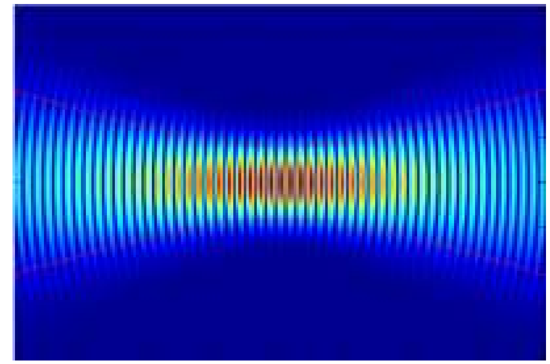
21. Propagation of Gaussian beams



How to propagate a Gaussian beam

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Gaussian beams

A laser beam can be described by this:

$$E(x,y,z,t) = u(x,y,z)\,e^{j(kz-\omega t)}$$

where $u(x,y,z)$ is a Gaussian transverse profile that varies slowly along the propagation direction (the z axis), and remains Gaussian as it propagates:

$$u(x,y,z) = \frac{1}{q(z)} \exp\left[-jk\frac{x^2+y^2}{2}\cdot\frac{1}{q(z)}\right]$$

The parameter q is called the "complex beam parameter."
It is defined in terms of two quantities:

w, the beam waist, and

R, the radius of curvature of the (spherical) wave fronts

$$\frac{1}{q(z)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi w^2(z)}$$

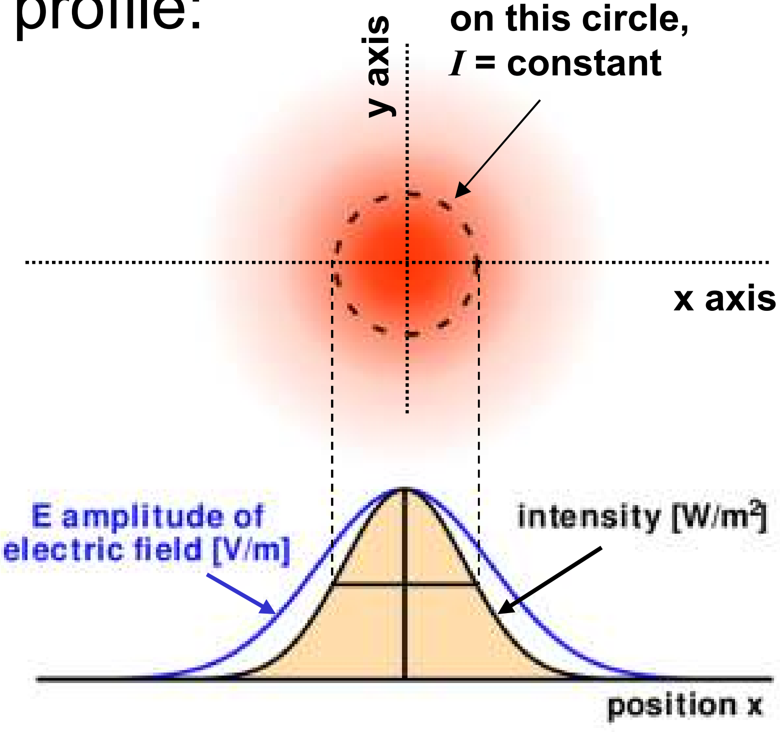
Gaussian beams

The magnitude of the electric field is therefore given by:

$$\left|E(x,y,z,t)\right| = \left|u(x,y,z)\right| = \left|\frac{1}{q(z)}\right| \exp\left[-\frac{x^2+y^2}{w^2}\right]$$

with the corresponding intensity profile:

$$I(x,y,z) \sim \left|\frac{1}{q(z)}\right|^2 \exp\left[-2\frac{x^2+y^2}{w^2}\right]$$



The spot is Gaussian:
And its size is
determined by
the value of w.

Propagation of Gaussian beams

At a given value of z, the properties of the Gaussian beam are described by the values of $q(z)$ and the wave vector.

So, if we know how $q(z)$ varies with z, then we can determine everything about how the Gaussian beam evolves as it propagates.

Suppose we know the value of $q(z)$ at a particular value of z.

e.g. $q(z) = q_0$ at $z = z_0$

Then we can determine the value of $q(z)$ at any subsequent point z_1 using:

$$q(z_1) = q_0 + (z_1 - z_0)$$

This determines the evolution of both $R(z)$ and $w(z)$.

Propagation of Gaussian beams - example

Suppose a Gaussian beam (propagating in empty space, wavelength λ) has an infinite radius of curvature (i.e., wave fronts with no curvature at all) at a particular location (say, $z = 0$).

Suppose, at that location ($z = 0$), the beam waist is given by w_0.

Describe the subsequent evolution of the Gaussian beam, for $z > 0$.

We are given that $R(0)$ = *infinity* and $w(0) = w_0$. So, we can determine $q(0)$:

$$\frac{1}{q(0)} = \frac{1}{R(0)} - j\frac{\lambda}{\pi w^2(0)}$$

$$= -j\frac{\lambda}{\pi w_0{}^2}$$

Thus $q(0) = j\dfrac{\pi w_0{}^2}{\lambda}$

NOTE: If $R = \infty$ at a given location, this implies that q is a pure imaginary number at that location: this location is a focal plane.

