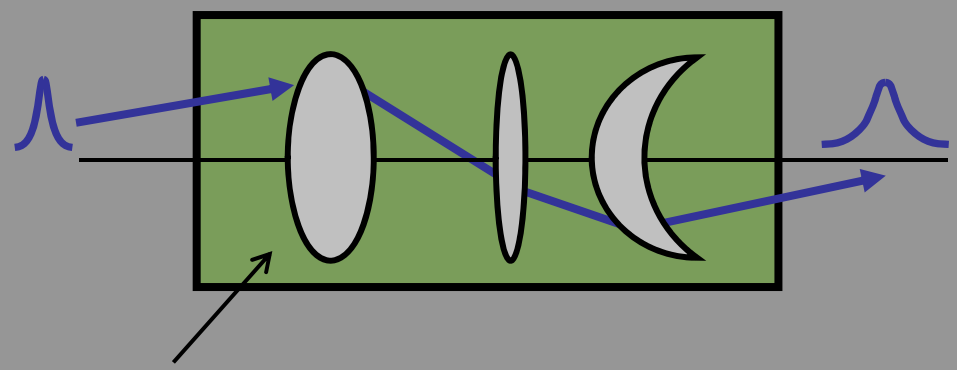
Propagation of laser beams

Ray optics and ray matrices

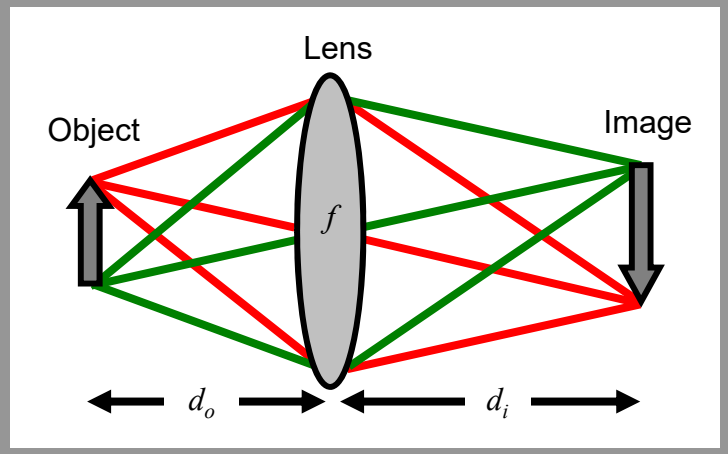
The Gaussian beam

Complex q and its propagation

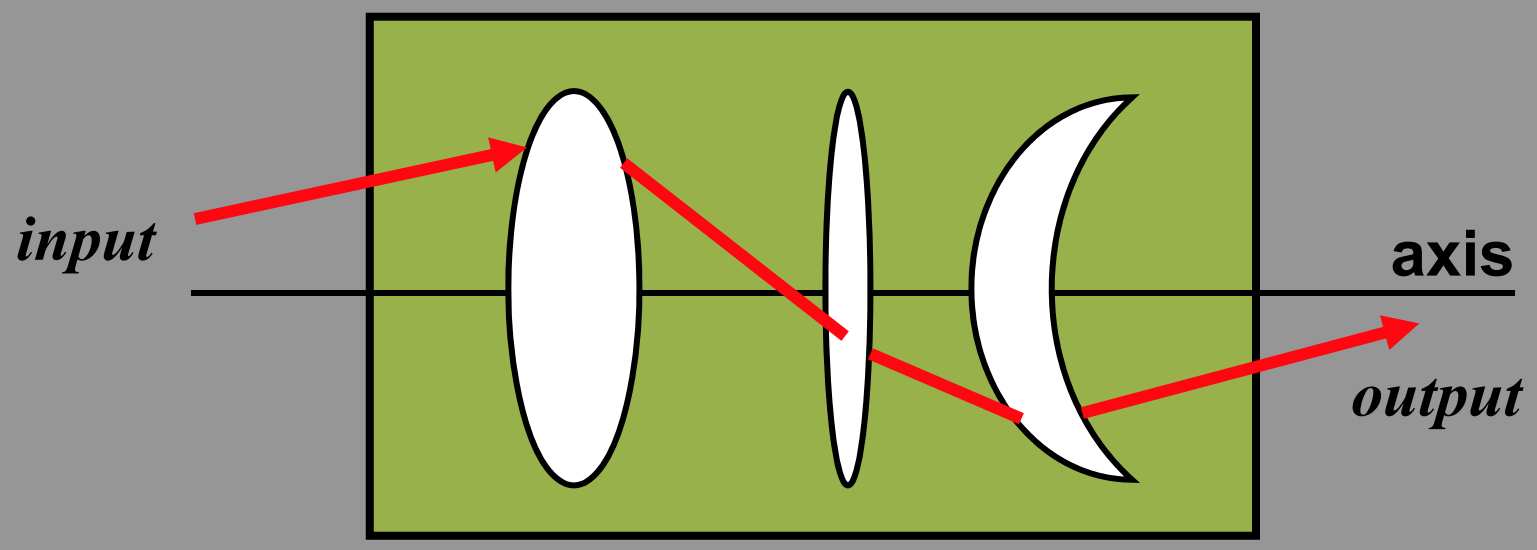
Laser resonators and stability



Optical system ↔ 2x2 Ray matrix



Ray Optics



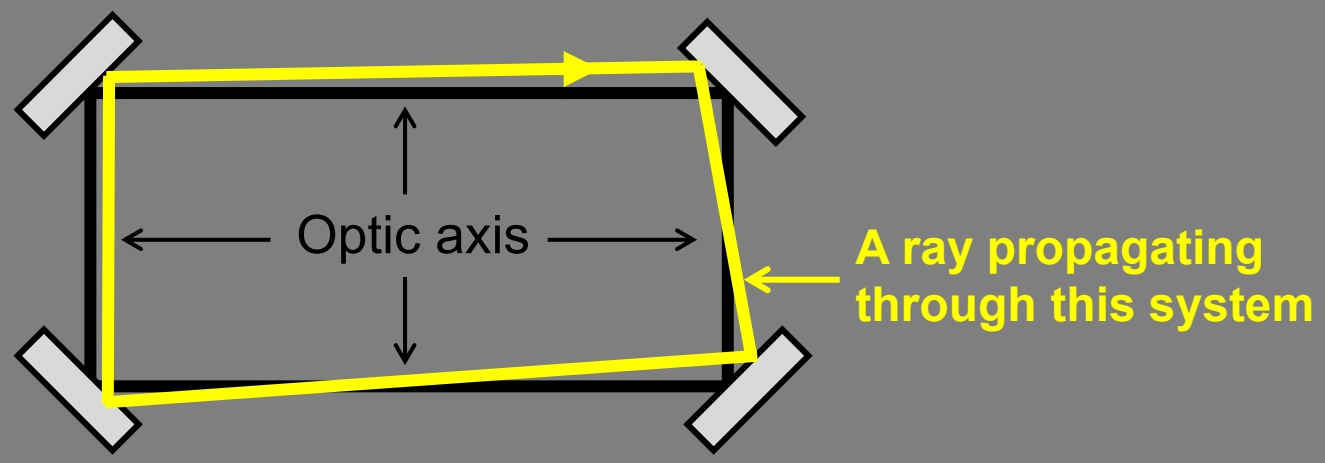
We'll define "light rays" as directions in space, corresponding, roughly, to k-vectors of light waves (but with direction only, no magnitude info).

Each optical system will have an axis, and all light rays will be assumed to propagate at small angles to the axis. This is called the *Paraxial Approximation*.

The Optic Axis

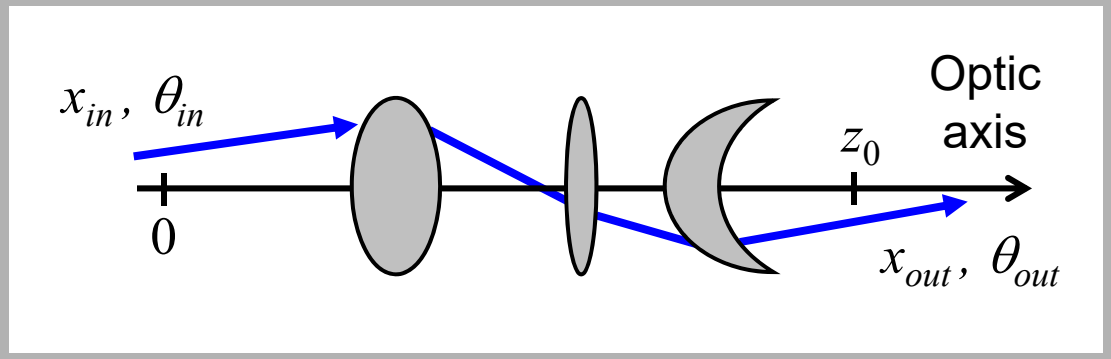
A mirror deflects the optic axis into a new direction with the angle of reflection equal to the angle of incidence.

This ring laser has an optic axis that scans out a rectangle.



We define all rays relative to the relevant optic axis.

