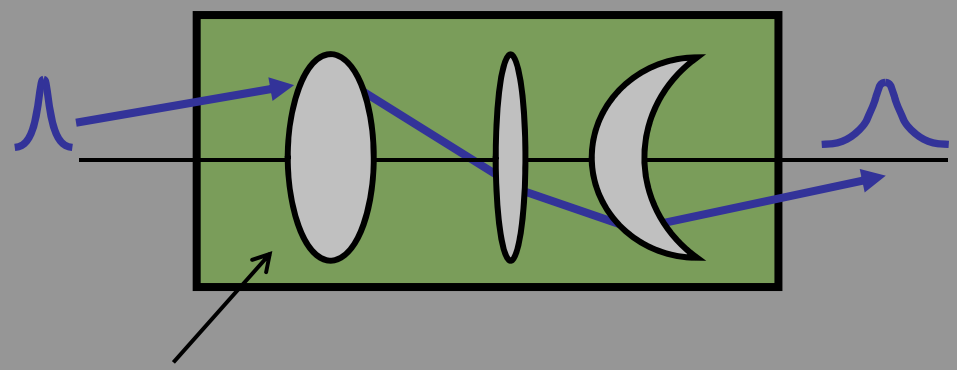
Propagation of laser beams, part II

Ray optics and ray matrices

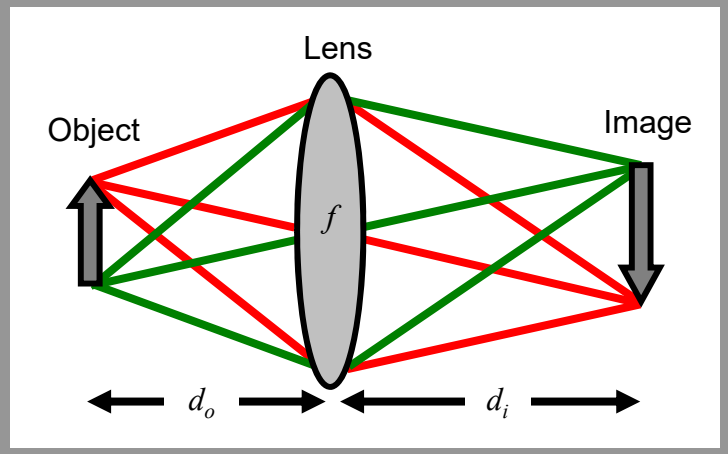
The Gaussian beam

Complex q and its propagation

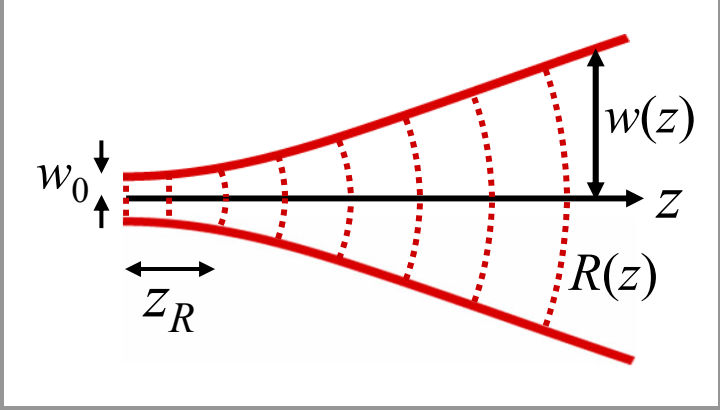
Laser resonators and stability



Optical system ↔ 2x2 Ray matrix



Gaussian Beam Spot, Radius, and Phase



The expressions for the spot size, radius of curvature, and phase shift:

$$w(z) = w_0 \sqrt{1 + \left(z / z_R\right)^2}$$

$$R(z) = z + \left(z_R^2 / z\right)$$

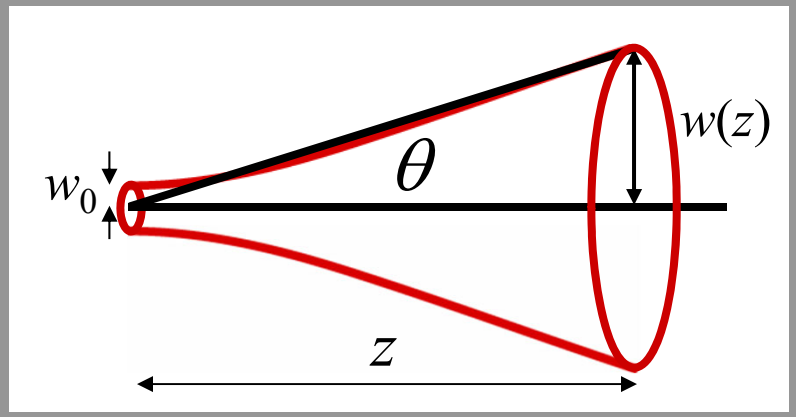
$$\psi(z) = \arctan(z / z_R)$$

where z_R is the **Rayleigh range**, and is given by:

