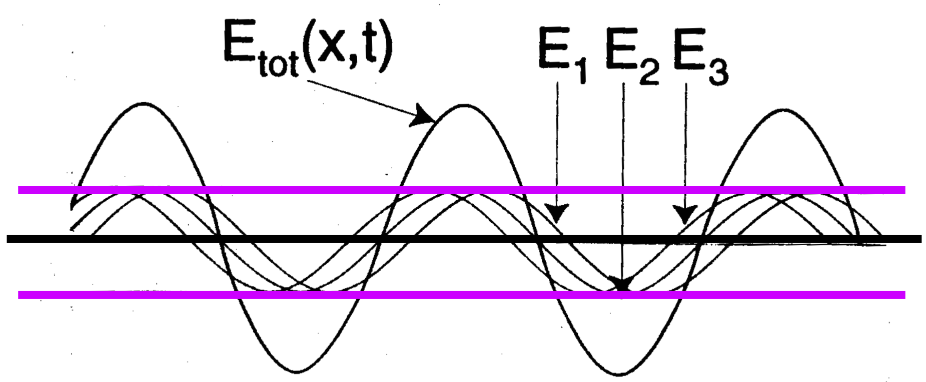


Adding waves of the same frequency, but different initial phase, yields a wave of the same frequency.

This isn't so obvious using trigonometric functions, but it's easy with complex exponentials:

$$\underset{\sim}{E}_{tot}(x,t) = \underset{\sim}{E}_1 \exp j(kx - \omega t) + \underset{\sim}{E}_2 \exp j(kx - \omega t) + \underset{\sim}{E}_3 \exp j(kx - \omega t)$$
$$= (\underset{\sim}{E}_1 + \underset{\sim}{E}_2 + \underset{\sim}{E}_3) \exp j(kx - \omega t)$$

where all the phases (other than the $kx-\omega t$) are lumped into E_1, E_2, and E_3.



Adding waves of the same frequency, but opposite direction, yields a "standing wave."

Waves propagating in opposite directions:

$$\underset{\sim}{E}_{tot}(x,t) = \underset{\sim}{E}_0 \exp j(kx - \omega t) + \underset{\sim}{E}_0 \exp j(kx + \omega t)$$

$$= \underset{\sim}{E}_0 \exp(jkx)[\exp(-j\omega t) + \exp(j\omega t)]$$

$$= 2\underset{\sim}{E}_0 \exp(jkx)\cos(\omega t)$$

Since we must take the real part of the field, this becomes:

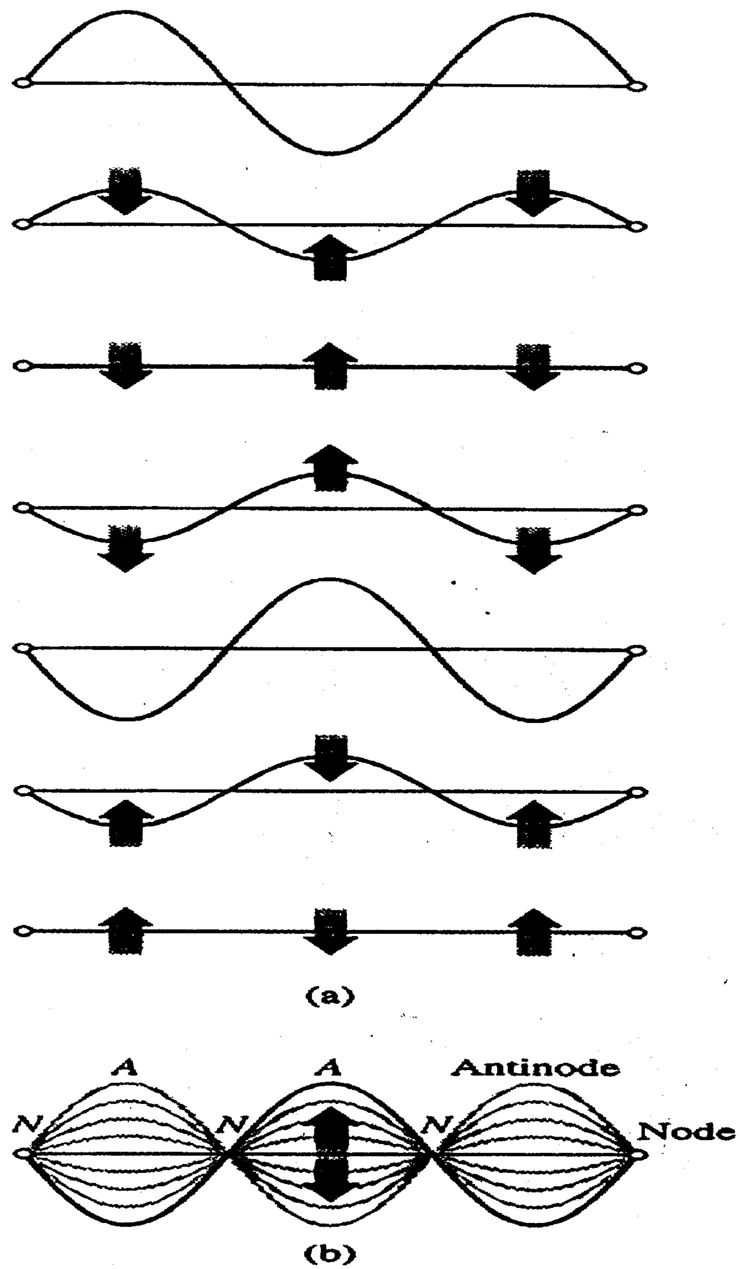
$$\boxed{E_{tot}(x,t) \;=\; 2E_0 \cos(kx)\cos(\omega t)}$$

(taking E_0 to be real)

Standing waves are important inside lasers, where beams are constantly bouncing back and forth.