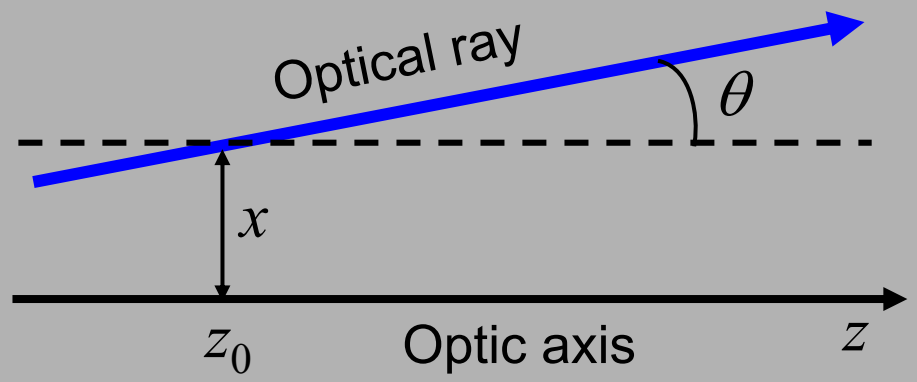
Ray Vectors



At every position, z, along the optic axis, a light ray can be defined by two co-ordinates:

its position, x

its slope, θ



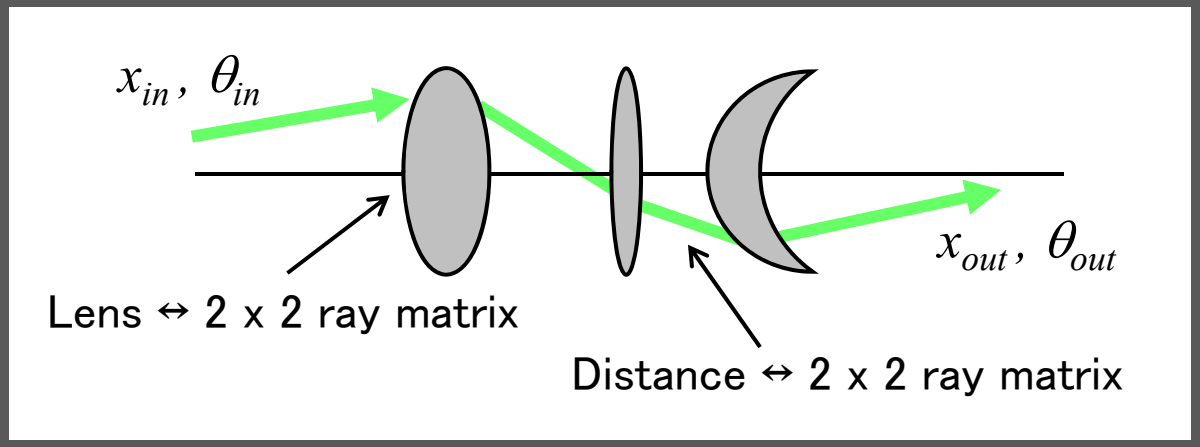
These parameters define a **ray vector**, which will change with distance, z, as the ray propagates through optics.

$$\begin{bmatrix} x \\ \theta \end{bmatrix}$$

Ray Matrices

For many optical components, we can define **2 x 2 ray matrices**.

An optical element's effect on a ray is found by multiplying the ray vector by the element's ray matrix.



$$\begin{bmatrix} x_{after\ lens} \\ \theta_{after\ lens} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_{before\ lens} \\ \theta_{before\ lens} \end{bmatrix}$$

Ray vector after lens

Ray matrix for lens

Ray vector before lens

We can do the same for the other lenses and the distances.

Ray Matrices as Derivatives

Since the displacements, x_{in} and x_{out}, and angles, θ_{in} and θ_{out}, are all assumed to be small, we can think in terms of partial derivatives.

$$x_{out} = \frac{\partial x_{out}}{\partial x_{in}} x_{in} + \frac{\partial x_{out}}{\partial \theta_{in}} \theta_{in}$$

$$\theta_{out} = \frac{\partial \theta_{out}}{\partial x_{in}} x_{in} + \frac{\partial \theta_{out}}{\partial \theta_{in}} \theta_{in}$$

Spatial magnification $\frac{\partial x_{out}}{\partial x_{in}}$

$\frac{\partial x_{out}}{\partial \theta_{in}}$

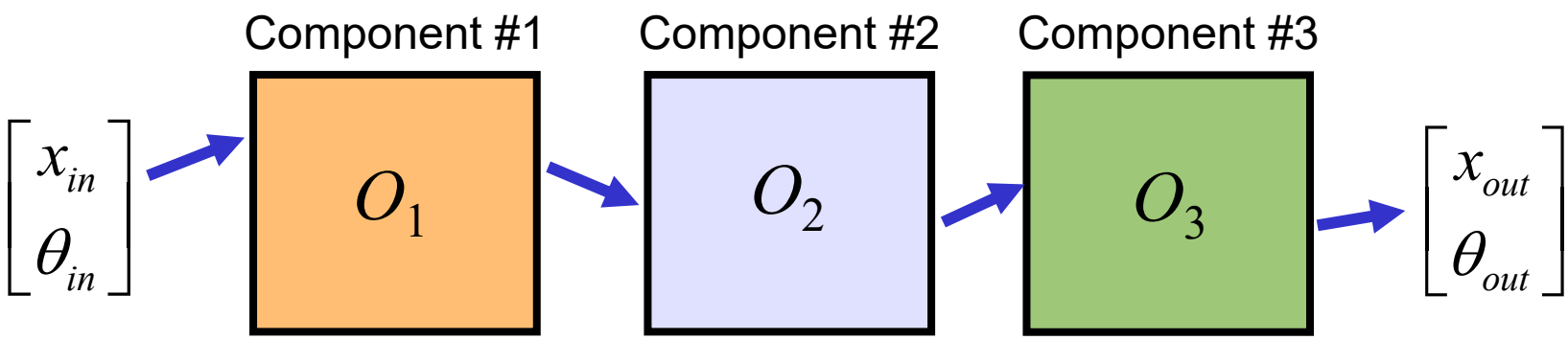
$$\begin{bmatrix} x_{out} \\ \theta_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix}$$

$\frac{\partial \theta_{out}}{\partial x_{in}}$

$\frac{\partial \theta_{out}}{\partial \theta_{in}}$ **Angular magnification**

We can write these equations in matrix form.

For cascaded elements, we simply multiply together all the individual ray matrices.

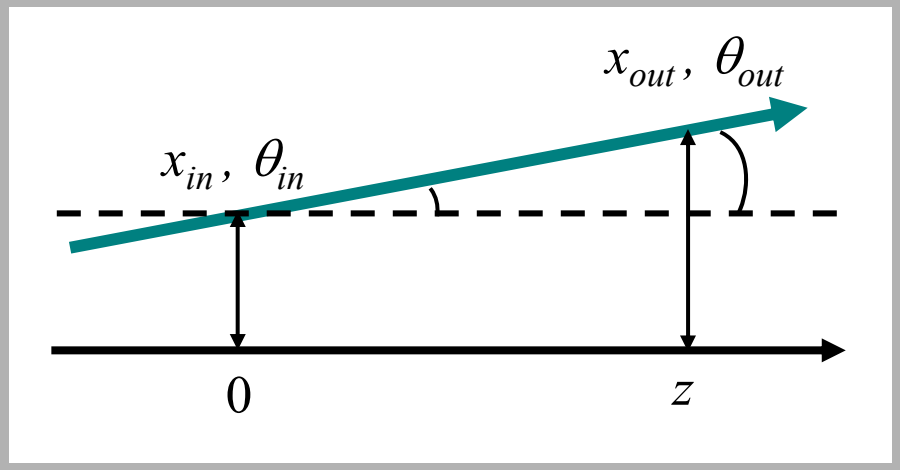


$$\begin{bmatrix} x_{out} \\ \theta_{out} \end{bmatrix} = O_3 \left\{ O_2 \left(O_1 \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix} \right) \right\} = O_3 \, O_2 \, O_1 \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix}$$

Notice that the order looks opposite to what it should be, but it actually does makes sense.

Ray Matrix for Free Space or a Medium

If x_{in} and θ_{in} are the position and slope upon entering, let x_{out} and θ_{out} be the position and slope after propagating an arbitrary distance, z.



$$x_{out} = x_{in} + z\,\theta_{in}$$
$$\theta_{out} = \theta_{in}$$

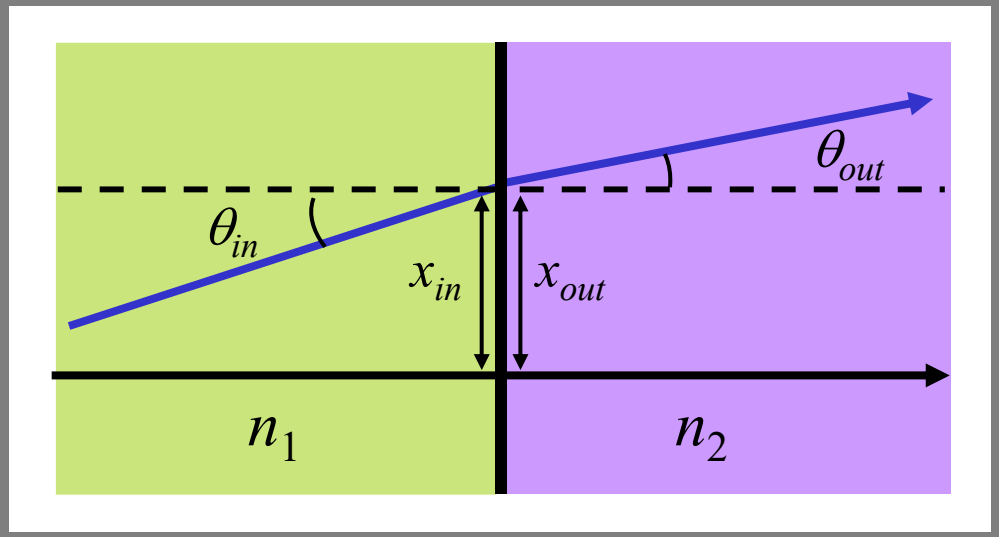
Small angle approximation: $\tan\theta \approx \theta$

