The Generation of Ultrashort Laser Pulses II

The phase condition

Trains of pulses – the Shah function

Laser modes and mode locking

Homogeneous vs. inhomogeneous gain media

Spatial modes
There are 3 conditions for steady-state laser operation.

Amplitude condition
  threshold
  slope efficiency

Phase condition
  axial modes
  homogeneous vs. inhomogeneous gain media

Transverse modes
  Hermite Gaussians
  the “donut” mode
Phase condition

Steady-state condition #2:

Phase is invariant after each round trip

$$\frac{\text{angle}(E_{\text{after}})}{\text{angle}(E_{\text{before}})} = \exp\left(-i\frac{\omega L_{rt}}{c}\right) = 1 \quad \Rightarrow \quad \omega_q = \frac{2\pi c}{L_{rt}} \cdot q \quad (q = \text{an integer})$$

Technically, this should be:

$$\left(L_{rt} - L_m\right)n_{\text{air}} + L_m n_{\text{sapphire}}$$

An integer number of wavelengths must fit in the cavity.
Longitudinal modes

\[ \omega_q = \frac{2\pi c}{L_{rt}} \cdot q \]

(q = an integer)

“axial” or “longitudinal” cavity modes

Mode spacing:
\[ \Delta \nu = \frac{c}{L_{rt}} \]

But how does this translate to the case of short pulses?
The spectrum of a **single** pulse

The uncertainty principle says that the product of the temporal and spectral pulse widths is greater than ~1. So a short pulse has a broad bandwidth.

But femtosecond lasers do not emit just one single pulse…
Femtosecond lasers emit trains of (nominally) identical pulses.

Every time the laser pulse hits the output mirror, some of it emerges.

The output of a typical ultrafast laser is a train of identical very short pulses:

where $I(t)$ represents a single pulse intensity vs. time and $T$ is the time between pulses.
The Shah Function

The Shah function, $\mathbb{I}(t)$, is an infinitely long train of equally spaced delta-functions.

The symbol $\mathbb{I}$ is pronounced *shah* after the Cyrillic character $\mathbb{I}$, which is said to have been modeled on the Hebrew letter $\daleth$ (shin) which, in turn, may derive from the Egyptian $\hbar\hbar\hbar$, a hieroglyph depicting papyrus plants along the Nile.
The Fourier Transform of the Shah Function

\[ \mathcal{F} \{ \text{III}(t) \} = \]

\[ = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(t - m) \exp(-i\omega t) \, dt \]

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So:

\[ \mathcal{F} \{ \text{III}(t) \} \propto \text{III}(\omega / 2\pi) \]

If \( \omega = 2n\pi \), where \( n \) is an integer, the sum diverges; otherwise, cancellation occurs and the sum vanishes.
The Shah Function and a Pulse Train

An infinite train of identical pulses can be written:

\[ E(t) = \sum_{m=-\infty}^{\infty} f(t - mT) \]

where \( f(t) \) is the shape of each pulse and \( T \) is the time between pulses.

But \( E(t) \) can also be written:

\[ E(t) = \mathcal{III}(t / T) \ast f(t) \]

Proof:

\[ \mathcal{III}(t / T) \ast f(t) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t' / T - m) f(t - t') \, dt' \]

To do the integral, set:

\( t'/T = m \) or \( t' = mT \)
The Fourier Transform of an Infinite Train of Pulses

An infinite train of identical pulses can be written:

\[ E(t) = \text{III}(t/T) \ast f(t) \]

The Convolution Theorem says that the Fourier Transform of a convolution is the product of the Fourier Transforms. So:

\[ \tilde{E}(\omega) \propto \text{III}(\omega T / 2\pi) F(\omega) \]

The spacing between frequencies (modes) is then \( \delta \omega = 2\pi/T \) or \( \delta \nu = 1/T \).
The Fourier Transform of a **Finite** Pulse Train

A **finite** train of identical pulses can be written:

\[ E(t) = \left[ \text{III}(t / T) \ g(t) \right] \ast f(t) \]

where \( g(t) \) is a finite-width envelope over the pulse train.

Use the fact that the Fourier transform of a product is a convolution…

\[ \tilde{E}(\omega) \propto \left[ \text{III}(\omega T / 2\pi) \ast G(\omega) \right] F(\omega) \]
A laser’s frequencies are often called **longitudinal modes**.

They’re separated by $1/T = c/2L$, where $L$ is the length of the laser.

Which modes lase depends on the gain and loss profiles.