The Rayleigh range

If w_0 is the beam waist at a focal point, then we can write q at that value of z as:

$$q_{focus} = j \cdot z_R$$

where z_R is the "Rayleigh range" - a key parameter in describing the propagation of beams near focal points.

$$z_R = \frac{\pi w_0^{\ 2}}{\lambda}$$

It is, roughly, the focal spot area divided by λ.

Now, how does q change as z increases?

Propagation of Gaussian beams - example

A distance z later, the new complex beam parameter is:

$$q(z) = q(0) + z = jz_R + z$$

$$\frac{1}{q(z)} = \frac{1}{z + jz_R} = \frac{z - jz_R}{z^2 + z_R^2}$$

Reminder:

$$\frac{1}{q(z)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi w^2(z)}$$

$$\mathrm{Re}\left\{\frac{1}{q(z)}\right\} = \frac{z}{z^2 + z_R^2} = \frac{1}{R(z)}$$

$$R(z) = \frac{z^2 + z_R^2}{z}$$

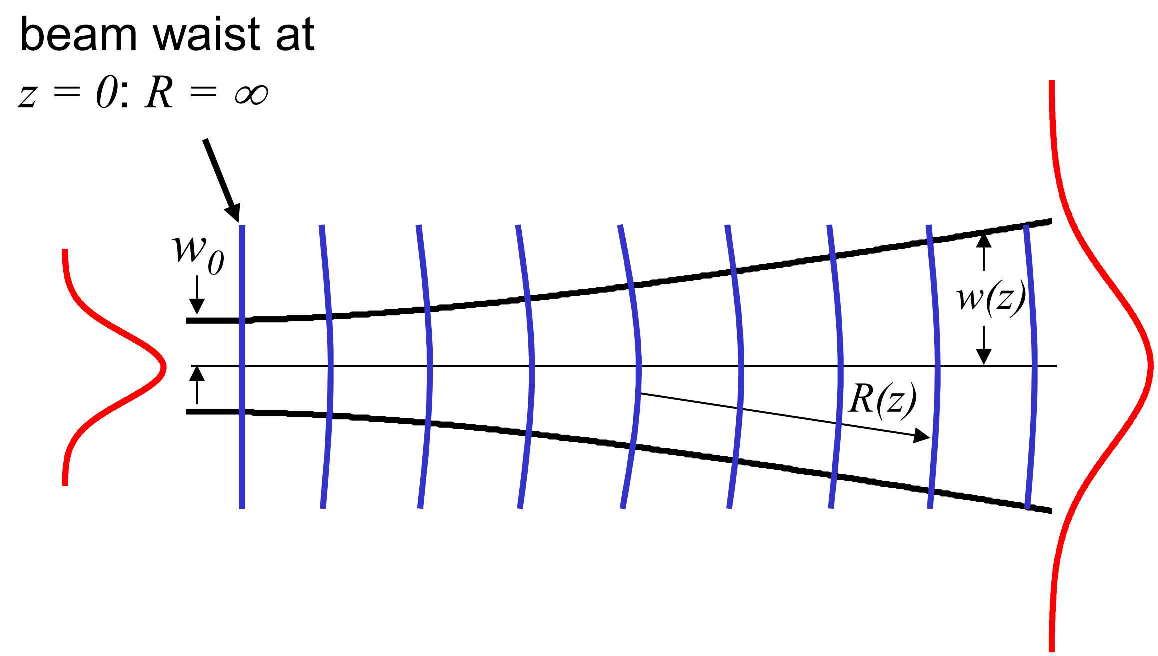
- At $z = 0$, R is infinite, as we assumed.
- As z increases, R first decreases from infinity, then increases.
- Minimum value of R occurs at $z = z_R$.

$$\mathrm{Im}\left\{\frac{1}{q(z)}\right\} = \frac{-z_R}{z^2 + z_R^2} = \frac{-\lambda}{\pi w^2(z)}$$

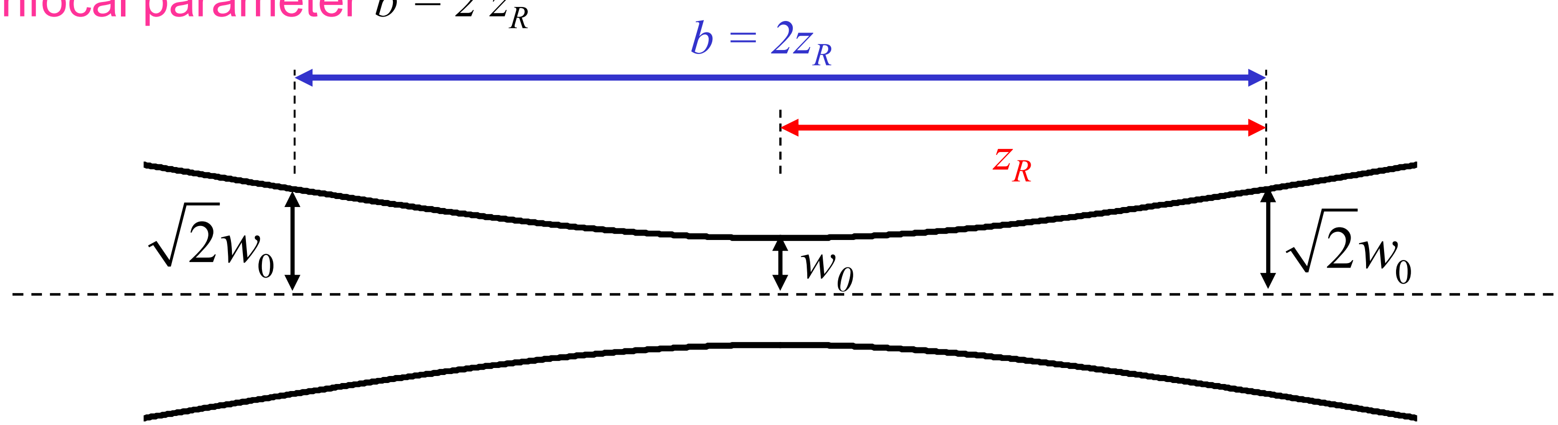
$$w(z) = w_0\sqrt{1 + \frac{z^2}{z_R^2}}$$