Rewriting these expressions in matrix notation:

$$\begin{bmatrix} x_{out} \\ \theta_{out} \end{bmatrix} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix}$$

$$\boxed{O_{space} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}}$$

Ray Matrix for an Interface



At the interface:

$$x_{out} = x_{in}$$

Now calculate θ_{out}:

Snell's Law says: $n_1 \sin(\theta_{in}) = n_2 \sin(\theta_{out})$

which, for small angles, becomes: $n_1 \theta_{in} = n_2 \theta_{out}$

$$\Rightarrow \theta_{out} = [n_1 / n_2]\, \theta_{in}$$

$$O_{interface} = \begin{bmatrix} 1 & 0 \\ 0 & n_1 / n_2 \end{bmatrix}$$

Ray Matrix for a Curved Interface

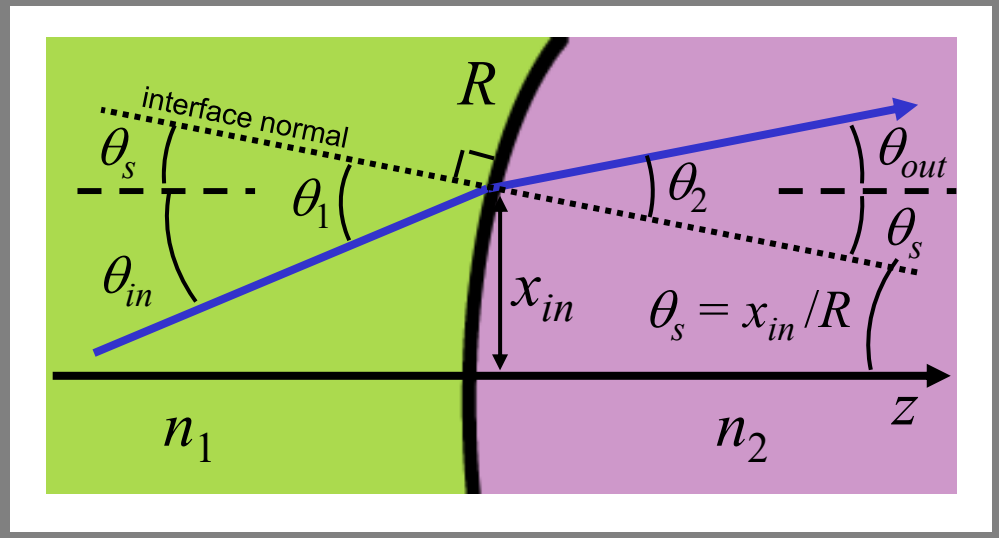
At the interface, again:

$$x_{out} = x_{in}.$$

To calculate θ_{out}, we must calculate θ_1 and θ_2.

θ_s is the surface slope at the height x_{in}.

$\theta_1 = \theta_{in} + \theta_s$ and $\theta_2 = \theta_{out} + \theta_s$



$$\theta_1 = \theta_{in} + x_{in}/R \quad \text{and} \quad \theta_2 = \theta_{out} + x_{in}/R$$

Snell's Law: $n_1\,\theta_1 = n_2\,\theta_2 \quad \Rightarrow n_1\,(\theta_{in} + x_{in}/R) = n_2\,(\theta_{out} + x_{in}/R)$

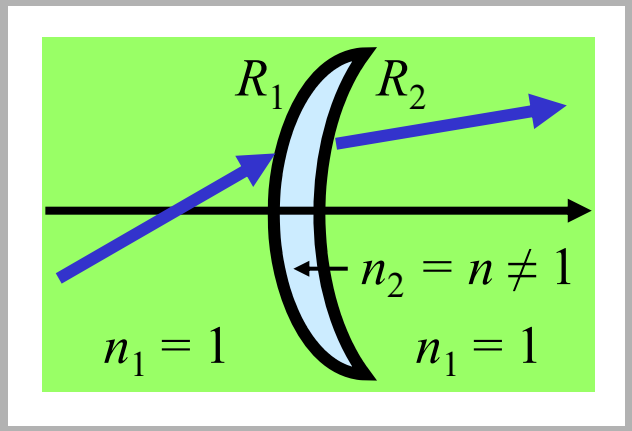
$$\Rightarrow \theta_{out} = (n_1/n_2)(\theta_{in} + x_{in}/R) - x_{in}/R$$

$$\Rightarrow \theta_{out} = (n_1/n_2)\theta_{in} + (n_1/n_2 - 1)\,x_{in}/R$$

$$O_{\substack{curved\\interface}} = \begin{bmatrix} 1 & 0 \\ (n_1/n_2 - 1)/R & n_1/n_2 \end{bmatrix}$$

A thin lens is just two curved interfaces.

We'll neglect the glass in between (it's a really thin lens!), and we'll take $n_1 = 1$.



$$O_{\substack{curved\\interface}} = \begin{bmatrix} 1 & 0 \\ (n_1/n_2 - 1)/R & n_1/n_2 \end{bmatrix}$$

$$O_{thin\ lens} = O_{\substack{curved\\interface\,2}}\, O_{\substack{curved\\interface\,1}} = \begin{bmatrix} 1 & 0 \\ (n-1)/R_2 & n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ [(1/n)-1]/R_1 & 1/n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ (n-1)/R_2 + n[(1/n)-1]/R_1 & n(1/n) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (n-1)/R_2 + (1-n)/R_1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ (n-1)(1/R_2 - 1/R_1) & 1 \end{bmatrix}$$

This can be written: $\boxed{\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}}$

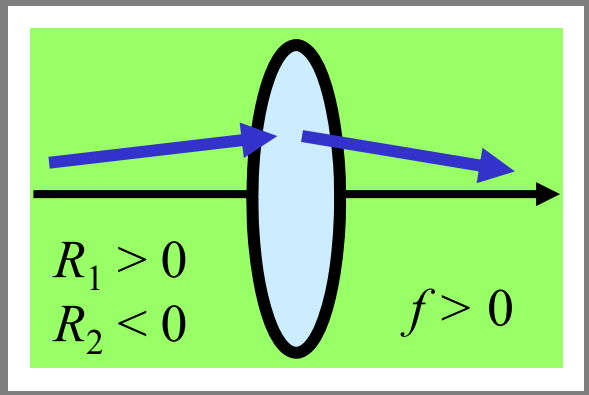
where: $\boxed{1/f = (n-1)(1/R_1 - 1/R_2)}$ **The Lens-Maker's Formula**

Ray Matrix for a Lens

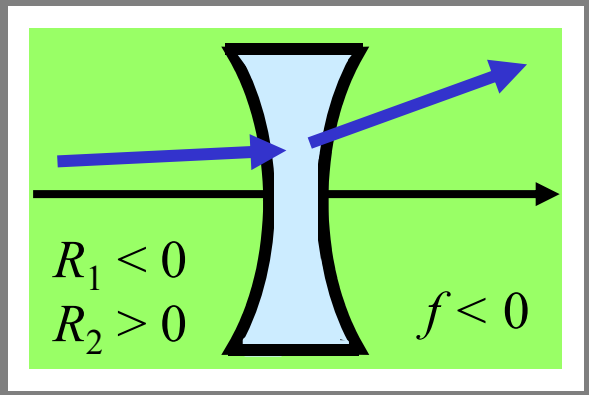
$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$O_{lens} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

The quantity, f, is the **focal length** of the lens. It's the most important parameter of a lens. It can be positive or negative (but never zero!).



If $f > 0$, the lens deflects rays toward the axis.