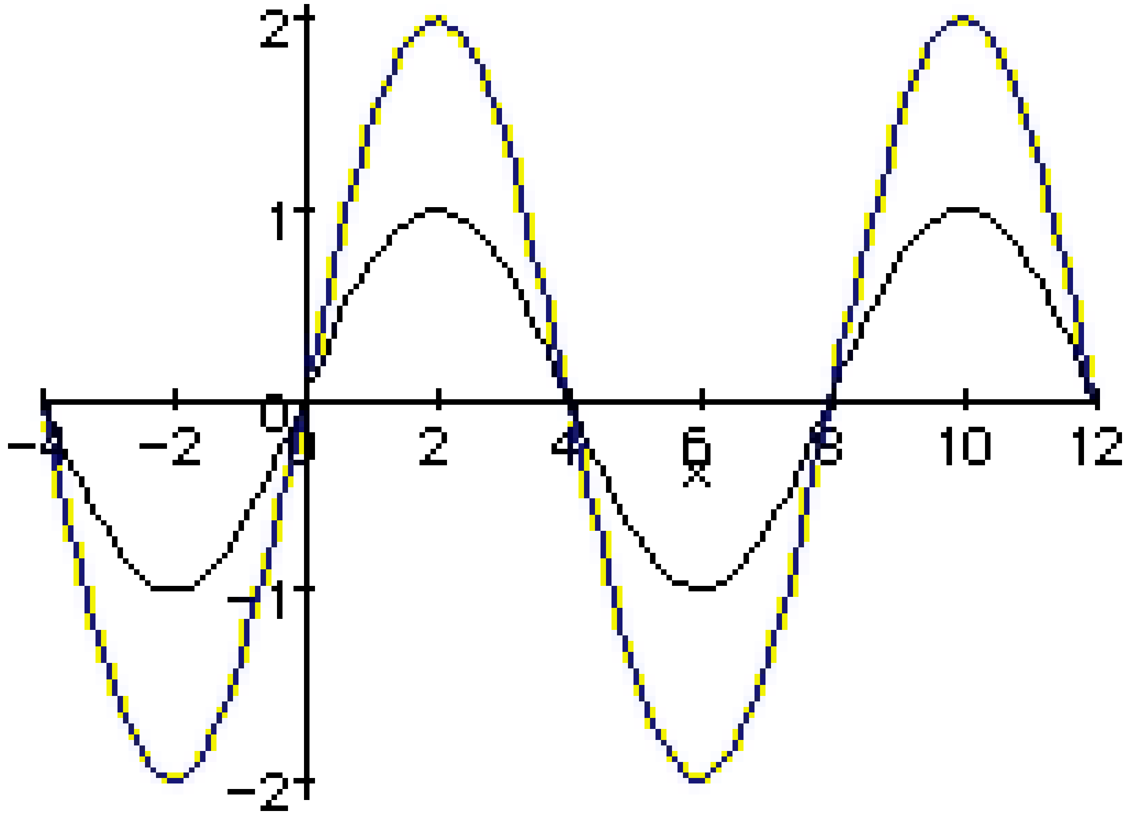
A Standing Wave

$$E_{tot}(x,t) \;=\; 2E_0 \cos(kx)\cos(\omega t)$$



A Standing Wave

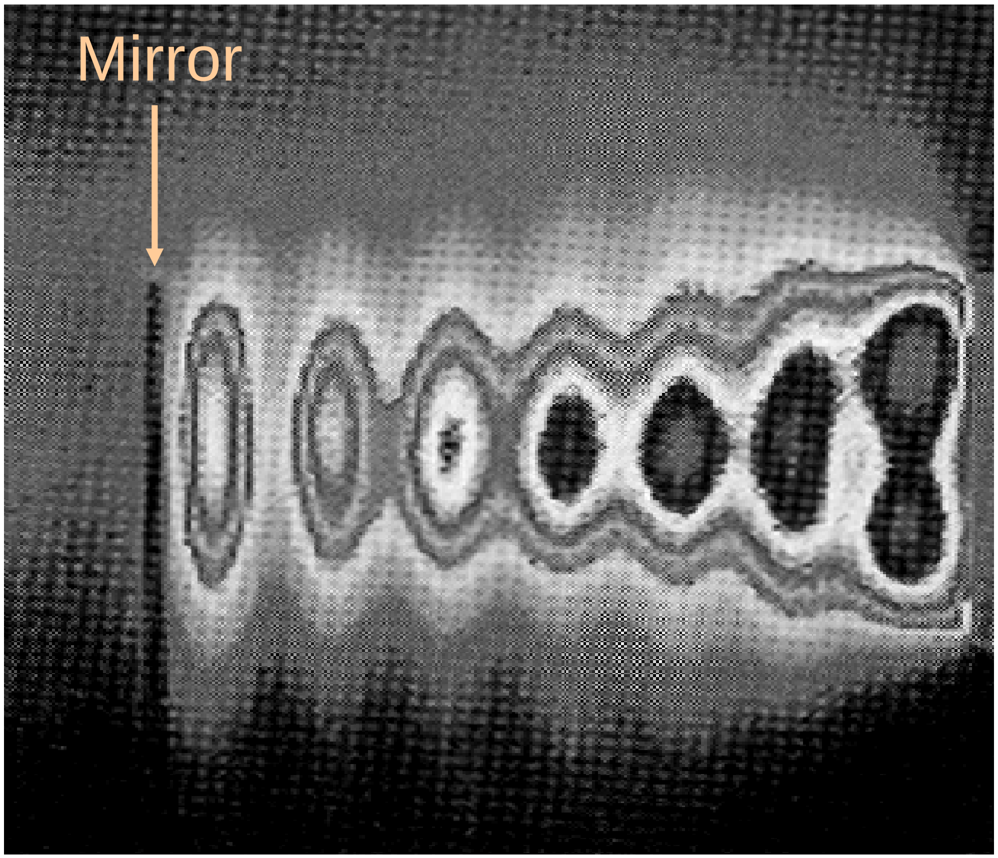
You've seen the previews.
Now, the movie!



Question: what is the speed of energy propagation here?