- At $z = 0$, $w = w_0$, as we assumed.
- As z increases, w increases.
- At $z = z_R$, $w(z) = sqrt(2)\times w_0$.

Rayleigh range and confocal parameter



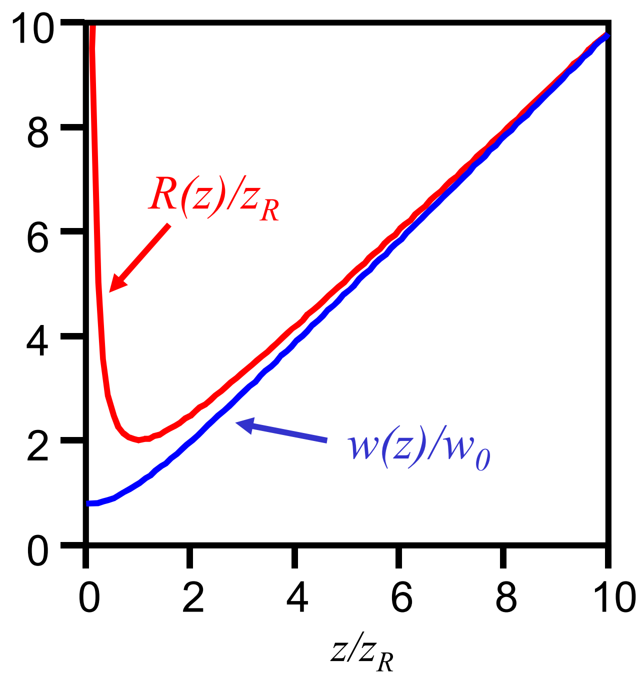
- Rayleigh range z_R = the distance from the focal point where the beam waist has increased by a factor of $\sqrt{2}$ (i.e., the beam area has doubled).
- Confocal parameter $b = 2\ z_R$



Propagation of Gaussian beams - example

When propagating away from a focal point at z = 0:

