18. Theory of Ultrashort Laser Pulse Generation

Active mode-locking

Passive mode-locking

Including dispersion and nonlinearity: The Landau-Ginzberg Equation

The Nonlinear Schrodinger Equation

Solitons

Gaussians?
Or hyperbolic secants?

The burning question: gaussian or sech²?

After all this talk about Gaussian pulses… what does Ti:sapphire really produce?

\[ e^{-\left(\frac{t}{\tau}\right)^2} \quad \text{or} \quad \text{sech}^2\left(\frac{t}{\tau}\right) \]
Recall that many frequencies ("modes") oscillate simultaneously in a laser, and when their phases are locked, an ultrashort pulse results.

Locking the modes of a laser requires nonlinear optics. There are numerous strategies.

Mode-locking yields ultrashort pulses
Two categories of Mode Locking

**Active Mode Locking**

We insert something into the laser cavity that sinusoidally modulates the amplitude of the pulse.

⇒ mode competition couples each mode to modulation sidebands
⇒ eventually, all the modes are coupled and phase-locked

**Passive Mode Locking**

We insert something into the laser cavity that favors high intensities.

⇒ strong maxima will grow stronger at the expense of weaker ones
⇒ eventually, all of the energy is concentrated in one packet
One option for active mode-locking: the electro-optic modulator

Applying a voltage to certain crystalline materials changes the refractive indices and introduces birefringence.

A few kV can turn a crystal into a half- or quarter-wave plate.

If \( V = 0 \), the pulse polarization doesn’t change.

If \( V = V_p \), the pulse polarization switches to its orthogonal state.

Applying a sinusoidal voltage yields sinusoidal modulation to the beam’s amplitude. Or, use it without a polarizer to simply introduce a phase modulation, which sinusoidally shifts the modes into and out of resonance with the actual cavity modes.
2\textsuperscript{nd} option for active mode-locking: the acousto-optic modulator

Here, an acoustic wave induces sinusoidal density, and hence sinusoidal refractive-index, variations in a medium. This will diffract away some of a light wave’s energy.

Such diffraction can be quite strong: \(~70\%.\) Sinusoidally modulating the acoustic wave amplitude yields sinusoidal modulation of the transmitted beam.
The Modulation Theorem: The Fourier Transform of $E(t)\cos(\omega_M t)$

$$\mathcal{F}\{E(t)\cos(\omega_M t)\} = \int_{-\infty}^{\infty} E(t) \cos(\omega_M t) \exp(-i\omega t) \, dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} E(t) \left[ \exp(i\omega_M t) + \exp(-i\omega_M t) \right] \exp(-i\omega t) \, dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} E(t) \exp(-i[\omega - \omega_M]t) \, dt + \frac{1}{2} \int_{-\infty}^{\infty} E(t) \exp(-i[\omega + \omega_M]t) \, dt$$

$$\mathcal{F}\{E(t)\cos(\omega_M t)\} = \frac{1}{2} \tilde{E}(\omega - \omega_M) + \frac{1}{2} \tilde{E}(\omega + \omega_M)$$

where $\tilde{E}(\omega)$ is the FT of the original unmodulated function $E(t)$

Multiplication by $\cos(\omega_M t)$ introduces side-bands.

If $E(t) = \text{sinc}^2(t)\exp(i\omega_0 t)$:

$$\mathcal{F}\{E(t)\}$$

$$\mathcal{F}\{E(t)\cos(\omega_M t)\}$$
For mode-locking, we adjust $\omega_M$ so that $\omega_M = \text{mode spacing}$.

This means that:
$$\omega_M = \frac{2\pi}{\text{cavity round-trip time}} = \frac{2\pi}{(2L/c)} = \frac{\pi c}{L}$$

Under this condition, each mode competes for gain with adjacent modes. Most efficient operation is for phases to lock. Result is global phase locking.

