# 20. The Fourier Transform in optics, II

Parseval's Theorem

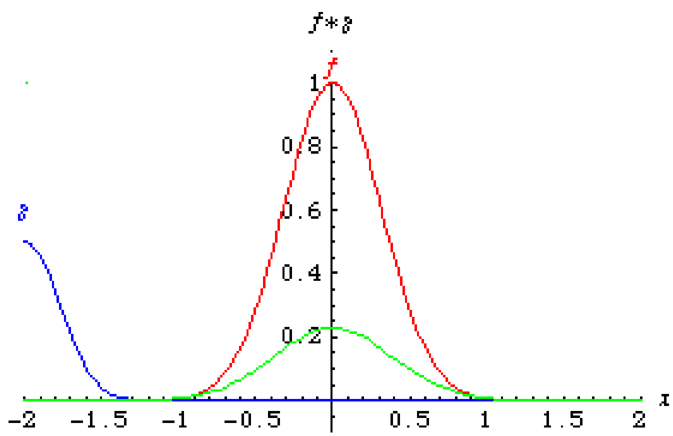
The Shift theorem

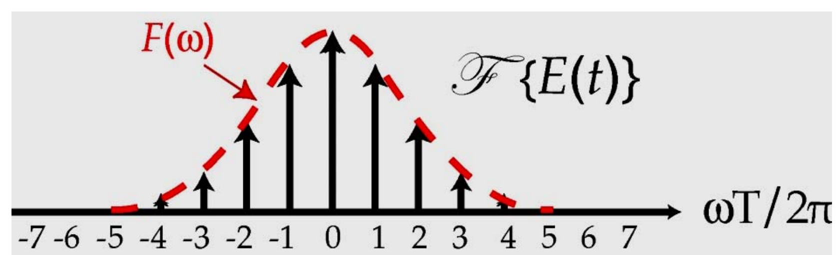
Convolutions and the Convolution Theorem

Autocorrelations and the Autocorrelation Theorem

The Shah Function in optics

The Fourier Transform of a train of pulses



# The spectrum of a light wave

The spectrum of a light wave is defined as:

$$S(\omega) = \left| F\{E(t)\} \right|^2$$

where $F\{E(t)\}$ denotes $E(\omega)$, the Fourier transform of $E(t)$.

The Fourier transform of $E(t)$ contains the same information as the original function $E(t)$. The Fourier transform is just a different way of representing a signal (in the frequency domain rather than in the time domain).

But the spectrum contains less information, because we take the magnitude of $E(\omega)$, therefore losing the phase information.

# Parseval's Theorem

Parseval's Theorem* says that the energy in a function is the same, whether you integrate over time or frequency:

$$\int_{-\infty}^{\infty} |\, f(t)|^2 \, dt \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} |\, F(\omega)\,|^2 \, d\omega$$

Proof:

$$\int_{-\infty}^{\infty} |\, f(t)|^2 \, dt = \int_{-\infty}^{\infty} f(t)\, f^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(j\omega t)\, d\omega \right] \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega') \exp(-j\omega' t)\, d\omega' \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega') \left[ \int_{-\infty}^{\infty} \exp(j[\omega - \omega']t)\, dt \right] d\omega'\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega')\, [2\pi\delta(\omega - \omega')]\, d\omega'\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, F^*(\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

* also known as Rayleigh's Identity.

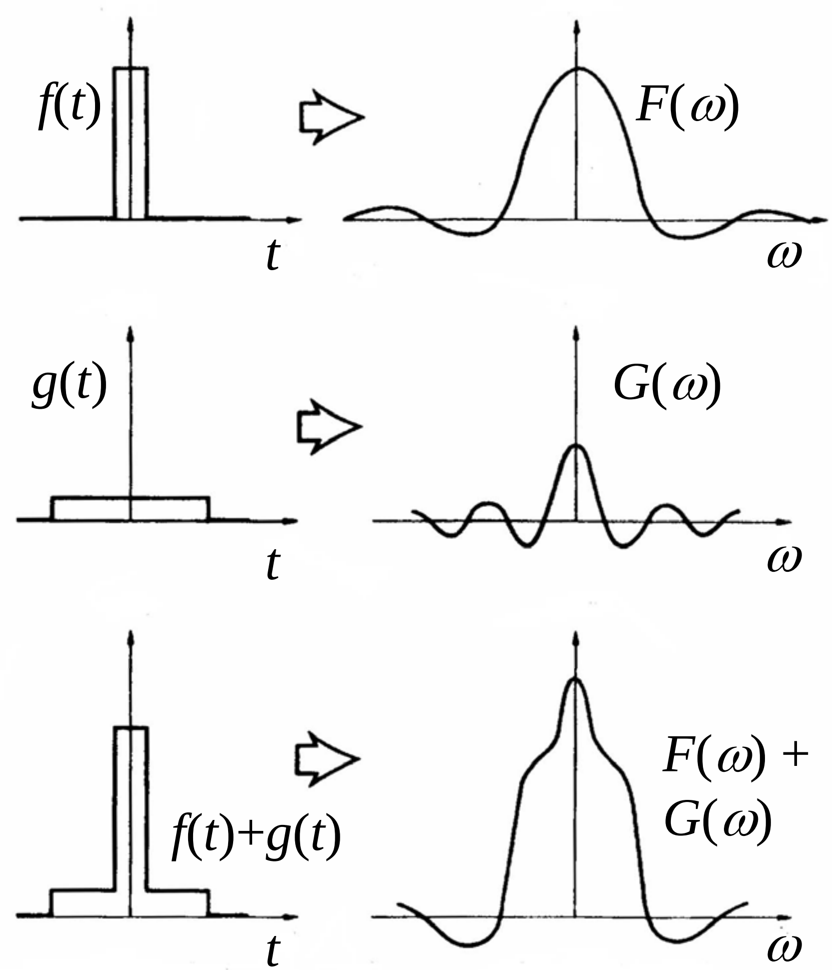
# The Fourier Transform of a sum of two functions