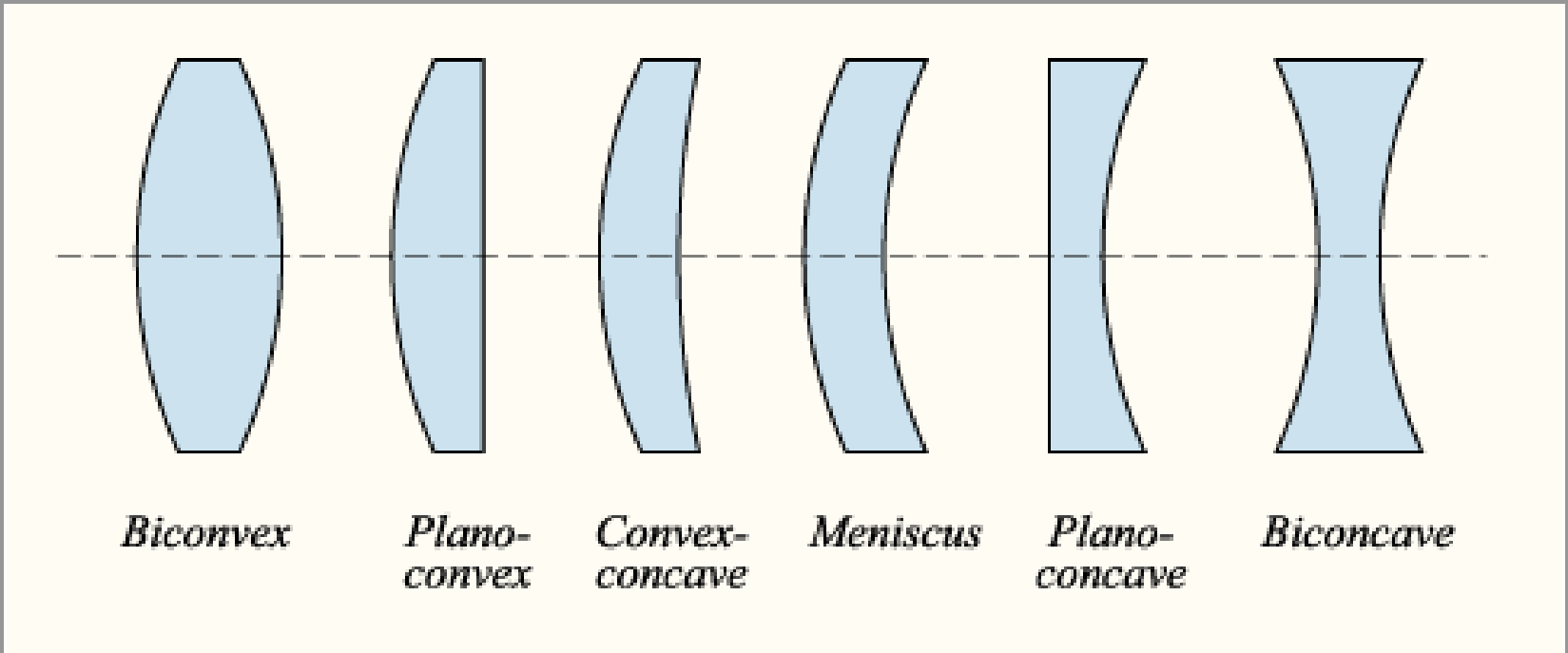
If $f < 0$, the lens deflects rays away from the axis.

Sign convention: $R > 0$ if the sphere center is to the right ($z > 0$), and $R < 0$ if the sphere center is to the left ($z < 0$).

It's easy to extend the Lens Maker's Formula to real lenses of greater thickness.

Types of lenses

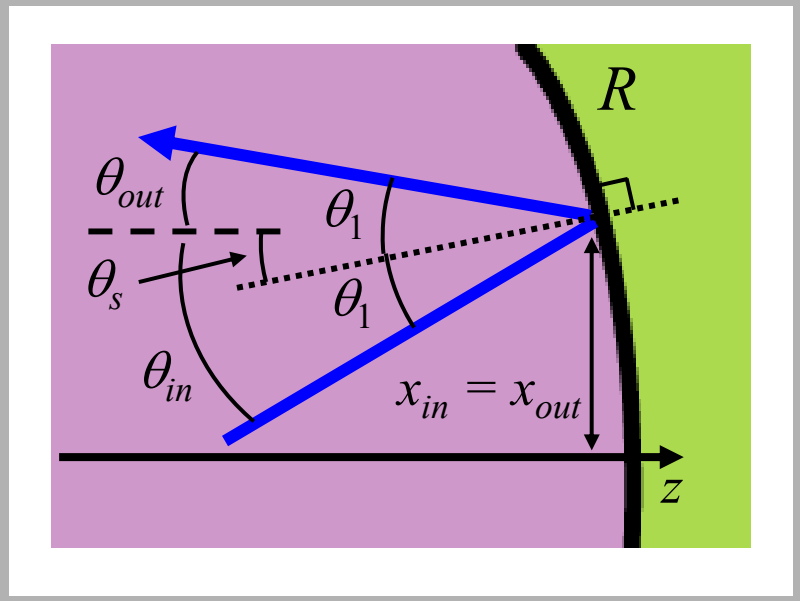
Lens nomenclature



Which type of lens to use (and how to orient it) depends on the application.

Ray Matrix for a Curved Mirror

Consider a mirror with radius of curvature, R, with its optic axis perpendicular to the mirror:



$$\theta_1 = \theta_{in} - \theta_s \qquad \theta_s \approx x_{in} / R$$

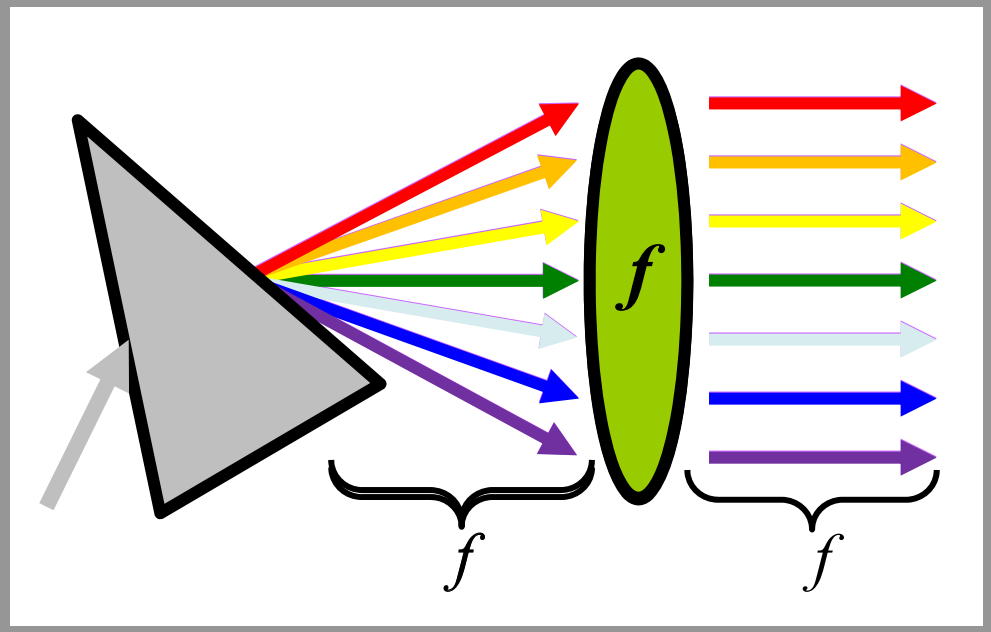
$$\theta_{out} = \theta_1 - \theta_s = (\theta_{in} - \theta_s) - \theta_s$$
$$\approx \theta_{in} - 2x_{in} / R$$

$$\Rightarrow \boxed{O_{mirror} = \begin{bmatrix} 1 & 0 \\ -2/R & 1 \end{bmatrix}}$$

Like a lens, a curved mirror will focus a beam. Its focal length is $R/2$.

Note that a flat mirror has $R = \infty$ and hence an identity ray matrix.

Lenses can simultaneously map angle to position and position to angle.



From input to output, use:

1) A distance f
2) A lens of focal length f
3) Another distance f

$$\begin{bmatrix} x_{out} \\ \theta_{out} \end{bmatrix} = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & f \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & f \\ -1/f & 0 \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix} = \begin{bmatrix} f\theta_{in} \\ -x_{in}/f \end{bmatrix}$$

So this arrangement maps input angle to position:

$$x_{out} \propto \theta_{in}$$

independent of input position

And it maps input position to angle:

$$\theta_{out} \propto x_{in}$$

independent of input angle

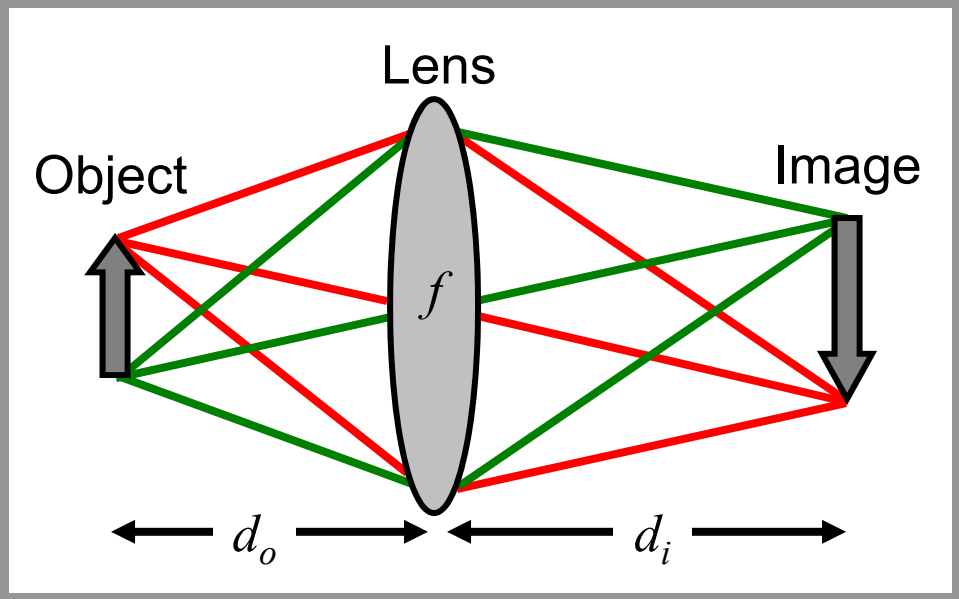
A system images an object when $B = 0$.