$$R(z) = \frac{z^2 + z_R^2}{z} \qquad w(z) = w_0\sqrt{1+\frac{z^2}{z_R^2}}$$

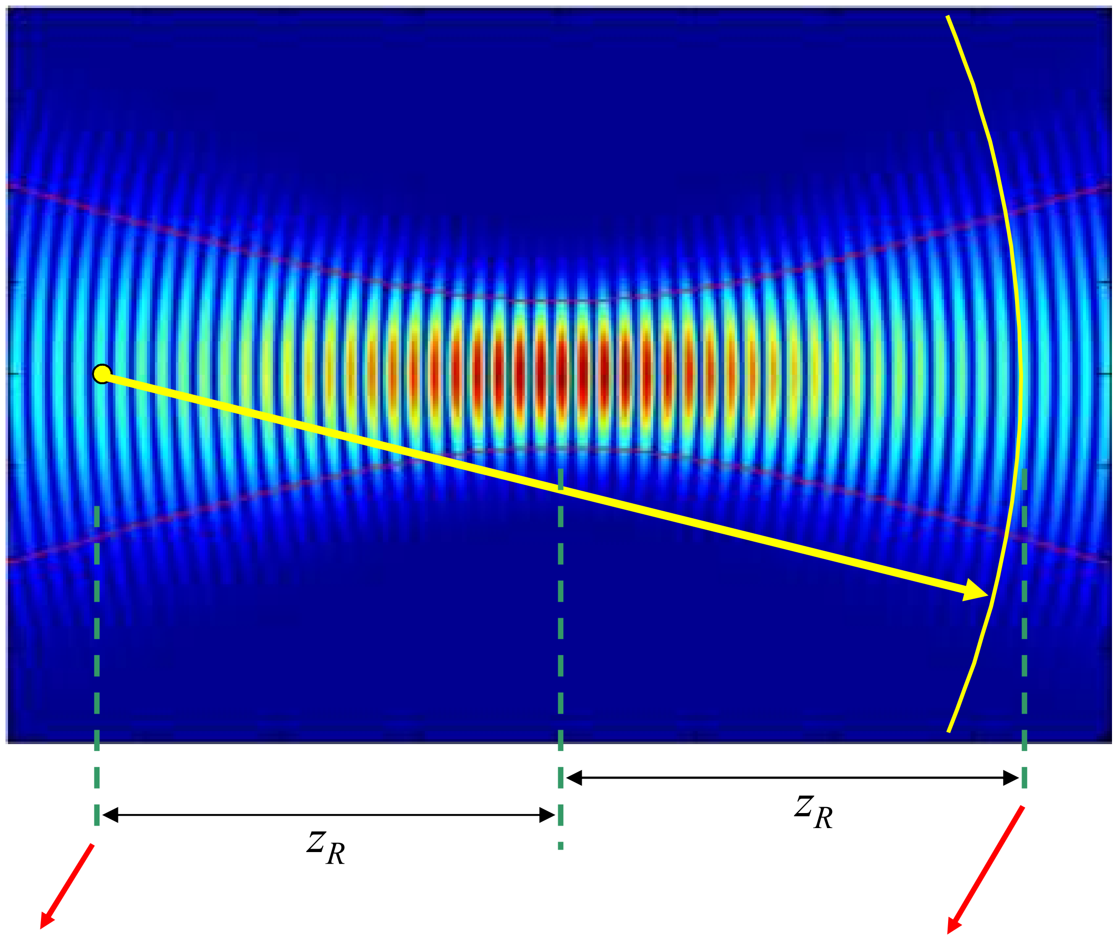


Note #1: for distances larger than a few times z_R, both the radius and waist increase linearly with increasing distance.

Note #2: positive values of R correspond to a diverging beam, whereas $R < 0$ would indicate a converging beam.

A focusing Gaussian beam

At a distance of one Rayleigh range from the focal plane, the wave front radius of curvature is equal to $2z_R$, which is its minimum value.

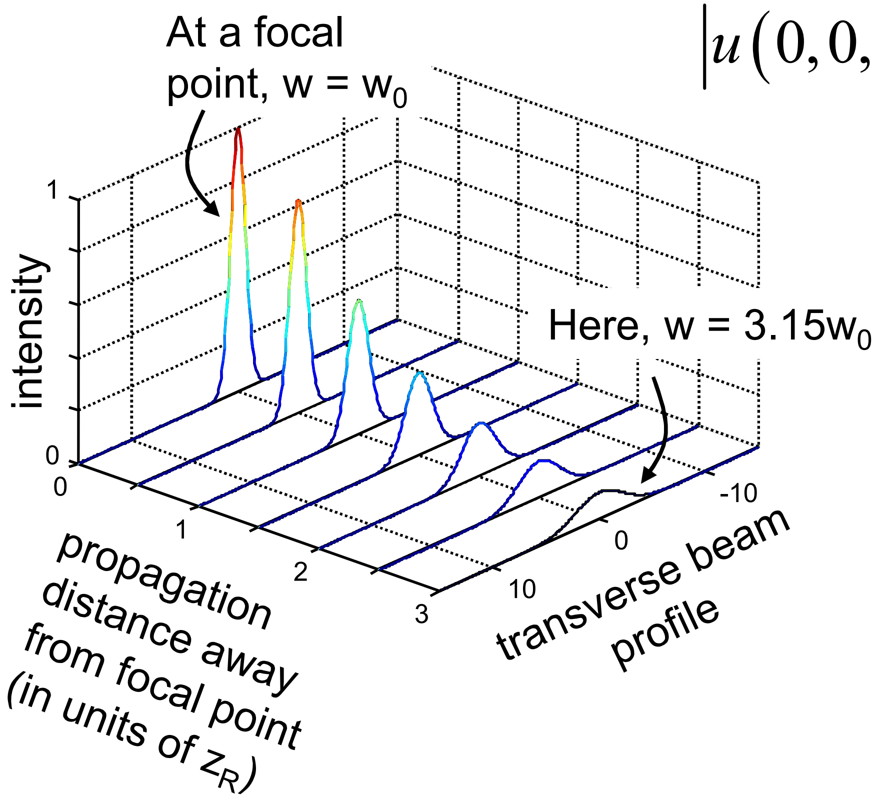


These are also the points at which the beam spot's area has doubled.

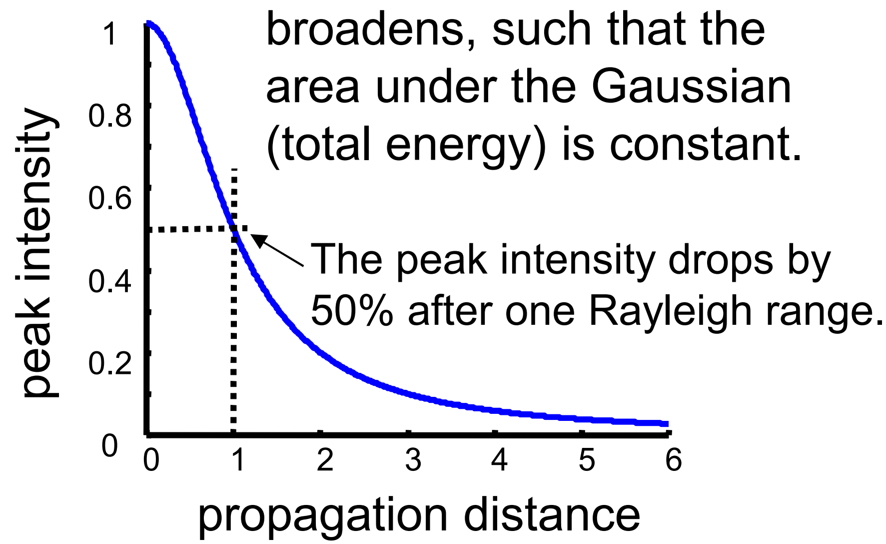
What about the on-axis intensity?

