21. Fourier transforms in optics, part 3

Magnitude and phase

some examples
amplitude and phase of light waves
what is the spectral phase, anyway?

The Scale Theorem

Defining the duration of a pulse
the uncertainty principle

Fourier transforms in 2D

x, k – a new set of conjugate variables
image processing with Fourier transforms

Fourier Transform Magnitude and Phase

For <u>any</u> complex quantity, we can decompose $f(t)$ and $F(\omega)$ into their magnitude and phase.

$f(t)$ can be written: $f(t) = Mag\{f(t)\} \exp[\, j\, Phase\{f(t)\}]$

where $Mag\{f(t)\}^2$ is called the **intensity, $I(t)$**,* and $Phase\{f(t)\}$ is called the ***temporal phase, $\phi(t)$***.

Analogously, $F(\omega) = Mag\{F(\omega)\} \exp[\, j\, Phase\{F(\omega)\}]$

The $Mag\{F(\omega)\}^2$ is called the **spectrum, $S(\omega)$**, and the $Phase\{F(\omega)\}$ is called the **spectral phase, $\varphi(\omega)$**.

Just as both the intensity and phase are required to specify $f(t)$, both the spectrum and spectral phase are required to specify $F(\omega)$.

*of course, in optics the intensity is $(1/2)\varepsilon_0 c_0 |E(t)|^2$ – the constants in front shouldn't be ignored!

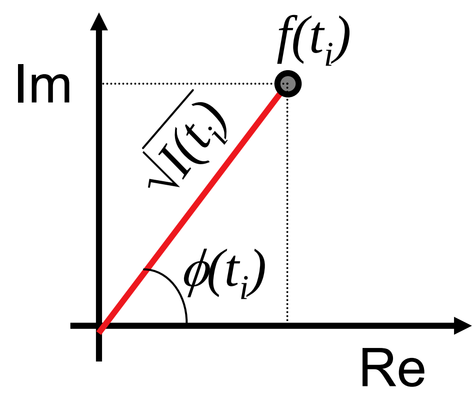
Calculating the Intensity and the Phase

It's easy to go back and forth between the function $f(t)$ and the intensity and phase.

The intensity: $I(t) \propto |f(t)|^2$

The phase:

$$\phi(t) = \arctan\left\{\frac{\mathrm{Im}[f(t)]}{\mathrm{Re}[f(t)]}\right\}$$



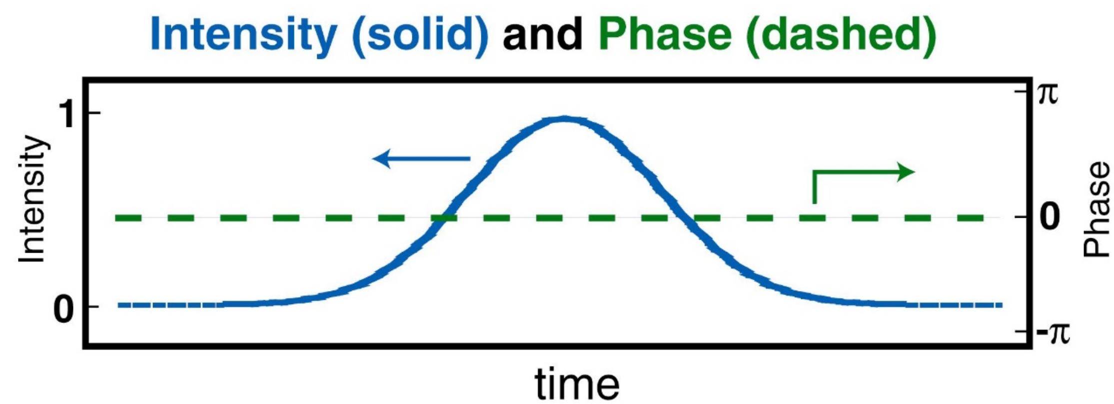
which is the same as:

$$\phi(t) = \mathrm{Im}\{\ \ln[f(t)]\ \}$$

Example: Intensity and Phase of a Gaussian

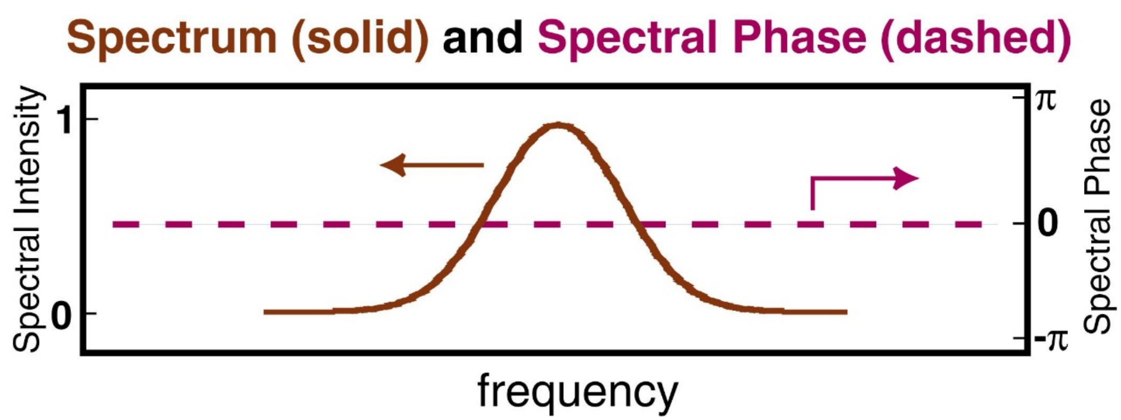
The Gaussian is real, so its phase is zero: $f(t) = Ae^{-at^2}$

Time domain:



The FT of a Gaussian is a Gaussian, so it also has zero spectral phase.