$$F\{af(t)+bg(t)\} =$$

$$a\cdot F\{f(t)\}+b\cdot F\{g(t)\}$$

The FT of a sum is the sum of the FT's.

Also, constants factor out.




This property reflects the fact that the Fourier transform is a linear operation.

# Shift Theorem

The Fourier transform of a shifted function, $f(t-a)$:

$$F\{f(t-a)\} = \exp(-j\omega a)F(\omega)$$

Proof :

$$F\{f(t-a)\} = \int_{-\infty}^{\infty} f(t-a)\exp(-j\omega t)dt$$

Change variables : $u = t - a$

$$\int_{-\infty}^{\infty} f(u)\exp(-j\omega[u+a])du$$

$$= \exp(-j\omega a)\int_{-\infty}^{\infty} f(u)\exp(-j\omega u)du$$

$$= \exp(-j\omega a)F(\omega)$$

QED

This theorem is important in optics, because we often encounter functions that are shifting (continuously) along the time axis – they are called waves!

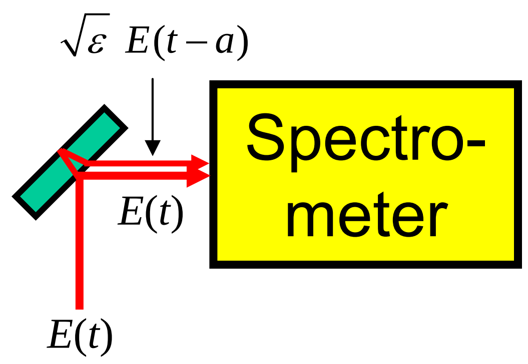
# An example of the Shift Theorem in optics

Suppose that we're measuring the spectrum of a light wave, *E(t)*, but a small fraction of the irradiance of this light, say ε, takes a different path that also leads to the spectrometer.

The extra light has the field, $\sqrt{\varepsilon}\; E(t-a)$, where $a$ is the extra path taken by the weak beam.

The measured spectrum is no longer what we expect it to be. It is contaminated! It is the magnitude-squared FT of the total field:

$$S(\omega) = \left| F\{E(t) + \sqrt{\varepsilon}\; E(t-a)\} \right|^2$$




Using the Shift Theorem:

$$= \left| E(\omega) + \sqrt{\varepsilon}\; \exp(-j\omega a)\, E(\omega) \right|^2$$

$$= \left|E(\omega)\right|^2 \left| 1 + \sqrt{\varepsilon}\; \exp(-j\omega a) \right|^2$$

$$= \left|E(\omega)\right|^2 \left[ 1 + 2\sqrt{\varepsilon}\; \cos(\omega a) + \varepsilon \right]$$

# Application of the Shift Theorem (cont'd)

$$S(\omega) = |E(\omega)|^2 \left\{ 1 + 2\sqrt{\varepsilon}\,\cos(\omega a) + \varepsilon \right\}$$

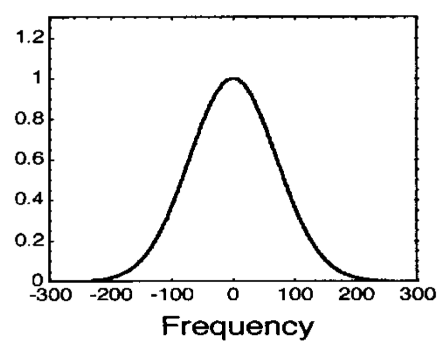
Neglecting ε compared to $\sqrt{\varepsilon}$ and 1:

$$S(\omega) \approx |E(\omega)|^2 \left\{ 1 + 2\sqrt{\varepsilon}\,\cos(\omega a) \right\}$$

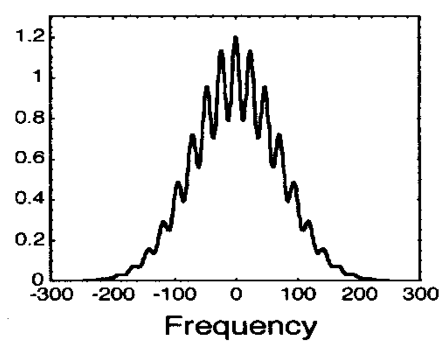
The contaminated spectrum will have ripples with a period of 2π/a.