When $B = 0$, all rays from a point x_{in} arrive at a point x_{out}, independent of angle.

$$\begin{bmatrix} x_{out} \\ \theta_{out} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix} = \begin{bmatrix} A x_{in} \\ C x_{in} + D \theta_{in} \end{bmatrix}$$

$$\boxed{x_{out} = A\, x_{in}}$$

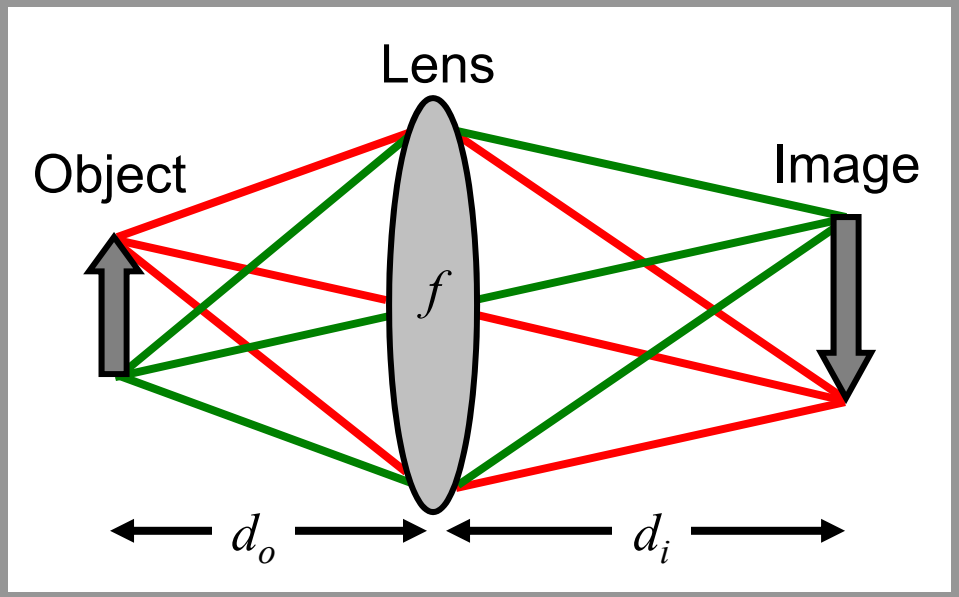
When $B = 0$, A is the **magnification**.



The Lens Law

From the object to the image, we have:

1) A distance d_o
2) A lens of focal length f
3) A distance d_i



$$O = \begin{bmatrix} 1 & d_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & d_o \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & d_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_o \\ -1/f & 1-d_o/f \end{bmatrix}$$

$$= \begin{bmatrix} 1-d_i/f & \boxed{d_o + d_i - d_o d_i / f} \\ -1/f & 1-d_o/f \end{bmatrix}$$

$$B = d_o + d_i - d_o d_i / f =$$

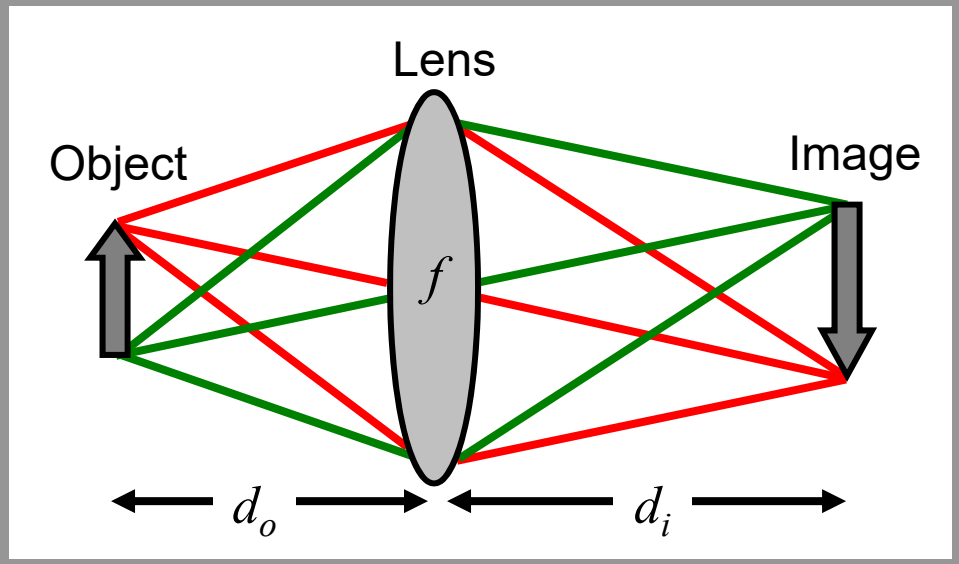
$$d_o d_i \left[1/d_o + 1/d_i - 1/f \right]$$

which equals zero if:

$$\boxed{\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}}$$

This is the Lens Law.

Imaging Magnification



If the imaging condition,

$$\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{f}$$

is satisfied, then:

$$O=\begin{bmatrix} 1-d_i/f & 0 \\ -1/f & 1-d_o/f \end{bmatrix}$$

So:

$$\boxed{O=\begin{bmatrix} M & 0 \\ -1/f & 1/M \end{bmatrix}}$$

$$A=1-d_i/f=1-d_i\left[\frac{1}{d_o}+\frac{1}{d_i}\right]$$

$$\Rightarrow \quad \boxed{=-\frac{d_i}{d_o}} \equiv M \text{ magnification}$$

$$D=1-d_o/f=1-d_o\left[\frac{1}{d_o}+\frac{1}{d_i}\right]$$

$$=-\frac{d_o}{d_i}=1/M$$

Angular magnification

If an optical system lacks cylindrical symmetry, we must analyze its x- and y-directions separately: cylindrical lenses.

A **spherical lens** focuses in both transverse directions.

A **cylindrical lens** focuses in **only one transverse direction.**

Focal power in vertical direction

Focal power in horizontal direction

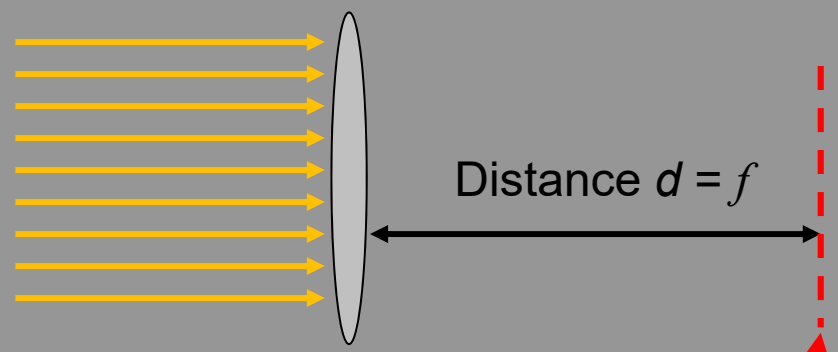
When using cylindrical lenses, we must perform two separate ray-matrix analyses, one for each transverse direction.

A failure of the ray optics approach

Input beam
(bundle of rays)

↓

Each ray has a unique x_{in}, but all have $\theta_{in} = 0$ (parallel rays)



Thin lens,
focal length f

What happens in this plane (the focal plane)?

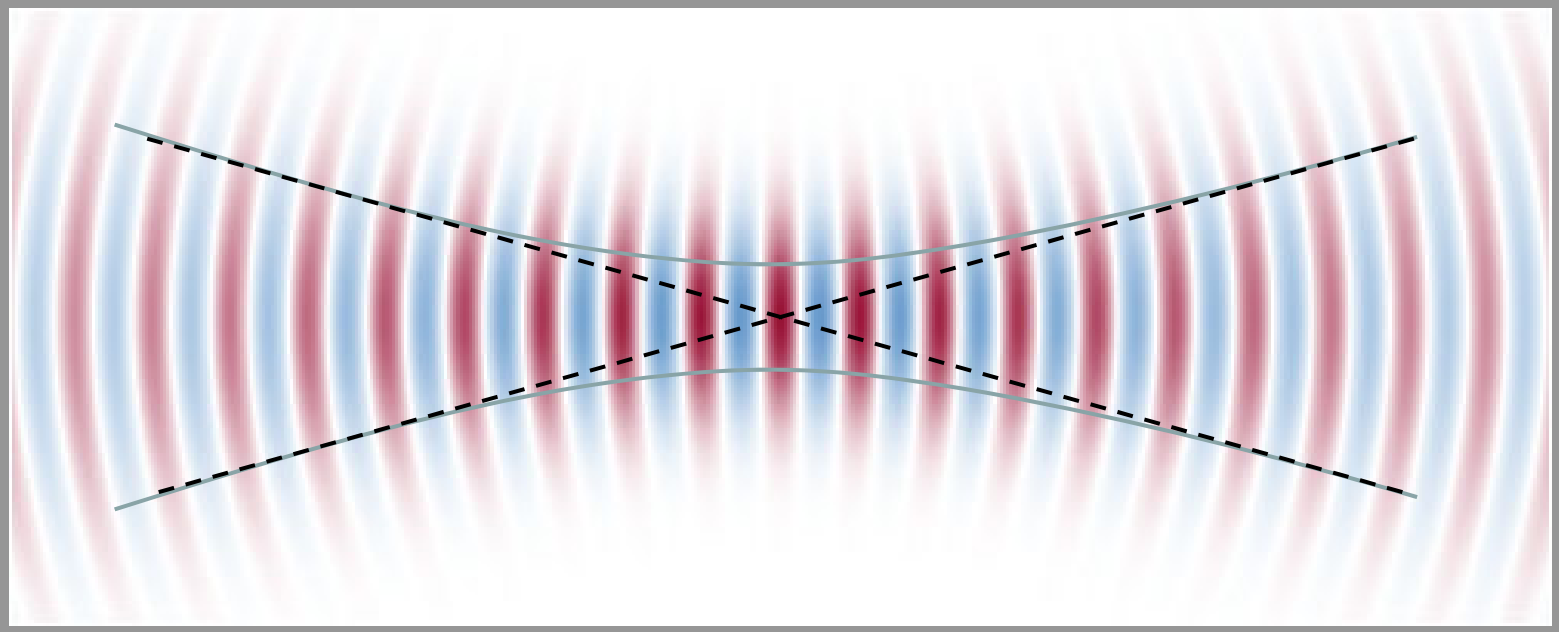
$$\begin{bmatrix} x_{out} \\ \theta_{out} \end{bmatrix} = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & f \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ 0 \end{bmatrix} \longrightarrow \begin{aligned} x_{out} &= 0 \\ \theta_{out} &= -x_{in}/f \end{aligned}$$

