30. Diffraction and the Fourier Transform

Diffraction examples
  Diffraction by an edge
  Arago spot

The far-field
  Fraunhofer Diffraction
  Some examples

Simeon Poisson
  (1781 - 1840)

Francois Arago
  (1786 - 1853)
Reminder: Fresnel-Kirchoff diffraction

Coordinates:
- the plane of the aperture: $x_1, y_1$
- the plane of observation: $x_0, y_0$
(a distance $z$ downstream)

$$r_{01} = \sqrt{z^2 + (x_0 - x_1)^2 + (y_0 - y_1)^2}$$

$$E(x_0, y_0) \propto \iint \exp \left\{ jk \left[ \frac{(x_1^2 - 2x_0x_1)}{2z} + \frac{(y_1^2 - 2y_0y_1)}{2z} \right] \right\} \text{Aperture}(x_1, y_1) E(x_1, y_1) \, dx_1 \, dy_1$$

Quadratics in the exponent: a messy integral
A plane wave incident on a sharp edge

Fresnel diffraction integral, in one dimension:

\[
E(x_0) = \exp\{jkz\} \int_{x_1=0}^{\infty} \frac{1}{j\lambda z} \exp\left\{ jk \left[ \frac{(x_0 - x_1)^2}{2z} \right] \right\} dx_1
\]

The irradiance exactly at the edge is 25% of the value far from the edge.
Diffraction by an Edge

Light passing by an edge

Electrons passing by an edge
An interesting manifestation of diffraction effects: The Spot of Arago

If a beam encounters a circular “stop”, it develops a hole, which fills in as it propagates and diffracts:

Interestingly, the hole fills in from the center first!
The Spot of Arago

Why does it happen?

According to Huygen’s principle, every point on the perimeter of the disc radiates a spherical wave.

Obviously, the distance from each spherical wave’s point of origin to a point on the central axis of the disc is the same.

Constructive interference on the center line!

In 1818, Poisson used Fresnel’s theory to predict this phenomenon. He regarded this as proof that Fresnel’s wave theory was nonsense, and that light must be a particle and not a wave. But almost immediately, Arago experimentally verified Poisson’s prediction, vindicating Fresnel.
Simplification of Fresnel diffraction

Recall the Fresnel diffraction result:

\[ E(x_0, y_0) \propto \iint \exp \left\{ jk \left[ \frac{-2x_0x_1 - 2y_0y_1}{2z} + \frac{x_1^2 + y_1^2}{2z} \right] \right\} \text{Aperture}(x_1, y_1) \, dx_1dy_1 \]

Let \( D \) be the largest dimension of the aperture: \( D^2 = \max(x_1^2 + y_1^2) \).

Our first step, which allowed us to obtain the Fresnel result, was the paraxial approximation:

\[ z \gg D \quad \text{or} \quad \frac{D}{z} \ll 1 \]

Note that this approximation does not contain the wavelength.

A more severe approximation is suggested by noticing that the integral simplifies A LOT if only we could neglect the quadratic terms \( x_1^2 \) and \( y_1^2 \).

If \( k \frac{D^2}{2z} \ll 1 \), then we could do that…
Approximation #2: involving the wavelength

This new approximation, which does contain \( \lambda \), is equivalent to:

\[
\frac{D}{z} \ll \frac{2}{kD} = \frac{\lambda}{\pi D}
\]

So \( \frac{D}{z} \) is not merely required to be less than 1, but is required to be less than \( \frac{\lambda}{\pi D} \), which is generally smaller than one for visible light.

Note: if the aperture is a slit of width \( D = 2b \), this condition becomes:

\[
\frac{2b}{z} \ll \frac{1}{kb} \quad \text{or} \quad \frac{4\pi b^2}{\lambda z} \ll 1
\]

Recall our definition of the Fresnel number for a slit of width \( 2b \): \( N = \frac{b^2}{\lambda z} \)

we see that this new approximation is equivalent to: \( 4\pi N \ll 1 \).
Approximation #2: Fraunhofer diffraction

Apply this approximation: \( \frac{kD^2}{2z} \ll 1 \) to the Fresnel diffraction result:

\[
E(x_0, y_0) \propto \int \int \exp \left\{ jk \left[ \frac{(-2x_0x_1 - 2y_0y_1)}{2z} + \frac{(x_1^2 + y_1^2)}{2z} \right] \right\} Aperture(x_1, y_1) \, dx_1 dy_1
\]

In this case, the quadratic terms are tiny, so we can ignore them.