Here, additional narrowband filtering has yielded a single mode.
Generating short pulses = Mode-locking

Locking vs. not locking the phases of the laser modes (frequencies)

Random phases of all laser modes:
- Out of phase
- Out of phase
- Out of phase

Locked phases of all laser modes:
- Out of phase
- In phase!
- Out of phase

Intensity vs. time:
- Random phases: Light bulb
- Locked phases: Ultrashort pulse!
Mode-locked vs. non-mode-locked light

Mode-locked pulse train:

\[ \tilde{E}(\omega) = F(\omega) \exp(i\omega T / 2\pi) \]

\[ = F(\omega) \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m / T) \]

A train of short pulses

Non-mode-locked pulse train:

\[ \tilde{E}(\omega) = \sum_{m=-\infty}^{\infty} F(\omega) \exp(i\phi_m) \delta(\omega - 2\pi m / T) \]

Random phase for each mode

\[ = F(\omega) \sum_{m=-\infty}^{\infty} \exp(i\phi_m) \delta(\omega - 2\pi m / T) \]

A mess…
Q: How many different modes can oscillate simultaneously in a 1.5 meter Ti:sapphire laser?

A: Gain bandwidth \( \Delta \lambda = 200 \text{ nm} \Rightarrow \Delta v = (c/\lambda^2) \Delta \lambda \approx 10^{14} \text{ Hz} \)

\[ \frac{\Delta v_{\text{bandwidth}}}{\Delta v_{\text{mode}}} = 10^6 \text{ modes} \]

That seems like a lot. Can this really happen?
Homogeneous gain media

We have seen that the gain is given by:

\[
g(\omega) = \frac{1}{2} \cdot \frac{\Delta N_0}{1 + I / I_{sat}} \cdot \frac{\sigma_0}{1 + \zeta^2}
\]

where \( \zeta = \frac{2(\omega - \omega_0)}{\gamma} \)

Suppose the gain is increased to a point where it equals the loss at a particular frequency, \( \omega_q \) which is one of the cavity mode frequencies.

Q: Suppose the gain is increased further. Can it be increased so that the mode at \( \omega_{q+1} \) oscillates in steady state?

A: In an ideal laser, NO! At \( \omega_q \), Gain = Loss!
Inhomogeneous gain media

Suppose that the collection of 4-level systems do not all share the same $\omega_0$

Consider a collection of sites, with fractional number between $\omega_0$ and $\omega_0 + d\omega_0$:

$$dN(\omega_0) = Ng(\omega_0)d\omega_0$$

The modified susceptibility is:

$$\chi(\omega) = \int d\omega_0 \chi_{h}(\omega; \omega_0)g(\omega_0)$$

“packets” are mutually independent - they can saturate independently!
Inhomogeneous broadening

\[ \chi(\omega) = \int d\omega_0 \chi_h(\omega; \omega_0) \cdot e^{-4 \ln(2) \left( \frac{\omega - \bar{\omega}_0}{\Delta \omega} \right)^2} \]

No closed-form solution

For strong inhomogeneous broadening (\(\Delta \omega \gg \gamma\)):

\[ \chi''(\omega) = \text{Gaussian, with width } = \Delta \omega \text{ (NOT } \gamma) \]

\[ \chi'(\omega): \text{no simple form, but it resembles } \chi_h'(\omega) \]

Examples:

Nd:YAG - weak inhomogeneity
Nd:glass - strong inhomogeneity
Ti:sapphire - absurdly strong inhomogeneity
Q: Suppose the gain is increased above threshold in an inhomogeneously broadened laser. Can it be increased so that the mode at $\omega_{q+1}$ oscillates cw?

A: Yes! Each homogeneous packet saturates independently.

“spectral hole burning”

- Multiple oscillating cavity modes
- Losses
- $a$ priori, these modes need not have any particular phase relationship to one another
- Laser gain profile
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   threshold
   slope efficiency

Phase condition
   axial modes
   homogeneous vs. inhomogeneous gain media

Transverse modes
   Hermite Gaussians
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\[ |E(x,y)|^2 \]
Steady-state condition #3:

"transverse modes": those which reproduce themselves on each round trip, except for overall amplitude and phase factors

How would we determine these modes?

An eigenvalue problem:

$$\beta_{nm} \cdot E_{nm}(x, y) = \iint K(x, y; x_0, y_0) \cdot E_{nm}(x_0, y_0) \, dx_0 \, dy_0$$

propagation kernel
Transverse modes

Solutions are the product of two functions, one for each transverse dimension:

$$E_{nm}(x, y) = u_n(x) \cdot u_m(y)$$

where the $u_n$’s are Hermite Gaussians:

$$|u_n(x)| = H_n \left( \frac{\sqrt{2}x}{w} \right) \cdot \exp \left( -\frac{x^2}{w^2} \right)$$

$w = \text{beam waist parameter}$

Notation:

"TEM" = transverse electric and magnetic

"nm" = number of nodes along two principal axes
Transverse modes - examples

A superposition of the 10 and 01 modes: the “donut mode”

|E(x,y)|^2

- 12 mode