20. Gaussian beams

More ray optics:
- Imaging and the lens law
- When ray optics fails

\[ |u(x,y)|^2 \]

Gaussian beam approach:
- Paraxial wave equation
- Complex beam parameter
We describe the propagation of rays through a paraxial optical system using ray matrices.

\[
\begin{bmatrix}
    x_{\text{out}} \\
    \theta_{\text{out}}
\end{bmatrix}
= \begin{bmatrix}
    A & B \\
    C & D
\end{bmatrix}
\begin{bmatrix}
    x_{\text{in}} \\
    \theta_{\text{in}}
\end{bmatrix}
\]

Something interesting occurs if this coefficient happens to be equal to zero.
A system images an object when \( B = 0 \).

When \( B = 0 \), we have:

\[
x_{\text{out}} = A x_{\text{in}}
\]

All rays from a point \( x_{\text{in}} \) arrive at a point \( x_{\text{out}} \), independent of the input angle \( \theta_{\text{in}} \). This tells us that an image is formed.

\( B = 0 \) is a necessary condition for imaging.

And when \( B = 0 \), then \( A \) is the magnification.

Let’s compute the ray matrix for this configuration, \( R_{\text{image}} \).
The Lens Law

From the object to the image, we have:

1) A distance \( d_o \)
2) A lens of focal length \( f \)
3) A distance \( d_i \)

\[
R_{\text{image}} = \begin{bmatrix} 1 & d_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & d_o \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & d_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_o \\ -1/f & 1-d_o/f \end{bmatrix} = \begin{bmatrix} 1-d_i/f & d_o+d_i-d_o d_i/f \\ -1/f & 1-d_o/f \end{bmatrix}
\]

\[
B = d_o + d_i - d_o d_i / f = d_o d_i [1/d_o + 1/d_i - 1/f]
\]

So \( B = 0 \) if:

\[
\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}
\]

This is called the “Lens Law.”

(only valid if the imaging system consists of a single lens)
If the imaging condition is satisfied, then:

\[
\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}
\]

is satisfied, then:

\[
R_{image} = \begin{bmatrix}
1 - d_i/f & 0 \\
-1/f & 1 - d_o/f
\end{bmatrix}
\]

So:

\[
R_{image} = \begin{bmatrix}
M & 0 \\
-1/f & 1/M
\end{bmatrix}
\]

\[
A = 1 - d_i/f = 1 - d_i \left[ \frac{1}{d_o} + \frac{1}{d_i} \right]
\]

\[
\Rightarrow \text{ (writing } A = M, \text{ magnification)}
\]

\[
M = -d_i/d_o
\]

\[
D = 1 - d_o/f = 1 - d_o \left[ \frac{1}{d_o} + \frac{1}{d_i} \right]
\]

\[
D = -d_o/d_i = 1/M
\]
Real images

When $d_o > f$, the object is *more* than one focal length away from the lens, then $d_i > 0$.

A positive lens projects a real image on the opposite side of the lens from the object.

![Diagram of a real image](image)

$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$

Projectors and camera lenses work this way. So does the lens in your retina. The inverted image projected onto your cornea is inverted again by your brain, so that things look right-side-up.
Virtual images

When $d_o < f$, the object is less than one focal length away from the lens, then $d_i < 0$.

No real image occurs. But a virtual image is observed if you look back through the lens.

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$$

This is a magnifying glass.
Failures of the ray optics approach

Input beam (bundle of rays)

Distance $d = f$

Each ray has a unique $x_{in}$, but all have $\theta_{in} = 0$ (parallel rays)

Thin lens, focal length $f$

What happens in this plane (focal plane)? In particular, how big is the illumination spot?

\[
\begin{bmatrix}
  x_{out} \\
  \theta_{out}
\end{bmatrix} = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 & f \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ 0 \end{bmatrix}
\]

\[x_{out} = 0 \quad \theta_{out} = -x_{in}/f\]

According to this analysis, the spot size in the focal plane is identically zero!