And, these ripples will have a surprisingly large amplitude:

Original spectrum:




Contaminated spectrum:




If ε =1% (a seemingly small amount), these ripples will have an amazingly large amplitude of $2\sqrt{\varepsilon} = 20\%!$

# The Convolution

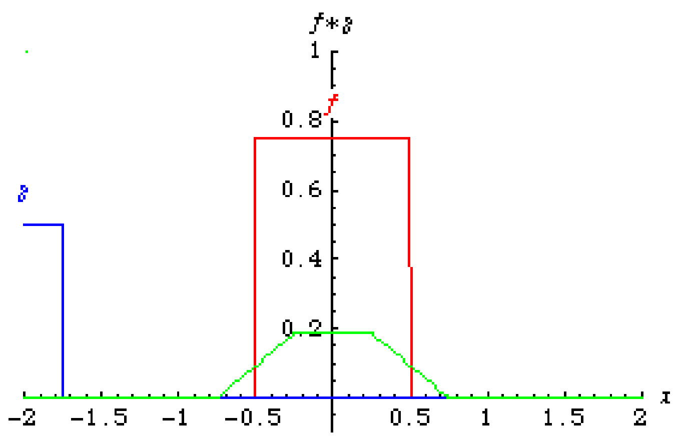
The convolution allows one function to smear or broaden another.

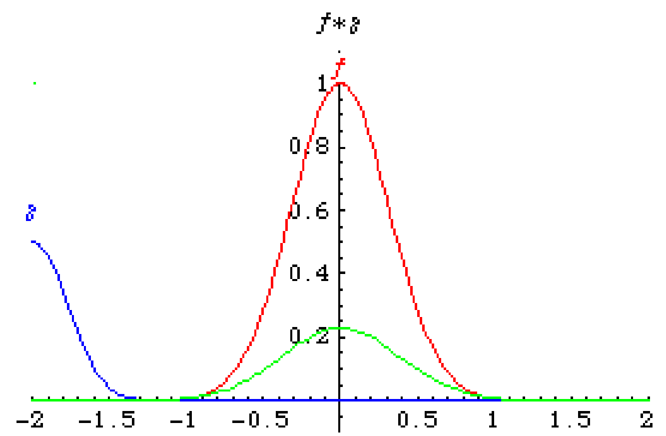
$$f(t) * g(t) \equiv \int_{-\infty}^{\infty} f(x)\, g(t-x)\, dx$$

$$= \int_{-\infty}^{\infty} f(t-x)\, g(x)\, dx$$

changing variables:

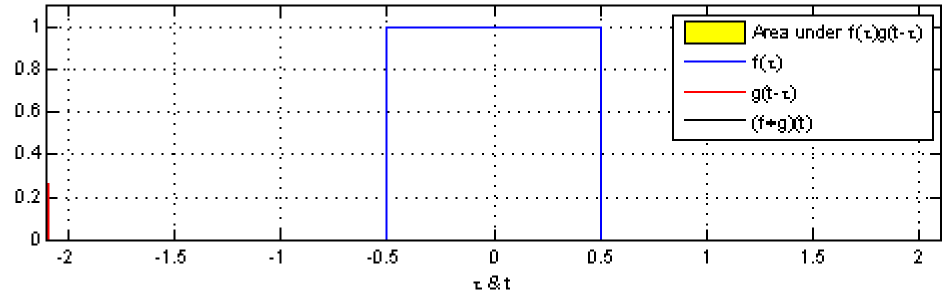
$x \rightarrow t - x$



# The convolution is sometimes easy to compute

Here, rect(x) * rect(x) = Triangle(x)




Convolution with a delta function simply shifts $f(t)$ so that it is centered on the delta-function, without changing its shape.

$$f(t) * \delta(t-a) = \int_{-\infty}^{\infty} f(t-u)\,\delta(u-a)\,du$$

$$= f(t-a)$$

This convolution does not smear out $f(t)$.

# The Convolution Theorem

The Convolution Theorem says that the FT of a convolution is the product of the Fourier Transforms:

$$F\{f(t) * g(t)\} = F(\omega) \cdot G(\omega)$$

Proof:

$$F\{f(t) * g(t)\} = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(x)\, g(t-x)\, dx \right\} \exp(-j\omega t)\, dt$$

$$= \int_{-\infty}^{\infty} f(x) \left\{ \int_{-\infty}^{\infty} g(t-x) \exp(-j\omega t)\, dt \right\} dx$$