$$z_R \equiv \pi w_0^2 / \lambda$$

One Rayleigh range away from the focal plane, the area of the beam spot is double the area at the focal point.

Gaussian beam divergence



Far away from the waist, the spot size of a Gaussian beam will be:

$$w(z) = w_0\sqrt{1+\left(z/z_R\right)^2} \approx w_0\sqrt{\left(z/z_R\right)^2} = w_0 z/z_R$$

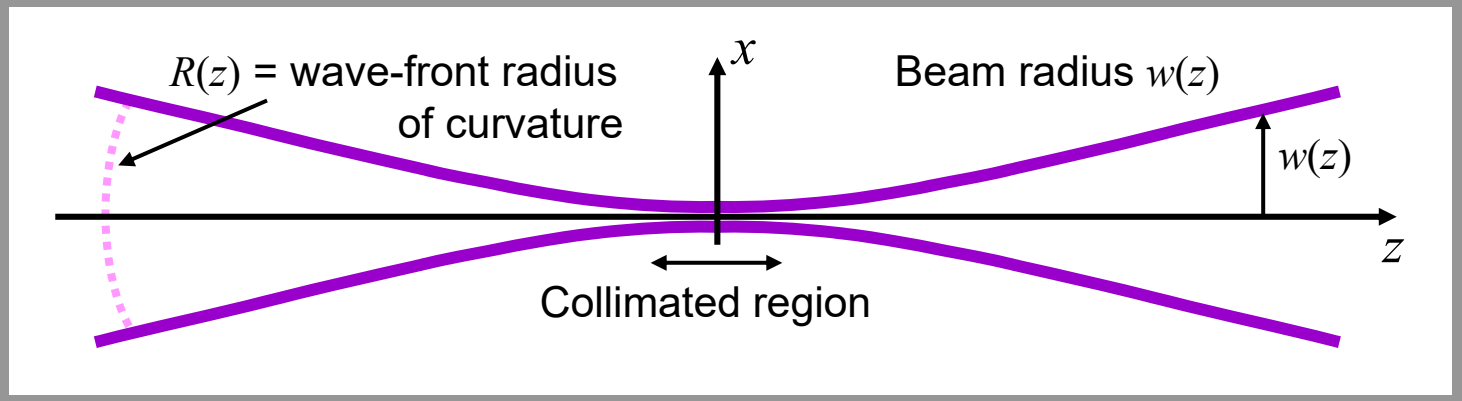
The beam 1/e divergence half angle is then $w(z)/z$ as $z \to \infty$:

$$\theta_{1/e} = \frac{w_0 z}{z_R z} = \frac{w_0}{z_R} = \frac{w_0}{\pi w_0^2/\lambda}$$

$$\Rightarrow \quad \boxed{\theta_{1/e} = \lambda/\left(\pi w_0\right)}$$

The smaller the waist and the larger the wavelength, the larger the divergence angle.

The Gaussian-beam complex-q parameter



$$E(x,y,z) \propto \exp\left[-i\frac{\pi}{\lambda}\frac{x^2+y^2}{R(z)}\right]\exp\left[-\frac{x^2+y^2}{w^2(z)}\right]$$

We can combine these two factors (they're both Gaussians):

$$E(x,y,z) \propto \exp\left[-i\frac{\pi}{\lambda}\frac{x^2+y^2}{q(z)}\right]$$

where: $$\boxed{\frac{1}{q(z)} \equiv \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}}$$ ← q completely determines the Gaussian beam.