Key question: how big is the illumination spot?

According to this analysis, the spot size in the focal plane is identically zero! This would imply that the intensity is infinite at that point.

In reality, the spot size of a focusing beam does not become zero in the focal plane.



The spot size gets smaller, then larger. But never reaches zero.

If we want to compute the spot size, then we need something better than ray optics!

What if ray optics is not good enough?

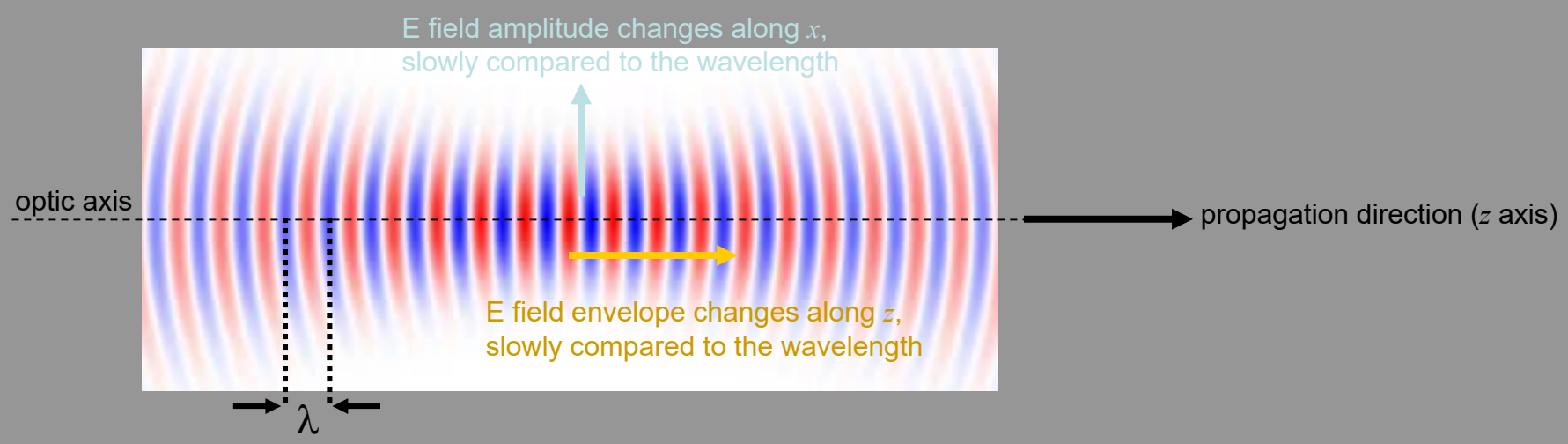
Start with the wave equation: $$\frac{\partial^2 E}{\partial x^2}+\frac{\partial^2 E}{\partial y^2}+\frac{\partial^2 E}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

Assume a harmonic time dependence, of the form $e^{j\omega t}$.

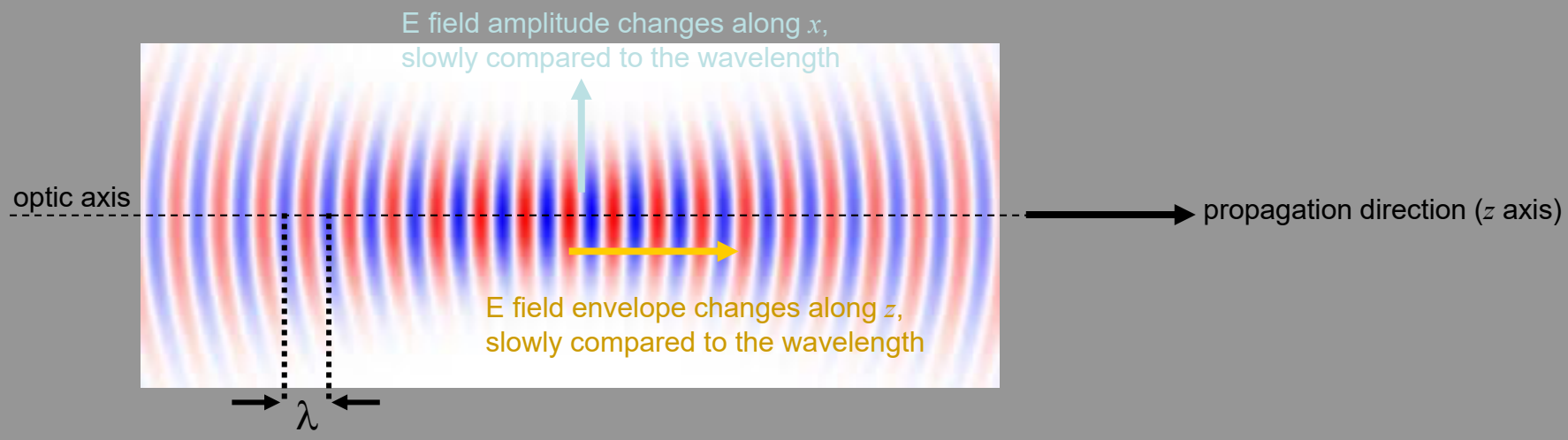
Also, assume that the variation along the direction of propagation (z axis) is given by:

e^{jkz} × (a function of *x*, *y and z* which varies slowly with respect to z)

(slowly varying with respect to both e^{jkz} and also the transverse variation)



What if ray optics is not good enough?



These assumptions imply: $E(x,y,z,t) = u(x,y,z) \cdot e^{j\omega t - jkz}$

and: $\left|\frac{\partial^2 u}{\partial z^2}\right| << \left|\frac{\partial^2 u}{\partial x^2}\right|, \left|\frac{\partial^2 u}{\partial y^2}\right|,$ and $\left|k\frac{\partial u}{\partial z}\right|$ $\quad (k = \omega/c)$

(Note: this is, essentially, the paraxial approximation)

Now, plug this form for $E(x,y,z,t)$ into the wave equation.

Paraxial wave equation

$$E(x,y,z,t) = u(x,y,z) \cdot e^{j\omega t - jkz}$$

x, y, and t derivatives are easy:

$$\frac{\partial^2 E}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} e^{j(\omega t - kz)}$$

$$\frac{\partial^2 E}{\partial y^2} = \frac{\partial^2 u}{\partial y^2} e^{j(\omega t - kz)}$$

$$\frac{\partial^2 E}{\partial t^2} = -\omega^2 u \; e^{j(\omega t - kz)}$$

z derivative:

$$\frac{\partial E}{\partial z} = -jk \times u e^{j(\omega t - kz)} + \frac{\partial u}{\partial z} e^{j(\omega t - kz)}$$

$$\frac{\partial^2 E}{\partial z^2} = -k^2 \times u e^{j(\omega t - kz)} - 2jk \times \frac{\partial u}{\partial z} e^{j(\omega t - kz)}$$

$$+ \cancel{\frac{\partial^2 u}{\partial z^2}} e^{j(\omega t - kz)}$$

paraxial approximation

The wave equation becomes:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \cancel{k^2 u} - 2jk \frac{\partial u}{\partial z} = -\cancel{\frac{\omega^2}{c^2} u}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 2jk \frac{\partial u}{\partial z} = 0}$$

This is known as the "paraxial wave equation"

The most useful solution to the paraxial wave equation

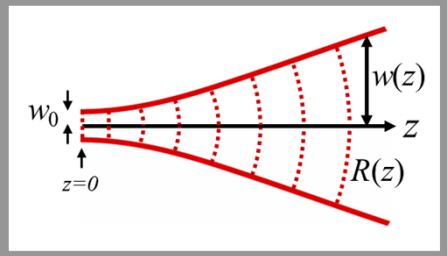
The solution we care about is an electric field whose amplitude and phase both vary quadratically with the transverse coordinates x and y – that is, like a Gaussian function:

$$\underset{\sim}{E}(x,y,z) \propto \frac{\exp\left[-ikz - i\psi(z)\right]}{w(z)} \exp\left[-\frac{x^2+y^2}{w^2(z)} - ik\frac{x^2+y^2}{2R(z)}\right]$$

$w(z)$ is the spot size at distance z

$R(z)$ is the beam radius of curvature at distance z, and

$\psi(z)$ is a distance-dependent phase shift.