A Standing Wave: Experiment

3.9 GHz microwaves



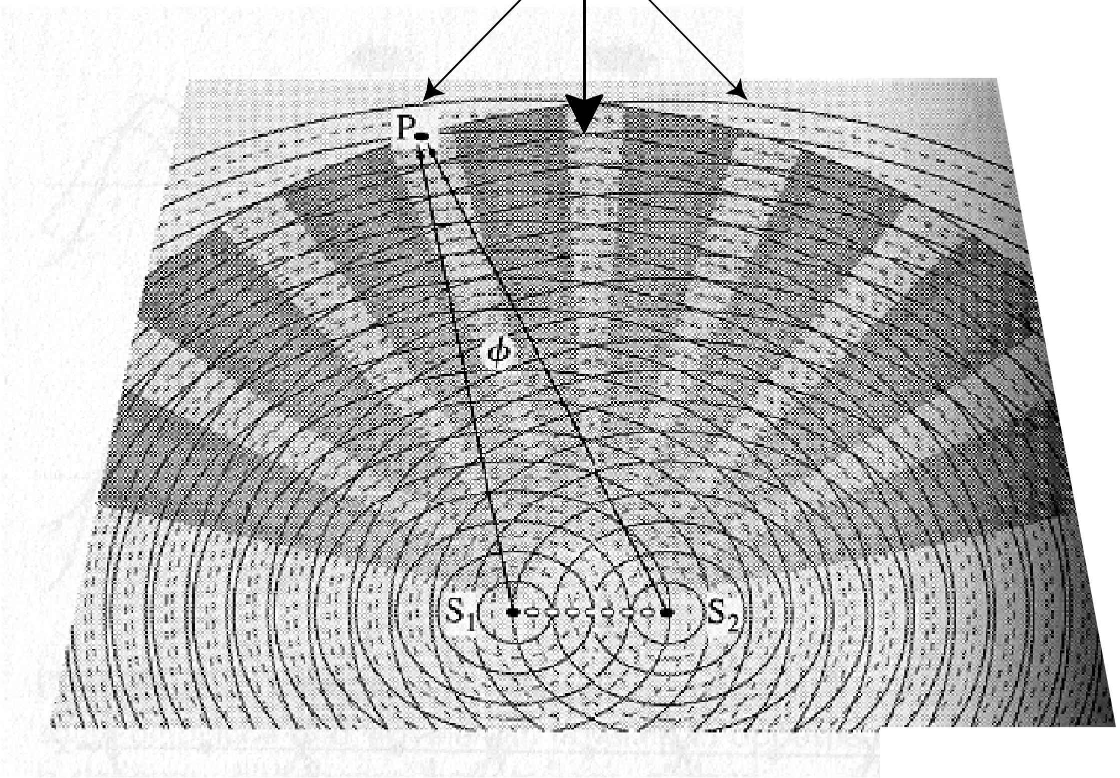
Input beam

Note the node at the reflector at left.

The same effect occurs in lasers.

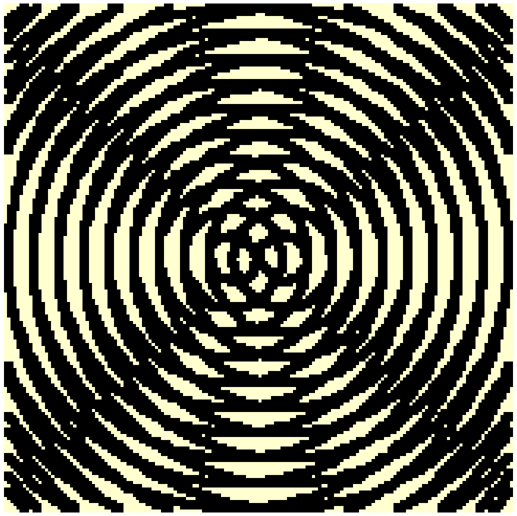
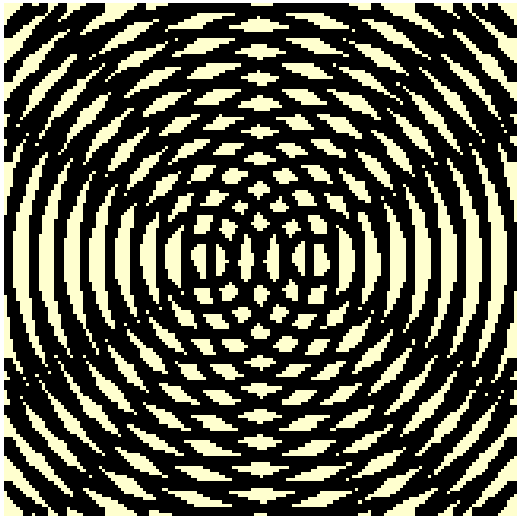
Interfering spherical waves also yield a standing wave

Antinodes



Two Point Sources

Different separations. Note the different node patterns.

When two waves of different frequency interfere, they produce beats.

$$\underset{\sim}{E}_{tot}(x,t) = \underset{\sim}{E}_0 \exp(j\omega_1 t) + \underset{\sim}{E}_0 \exp(j\omega_2 t)$$

Let $\omega_{ave} = \dfrac{\omega_1 + \omega_2}{2}$ and $\Delta\omega = \dfrac{\omega_1 - \omega_2}{2}$