We have seen how the beam radius and beam waist evolve as a function of z, moving away from a focal point. But how about the intensity of the beam at its center (that is, at x = y = 0)?

$$\left|u(0,0,z)\right| = \frac{1}{q(z)} \longrightarrow I(0,0,z) \propto \left|\frac{1}{q(z)}\right|^2$$



The peak drops as the width broadens, such that the area under the Gaussian (total energy) is constant.



Propagation of Gaussian beams - example #2

Suppose, at $z = 0$, a Gaussian beam has a waist of 50 μm and a radius of curvature of –1 cm. The wavelength is 786 nm. Where is the focal point, and what is the spot size at the focal point?

At $z = 0$, we have: $$\frac{1}{q_0} = \frac{1}{-10^4\ \mu m} - j\frac{786\ nm}{\pi(50\ \mu m)^2} = -10^{-4} - j\cdot 10^{-4}$$

So: $$q_0 = \frac{-10^4\ \mu m}{1+j} = -\frac{10^4\ \mu m}{\sqrt{2}}(1-j)$$

The focal point is the point at which the radius of curvature is infinite, which (as we have seen) implies that q is imaginary at that point.

$q(z) = q_0 + z$ Choose the value of z such that $Re\{q\} = 0$

At $$z = \frac{10^4\ \mu m}{\sqrt{2}} \longrightarrow q(z) = \frac{10^4\ \mu m}{\sqrt{2}} j$$

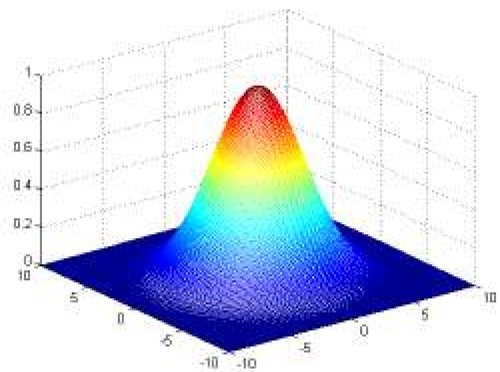
$$\frac{1}{q(z)} = -j\frac{\sqrt{2}}{10^4\ \mu m} = -j\frac{0.786\ \mu m}{\pi w^2}$$

This gives w = 42 μm.

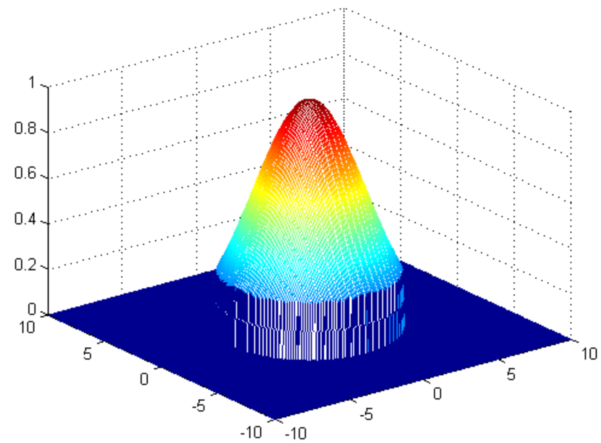
Aperture transmission

The irradiance of a Gaussian beam drops dramatically as one moves away from the optic axis. How large must a circular aperture be so that it does not significantly truncate a Gaussian beam?

Before aperture



After aperture



Before the aperture, the radial variation of the irradiance of a beam with waist w is:

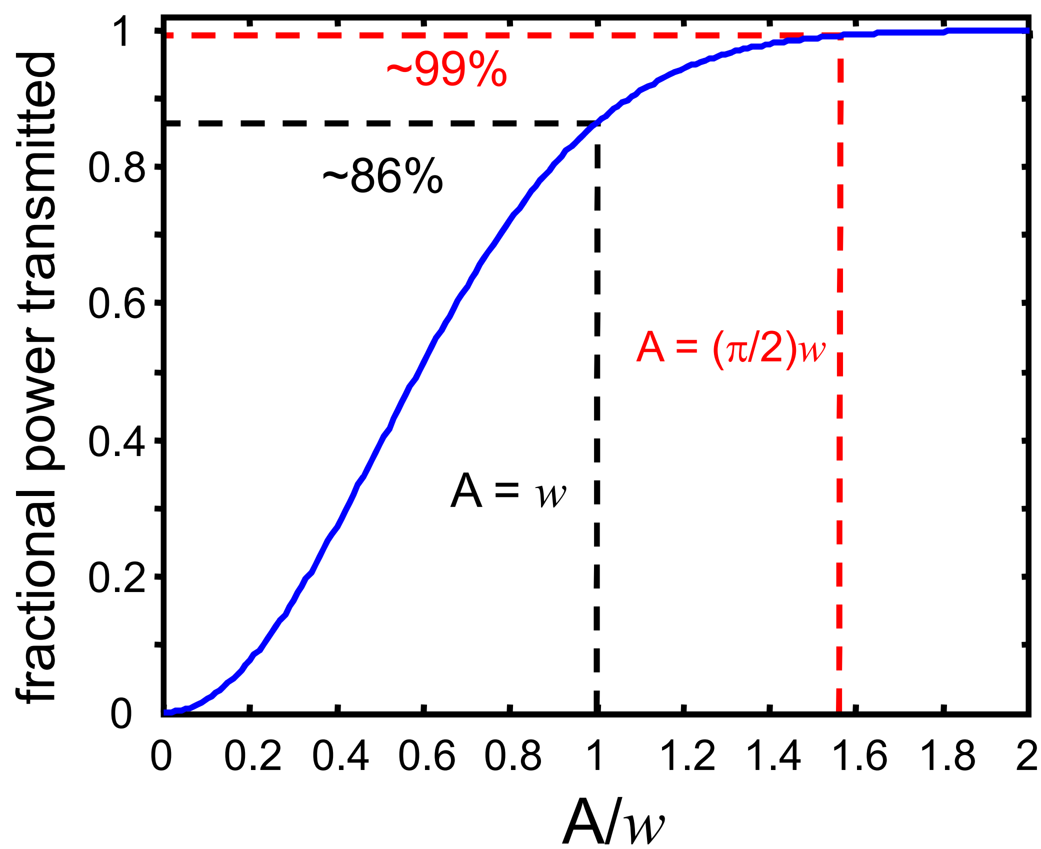
$$I(r) = \frac{2P}{\pi w^2} e^{-2r^2/w^2}$$