In this case, the integral simplifies quite a lot. Still using all the other approximations that we discussed last time, we now have:

\[
E(x_0, y_0) \propto \int \int \exp \left\{ -\frac{jk}{z} (x_0 x_1 + y_0 y_1) \right\} Aperture(x_1, y_1) \, dx_1 dy_1
\]

Fraunhofer Diffraction: the situation that occurs when the quadratic terms can be ignored.
Joseph von Fraunhofer

Fraunhofer achieved fame by developing recipes for the world’s finest optical glass. He also invented precise methods for measuring dispersion of glass, and discovered more than 500 different absorption lines in sunlight, most due to specific atomic or molecular species at the sun’s surface.

These are still known as Fraunhofer lines.

He had almost nothing to do with Fraunhofer diffraction that we’re discussing today. He did, however, invent the diffraction grating, which we will discuss next lecture.

He was almost an exact contemporary of Augustin Fresnel (1788 – 1827). But it is unlikely that they ever met.
The Fraunhofer regime

How far away is far enough? We must have both $z \gg D$ and $z \gg \pi D^2/\lambda$.

Example #1: green light ($\lambda = 0.5 \, \mu m$)

If $D = 1$ millimeter, then:

$$z \gg \frac{\pi D^2}{\lambda} = \frac{\pi (1000)^2}{0.5} = 6.3 \text{ meters}$$

If $D = 10$ microns, then:

$$z \gg \frac{\pi D^2}{\lambda} = \frac{\pi (10)^2}{0.5} = 630 \text{ microns}$$

Example #2: microwaves ($\lambda = 3 \, \text{cm}$)

If $D = 10$ centimeters, then:

$$z \gg \frac{\pi D^2}{\lambda} = \frac{\pi (10)^2}{3} = 1 \text{ meter}$$

If $D = 1$ millimeter, then:

$$z \gg \frac{\pi D^2}{\lambda} = \frac{\pi (1)^2}{30} = 0.1 \text{ millimeters}$$

But notice that $z \gg 1 \, \text{mm}$ is required in this case for the paraxial approximation to also be true.
Fraunhofer diffraction is a Fourier transform

\[ E(x_0, y_0) \propto \iiint \exp \left\{ -\frac{jk}{z} (x_0 x_1 + y_0 y_1) \right\} \text{Aperture}(x_1, y_1) \, dx_1 dy_1 \]

This is just a Fourier Transform! (actually, two of them, in two variables)

Interestingly, it’s a Fourier Transform from position, \( x_1 \), to another position variable, \( x_0 \) (in another plane, i.e., a different \( z \) position).

Usually, the Fourier “conjugate variables” have reciprocal units (e.g., \( t \) and \( \omega \), or \( x \) and \( k \)).

The conjugate variables here are really \( x_1 \) and \( kx_0/z \), which do have reciprocal units.
Fraunhofer diffraction is a Fourier transform

In one dimension:

\[ E(x_0) \propto \int \exp \left\{ -j \left( \frac{kx_0}{z} \right) x_1 \right\} \text{Aperture}(x_1)dx_1 \]

So, the light in the Fraunhofer regime (the “far field”) is simply the Fourier Transform of the apertured field!

Knowing this makes the calculations a lot easier…
Fraunhofer diffraction pattern for a slit

In this case, the problem is a single Fourier transform (in $x$), rather than two of them (in $x$ and $y$):

$$E(x_0) \propto \int \exp\left\{-\frac{jk}{z}(x_0 x_1)\right\} \text{Aperture}(x_1) dx_1$$

The aperture function is simple:

$$\text{Aperture}(x_1) = \begin{cases} 1 & \text{if } -b < x_1 < b \\ 0 & \text{otherwise} \end{cases}$$