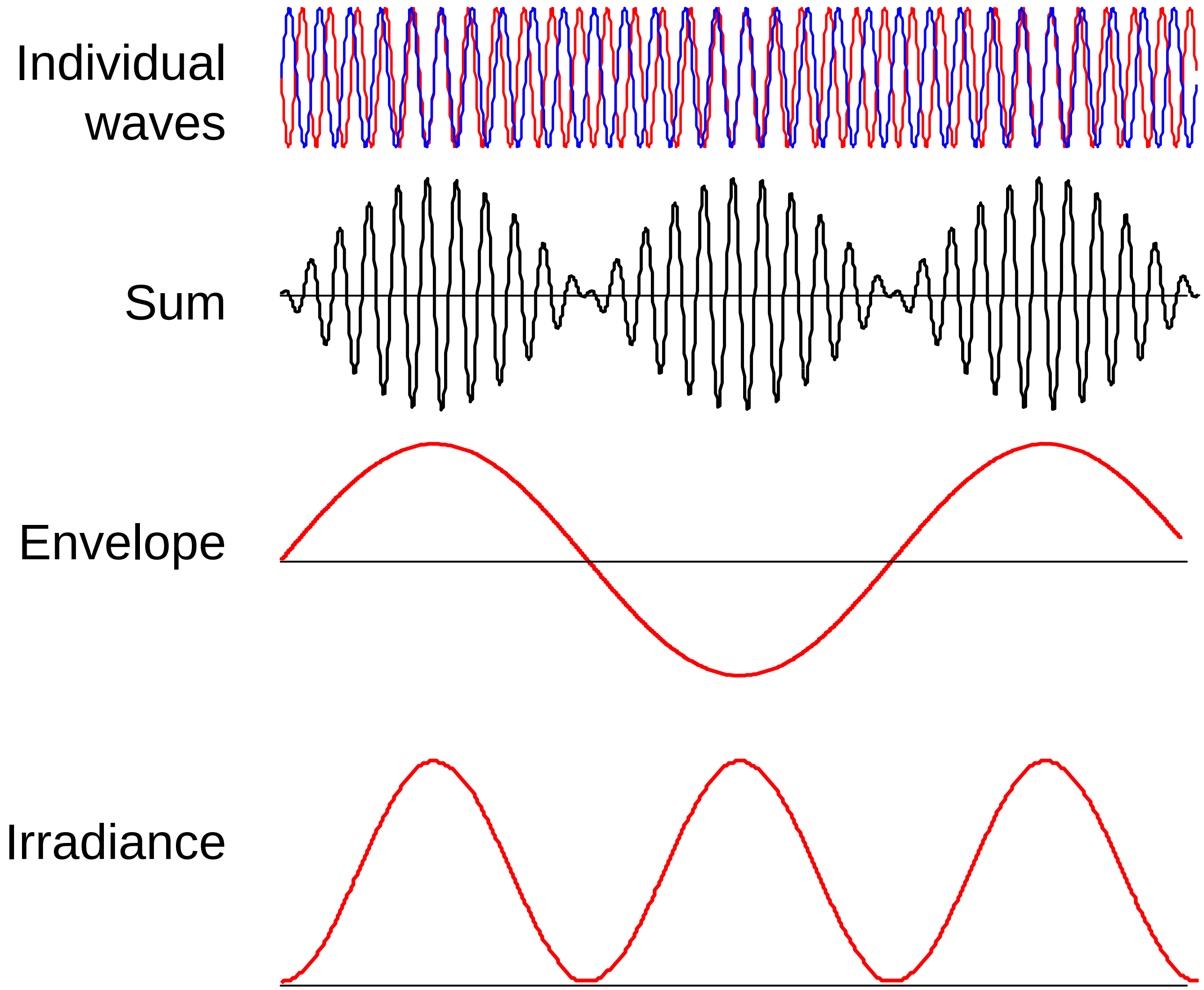
So: $\underset{\sim}{E}_{tot}(x,t) = \underset{\sim}{E}_0 \exp j(\omega_{ave} t + \Delta\omega t) + \underset{\sim}{E}_0 \exp j(\omega_{ave} t - \Delta\omega t)$

$$= \underset{\sim}{E}_0 \exp(j\omega_{ave} t)[\exp(j\Delta\omega t) + \exp(-j\Delta\omega t)]$$

$$= 2\underset{\sim}{E}_0 \exp(j\omega_{ave} t)\cos(\Delta\omega t)$$

Taking the real part yields the product of a rapidly varying cosine (ω_{ave}) and a slowly varying cosine ($\Delta\omega$).

When two waves of different frequency interfere, they produce "beats."



When two light waves of different frequency interfere, they produce beats.

$$\underset{\sim}{E}_{tot}(x,t) = \underset{\sim}{E}_0 \exp j(k_1 x - \omega_1 t) + \underset{\sim}{E}_0 \exp j(k_2 x - \omega_2 t)$$

Let $\quad k_{ave} = \dfrac{k_1 + k_2}{2}$ and $\Delta k = \dfrac{k_1 - k_2}{2}$

Similiarly, $\quad \omega_{ave} = \dfrac{\omega_1 + \omega_2}{2}$ and $\Delta\omega = \dfrac{\omega_1 - \omega_2}{2}$

So:

$$\begin{aligned}\underset{\sim}{E}_{tot}(x,t) &= \underset{\sim}{E}_0 \exp j(k_{ave}x + \Delta kx - \omega_{ave}t - \Delta\omega t) + \underset{\sim}{E}_0 \exp j(k_{ave}x - \Delta kx - \omega_{ave}t + \Delta\omega t) \\ &= \underset{\sim}{E}_0 \exp j(k_{ave}x - \omega_{ave}t)\left[\exp j(\Delta kx - \Delta\omega t) + \exp\{-j(\Delta kx - \Delta\omega t)\}\right] \\ &= 2\underset{\sim}{E}_0 \exp j(k_{ave}x - \omega_{ave}t)\cos(\Delta kx - \Delta\omega t)\end{aligned}$$

Real part: $\boxed{2E_0 \cos(k_{ave}x - \omega_{ave}t)\cos(\Delta kx - \Delta\omega t)}$

Group velocity

Light-wave beats (continued):

$$E_{tot}(x,t) = 2E_0 \cos(k_{ave}x - \omega_{ave}t) \cos(\Delta k x - \Delta\omega t)$$

This is a rapidly oscillating wave: $[\cos(k_{ave}x - \omega_{ave}t)]$

with a slowly varying amplitude: $[2E_0 \cos(\Delta k x - \Delta\omega t)]$

The phase velocity comes from the rapidly varying part: $\mathrm{v} = \omega_{ave} / k_{ave}$

What about the other velocity—**the velocity of the amplitude**?