where P is the total power in the beam: $P = \iint \left| u(x,y) \right|^2 dxdy$

Aperture transmission

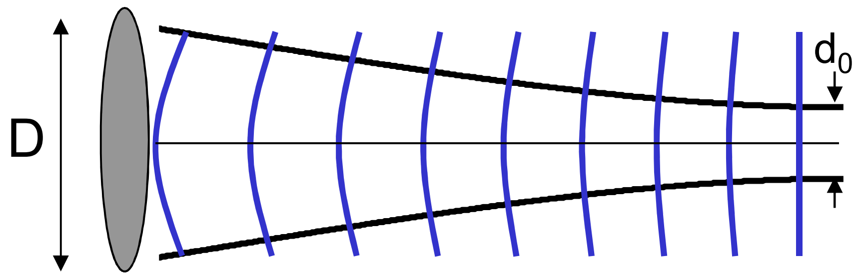
If this beam (waist w) passes through a circular aperture with radius A (and is centered on the aperture), then:

$$\text{fractional power transmitted} = \frac{2}{\pi w^2}\int_0^A 2\pi r\, e^{-2r^2/w^2}\, dr = 1 - e^{-2A^2/w^2}$$



Focusing a Gaussian beam

The focusing of a Gaussian beam can be regarded as the reverse of the propagation problem we did before.



A Gaussian beam focused by a thin lens of diameter D to a spot of diameter d_0.

How big is the focal spot?

Well, of course this depends on how we define the size of the focal spot. If we define it as the circle which contains 86% of the energy, then $d_0 = 2w_0$.

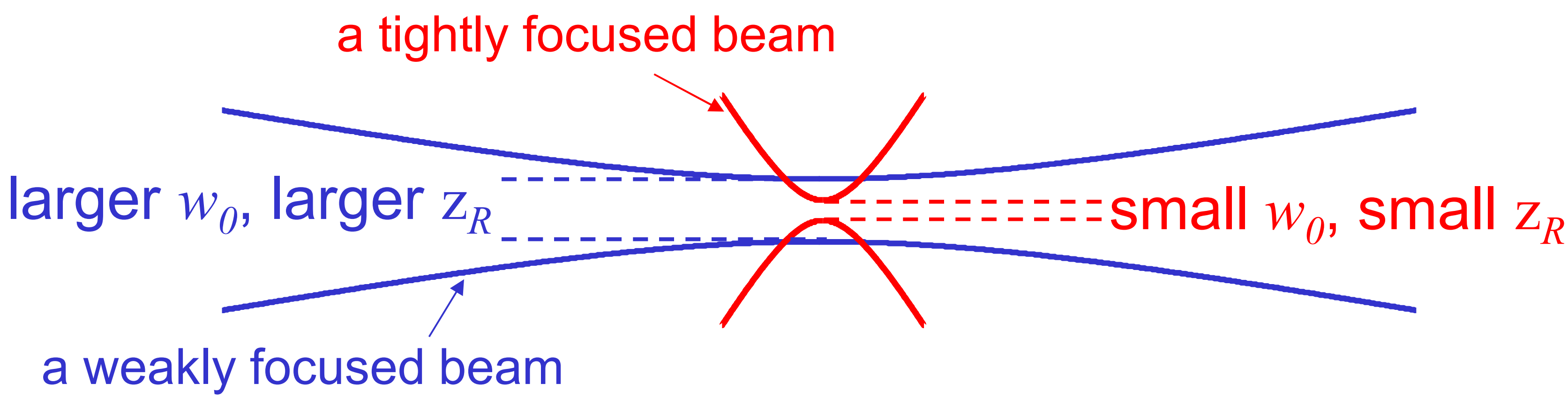
Then, if we assume that the input beam completely fills the lens (so that its diameter is D), we find:

$$d_0 \approx \frac{2 f \lambda}{D} = 2 (f\,\#) \lambda$$

where $f\# = f/D$ is the f-number of the lens.