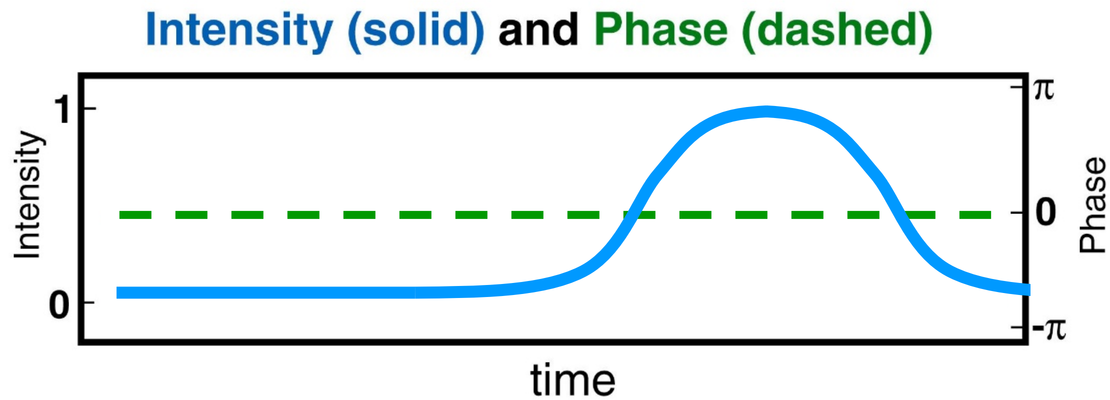
Frequency domain:



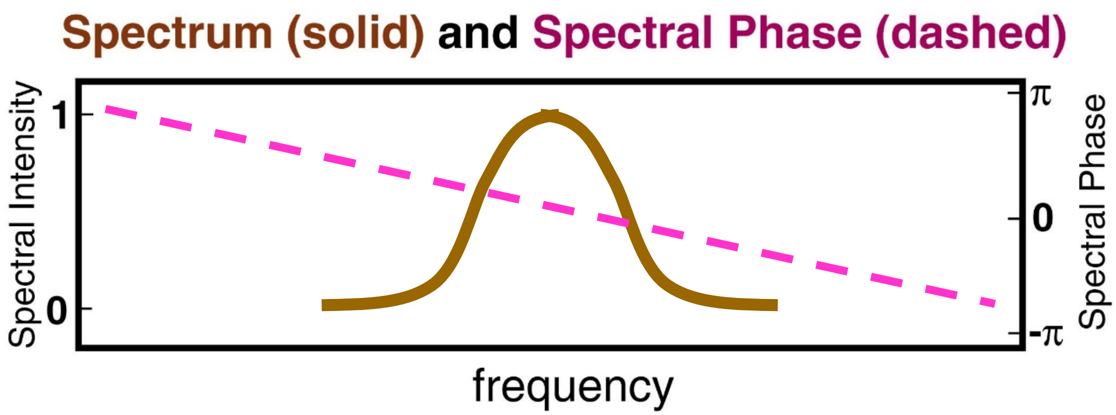
The spectral phase of a time-shifted pulse

Recall the Shift Theorem: $F\left\{f(t-t_0)\right\} = \exp(-j\omega t_0)F(\omega)$

Time-shifted Gaussian pulse (with zero phase):



A time-shift simply adds some linear spectral phase in the frequency domain!



Light has intensity and phase also

For a single-frequency light wave, we have written all semester long:

$$E(t) = \mathrm{Re}\left\{ \underset{\sim}{E}_0 \exp\left[j\omega_0 t \right] \right\} \longleftarrow$$

neglecting the spatial dependence, for now

where this is the complex amplitude, given by: $\underset{\sim}{E}_0 = \left|\underset{\sim}{E}_0\right| e^{j\theta}$

initial phase θ

For a light wave with many superposed frequencies, we cannot assume that the initial phase is the same for all of them!

It is easiest to envision this in the frequency domain:

$$\tilde{E}(\omega) = \sum E_0\left(\omega_i\right) \exp\left[-j\varphi(\omega_i)\right] \longleftarrow$$

$$\rightarrow \sqrt{S(\omega)} \exp\left[-j\varphi(\omega)\right]$$

A Fourier sum of frequency components ω_i, each with its own phase $\varphi(\omega_i)$

Light has intensity and phase also

A light wave has the frequency-domain electric field:

$$\tilde{E}(\omega) = \sqrt{S(\omega)}\, \exp\left[-j\varphi(\omega)\right]$$

Spectrum

Spectral Phase

The minus signs are just conventions.

Equivalently, in the time domain:

$$E(t) = \mathrm{Re}\left\{\sqrt{I(t)}\, \exp\left[j(\omega_0 t - \phi(t))\right]\right\}$$