Described by a system of $N$ coupled equations: $E_n \Leftrightarrow E_{n+1}, E_{n-1}$
Modeling laser modes and gain

Lasers have a mode spacing:

\[ \omega_M = \frac{2\pi}{T_R} = \frac{\pi c}{L} \]

Let the zero\(^{th}\) mode be at the center of the gain, \(\omega_0\). The \(n^{th}\) mode frequency is then:

\[ \omega_n = \omega_0 + n \omega_M \]

where \(n = \ldots, -1, 0, 1, \ldots\)

Let \(a_n\) be the amplitude of the \(n^{th}\) mode and assume a Lorentzian gain profile, \(G(n)\). Then, the amplitude increases after one round trip according to:

\[ a_n^{(k+1)} = G(n) a_n^{(k)} = \left[ 1 + \frac{g}{1 + \left( n\omega_M \right)^2 / \Omega_g^2} \right] a_n^{(k)} \]

where \(\Omega_g = \text{gain bandwidth}\)

and where the superscript \((k)\) indicates the round trip number.
Modeling an amplitude modulator

An amplitude modulator uses the electro-optic or acousto-optic effect to deliberately cause losses at the laser round-trip frequency, $\omega_M$.

A modulator multiplies the laser light (i.e., each mode) by $M[1-\cos(\omega_M t)]$

$$M \left[1 - \cos(\omega_M t)\right] a_n e^{i(\omega_o + n\omega_M)t} = M \left[-\frac{1}{2} \exp(-i\omega_M t) + 1 - \frac{1}{2} \exp(i\omega_M t)\right] a_n e^{i(\omega_o + n\omega_M)t}$$

$$= -Ma_n e^{i\omega_o t} \left[\frac{1}{2} e^{i(n-1)\omega_M t} - e^{in\omega_M t} + \frac{1}{2} e^{i(n+1)\omega_M t}\right]$$

Notice that this spreads the energy from the $n^{th}$ to the $(n+1)^{st}$ and $(n-1)^{st}$ modes. Including the passive loss, $\ell$, we can write this as:

$$a_n^{(k+1)} = a_n^{(k)} + g \left(1 + \frac{(n\omega_M)^2}{\Omega_g^2}\right)^{-1} a_n^{(k)} - \ell a_n^{(k)} + \frac{M}{2} \left(a_n^{(k+1)} - 2a_n^{(k)} + a_n^{(k-1)}\right)$$

Reminder: the superscript indicates the $k^{th}$ round trip.
Solve for the steady-state solution

Approximate: modes near the center of the Lorentzian gain, so $n\omega_M \ll \Omega_g$

thus:

$$\frac{g}{1 + (n\omega_M)^2 / \Omega_g^2} \approx g \left[ 1 - \left( \frac{n\omega_M}{\Omega_g} \right)^2 \right]$$

which results in:

$$a_n^{(t+1)} = a_n^{(t)} + g \left( 1 - \frac{(n\omega_M)^2}{\Omega_g^2} \right) a_n^{(k)} - \ell a_n^{(k)} + \frac{M}{2} \left( a_{n+1}^{(k)} - 2a_n^{(k)} + a_{n-1}^{(k)} \right)$$

In steady state, $a_n^{(k+1)} = a_n^{(k)}$

Also, the finite difference becomes a second derivative when the modes are many and closely spaced:

$$a_n^{(k)} - 2a_n^{(k)} + a_{n-1}^{(k)} \rightarrow \omega_M^2 \frac{d^2}{d\omega^2} a(\omega)$$

where, in this continuous limit,

$$a_n^{(k)} \rightarrow a(\omega)$$

where: $\omega = n \omega_M$
Solve for the steady-state solution

Thus we have:

\[
0 = \left[ g \left( 1 - \frac{\omega^2}{\Omega_g^2} \right) - \ell + \frac{M \omega_M^2}{2} \frac{d^2}{d\omega^2} \right] a(\omega)
\]

This differential equation has the solution:

\[
a(\omega) = H_v(\omega \tau) e^{-\omega^2 \tau^2 / 2} \quad \text{(Hermite Gaussians)}
\]

with the constraints:

\[
\frac{1}{\tau^4} = \frac{M \omega_M^2 \Omega_g^2}{2g} \quad \quad g - \ell = M \omega_M^2 \tau^2 \left( v + \frac{1}{2} \right)
\]