The complex-q parameter at a focal point

$$w(z) = w_0 \sqrt{1 + \left(z / z_R \right)^2}$$

$$R\left(z\right) = z + \left(z_R^2 / z \right)$$

At a focal point (z = 0), these become:

$$w(z) = w_0$$

$$R\left(z\right) = \infty$$

Thus at a focal point (z = 0), the q parameter is:

$$\frac{1}{q(z)} = \frac{1}{\infty} - i \frac{\lambda}{\pi {w_0}^2}$$

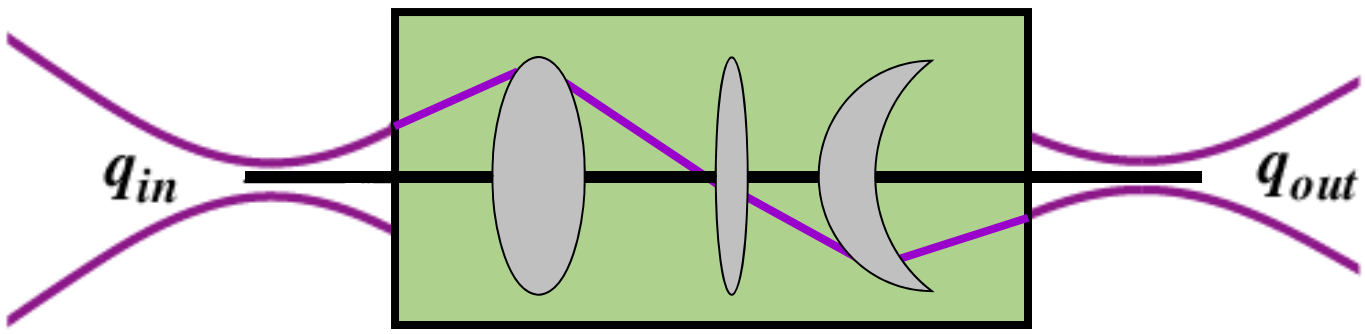
$$q(z) = i \frac{\pi {w_0}^2}{\lambda} = i\, z_R$$

The q parameter is pure imaginary at a focal point, where the wave front is planar.

Ray matrices and the propagation of q

We'd like to be able to follow Gaussian beams through optical systems. Remarkably, ray matrices can be used to propagate the q-parameter.

$$O = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$



Optical system

This relation holds for all systems for which ray matrices hold:

$$q_{out} = \frac{Aq_{in} + B}{Cq_{in} + D}$$

Just multiply all the matrices first and use this result to obtain q_{out} for the relevant q_{in}!

Propagating q: an example

$$q_{out} = \frac{Aq_{in} + B}{Cq_{in} + D}$$

Free-space propagation through a distance z:

The ray matrix for free-space propagation is: $O_{space} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$

Then: $q(z) = \dfrac{1\,q(0) + z}{0\,q(0) + 1} = q(0) + z$

Propagating q: an example (cont'd)

Does $q(z) = q_0 + z$? This is equivalent to: $1/q(z) = 1/(q_0 + z)$.

LHS: $$\frac{1}{q(z)} \equiv \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}$$

$$\frac{1}{q(z)} \equiv \frac{1}{z + z_R^2/z} - i\frac{1}{z_R(1 + z^2/z_R^2)} = \frac{1}{z + z_R^2/z} - i\frac{1}{z_R + z^2/z_R}$$

RHS: $q(0) = i\dfrac{\pi w_0^2}{\lambda} = i z_R$ so $q(0) + z = iz_R + z$

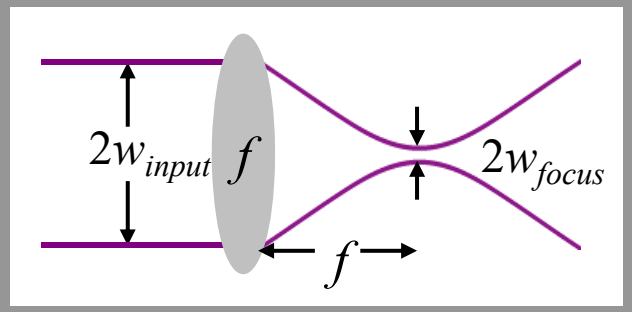
So: $$\frac{1}{q(0)+z} = \frac{1}{z + iz_R} = \frac{z - iz_R}{z^2 + z_R^2} = \frac{z}{z^2 + z_R^2} - \frac{iz_R}{z^2 + z_R^2}$$

$$= \frac{1}{z + z_R^2/z} - i\frac{1}{z^2/z_R + z_R}$$ which is just this.

$$= \frac{1}{q(z)}$$

So: $$\boxed{q(z) = q(0) + z}$$

Propagating q: another example



Focusing a collimated beam (i.e., a lens, f, followed by a distance, f):