Intensity

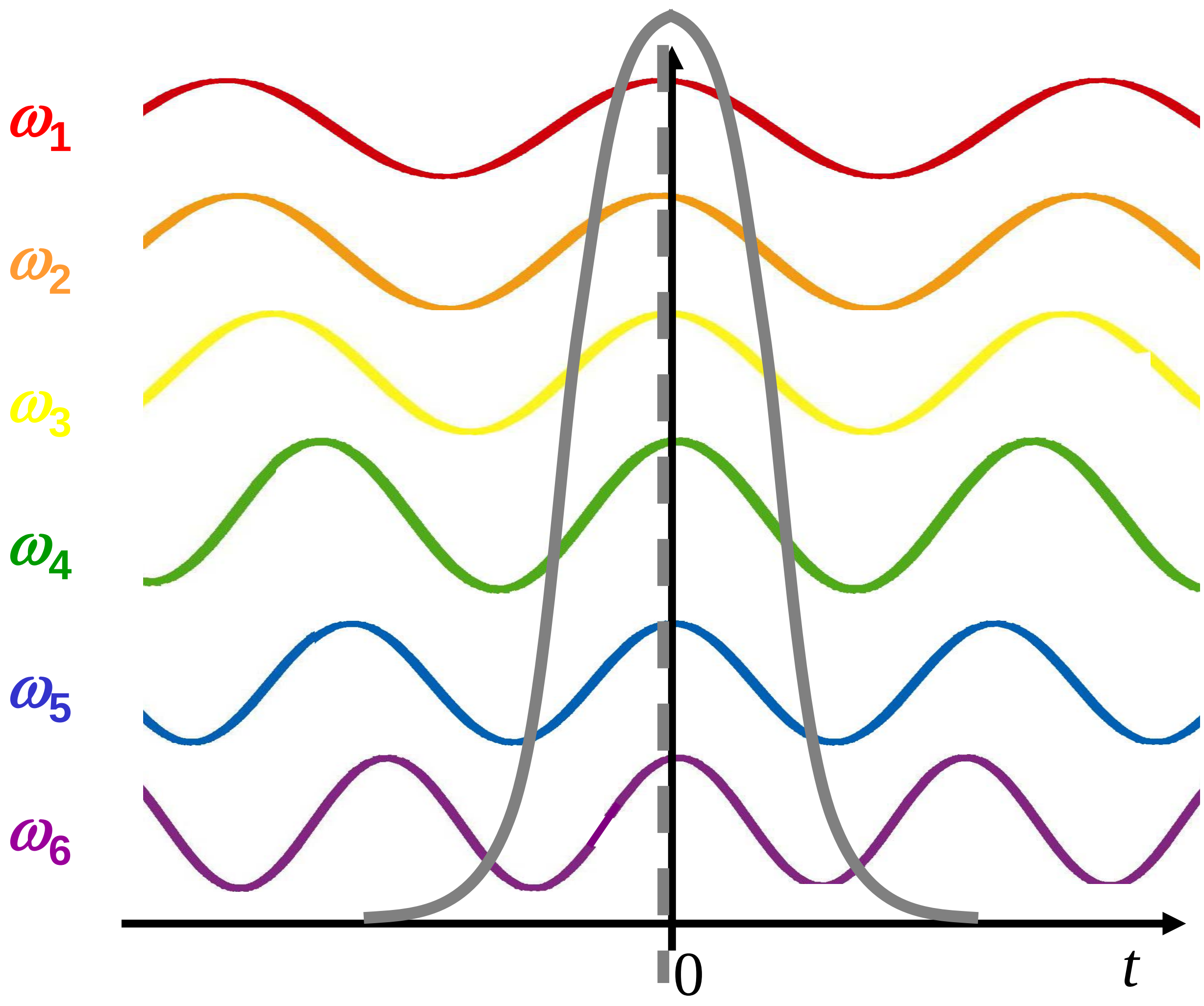
Temporal phase

Usually, in the time domain, we separate out the part that is linear in t.

Once again, knowledge of the intensity and temporal phase or of the spectrum and spectral phase is sufficient to determine the light wave.

What is the spectral phase anyway?

The spectral phase is the absolute phase of each frequency component.



All six of these frequencies have zero phase. So a light wave which is the sum of these 6 waves has:

$$\varphi(\omega) = 0$$

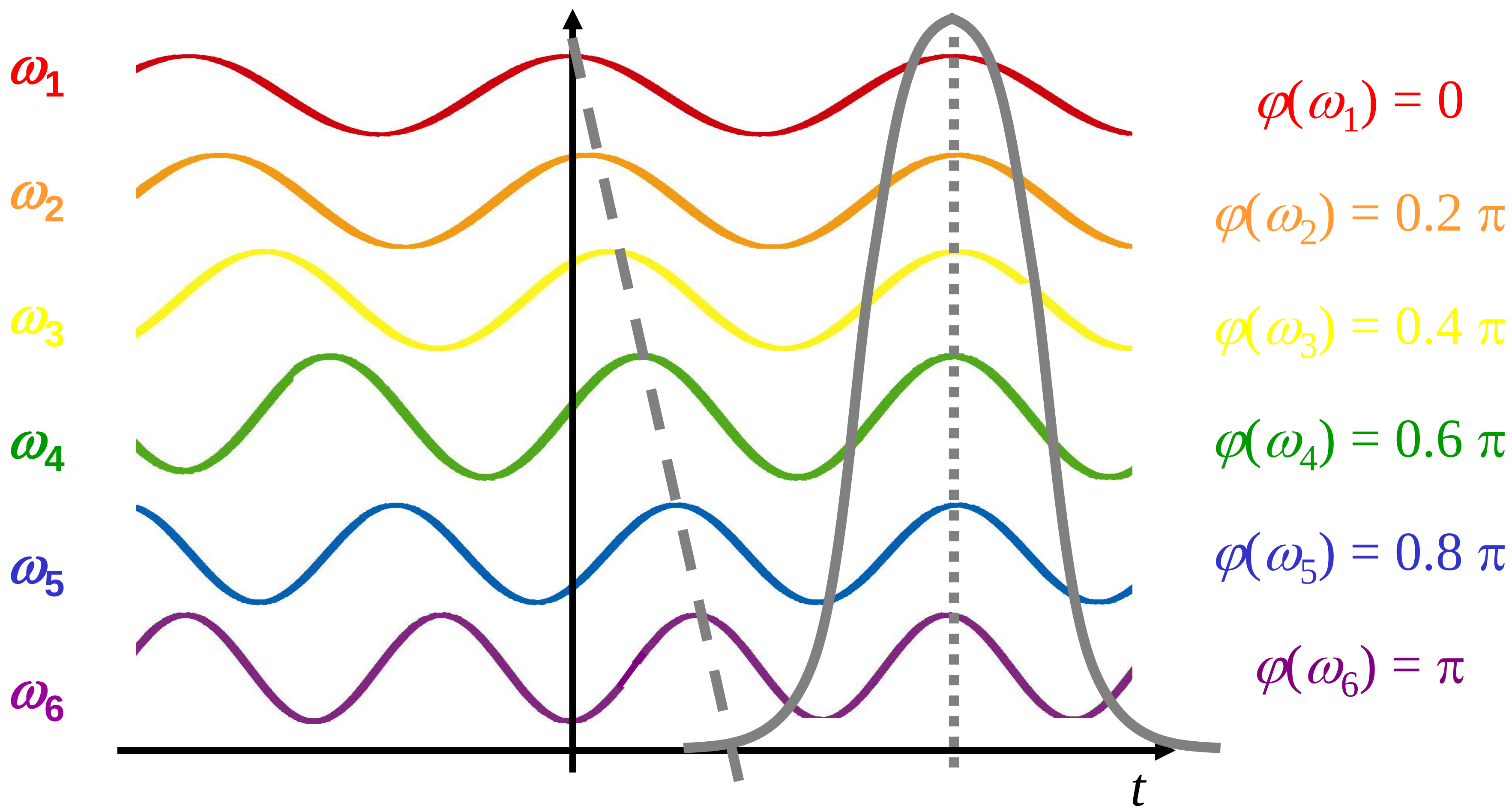
Note that this wave has constructive interference, and hence a peak, at $t = 0$.