In practice, the lowest-order \( (v = 1) \) mode occurs:

\[
a(\omega) = Ae^{-\omega^2 \tau^2 / 2}
\]

Active sinusoidal modulation of the laser modes at the round-trip frequency produces a \textbf{Gaussian spectrum}!
AM mode locking: pulse duration

The solution:

\[ a(\omega) = H_v(\omega \tau) e^{-\omega^2 \tau^2 / 2} \]

with the constraints:

\[
\frac{1}{\tau^4} = \frac{M \omega_m^2 \Omega_g^2}{2g}
\]

\[ g - \ell = M \omega_m^2 \tau^2 \left( \nu + \frac{1}{2} \right) \]

The parameter \( \tau \) determines the spectral bandwidth, which in turn determines the shortest possible pulse duration:

\[ \tau = \frac{1}{\sqrt{\Omega_g}} \cdot 4 \sqrt{\frac{2g}{M \omega_m^2}} \]

In other words, the spectral bandwidth of the mode-locked pulse varies as the square root of the gain bandwidth.

AM mode-locking does not exploit the full bandwidth of an inhomogeneous medium!
Fourier transforming to the time domain

Recalling that multiplication by $-\omega^2$ in the frequency domain is just a second derivative in the time domain (and vice versa).

So this:

$$a_n^{(k+1)} - a_n^{(k)} = \left[ g \left( 1 - \frac{\omega^2}{\Omega_g^2} \right) - \ell + \frac{M \omega_M^2}{2} \frac{d^2}{d\omega^2} \right] a(\omega)$$

becomes this:

$$a^{(k+1)}(t) - a^{(k)}(t) = \left[ g \left( 1 + \frac{1}{\Omega_g^2} \frac{d^2}{dt^2} \right) - \ell - \frac{M}{2} \omega_M^2 t^2 \right] a^{(k)}(t)$$

which (in the continuous limit) has the solution:

$$a(t) = \frac{(i)^v}{\sqrt{2\pi}} H_v \left( \frac{t}{\tau} \right) e^{-t^2/2\tau^2}$$

This makes sense because Hermite-Gaussians are their own Fourier transforms.

The time-domain will prove to be better for modeling passive mode-locking.
Other active mode-locking techniques

FM mode-locking
produce a phase shift per round trip
implementation: electro-optic modulator
similar results in terms of steady-state pulse duration

Synchronous pumping
gain medium is pumped with a pulsed laser, at a rate of 1 pulse per round trip
requires an actively mode-locked laser to pump your laser ($$
requires the two cavity lengths to be accurately matched
useful for converting long AM pulses into short AM pulses
(e.g., 150 psec argon-ion pulses ⇒ sub-psec dye laser pulses)

Additive-pulse or coupled-cavity mode-locking
external cavity that feeds pulses back into main cavity synchronously
requires two cavity lengths to be matched
can be used to form sub-100-fsec pulses
Passive mode-locking

Saturable absorption:
absorption saturates during the passage of the pulse
leading edge is selectively eroded

Saturable gain:
gain saturates during the passage of the pulse
leading edge is selectively amplified

Fast absorber

Slow absorber
Kerr lensing is a type of saturable absorber

If a pulse experiences additional focusing due to the Kerr lens nonlinearity, and we align the laser for this extra focusing, then a high-intensity beam will have better overlap with the gain medium.

This is a type of saturable loss, with the same saturation behavior we’ve seen before:

\[
\frac{\Delta I}{I} \sim \frac{1}{1 + I/I_{sat}}
\]
Saturable-absorber mode-locking

Neglect gain saturation, and model a fast saturable absorber:

\[ \alpha \left( |a(t)|^2 \right) = \frac{\alpha_0}{1 + \frac{|a(t)|^2}{I_{sat}}} \]

The transmission through a fast saturable absorber of length \( L_a \):

\[ e^{-\alpha L_a} \approx 1 - \alpha L_a \approx 1 - \alpha_0 L_a \left( 1 - \frac{|a(t)|^2}{I_{sat}} \right) = 1 - \alpha_0 L_a + \gamma |a|^2 \]

where:

\[ \gamma = \frac{\alpha_0 L_a}{I_{sat}} \]