Shift Theorem

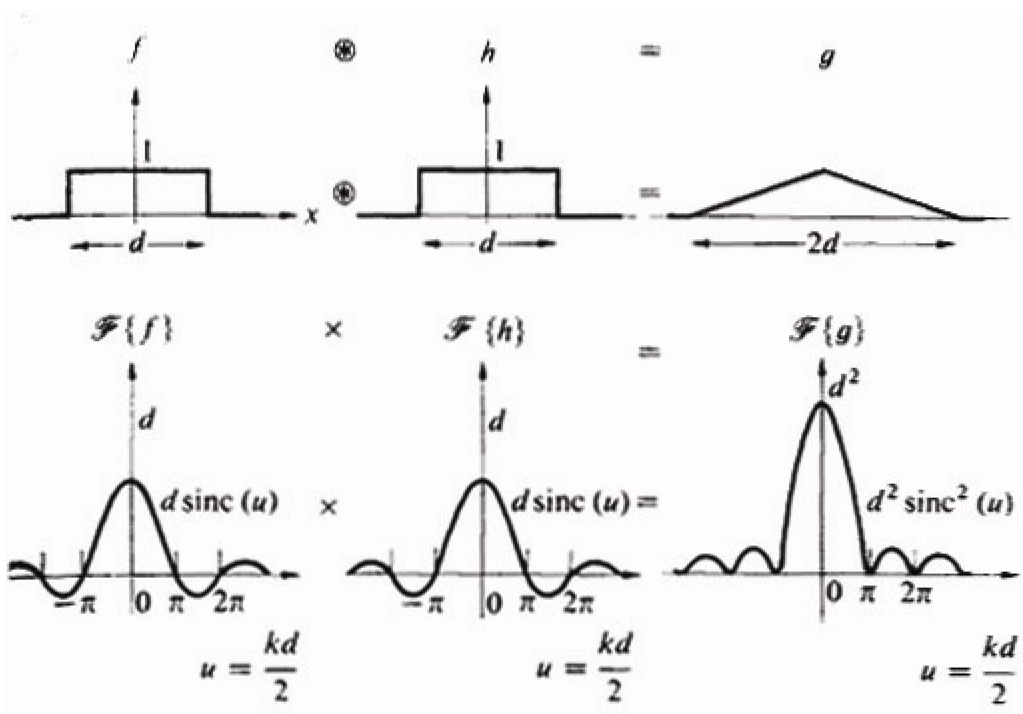
$$= \int_{-\infty}^{\infty} f(x) \left\{ G(\omega) \exp(-j\omega x) \right\} dx$$

$$= \int_{-\infty}^{\infty} f(x) \exp(-j\omega x)\, dx\, G(\omega) = F(\omega) G(\omega)$$

# The Convolution Theorem in action

$$\mathrm{rect}(x) * \mathrm{rect}(x) = \mathrm{Triangle}(x)$$

We saw last lecture that: $F\{\mathrm{rect}(x)\} = \mathrm{sinc}(k)$



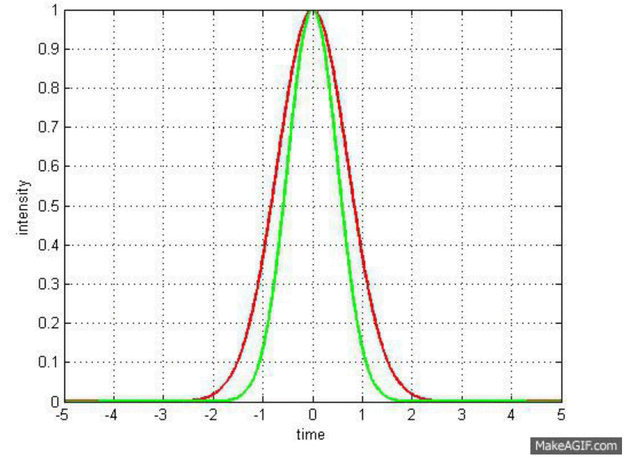

Therefore: $F\{\mathrm{Triangle}(x)\} = \mathrm{sinc}^2(k)$

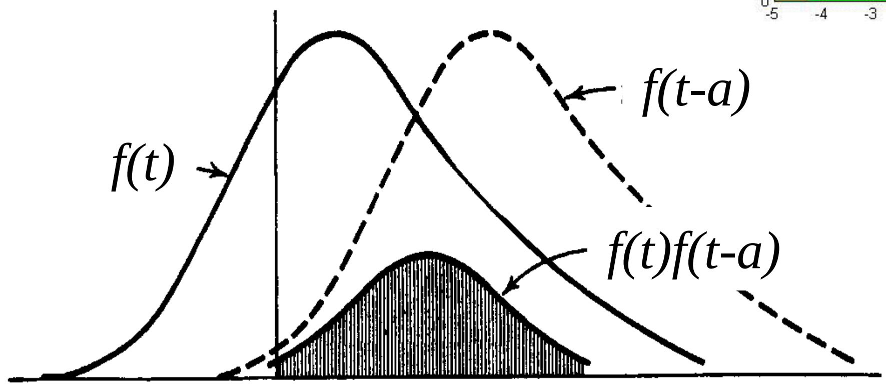
# The Autocorrelation

The ‘autocorrelation’ of a function $f(x)$ is given by the convolution of the function with itself:

$$g(t) = f(t) * f(t) = \int_{-\infty}^{\infty} f(t')\, f(t-t')\, dt'$$

A Gaussian convolved with itself




The shaded area is the value of the autocorrelation for one particular value of the displacement $a$. This represents one point in the autocorrelation $g(t)$.