And it has cancellation elsewhere.

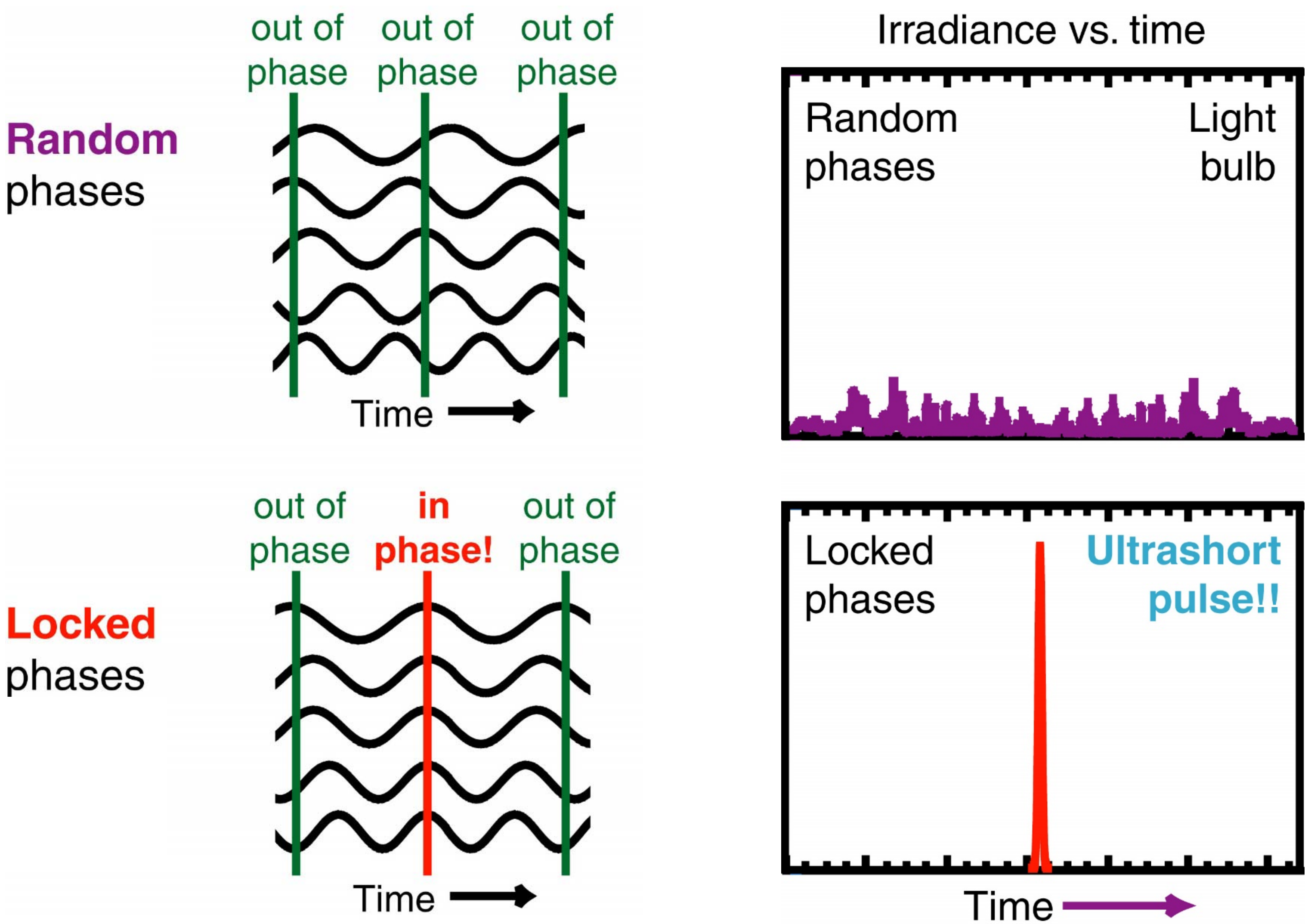
Now try a linear spectral phase: $\varphi(\omega) = a\omega$.

By the Shift Theorem, a linear spectral phase is just a delay in time.

And this diagram shows why it works that way!



The spectral phase is what distinguishes a light bulb from a short light pulse.



Scale Theorem

The Fourier transform of a scaled function, $f(\alpha\, t)$ (assume $\alpha > 0$):

$$F\{f(\alpha t)\} = \int_{-\infty}^{\infty} f(\alpha t) \exp(-j\omega t)\, dt$$

Change variables: $u = \alpha\, t$

$$F\{f(u)\} = \int_{-\infty}^{\infty} f(u) \exp(-j\omega[u/\alpha])\, \frac{du}{\alpha}$$

$$= \int_{-\infty}^{\infty} f(u) \exp(-j[\omega/\alpha]\, u)\, \frac{du}{\alpha}$$