Including saturable absorption in the mode-amplitude equation (and removing the active mode-locking term proportional to \( M \)):

\[ a^{(k+1)} - a^{(k)} = \left\{ g \left( 1 + \frac{1}{\Omega_g^2} \frac{d^2}{dt^2} \right) - \ell + \gamma |a^{(k)}|^2 \right\} a^{(k)} \]

Here, we lumped the constant loss into \( \ell \).
The sech pulse shape

In steady state, this equation has the solution:

\[ a(t) = A_0 \operatorname{sech}\left(\frac{t}{\tau}\right) \]

where the conditions on \( \tau \) and \( A_0 \) are:

\[ \gamma |A_0|^2 = \frac{2g}{\Omega_g^2 \tau^2} \]

\[ g - \ell + \frac{g}{\Omega_g^2 \tau^2} = 0 \]

\[ \tau = \frac{1}{\Omega_g} \sqrt{\frac{2g}{\gamma |A_0|^2}} \]

Note: the pulse duration is now proportional to the inverse of the gain bandwidth. Passive mode locking produces shorter pulses!
The Master Equation: including GVD

Expand $k$ to second order in $\omega$:  
$$k(\omega) = k(\omega_0) + k' \Delta \omega + \frac{1}{2} k'' \Delta \omega^2$$

After propagating a distance $L_d$, the amplitude becomes:

$$a(L_d, \omega) = \exp\left\{-i \left[k(\omega_0) + k' \Delta \omega + \frac{1}{2} k'' \Delta \omega^2 \right]L_d \right\} a(0, \omega)$$

Ignore the constant phase and $v_g$, and approximate the 2nd-order phase:

$$\exp\left\{-\frac{1}{2} i k'' \Delta \omega^2 L_d \right\} a(0, \omega) \approx \left(1 - \frac{1}{2} i k'' \Delta \omega^2 L_d \right) a(0, \omega)$$

Inverse-Fourier-transforming:

$$a(L_d, t) = \left(1 + \frac{1}{2} i k''L_d \frac{d^2}{dt^2} \right) a(0, t) \equiv \left(1 + iD \frac{d^2}{dt^2} \right) a(0, t)$$

where:  
$$D \equiv \frac{1}{2} k''L_d$$
The Master Equation (continued): including the Kerr effect

The Kerr Effect: \( n = n_0 + n_2 I \) so: \[ \Phi = \frac{2\pi}{\lambda} n_2 I_k |a(t)|^2 \equiv \delta |a(t)|^2 \]

The master equation (assuming small effects) becomes:

\[
\begin{align*}
   a^{(k+1)} - a^{(k)} &= \left\{ g\left(1 + \frac{1}{\Omega_g^2} \frac{d^2}{dt^2}\right) + iD \frac{d^2}{dt^2} - \ell + (\gamma - i\delta)|a^{(k)}|^2 \right\} a^{(k)} \\
   \end{align*}
\]

In steady state: \( a^{(k+1)} - a^{(k)} = i\psi a(t) \) where \( \psi \) is the phase shift per round trip.

\[
\left\{ -i\psi + (g - \ell) + \left( \frac{g}{\Omega_g^2} + iD \right) \frac{d^2}{dt^2} + (\gamma - i\delta)|a|^2 \right\} a(t) = 0
\]

This important equation is called the Landau-Ginzberg Equation.
Solution to the Master Equation

It has an analytic solution:

$$a(t) = A_0 \left[ \text{sech} \left( \frac{t}{\tau} \right) \right]^{(1+i\beta)}$$

where:

$$-i\psi + g - \ell + \frac{(1+i\beta)}{\tau^2} \left( \frac{g}{\Omega_g^2} + iD \right) = 0$$

$$\frac{1}{\tau^2} \left( \frac{g}{\Omega_g^2} + iD \right) \left( 2 + 3i\beta - \beta^2 \right) = (\gamma - i\delta) A_0^2$$

The complex exponent yields chirp.
The pulse length and chirp parameter

\[ D_n = \frac{\Omega_g^2}{g} D \]