Define the "group velocity:" $v_g \equiv \Delta\omega / \Delta k$

In general, we define the group velocity as:

$$\boxed{v_g = \frac{d\omega}{dk}}$$

Usually, group velocity is not equal to phase velocity, except in empty space.

For our example, $v_g \equiv \dfrac{\Delta\omega}{\Delta k}$

$$= \frac{c_0 k_1 - c_0 k_2}{n_1 k_1 - n_2 k_2}$$

where the subscripts 1 and 2 refer to the values at ω_1 and at ω_2. k_1 and k_2 are the k-vectors in vacuum.

If $n_1 = n_2 = n,\qquad v_g = \dfrac{c_0}{n}\dfrac{k_1 - k_2}{k_1 - k_2} = \dfrac{c_0}{n} =$ phase velocity

If $n_1 \neq n_2,\qquad v_g \neq$ phase velocity

Calculating the Group velocity

$$v_g \equiv d\omega / dk$$

Now, ω is the same in or out of the medium, but $k = k_0 n$, where k_0 is the k-vector in vacuum, and n is what depends on the medium.
So it's easier to think of ω as the independent variable:

$$\boxed{v_g \equiv \left[dk / d\omega \right]^{-1}}$$

Using $k = \omega\, n(\omega) / c_0$, calculate: $\dfrac{dk}{d\omega} = \dfrac{d}{d\omega}\left[\dfrac{n(\omega)\,\omega}{c_0}\right] = \dfrac{1}{c_0}\left[n(\omega) + \omega \dfrac{dn}{d\omega}\right]$

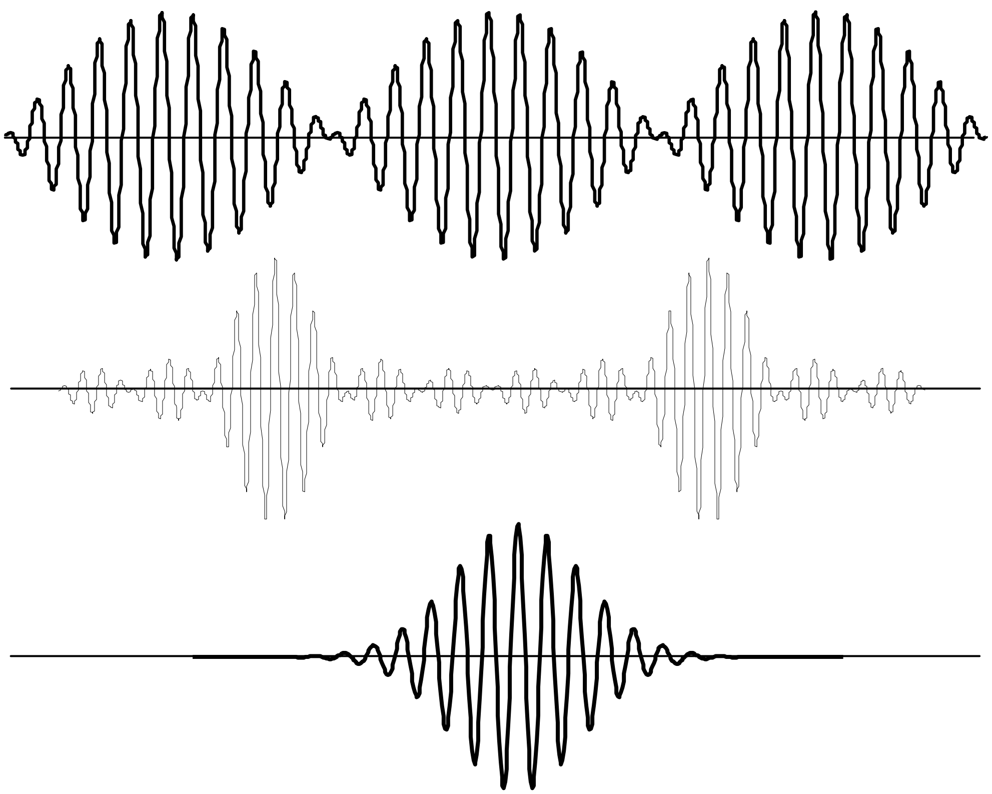
So $\quad v_g = \dfrac{c_0}{\left(n + \omega \dfrac{dn}{d\omega}\right)} \quad$ or $\quad v_g = \dfrac{v_\phi}{1 + \dfrac{\omega}{n}\dfrac{dn}{d\omega}}$

So, the group velocity equals the phase velocity when $dn/d\omega = 0$, such as in vacuum. Otherwise, since n usually increases with ω (normal dispersion), $dn/d\omega > 0$ and so usually $v_g < v_\phi$.

Why is this important?

You cannot send information using a wave, unless you make it into some kind of pulse.

You cannot make a pulse without superposing different frequencies.
Pulses travel at the group velocity.



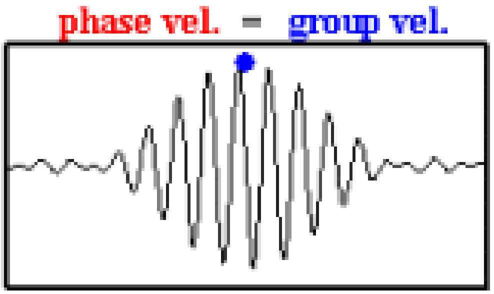
sum of 2 different frequencies

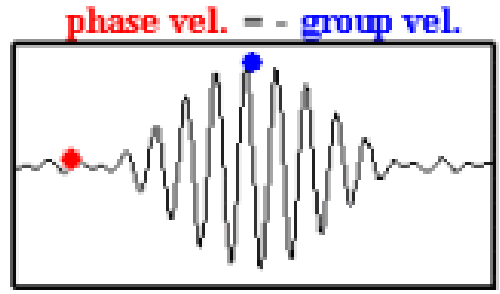
sum of 6 different frequencies

sum of many different frequencies

Group velocity (v_g) vs. phase velocity (v_ϕ)