It is very difficult to construct an optical system with $f\# < 0.5$, so $d_0 > \lambda$.

Depth of field



Depth of field = range over which the beam remains approximately collimated
= confocal parameter ($2z_R$)

Depth of field $\approx 2z_R = 2\pi\left(f\,\#\right)^2\lambda$

If a beam is focused to a spot N wavelengths in diameter, then the depth of field is approximately N^2 wavelengths in length.

What about my laser pointer? λ = 0.532 μm, and f# ~ 1000

So depth of field is about 3.3 meters.

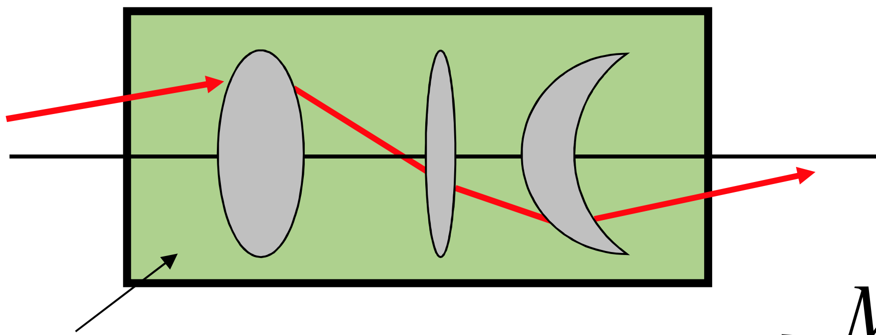
Depth of field: example

f/# = 32 (large depth of field)

f/# = 5 (smaller depth of field)

Gaussian beams and ABCD matrices

The q parameter for a Gaussian beam evolves according to the same parameters used for the ABCD matrices!



Optical system ↔ 2x2 Ray matrix

$$M_{system} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

If we know q_{in}, the q parameter at the input of the optical system, then we can determine the q parameter at the output:

$$\boxed{q_{out} = \frac{Aq_{in} + B}{Cq_{in} + D}}$$

Gaussian beams and ABCD matrices

Example: propagation through a distance d of empty space

$$M = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

The q parameter after this propagation is given by:

$$q_{out} = \frac{Aq_{in} + B}{Cq_{in} + D} = q_{in} + d$$

which is the same as what we've seen earlier: propagation through empty space merely adds to q by an amount equal to the distance propagated.

But this works for more complicated optical systems too.

Gaussian beams and lenses

Example: propagation through a thin lens: $M = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$

$$q_{out} = \frac{Aq_{in} + B}{Cq_{in} + D} = \frac{q_{in}}{-\frac{1}{f} q_{in} + 1}$$

$$\frac{1}{q_{out}} = \frac{-\frac{1}{f} q_{in} + 1}{q_{in}} = \frac{1}{q_{in}} - \frac{1}{f}$$

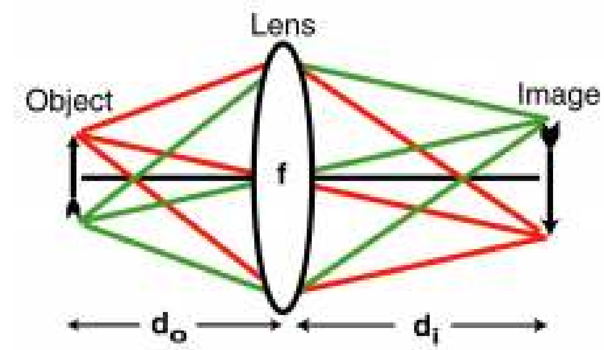
So, the lens does not modify the imaginary part of $1/q$.
The beam waist w is unchanged by the lens.

The real part of $1/q$ decreases by $1/f$. The lens changes the radius of curvature of the wave front:

$$\frac{1}{R_{out}} = \frac{1}{R_{in}} - \frac{1}{f}$$

Gaussian beams and imaging

Example: an imaging system



$$\text{ray matrix} = \begin{bmatrix} -d_i/d_o & 0 \\ -1/f & -d_o/d_i \end{bmatrix} = \begin{bmatrix} M & 0 \\ -1/f & 1/M \end{bmatrix}$$

(matrix connecting the object plane to the image plane)

(M = magnification)

Assume that the Gaussian beam has a focal spot located a distance d_o before the lens (i.e., at the position of the "object").

Then $$q_{in} = j \cdot z_R = j\frac{\pi w_{in}^{\ 2}}{\lambda}$$

w_{in} = beam waist at the input of the system

Question: what is w_{out}? (the beam waist at the image plane)

Gaussian beams and imaging

$$q_{in} = j \cdot z_R$$

$$q_{out} = \frac{M q_{in}}{-q_{in}/f + 1/M} = \frac{j z_R M}{-j z_R/f + 1/M}$$

If we want to find the beam waist at the output plane, we must compute the imaginary part of $1/q_{out}$:

$$\mathrm{Im}\left\{\frac{1}{q_{out}}\right\} = -\frac{1}{M^2 z_R} = -\frac{1}{M^2} \times \frac{\lambda}{\pi w_{in}{}^2}$$

This quantity is related to the beam waist at the output, because:

$$\mathrm{Im}\left\{\frac{1}{q_{out}}\right\} = -\frac{\lambda}{\pi w_{out}{}^2} \longrightarrow w_{out} = M\, w_{in}$$

So the beam's spot size is magnified by the lens, just as we would have expected from our ray matrix analysis of the imaging situation.