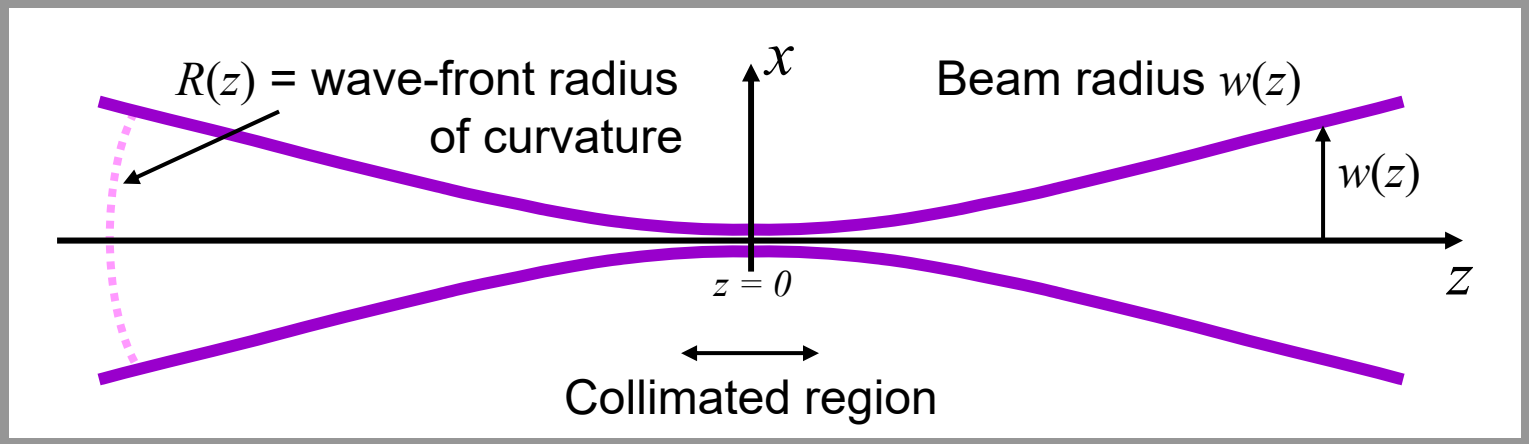


Note that the spot size of this beam never reaches zero unless $w(z)$ goes to zero at some value of z (which it never does).

This solution to the paraxial wave equation is a nearly perfect description of the beam produced by most lasers.

Laser beams are Gaussian beams, not rays.

Real laser beams always have a finite spot size. Even if you focus the beam, the minimum size is never zero.



At any value of z, the intensity vs. x (or y) is a Gaussian function, with a half-width of w. This width varies with z, as shown above.

The beam has a waist (minimum radius) at the focal point ($z = 0$, in the above diagram) where the spot size is w_0. It then expands to $w = w(z)$ at a distance z away from the focus.

The wave front radius of curvature, $R(z)$, also changes with distance.

Gaussian beam spot

The magnitude of the electric field is therefore given by:

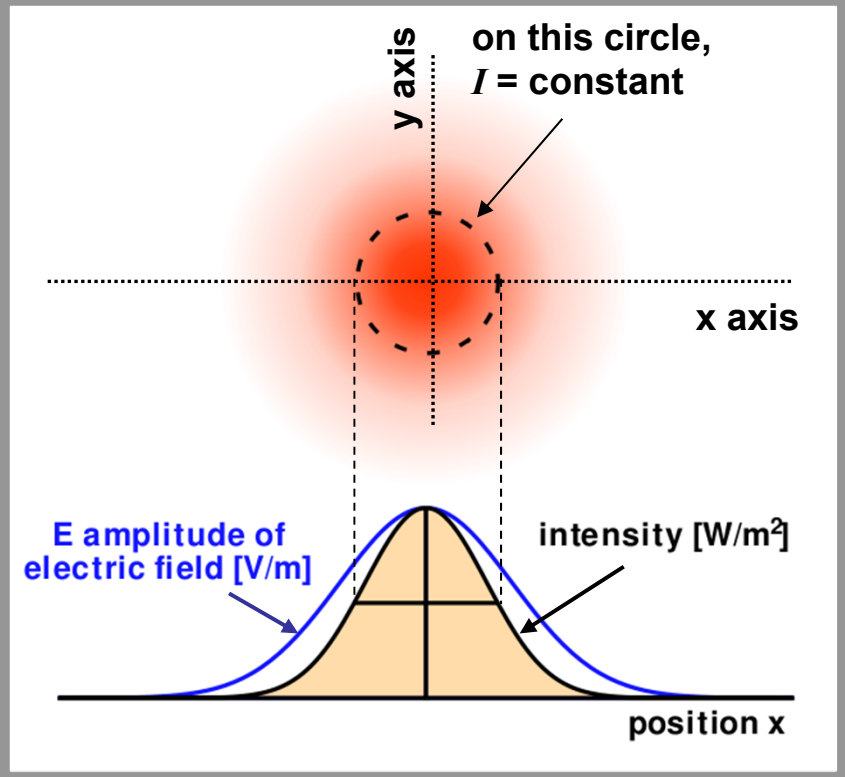
$$\left|E(x,y,z,t)\right| = \left|u(x,y,z)\right| \propto \exp\left[-\frac{x^2+y^2}{w(z)^2}\right]$$

with the corresponding intensity profile:

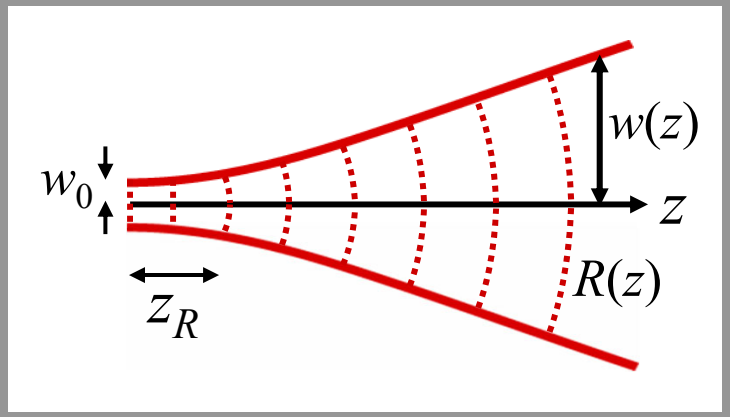
$$I(x,y,z) \propto \exp\left[-2\frac{x^2+y^2}{w(z)^2}\right]$$

The spot is a 2D Gaussian:

Its width is determined by the value of w.



Gaussian Beam Spot, Radius, and Phase dependence on z



The dependence of the spot size, radius of curvature, and phase shift on propagation distance z:

These expressions assume that the minimum spot size (focal point) is located at z = 0.

$$w(z) = w_0 \sqrt{1 + \left(z / z_R\right)^2}$$

$$R(z) = z + \left(z_R^2 / z\right)$$

$$\psi(z) = \arctan(z / z_R)$$

Also assuming that the beam is propagating in empty space and not encountering any lenses or other optical components

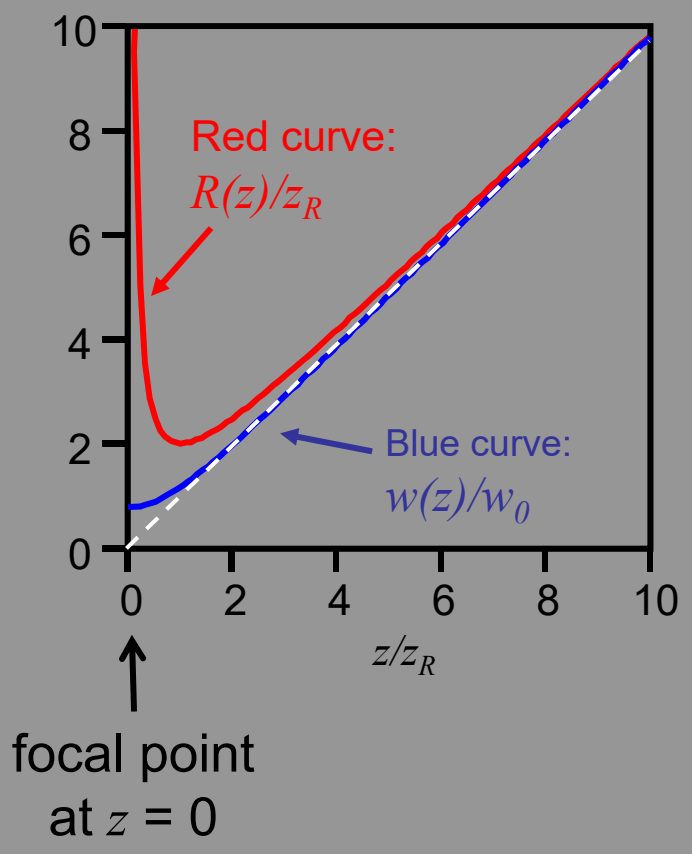
where z_R is the **Rayleigh range**, and is given by:

$$z_R \equiv \pi w_0^2 / \lambda$$

Variation of R(z) and w(z) with distance

When propagating away from a focal point at z = 0:

$$R(z) = \frac{z^2 + z_R^{\ 2}}{z} \qquad w(z) = w_0\sqrt{1 + \frac{z^2}{z_R^{\ 2}}}$$