In optics, we define the autocorrelation with a complex conjugate:

$$\int_{-\infty}^{\infty} f(t')\, f^*(t'-t)\, dt'$$

# The Autocorrelation Theorem

The Fourier transform of the autocorrelation of a function is equal to the spectrum of the function:

$$F\left\{\int_{-\infty}^{\infty} f(t')\, f^*(t'-t)\, dt'\right\} = \left|F\{f(t)\}\right|^2$$

The proof follows directly from:

a) the convolution theorem:

$$F\{f(t) * f^*(t)\} = F(\omega)\cdot F^*(\omega)$$

b) the definition of the spectrum:

$$S(\omega) = F(\omega)\cdot F^*(\omega) = \left|F(\omega)\right|^2$$

# The Autocorrelation Theorem in optics

$$F\left\{\int_{-\infty}^{\infty} E(t')E^*(t'-t)\,dt'\right\} = \left|F\{E(t)\}\right|^2 = \left|\tilde{E}(\omega)\right|^2 = S(\omega)$$

= the spectrum of the light!

The Fourier transform of a light-wave field's autocorrelation is its spectrum!

This relation yields an alternative technique for measuring a light wave's spectrum. It is used <u>extensively</u> for measuring the spectrum of light in the infrared, a technique known as "Fourier-transform infrared spectroscopy" (FTIR).

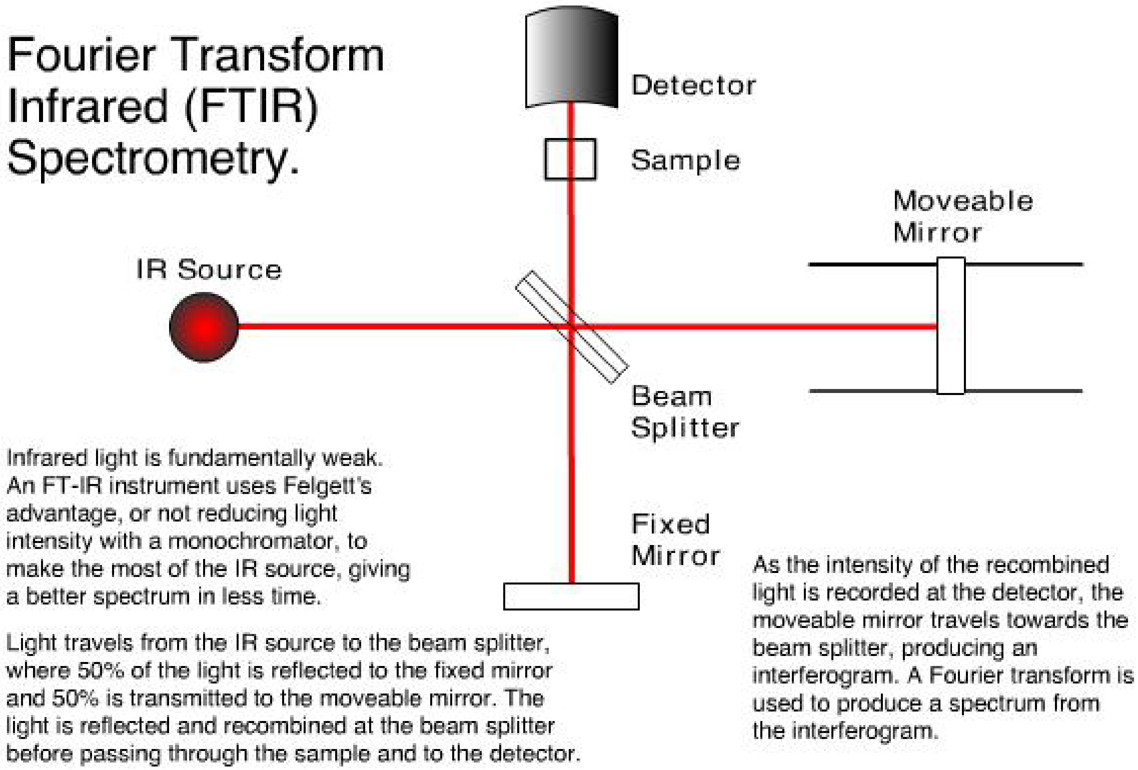
This version of the Autocorrelation Theorem is known as the "Wiener-Khinchin Theorem."