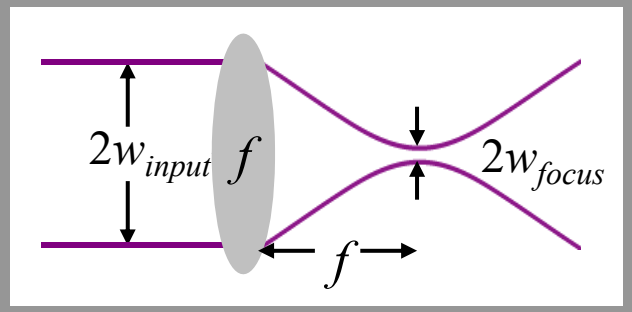
$$O_{space}O_{lens} = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} = \begin{bmatrix} 0 & f \\ -1/f & 1 \end{bmatrix}$$

A collimated beam has a big spot size (w_{input}) and Rayleigh range (z_R^{input}), and an infinite radius of curvature (R), so: $q_{in} = i\, z_R^{input}$

$$\frac{1}{q_{out}} = \frac{Cq_{in} + D}{Aq_{in} + B} \quad \Rightarrow \quad \frac{1}{q_{focus}} = \frac{(-1/f)(iz_R^{in}) + 1}{0\,(iz_R^{in}) + f} = \frac{1 - iz_R^{in}/f}{f}$$

So: $$\mathrm{Im}\left[1/q_{focus}\right] = -\frac{z_R^{in}}{f^2} = -\frac{\pi w_{input}^2}{\lambda f^2}$$

Propagating q: another example



$$\mathrm{Im}\left[1/q_{focus}\right] = -\frac{\pi w_{input}^2}{\lambda f^2}$$

But (from the definition of q): $$\mathrm{Im}\left[1/q_{focus}\right] \equiv -\frac{\lambda}{\pi w_{focus}^2}$$

By comparing these two expressions, we find:

$$w_{focus} = \frac{\lambda\ f}{\pi\ w_{input}}$$

or:

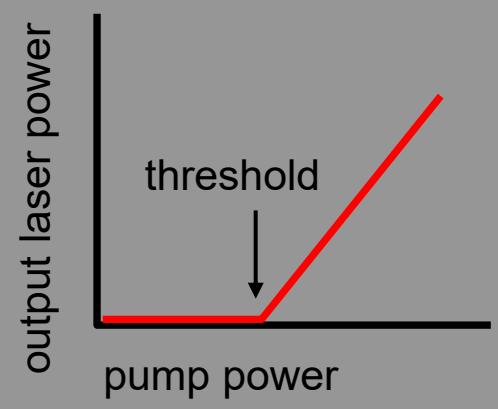
$$\boxed{d_{focus} = \frac{4}{\pi}\frac{\lambda\ f}{D_{input}}}$$

The well-known result for the focusing of a Gaussian beam

Remember these three criteria?

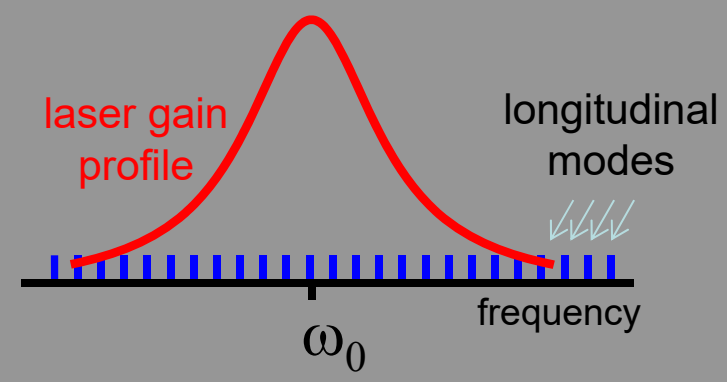
1. The amplitude must reproduce itself after each round trip. Gain = Loss.

Consequence: lasers have a threshold.



2. The phase must reproduce itself after each round trip.

Consequence: lasers have longitudinal modes: specific frequencies of operation.

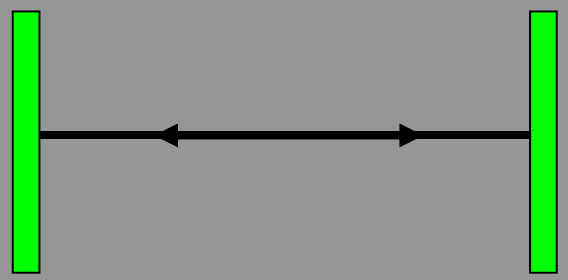


3. The transverse distribution of intensity must reproduce itself after each round trip.

Consequence: lasers have spatial modes: Gaussian beams! The details of the spatial mode are determined by the **laser resonator**.