This is obviously wrong: if we want to compute the spot size, then we need something better than ray optics!
What if ray optics is not good enough?

Start with the wave equation:
\[
\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}
\]

Assume a harmonic time dependence, of the form \( e^{j\omega t} \).

Also, assume that the variation along the direction of propagation (z axis) is given by:
\[ e^{jkz} \times \text{(a slowly varying function of } z) \]

(slowly varying with respect to both \( e^{jkz} \) and also the transverse variation)
These assumptions imply:  \( E(x,y,z,t) = u(x,y,z) \cdot e^{j\omega t - jkz} \)

and:  \( \left| \frac{\partial^2 u}{\partial z^2} \right| \ll \left| \frac{\partial^2 u}{\partial x^2} \right|, \left| \frac{\partial^2 u}{\partial y^2} \right|, \text{ and } \left| k \frac{\partial u}{\partial z} \right| \)  \( (k = \omega/c) \)

(Note: this is, essentially, the paraxial approximation)

Now, plug this form for \( E(x,y,z,t) \) into the wave equation.
Paraxial wave equation

\[ E(x,y,z,t) = u(x,y,z) \cdot e^{j\omega t - jkz} \]

x, y, and t derivatives are easy:

\[
\begin{align*}
\frac{\partial^2 E}{\partial x^2} &= \frac{\partial^2 u}{\partial x^2} e^{j(\omega t - kz)} \\
\frac{\partial^2 E}{\partial y^2} &= \frac{\partial^2 u}{\partial y^2} e^{j(\omega t - kz)} \\
\frac{\partial^2 E}{\partial t^2} &= -\omega^2 u e^{j(\omega t - kz)}
\end{align*}
\]

z derivative:

\[
\begin{align*}
\frac{\partial E}{\partial z} &= -jk \times ue^{j(\omega t - kz)} + \frac{\partial u}{\partial z} e^{j(\omega t - kz)} \\
\frac{\partial^2 E}{\partial z^2} &= -k^2 \times ue^{j(\omega t - kz)} - 2jk \frac{\partial u}{\partial z} e^{j(\omega t - kz)} + \frac{\partial^2 u}{\partial z^2} e^{j(\omega t - kz)}
\end{align*}
\]

wave equation becomes:

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - k^2 u - 2jk \frac{\partial u}{\partial z} = -\frac{\omega^2}{c^2} u
\]

the “paraxial wave equation”
Paraxial and exact wave equations

**exact wave equation**

\[
\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}
\]

A well-known solution to the exact equation:

\[
E(r) = \frac{e^{-jk(r-r_0)}}{(r-r_0)}
\]

a spherical wave emerging from the point \( r = r_0 \).

**Paraxial wave equation**

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 2jk \frac{\partial u}{\partial z} = 0
\]

\[
E(x,y,z,t) = u(x,y,z) \cdot e^{j\omega t - jkz}
\]

Can we use this as a guide to find a solution to the (approximate) paraxial equation?
Paraxial approximation to a spherical wave

Note that:
\[
(r - r_0) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}
\]

\[
= (z - z_0) \cdot \sqrt{1 + \frac{(x - x_0)^2 + (y - y_0)^2}{(z - z_0)^2}}
\]

In the spirit of the paraxial approximation, we assume that the relevant values of \((x - x_0)\) and \((y - y_0)\) are always small compared to \((z - z_0)\).