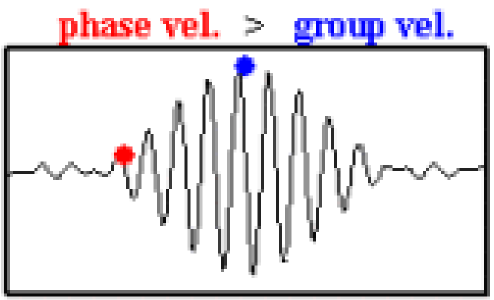
$$v_g = v_\phi$$

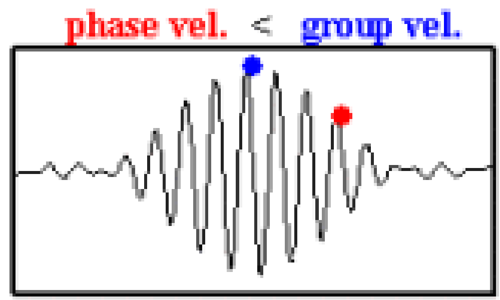




$$v_g = -v_\phi$$

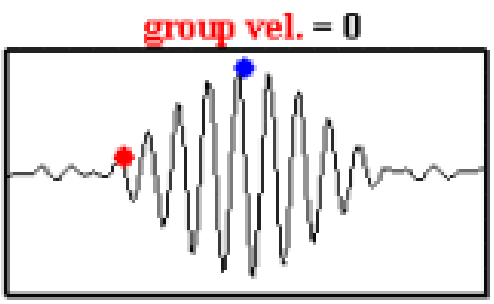
$$v_g < v_\phi$$

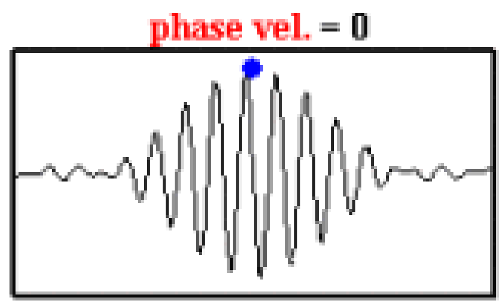




$$v_g > v_\phi$$

$$v_g = 0$$





$$v_\phi = 0$$

Source:
http://web.bryanston.co.uk/physics/Applets/Wave%20animations/Sound%20waves/Dispersive%20waves.htm

Calculating Group Velocity vs. Wavelength

We more often think of the refractive index in terms of wavelength,so let's write the group velocity in terms of the vacuum wavelength λ_0.

Use the chain rule: $$\frac{dn}{d\omega} = \frac{dn}{d\lambda_0}\frac{d\lambda_0}{d\omega}$$

Now, $\lambda_0 = \dfrac{2\pi c_0}{\omega}$, so: $$\frac{d\lambda_0}{d\omega} = \frac{-2\pi c_0}{\omega^2} = \frac{-2\pi c_0}{(2\pi c_0/\lambda_0)^2} = \frac{-\lambda_0^2}{2\pi c_0}$$

Recalling that: $$v_g = \left(\frac{c_0}{n}\right) \Big/ \left[1 + \frac{\omega}{n}\frac{dn}{d\omega}\right]$$

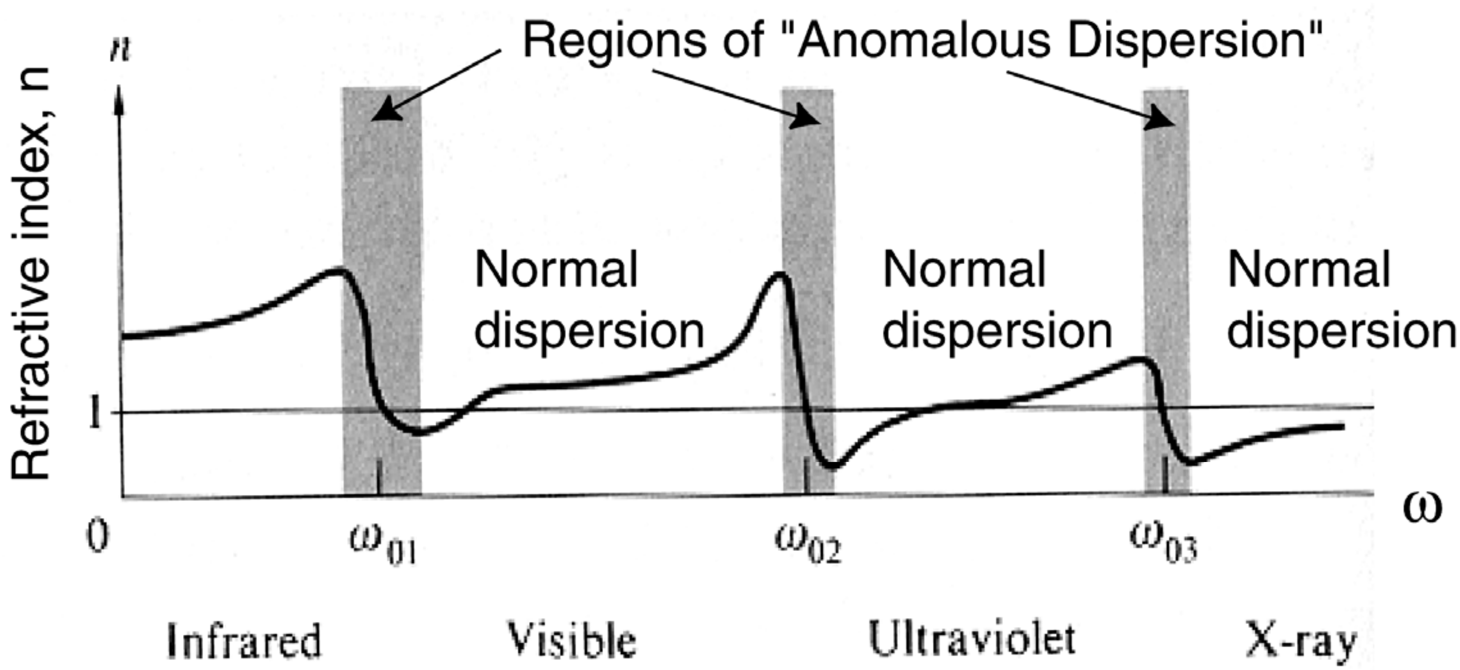
we have: $$v_g = \left(\frac{c_0}{n}\right) \Big/ \left[1 + \frac{\cancel{2\pi c_0}}{n\cancel{\lambda_0}}\left\{\frac{dn}{d\lambda_0}\left(\frac{-\lambda_0^{\cancel{2}}}{\cancel{2\pi c_0}}\right)\right\}\right]$$