But we know that the Fourier transform of a rectangle function (of width $2b$) is a sinc function:

$$\text{FT}\left[ \text{Aperture}(x_1) \right] \propto \frac{\sin\left(\frac{kx_0 b}{z}\right)}{kx_0 b / z} = \frac{\sin\left(2\pi N x_0 / b\right)}{2\pi N x_0 / b}$$

How very satisfying! This is exactly the answer we saw last lecture, for the Fresnel diffraction result in the limit of very large $z$. 

written here in terms of the Fresnel number $N = \frac{b^2}{\lambda z}$
Diffraction from two slits

The aperture function is a convolution of a single slit (a rect function) with two delta functions:

\[
rect(x_1) = \begin{cases} 
1 & -b < x_1 < b \\
0 & \text{otherwise}
\end{cases}
\]

\[
Aperture(x_1) = rect(x_1) * \left[ \delta(x_1 - a) + \delta(x_1 + a) \right]
\]

The far-field diffraction pattern is the FT of this convolution, which is the product of the FTs:

\[
FT\left[ Aperture(x_1) \right] \propto \frac{\sin(kx_0b/z)}{kx_0b/z} \left( e^{jk(a/z)x_0} + e^{-jk(a/z)x_0} \right)
\]

\[
= \frac{\sin(kx_0b/z)}{kx_0b/z} \cos(kx_0a/z)
\]

The result for the case of \( b = 10\lambda \) and \( a = 30\lambda \).
Fraunhofer Diffraction from a Square Aperture

A square aperture (edge length = 2b) just gives the product of two sinc functions in x and in y. Just as if it were two slits, orthogonal to each other.

\[
FT\left[\text{Square Aperture}(x_1, y_1)\right] \propto \frac{\sin(kx_0 b/z) \cdot \sin(ky_0 b/\zeta)}{kx_0 b/\zeta \cdot ky_0 b/\zeta}
\]
Fraunhofer diffraction from a circular aperture

The 2D Fourier transform of a circular aperture, radius = \( b \), is given by a Bessel function of the first kind:

\[
FT \left[ \text{Circular aperture} (x_1, y_1) \right] \propto \frac{J_1(k \rho b/z)}{k \rho b/z}
\]

where \( \rho = \sqrt{x_1^2 + y_1^2} \) is the radial coordinate in the \( x_1 - y_1 \) plane.

Most of the energy falls in the central region, for values \( k \rho b/z < 3.83 \).
The Airy pattern

A circular aperture yields a diffracted pattern known as an “Airy pattern” or an “Airy disc”.

\[
\text{Diffracted Irradiance} \propto \frac{J_1(k\rho b/z)^2}{k\rho b/z}
\]

The central spot contains about 84% of the total energy:

\[
\int_0^\infty \frac{J_1(r)^2}{r} dr = 0.838
\]

\[
\int_0^\infty \frac{J_1(r)^2}{r} dr = 0.838
\]
The size of the Airy pattern

How big is the central spot, where most of the energy is found?

Define the size of the spot as $\rho_{\text{spot}}$, the radial distance from the center to the first zero:

$$k \rho_{\text{spot}} b/z = 3.83$$

$$\rho_{\text{spot}} = \frac{3.83 z}{kb} = 3.83 \frac{\lambda z}{2\pi b}$$

The diameter of the aperture is $D = 2b$, so we can write:

$$\rho_{\text{spot}} = \frac{3.83 \lambda z}{\pi D} = 1.22 \frac{\lambda z}{D}$$
Diffraction from small and large circular apertures

This is a good illustration of the Scale Theorem!

\[ \rho_{\text{spot}} \propto \frac{1}{D} \]
Laser speckle is a diffraction pattern. When a laser illuminates a rough surface or passes through a region where it can scatter a little bit, the result is a “speckle” pattern. It is a diffraction pattern from the very complex surface.

Don’t try to do this Fourier Transform at home.

But people do. Computing the inverse FT of a speckle pattern can give information about the degree of roughness of a surface.
There are situations where this Fourier transform idea is not so useful.

Example: light passing by an edge. In this case, the effective “width” of the slit, $D$, is infinity/2. It is impossible to reach the Fraunhofer regime of $z >> \pi D^2/\lambda$. 

![Diagram of light passing by an edge](image)

- geometrical shadow
- exact diffraction result

Intensity vs. position $x_0$