Interestingly, the beam does <u>not</u> have $R = \infty$ at the output plane.

Higher order Gaussian modes

The Gaussian beam is only the lowest order (i.e., simplest) solution. The more general solution involves Hermite polynomials:

$$\left|u_n(x,z)\right| = H_n\left(\frac{\sqrt{2}x}{w(z)}\right)\exp\left(-\frac{x^2}{w^2(z)}\right)$$

where the first few Hermite polynomials are:

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

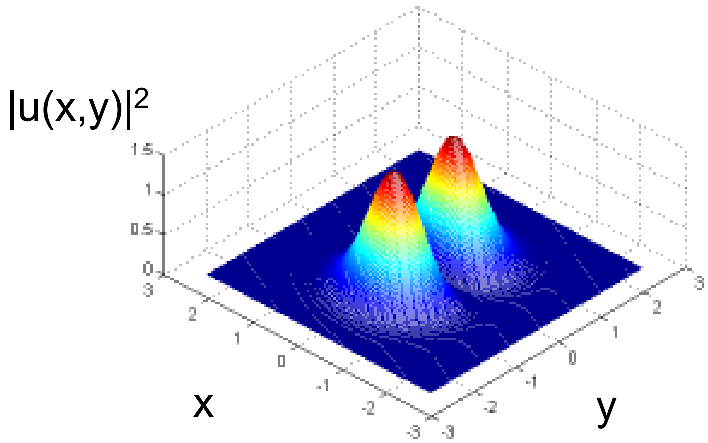
$$H_2(x) = 4x^2 - 2$$

And, in general, the x and y directions can be different!

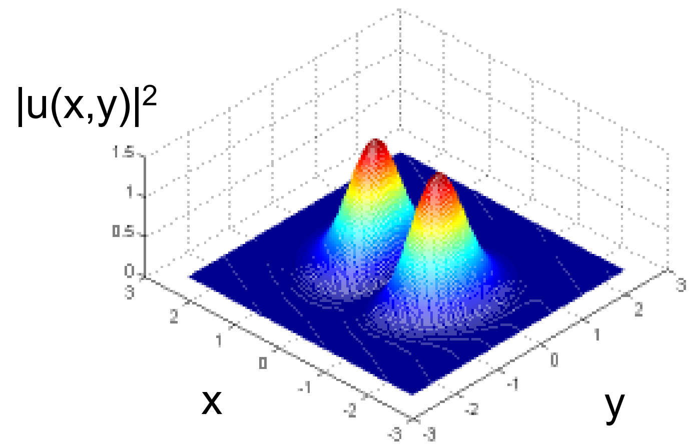
$$u_{nm}(x,y,z) = u_n(x,z)\cdot u_m(y,z)$$

The Gaussian beam we have discussed corresponds to the values n = 0, m = 0 - the so-called “zero-zero” mode.

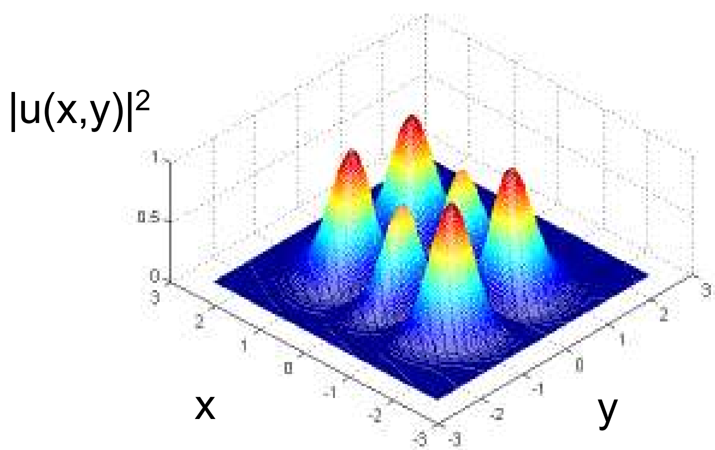
Higher order Gaussian modes - examples



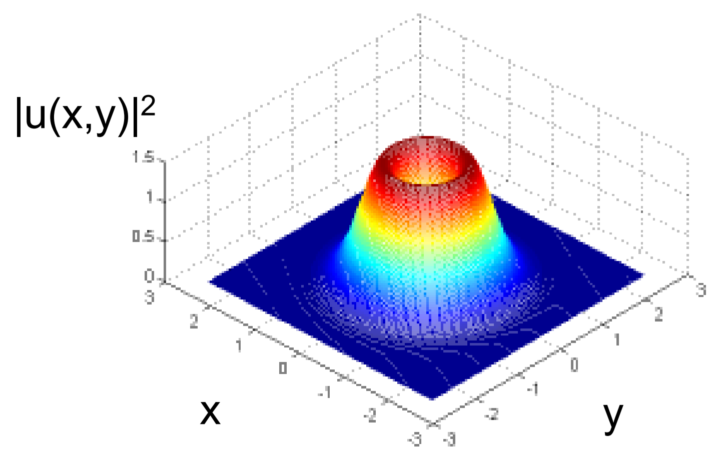
10 mode



01 mode



12 mode



A superposition of the 10 and 01 modes: a “donut mode”