or: $$\boxed{v_g = \left(\frac{c_0}{n}\right) \Big/ \left(1 - \frac{\lambda_0}{n}\frac{dn}{d\lambda_0}\right)} = \frac{c_0}{n - \lambda_0 \dfrac{dn}{d\lambda_0}}$$

The group velocity is less than the phase velocity in regions of normal dispersion

$$v_g = \frac{c_0}{n + \omega \dfrac{dn}{d\omega}}$$

In regions of normal dispersion, $dn/d\omega$ is positive. So $v_g < c_0/n < c_0$ for these frequencies.



The group velocity often depends on frequency

We have seen that the phase velocity depends on ω, because n does.

$$v_\phi = \frac{c_0}{n(\omega)}$$

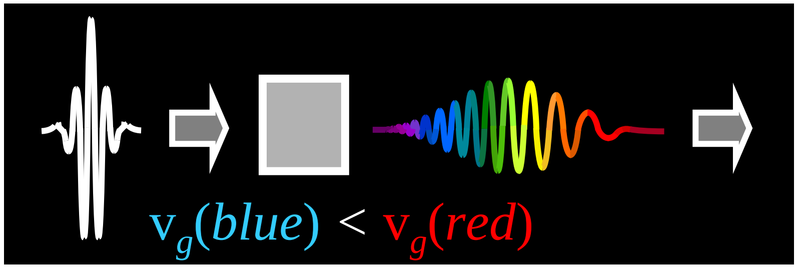
It should not be surprising that the group velocity also depends on ω.

$$v_g = \frac{c_0}{n(\omega) + \omega \frac{dn}{d\omega}}$$

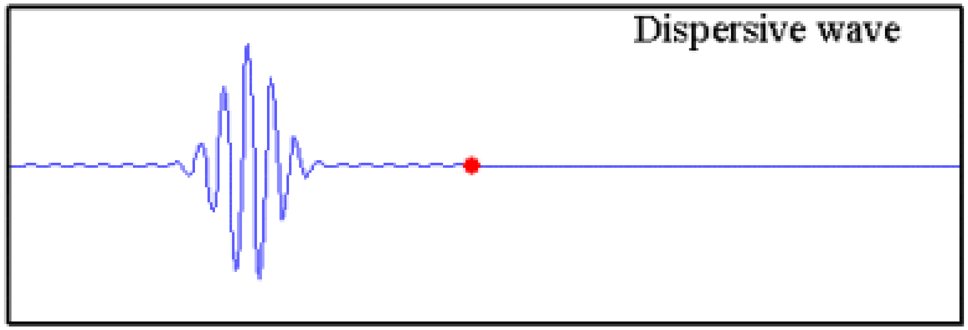
When the group velocity depends on frequency, this is known as *group velocity dispersion*, or GVD.

Just as essentially all solids and liquids exhibit dispersion, they also all exhibit GVD. This property is crucially important in the design of, e.g., optical data transfer systems that use fiber optics.

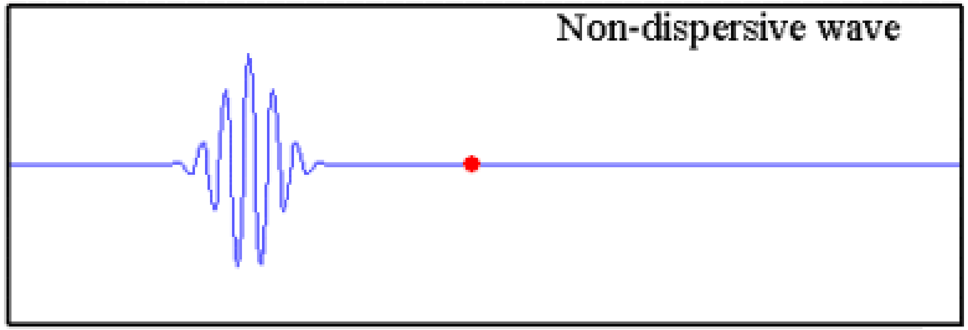
GVD distorts the shape of a pulse as it propagates in a medium



GVD means that the group velocity will be different for different wavelengths in a pulse.