In that case:
\[
\frac{(x - x_0)^2 + (y - y_0)^2}{(z - z_0)^2} \ll 1
\]

\[
(r - r_0) = (z - z_0) \cdot \sqrt{1 + \varepsilon}
\]

where
\[
\varepsilon = \frac{(x - x_0)^2 + (y - y_0)^2}{(z - z_0)^2}
\]
Paraxial approximation to a spherical wave

Taylor expansion of a square root:

$$\sqrt{1 + \varepsilon} \approx 1 + \frac{1}{2} \varepsilon + \ldots$$

Therefore:

$$r - r_0 \approx (z - z_0) + \frac{(x - x_0)^2 + (y - y_0)^2}{2(z - z_0)} + \ldots$$

Plug this approximate form for \( r - r_0 \) into the expression for the spherical wave:

$$E(x, y, z, t) = \frac{e^{-jk(z-z_0)}}{r} e^{-jk\left(\frac{(x-x_0)^2 + (y-y_0)^2}{2(z-z_0)}\right)} e^{j\omega t}$$

We will ignore this term because it is small.
Paraxial spherical waves

A paraxial spherical wave:

\[
\begin{align*}
\exp \left\{ -jk \frac{(x-x_0)^2 + (y-y_0)^2}{2(z-z_0)} \right\}
\end{align*}
\]

\[
\begin{align*}
u(x, y, z) &= \frac{\exp \left\{ -jk \frac{(x-x_0)^2 + (y-y_0)^2}{2(z-z_0)} \right\}}{(z-z_0)}
\end{align*}
\]

This is an approximation of the solution to the exact equation. Is it also an exact solution to the approximate equation?

Well, it’s a bit tedious to prove, but YES it is.

\[
\begin{align*}
E(x, y, z, t) &= u(x, y, z)e^{-jkt + j\omega t} = \frac{\exp \left\{ -jk \frac{(x-x_0)^2 + (y-y_0)^2}{2(z-z_0)} \right\}}{(z-z_0)}e^{-jkt + j\omega t}
\end{align*}
\]
Gaussian spherical waves

To make things a bit more compact, we define: \( R(z) = z - z_0 \)

\[
u(x, y, z) = \frac{\exp \left\{-jk \frac{\left(x-x_0\right)^2 + \left(y-y_0\right)^2}{2R(z)}\right\}}{R(z)}\]

We can then see that the phase of the wave, in a fixed transverse plane (i.e., at fixed \( z \)), varies as:

\[
\phi(x, y) = k \frac{\left(x-x_0\right)^2 + \left(y-y_0\right)^2}{2R(z)}
\]

Just as with the spherical wave we started with, the surfaces of constant phase are spherical. The radius of these spheres is \( R(z) \).

The radius of curvature increases linearly with increasing propagation distance.
If the radius of curvature at \( z = z_0 \) is \( R_0 \), then we should have defined \( R(z) = R_0 + (z - z_0) \).

\[
R_0 = \text{radius of curvature at } z = z_0
\]
The size of the beam is still infinite

What have we gained from all of this manipulation?

\[
E(x, y, z, t) = \frac{\exp\left\{-jk \frac{(x - x_0)^2 + (y - y_0)^2}{2R(z)}\right\}}{R(z)} e^{jkz - j\omega t}
\]

Seemingly not much! This is still not a very useful expression!

How does this E-field vary at a particular value of z, say \(z = 0\)? In other words, what is the transverse profile of the field?

\[
|E(x, y, 0, t)| = \frac{1}{R(0)}
\]

It has a constant transverse profile, even for \(x, y \to \infty\)

→ Not a particularly realistic description of a “beam”
Complex source point

A simple modification to fix this: let the source location be complex!

(Is this legal? YES - this parameter cancels out when this form is inserted into the wave equation - it has no physical meaning!)