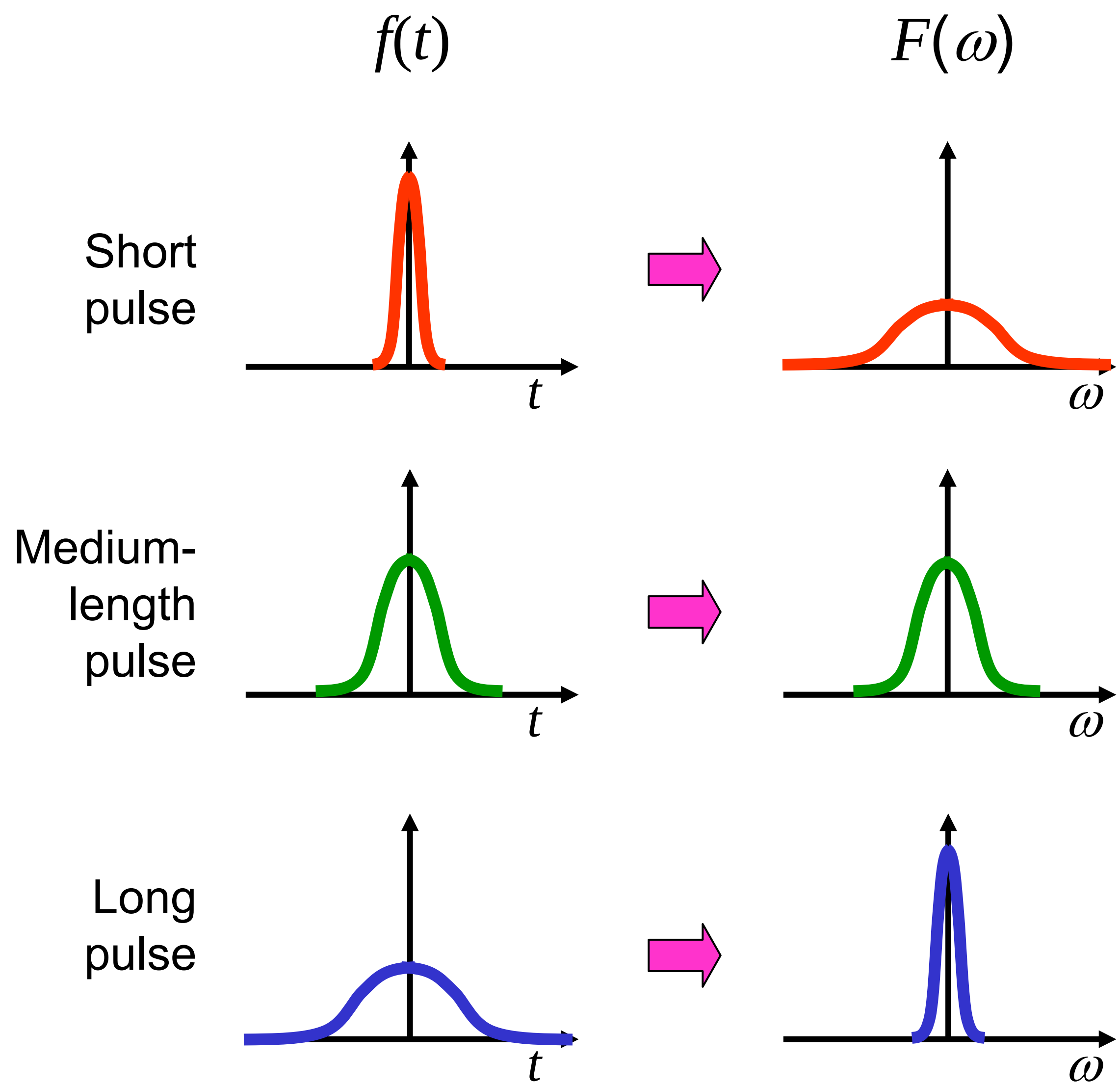
$$= \frac{F(\omega/\alpha)}{\alpha}$$

The Fourier transform of a function that has been scaled by a certain factor in the time domain is the Fourier transform of the unscaled function, scaled by the inverse factor.

The Scale Theorem in action

The shorter the pulse, the broader the spectrum!

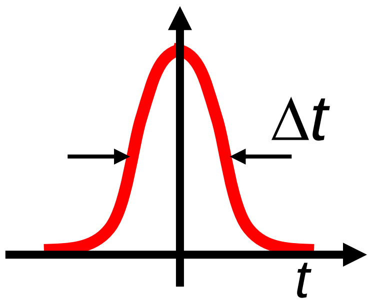
This is the essence of the Uncertainty Principle!



But be careful:

- A short pulse **requires** a broad spectrum;
- But a long pulse **does not require** a narrow spectrum!

The Width of a Pulse



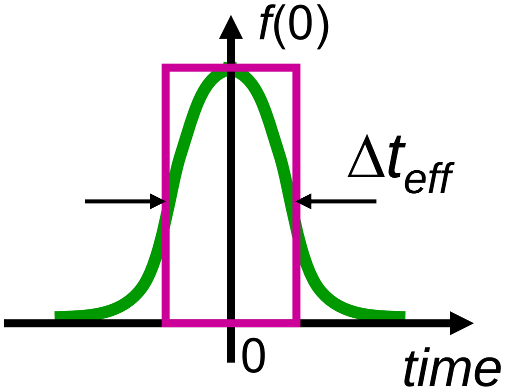
There are many definitions of the "width" or "length" of a wave or pulse.

The "**effective width**" is the width of a rectangle whose ***height*** and ***area*** are the same as those of the pulse.