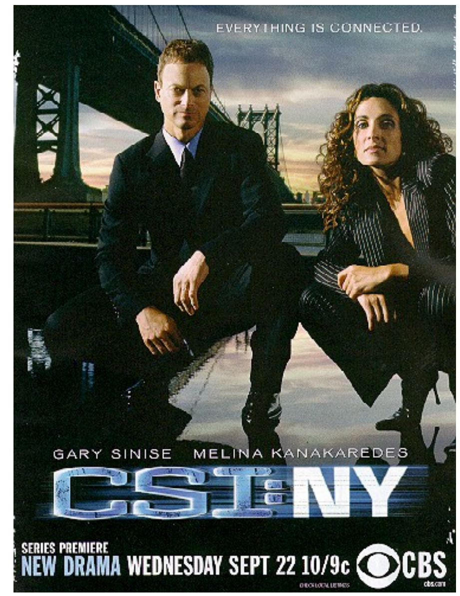
Norbert Wiener
1894-1964

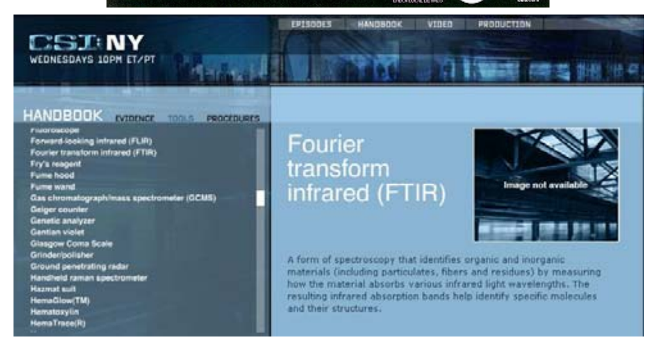
Aleksandr Khinchin
1894-1959

# Fourier-transform spectroscopy

FT spectroscopy is one of the most widely used techniques in chemical analysis.



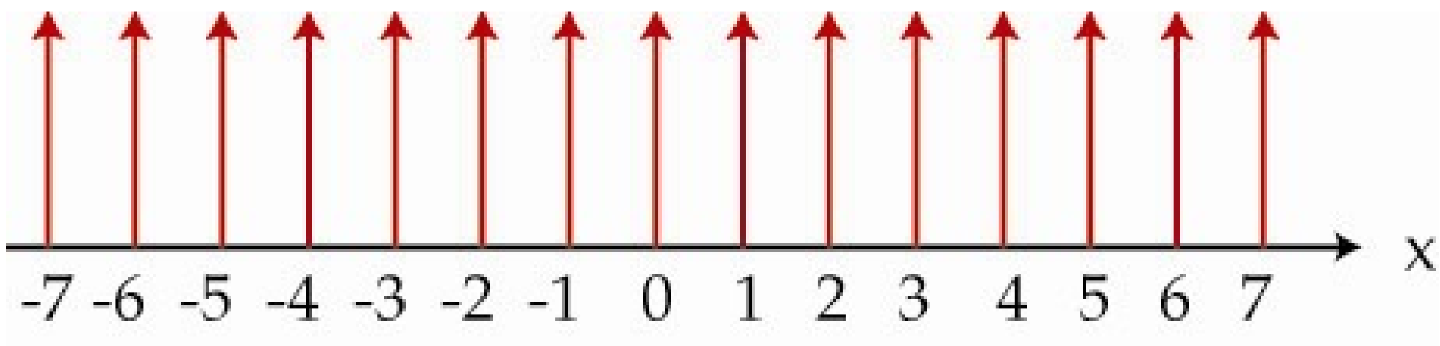




We will discuss FT spectroscopy in more detail in lecture 36.

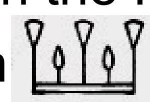
See also: http://mmrc.caltech.edu/FTIR/FTIRintro.pdf

# The Shah Function

The Shah function, III($x$), is an infinitely long train of equally spaced delta-functions.




$$III(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$

The symbol $III$ is pronounced *shah* after the Cyrillic character ш, which is said to have been modeled on the Hebrew letter ש (shin) which, in turn, may derive from the Egyptian  a hieroglyph depicting papyrus plants along the Nile.

# The Fourier Transform of the Shah Function

$$F\left\{III\left(t\right)\right\} = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(t-m)\exp(-j\omega t)dt$$

$$= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t-m)\exp(-j\omega t)dt$$

$$= \sum_{m=-\infty}^{\infty} \exp(-j\omega m)$$

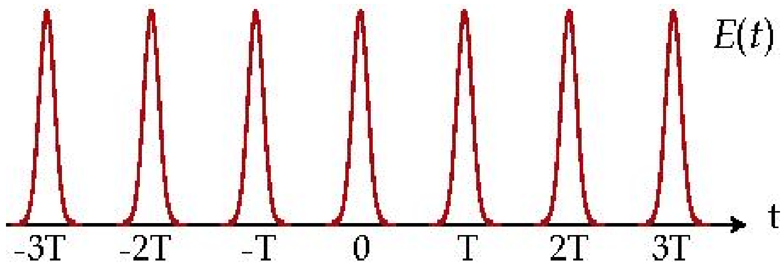
If $\omega = 2n\pi$, where *n* is an integer, the sum diverges; otherwise, cancellation occurs, and the sum vanishes.

So: $$F\left\{\mathrm{III}\left(t\right)\right\} \propto \mathrm{III}\left(\omega/2\pi\right)$$

The Fourier transform of the Shah function is another Shah function.