Replace the real $R(z)$ by the complex quantity $q(z) = q_0 + (z - z_0)$

Also, without loss of generality, we can set $x_0 = y_0 = 0$.

$$u(x, y, z) = \frac{1}{q(z)} \exp \left( -jk \frac{x^2 + y^2}{2q(z)} \right)$$

(Note: $q$ has units of length)

If $q$ is complex, then so is $1/q$: $\frac{1}{q(z)} = \text{Re} \left\{ \frac{1}{q(z)} \right\} + j \text{Im} \left\{ \frac{1}{q(z)} \right\}$
Complex beam parameter $q(z)$

$$u(x, y, z) = \frac{1}{q(z)} \exp \left( -jk \frac{x^2 + y^2}{2} \frac{1}{q(z)} \right)$$

$$= \frac{1}{q(z)} \exp \left[ -jk \frac{x^2 + y^2}{2} \left( \text{Re} \left\{ \frac{1}{q(z)} \right\} + j \text{Im} \left\{ \frac{1}{q(z)} \right\} \right) \right]$$

This part plays the same role as $R(z)$, the phase front radius of curvature.

This part is $\exp\{\text{real number}\}$! It gives us the finite extent in the transverse dimension.

$$u(x, y, z) = \frac{1}{q(z)} \exp \left[ -jk \frac{x^2 + y^2}{2} \text{Re} \left\{ \frac{1}{q(z)} \right\} \right] \exp \left( k \frac{x^2 + y^2}{2} \text{Im} \left\{ \frac{1}{q(z)} \right\} \right)$$

This part defines the phase, i.e., the wave fronts

This part is real! It tells us the $x$ and $y$ dependence of the amplitude!
Constraints on \( q(z) \)

\[ u(x, y, z) = \frac{1}{q(z)} \exp \left[ -jk \frac{x^2 + y^2}{2} \Re \left\{ \frac{1}{q(z)} \right\} \right] \exp \left( \frac{\pi}{\lambda} (x^2 + y^2) \Im \left\{ \frac{1}{q(z)} \right\} \right) \]

- \( \Re\{1/q(z)\} \) can take on any value, except infinity (since that would correspond to a zero radius of curvature, which makes no sense!).

- the imaginary part of \( 1/q(z) \) MUST be negative! Otherwise, \( u(x,y,z) \) diverges to infinity as \( x,y \) increase.

Redefine \( R(z) \) and define \( w(z) \) such that

\[
\frac{1}{q(z)} = \frac{1}{R(z)} - j \frac{\lambda}{\pi w^2(z)}
\]

In that case, we have:

\[
u(x, y, z) = \frac{1}{q(z)} \exp \left( -jk \frac{x^2 + y^2}{2R(z)} \right) \exp \left( -\frac{x^2 + y^2}{w^2(z)} \right)\]
Properties of the complex beam parameter

By replacing the (real valued) radius of curvature $R(z)$ for a spherical wave emerging from a real source point $z_0$ with a complex radius of curvature $q(z)$, we convert the paraxial spherical wave into a Gaussian beam. This is still an exact solution to the paraxial wave equation, but with the desirable property of finite extent in the x-y plane, and therefore finite total energy.

$$u(x, y, z) = \frac{1}{q(z)} \exp \left( -jk \frac{x^2 + y^2}{2R(z)} \right) \exp \left( -\frac{x^2 + y^2}{w^2(z)} \right)$$

radius of curvature \hspace{1cm} (beam waist, or spot size)$^2$

Note: this is not the same $R(z)$ as we had before, although it still plays the role of the radius of curvature of the phase fronts. But it is not true that $R(z)$ evolves with propagation distance according to $R(z) = R_0 + (z - z_0)$. Instead, $q(z)$ evolves:

$$q(z) = q(z_0) + (z - z_0)$$

...and $1/R(z) = \text{Re}\{1/q(z)\}$
Gaussian beams

\[ u(x, y, z) = \frac{1}{q(z)} \exp \left( -j k \frac{x^2 + y^2}{2R(z)} \right) \exp \left( -\frac{x^2 + y^2}{w^2(z)} \right) \]

Plots of \(|u|^2\) (which is proportional to the irradiance) versus \(x\) and \(y\):

\[ |u(x,y)|^2 \]

\( w = 3 \)

\[ |u(x,y)|^2 \]

\( w = 1 \)

The value of \( w \) determines the width of the beam.