Effective width ≡ Area / height:

$$\Delta t_{eff} \equiv \frac{1}{f(0)} \int_{-\infty}^{\infty} |f(t)| \, dt$$

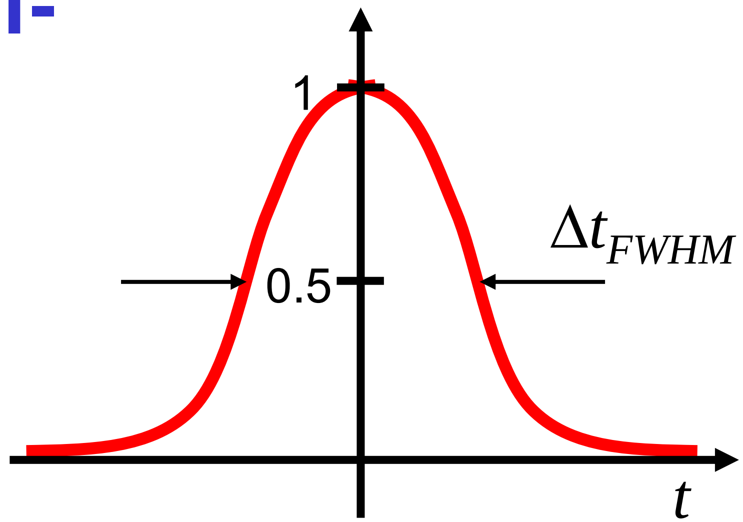
(Abs value is unnecessary for intensity.)



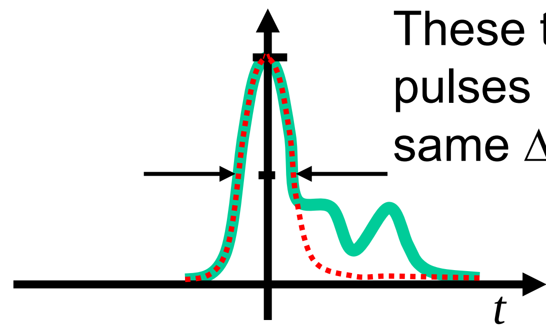
More often, people use a different measure of pulse width, one which is easier to compute.

The Full Width at Half-Maximum (FWHM)

"**Full-width at half-maximum**" is the distance between the half-maximum points. It is a convenient measure of the duration of a pulse.



Advantage: It is experimentally easy to determine.
Disadvantage: It doesn't tell us anything about the structure of the pulse.



These two pulses have the same Δt_{FWHM}

We can define this width in terms of f(t) or (more often) of its intensity, $|f(t)|^2$.

And we can define *spectral* widths ($\Delta\omega$ or $\Delta\nu$) similarly in the frequency domain, in terms of the spectrum $|F(\omega)|^2$.

The Uncertainty Principle

The Uncertainty Principle says that the product of a function's widths in the time domain (Δt) and the frequency domain ($\Delta\omega$) has a minimum value. And that value must be greater than zero.

<u>A demonstration (not a proof)</u>

Define the widths as: $\Delta t \equiv \frac{1}{f(0)} \int_{-\infty}^{\infty} |f(t)| \, dt \qquad \Delta\omega \equiv \frac{1}{F(0)} \int_{-\infty}^{\infty} |F(\omega)| \, d\omega$

(assuming $f(t)$ and $F(\omega)$ peak at t = 0 and ω = 0, respectively)

$$\Delta t \ge \frac{1}{f(0)} \int_{-\infty}^{\infty} f(t) \, dt = \frac{1}{f(0)} \int_{-\infty}^{\infty} f(t) \exp(-j[0]t) \, dt = \frac{F(0)}{f(0)}$$

$$\Delta\omega \ge \frac{1}{F(0)} \int_{-\infty}^{\infty} F(\omega) \, d\omega = \frac{1}{F(0)} \int_{-\infty}^{\infty} F(\omega) \exp(j\omega[0]) \, d\omega = \frac{2\pi f(0)}{F(0)}$$

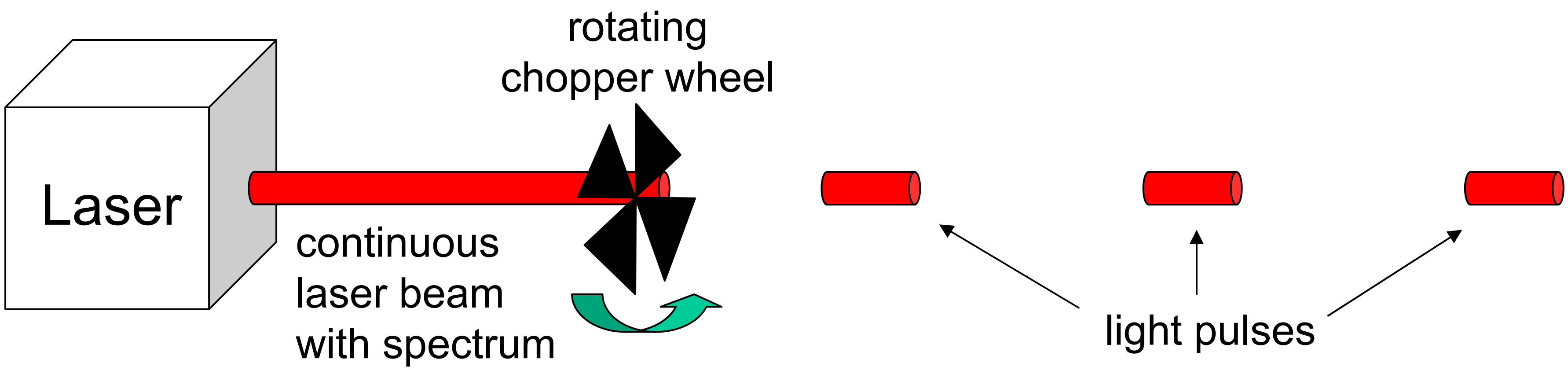
$$\longrightarrow \quad \Delta\omega \, \Delta t \ge 2\pi \frac{f(0)}{F(0)} \frac{F(0)}{f(0)}$$

$$\boxed{\Delta\omega \, \Delta t \ge 2\pi}$$

$$\boxed{\Delta\nu \, \Delta t \ge 1}$$

The Uncertainty Principle in optics

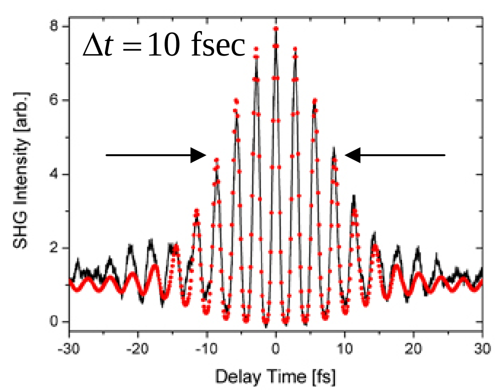
Suppose we start with a continuous laser, with $E(t) = E_0 e^{-j\omega_0 t}$