# The Shah Function in optics

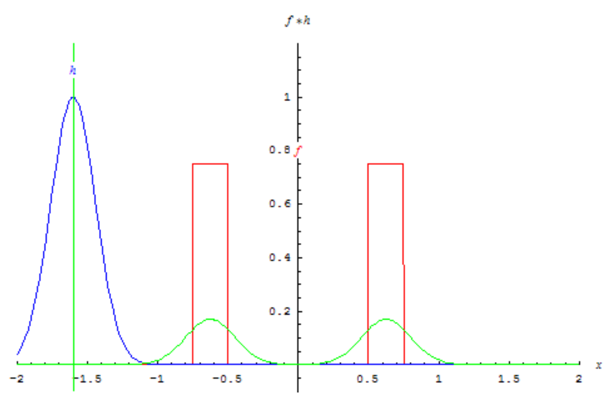
An infinite train of identical functions *f(t)* can be written as a convolution:




$$E(t) = \mathrm{III}(t/T) * f(t)$$

$$= \sum_{m=-\infty}^{\infty} f(t/T - m)$$

where *f(t)* is the shape of each pulse and $T$ is the time between pulses.

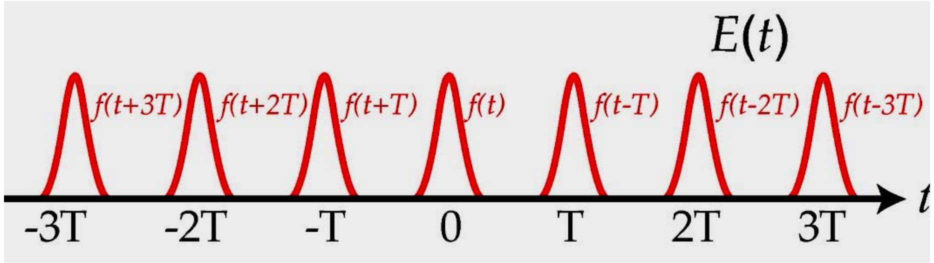



Here, a pulsed function (blue) is convolved with two narrow rectangles (red), which results in a reproduction (green) of the pulsed function centered on the locations of each of the rectangles.

Picture this repeating infinitely along the time axis, and you have the idea.

# The spectrum of an infinite train of pulses

$$E(t) = \text{III}(t/T) * f(t)$$

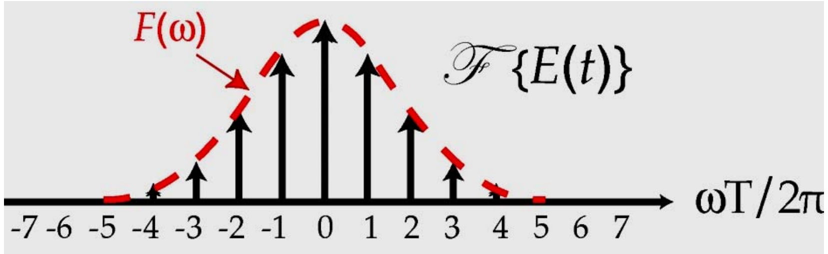



Here *f(t)* represents a single pulse and $T$ is the time between pulses.

What is the spectrum of this pulse train?

Applying the Convolution theorem:

$$E(\omega) \propto \text{III}(\omega T / 2\pi) F(\omega)$$

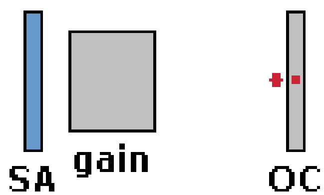



The spectrum of a train of identical pulses is a series of delta functions, with an envelope determined by $F(\omega)$, the spectrum of just one of the pulses.

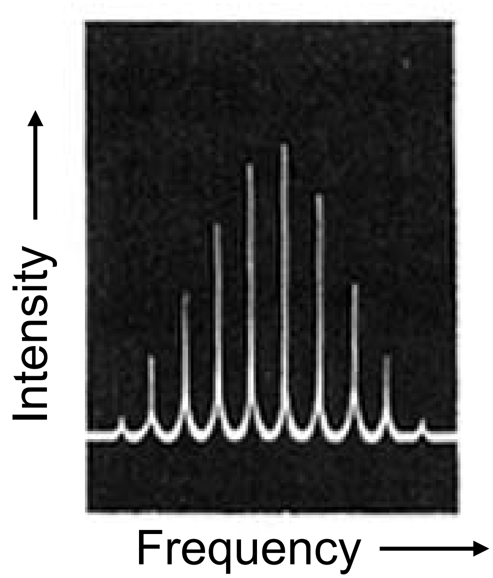
# Lasers often produce train of pulses

A single pulse bouncing back and forth inside a laser cavity, with round-trip time $T$, will produce such a pulse train.

The spacing between frequencies is then $\delta\omega = 2\pi/T$ or $\delta\nu = 1/T$.




So if you measure the spectrum of a laser like this, you'll find a series of spikes:




These are known as the "longitudinal modes" of the laser.

In fact, all lasers have these discrete modes, whether or not they are pulsed lasers.