Figure 1.7. The plots of pulsewidth \( \tau \) and chirp parameter \( \beta \) as functions of dispersion, with the SPM coefficient as parameter. 
\( \beta \), chirp parameter; \( D_n \), normalized GVD parameter; 
\( \gamma_n \), normalized equivalent fast saturable absorber, \( \gamma_n = \gamma_n \frac{W}{2g} \Omega_g \); 
\( \delta_n \), normalized self-phase modulation parameter, \( \delta_n = \delta_n \frac{W}{2g} \Omega_g \); 
\( \tau_n \) normalized pulse width \( \tau_n = \Omega_g \tau \).
The spectral width

The spectral width vs. dispersion for various SPM values.

A broader spectrum is possible if some positive chirp is acceptable.
The Nonlinear Schrödinger Equation

Recall the master equation:

\[ a^{(k+1)} - a^{(k)} = \left\{ g \left( 1 + \frac{1}{\Omega_g^2} \frac{d^2}{dt^2} \right) + iD \frac{d^2}{dt^2} - \ell + (\gamma - i\delta)|a^{(k)}|^2 \right\} a^{(k)} \]

If you were interested in light pulses not propagating inside a laser cavity, but in a medium outside the laser, then you would change this to a continuous differential equation (rather than a difference equation with a discrete round-trip index \(k\)).

You would also ignore gain and loss (both linear loss and saturable loss).

Then you would have the Nonlinear Schrödinger Equation:

\[
\frac{\partial a}{\partial z} = \left( iD \frac{\partial^2}{\partial t^2} - i\delta|a|^2 \right) a
\]

\(D\) – GVD parameter
\(\delta\) – Kerr nonlinearity
The solution to the nonlinear Schrödinger equation is:

\[ a(z,t) = A_0 \text{sech}\left(\frac{t}{\tau}\right) \exp\left[ -i \frac{\delta |A_0|^2}{2} \right] \]

where:

\[ \frac{1}{\tau^2} = -\frac{\delta |A_0|^2}{2D} \]

Note that \( \delta/D \), must be negative, or no solution exists.

But note that, despite dispersion, the pulse length and shape do not vary with distance: **this predicts solitons**
Collision of two solitons
Evolution of a soliton from a square wave
So then do all mode-locked lasers produce sech(t) pulses?

\[
\frac{\partial a}{\partial z} = \left( iD \frac{\partial^2 }{\partial t^2} - i\delta |a|^2 \right) a(z)
\]

The Master Equation assumed that the dispersion is uniform throughout the laser cavity, so that the pulse is always experiencing a certain (constant) GVD as it propagates through one full round-trip.

But that is far from being true.
Distributed dispersion within the laser cavity

So the dispersion $D$ should depend on position within the laser cavity, $D = D(z)$. In principle, so should the Kerr nonlinearity, $\delta = \delta(z)$ since this nonlinearity only exists inside the gain medium.
Perturbed nonlinear Schrödinger equation

\[
\frac{\partial a}{\partial z} = \left( iD(z) \frac{\partial^2}{\partial t^2} - i\delta(z)|a|^2 \right) a(z)
\]

\(\delta(z) = \) zero except inside the Ti:sapphire crystal

\(D(z) = \) cavity dispersion map (positive inside the crystal, negative between the prisms)

The pulse experiences these perturbations periodically, once per cavity round-trip.

Solution: requires numerical integration except in the asymptotic limit (infinite \(|D|\))

Result: the pulse shape is not invariant! It varies during the round trip. But it is still stable at any particular location in the laser.

“Dispersion-managed solitons”

In real lasers, the pulse shape is complicated

Dispersion management can produce many different pulse shapes.

- small dispersion swing: sech pulses
- moderate dispersion: Gaussian pulses
- large dispersion: neither; the result depends sensitively on gain filtering

Also:
The perturbed NLSE neglects gain and saturable loss, both of which are required for mode-locked operation.

In principle, both would also need to be included in a distributed way:
\[
g = g(z) \quad \text{and} \quad \gamma = \gamma(z)
\]

Bottom line: it’s complicated.