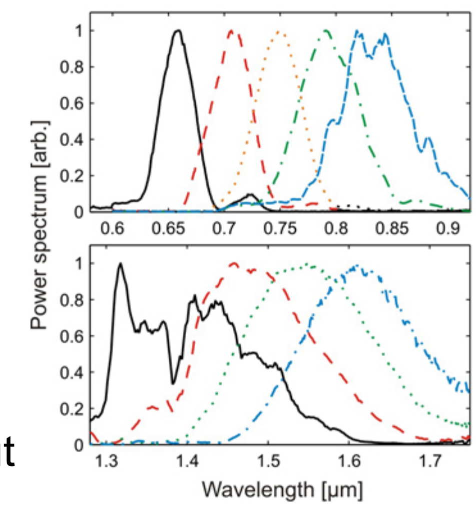


$S(\omega) = \delta(\omega - \omega_0)$

The wheel changes the spectrum of the light!

To make a short light pulse,
you need a broad spectrum.

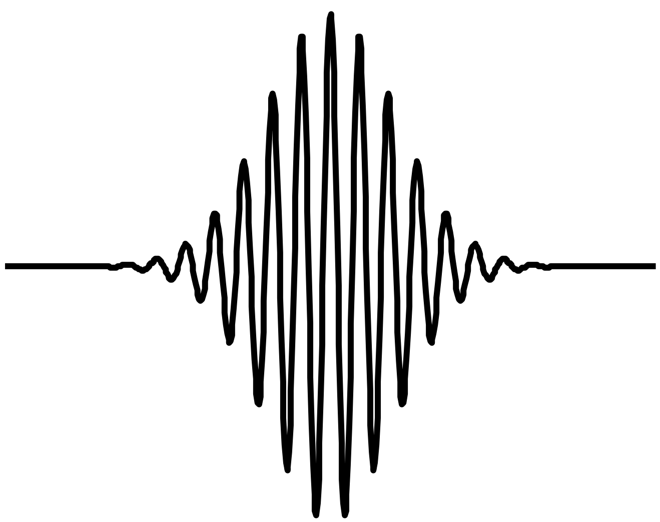




The spectrum of a short light pulse is always broad, but can be centered at any wavelength.

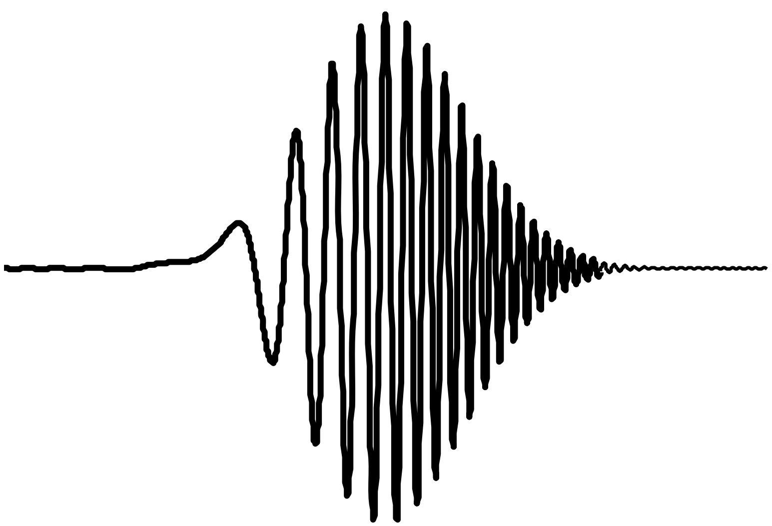
The spectral phase… again

These two pulses have the same spectra.
But they are obviously not the same.



$$E(t) = E_0 e^{-at^2} e^{-j\omega_0 t}$$

temporal phase $\varphi(t) = 0$



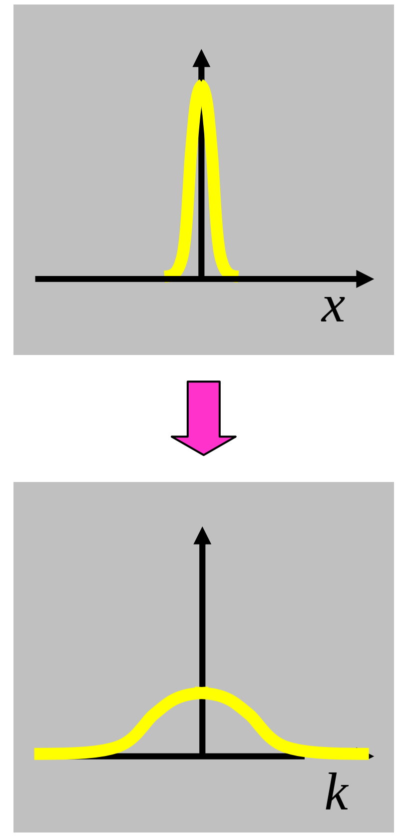
$$E(t) = E_0 e^{-at^2} e^{-j\omega_0 t - j\beta t^2}$$

temporal phase $\varphi(t) = \beta t^2$

In ultrafast optics, measuring the phase of your laser pulse is often very important, and also very challenging.

Another set of conjugate variables: x, k

If $f(x)$ is a function of position,



$$G(k) = \int_{-\infty}^{\infty} f(x) \exp(-jkx)\, dx$$

$$F\{f(x)\} = G(k)$$

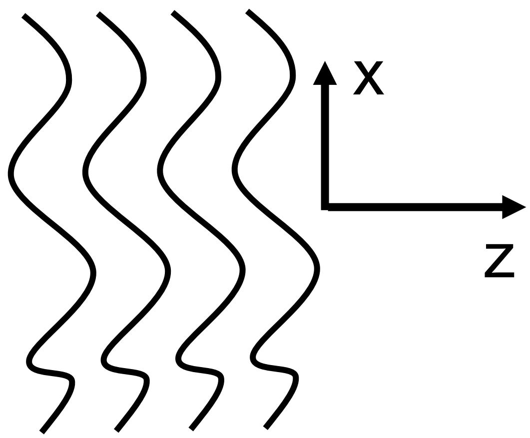
We refer to k as the spatial frequency.