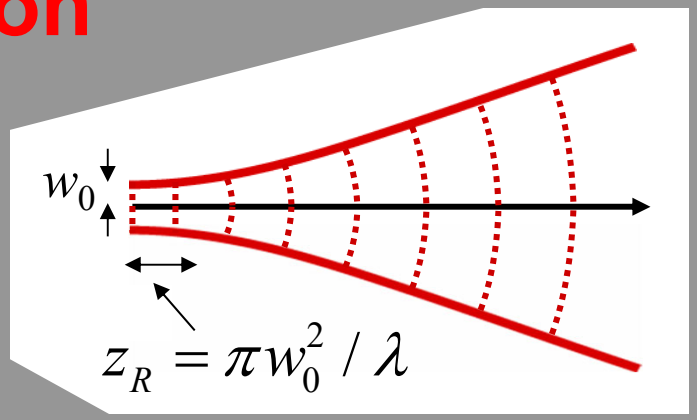


Note #1: for distances larger than a few times z_R, both the radius and waist increase linearly with increasing distance. This is also what ray optics (white dashed line) would predict.

Note #2: positive values of R correspond to a diverging beam, whereas $R < 0$ would indicate a converging beam.

Gaussian Beam Collimation

Twice the Rayleigh range is the distance over which the beam remains about the same size, that is, remains ***collimated.***



Waist spot size w_0	Collimation Distance λ = 10.6 μm	Collimation Distance λ = 0.633 μm
.225 cm	0.003 km	0.045 km
2.25 cm	0.3 km	5 km
22.5 cm	30 km	500 km

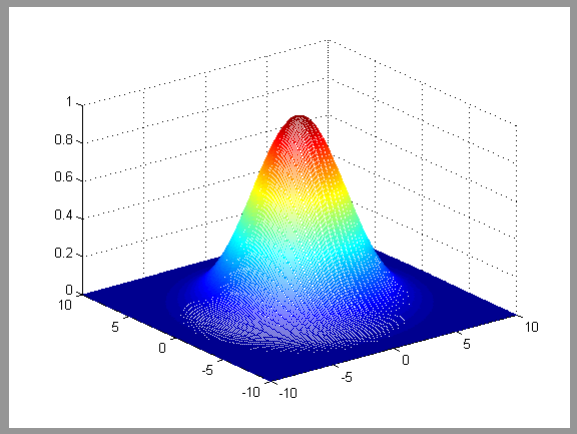
Longer wavelengths and smaller waists expand faster than shorter ones.

Tightly focused laser beams expand quickly.
Weakly focused beams expand less quickly, but still expand.
As a result, it's very difficult to shoot down a missile with a laser.

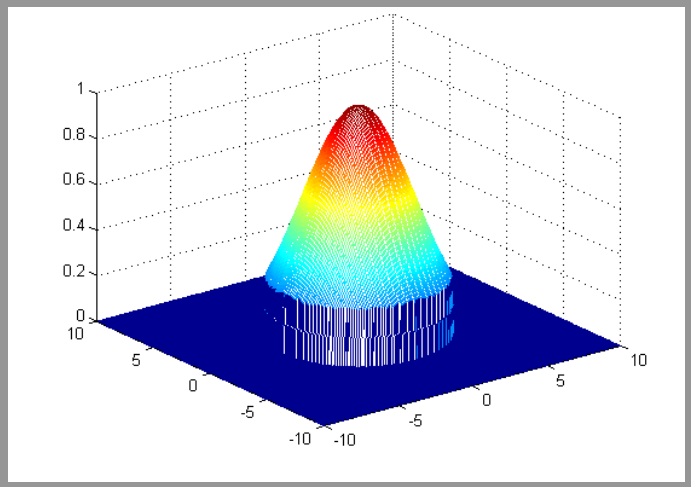
Aperture transmission

The irradiance of a Gaussian beam drops dramatically as one moves away from the optic axis. How large must a circular aperture be so that it does not significantly truncate a Gaussian beam?

Before aperture



After aperture



Before the aperture, the radial variation of the irradiance of a beam with waist w is:

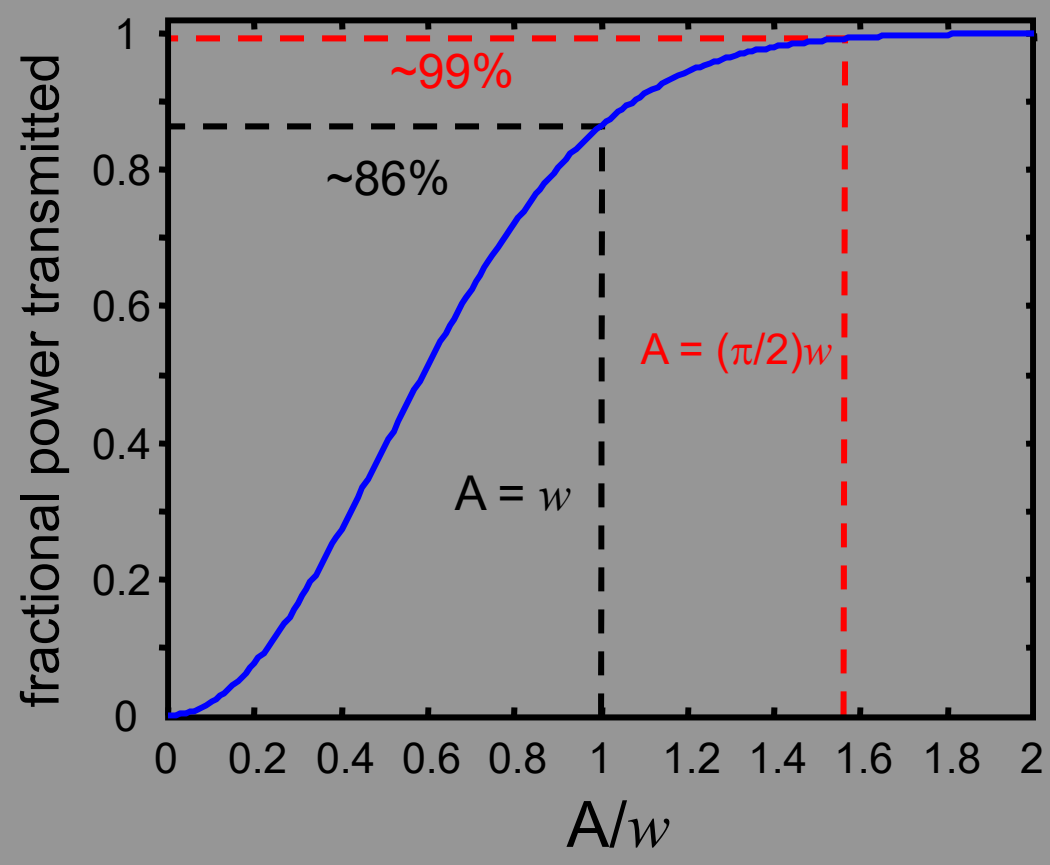
$$I(r) = \frac{2P}{\pi w^2} e^{-2r^2/w^2}$$

where P is the total power in the beam: $P = \iint \left| u(x,y) \right|^2 dxdy$

Aperture transmission

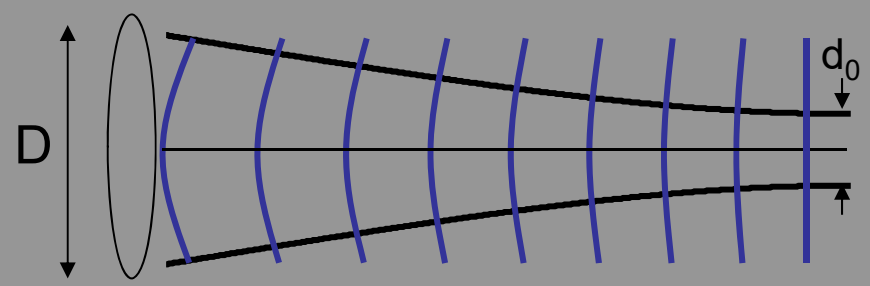
If this beam (waist w) passes through a circular aperture with radius A (and is centered on the aperture), then:

$$\text{fractional power transmitted} = \frac{2}{\pi w^2}\int_0^A 2\pi r\, e^{-2r^2/w^2}\, dr = 1 - e^{-2A^2/w^2}$$



Focusing a Gaussian beam

The focusing of a Gaussian beam can be regarded as the reverse of the propagation problem we did before.



A Gaussian beam focused by a thin lens of diameter D to a spot of diameter d_0.

How big is the focal spot?

Well, of course this depends on how we define the size of the focal spot. If we define it as the circle which contains 86% of the energy, then $d_0 = 2w_0$.

Then, if we assume that the input beam completely fills the lens (so that its diameter is D), we find:

$$d_0 \approx \frac{2f\lambda}{D} = 2(f\#)\lambda$$

where $f\# = f/D$ is the f-number of the lens.

It is very difficult to construct an optical system with $f\# < 0.5$, so $d_0 > \lambda$.