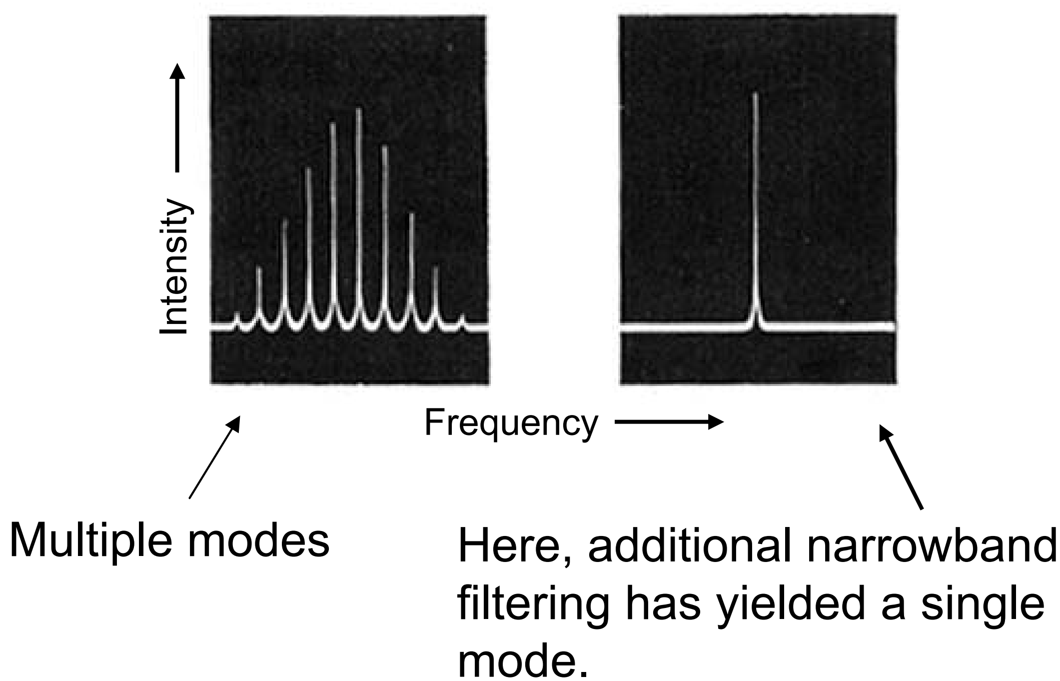
# Modes of a laser

A laser's frequencies, its "longitudinal modes," are the frequencies at which it can operate. It cannot operate at other frequencies.

The modes are separated in frequency by $1/T = c/L$, where $L$ is the round-trip length of the laser.

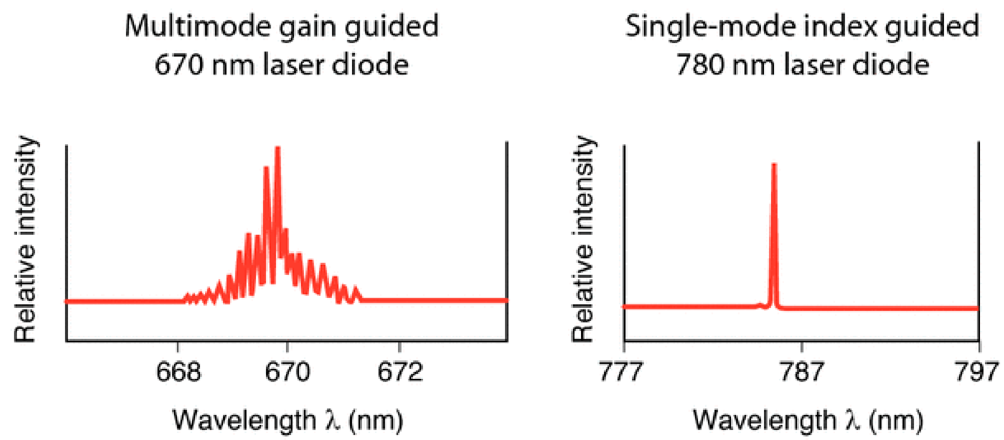
Most lasers are "multi-mode" - emitting light at more than one frequency at a time.

Some are "single mode" - only one longitudinal mode oscillates at a time.



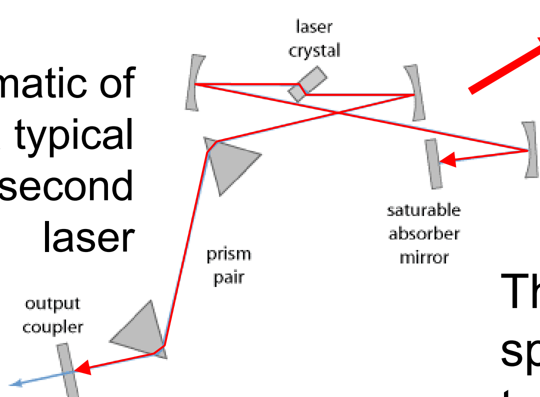
# Modes of a laser - examples

If the laser is small, then the mode spacing is large, and easy to measure.




If the laser is larger, then the mode spacing can be quite small, and hard to resolve.

schematic of a typical femtosecond laser




end-to-end cavity length: 1.5 meters
So the mode spacing is:

$$\delta\nu = \frac{c_0}{L_{RT}} = 100 \text{ MHz}$$

The spectral resolution of typical spectrometers is MUCH too coarse to resolve such closely spaced lines.