Everything we've said about Fourier transforms between the t and ω domains also applies to the x and k domains.

Note that the units of x and k are inverses, just like with t and ω.

The x,k domain is relevant in optics

Any arbitrarily complex wave front can be written as a superposition of plane waves, each traveling at a slightly different angle:



A complicated wave front propagating in the z direction

$$E(x,z) = \int A(k_x, z)\; e^{-jk_x x} dk_x$$

where the coefficients $A(k_x,z)$ can be found by the inverse FT:

$$A(k_x, z) = \frac{1}{2\pi} \int E(x,z) e^{jk_x x} dx$$

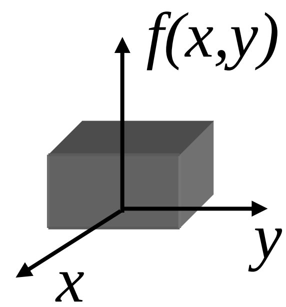
In analogy with the “frequency components” of a spectrum, we refer to the quantity k_x as a “spatial frequency component”.

The 2D Fourier Transform

If we can expand in x, why not also y?

$$\mathrm{F}^{(2)}\{f(x,y)\} = F(k_x,k_y)$$

$$= \iint f(x,y)\,\exp[-j(k_x x+k_y y)]\,dx\,dy$$

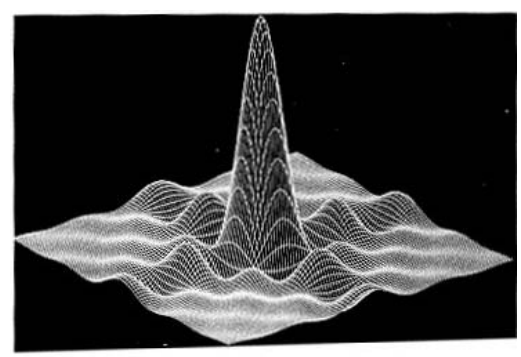


If $f(x,y) = f_x(x)\,f_y(y)$,

then the 2D FT splits into two 1D FT's.

But this does not usually happen.

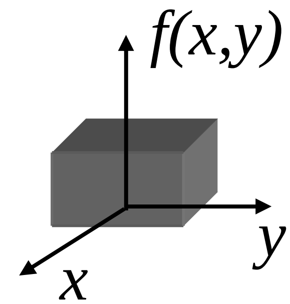
$\mathrm{F}^{(2)}\{f(x,y)\}$



A 2D Fourier Transform: a square function

Consider a square function in the *xy* plane:

$$f(x,y) \;=\; \mathrm{rect}(x)\,\mathrm{rect}(y)$$

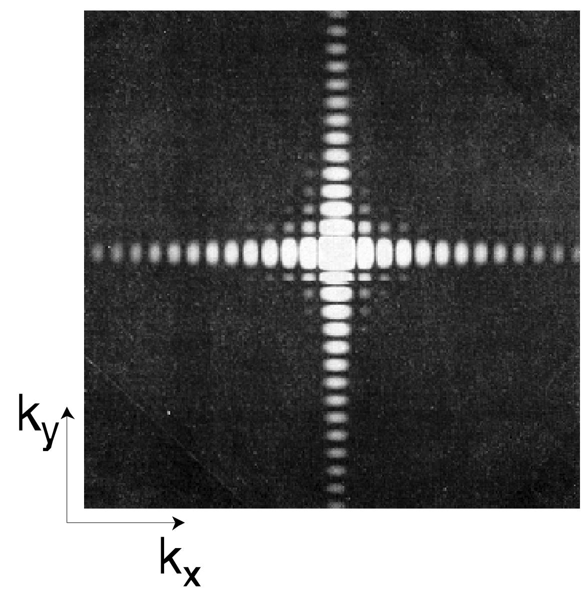


The 2D Fourier transform splits into the product of two 1D Fourier transforms:

$$F^{(2)}\{f(x,y)\} \;=\; \mathrm{sinc}(k_x)\,\mathrm{sinc}(k_y)$$

This picture is an optical determination of the Fourier transform of the 2D square function!

$F^{(2)}\{f(x,y)\}$



Fourier Transform Magnitude and Phase

cameraman *dog and cat*

Pictures reconstructed using the Fourier phase of another picture

Mag{dog and cat}
Phase{cameraman}

Mag{cameraman}
Phase{dog and cat}

The phase of the Fourier transform is much more important than the magnitude in reconstructing an image.

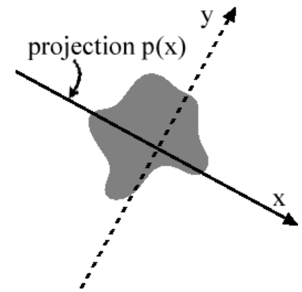
More information: http://homepages.inf.ed.ac.uk/rbf/HIPR2/fourier.htm

The Projection Slice Theorem

Suppose we have a 2D image, defined by pixel values $f(x,y)$, and its 2D Fourier transform $F(k_x,k_y)$.

We can define the projection of this image onto any axis (say, the x axis) as the sum along lines perpendicular to the x axis:

$$p(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$



The Projection Slice theorem says that the Fourier transform of $p(x)$ is one slice through $F(k_x, k_y)$, along the k_x axis which is parallel to the projection axis (the x axis).

$$s(k_x) = F(k_x, 0) = \int_{-\infty}^{\infty} p(x)\, e^{-j2\pi x k_x}\, dx$$

This is the basis for tomographic image reconstruction, as in CAT scans.