GVD $\neq$ 0



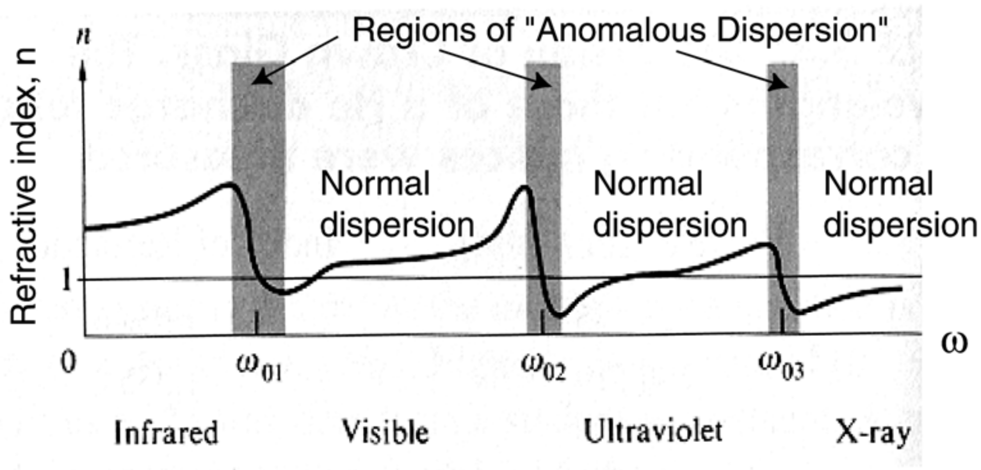
GVD = 0

Source:
http://web.bryanston.co.uk/physics/Applets/Wave%20animations/Sound%20waves/Dispersive%20waves.htm

The group velocity can exceed c_0 when dispersion is anomalous

$$v_g = \frac{c_0}{n + \omega \dfrac{dn}{d\omega}}$$

$dn/d\omega$ is negative in regions of anomalous dispersion, that is, near a resonance. So v_g exceeds v_ϕ, and can even exceed c_0 in these regions!

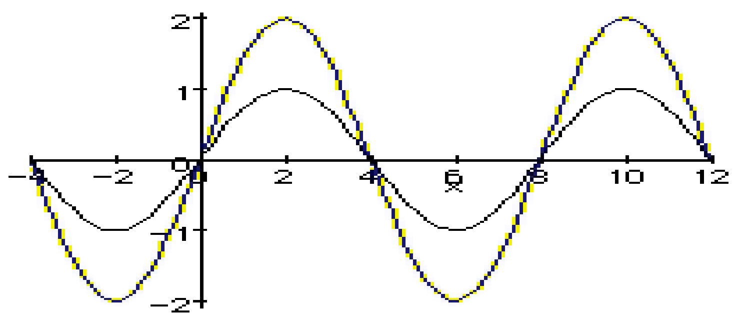


We note that absorption is strong in these regions. $dn/d\omega$ is only steep when the resonance is narrow, so only a narrow range of frequencies has $v_g > c_0$. Frequencies outside this range have $v_g < c_0$.

The group velocity can exceed c_0 when dispersion is anomalous

There is a more fundamental reason why $v_g > c_0$ doesn't necessarily bother us.

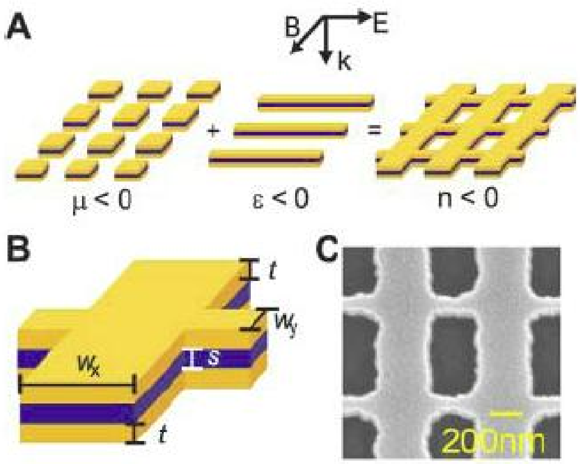
The interpretation of the group velocity as the speed of energy propagation is only valid in the case of normal dispersion! In fact, mathematically we can superpose waves to make any group velocity we desire - even zero!



For discussion, see: http://www.mathpages.com/home/kmath210/kmath210.htm

In artificially designed materials, almost any behavior is possible

Here's one recent example:



A metal/dielectric composite structure

In this material, a light pulse appears to exit the medium before entering it.

Of course, relativity and causality are *never* violated.

Simultaneous Negative Phase and Group Velocity of Light in a Metamaterial

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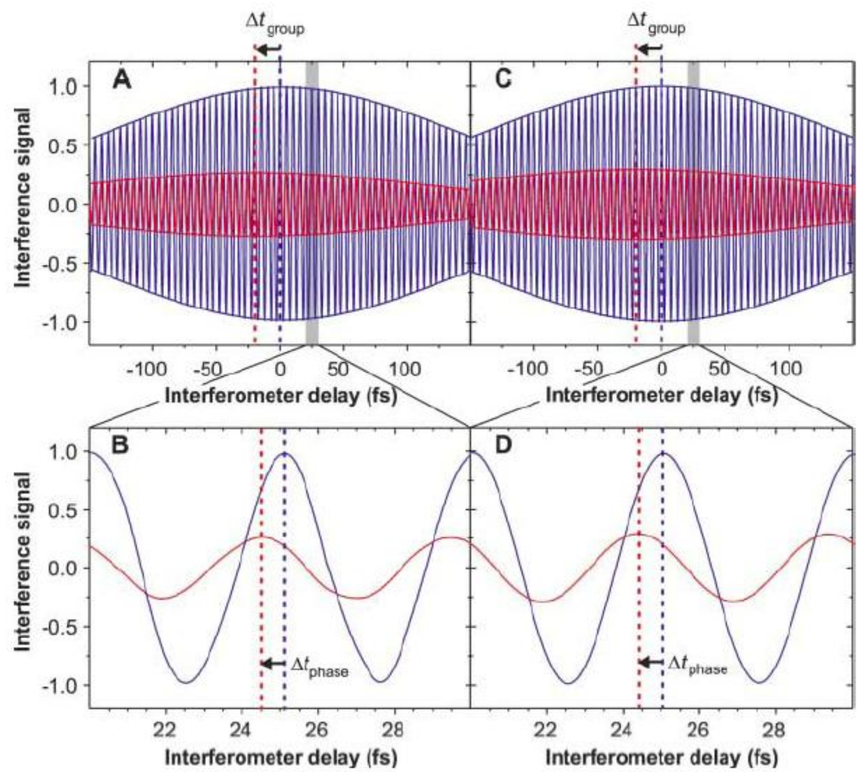

experiment theory