Laser resonators

The resonator is the mirrors (plus all other optical components) that act to confine the EM wave.



A laser resonator with flat parallel mirrors

A careful analysis of the resonator will be important in understanding the behavior of lasers, particularly their transverse intensity patterns.

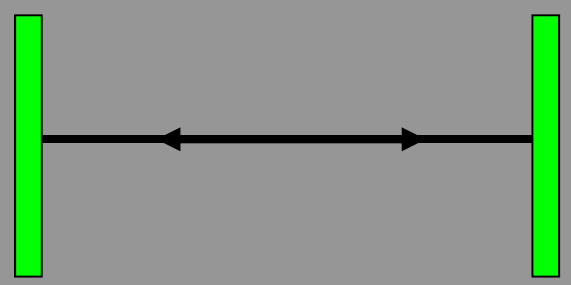
Empty cavity analysis: we assume that there is no gain medium or optics inside the laser.

This allows us to consider the resonator as a separate problem from the consideration of the laser medium.

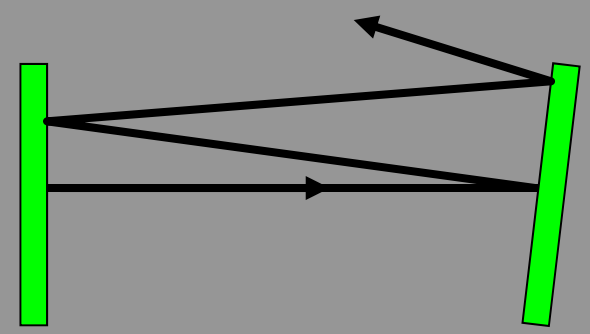
Stability of a resonator

Within the ray matrix formalism, we can define the stability of a laser resonator in a simple and intuitive way.

Consider a paraxial ray propagating inside the resonator.



A stable resonator



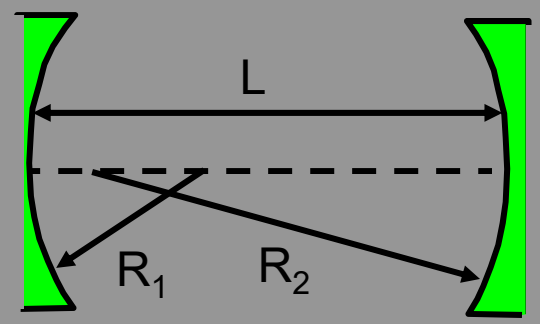
An unstable resonator

If the ray escapes in a finite number of bounces, then the resonator is unstable.

If the ray is still trapped after an infinite number of bounces, then the resonator is stable.

Stability of a resonator - ray matrix analysis

Let's consider a general two-mirror cavity:



Note: these are both concave mirrors. By convention, R > 0 for concave, and R < 0 for convex.

ABCD matrix analysis for a single round trip:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2/R_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2/R_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2L/R_2 & 2L-2L^2/R_2 \\ 4L/R_1R_2 - 2/R_1 - 2/R_2 & 1-2L/R_2 - 4L/R_1 + 4L^2/R_1R_2 \end{bmatrix}$$

Ray matrix: many bounces

After N round trips, the output ray is related to the initial ray by:

$$\begin{bmatrix} x_{out} \\ \theta_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^N \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix}$$

If we define a new variable, χ, such that: $\cos\chi = 1 - \frac{2L}{R_1} - \frac{2L}{R_2} + \frac{2L^2}{R_1 R_2}$

then it can be shown that:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^N = \frac{1}{\sin\chi} \begin{bmatrix} A\sin(N\chi) - \sin[(N-1)\chi] & B\sin(N\chi) \\ C\sin(N\chi) & D\sin(N\chi) - \sin[(N-1)\chi] \end{bmatrix}$$

Ray matrix: stability criterion

$$x_{out} = \frac{1}{\sin\chi}\left[\left(A\sin(N\chi) - \sin((N-1)\chi)\right)x_{in} + \left(B\sin(N\chi)\right)\theta_{in}\right]$$

We note that the output ray position x_{out} remains finite when N goes to infinity, as long as χ is a real number.

If χ becomes complex, then $\sin N\chi = \frac{1}{2j}\left(e^{jN\chi} - e^{-jN\chi}\right)$

As $N \to \infty$, one of these exponential terms grows to infinity.

Thus, the condition for resonator stability is χ = real, or $|\cos\chi| \le 1$

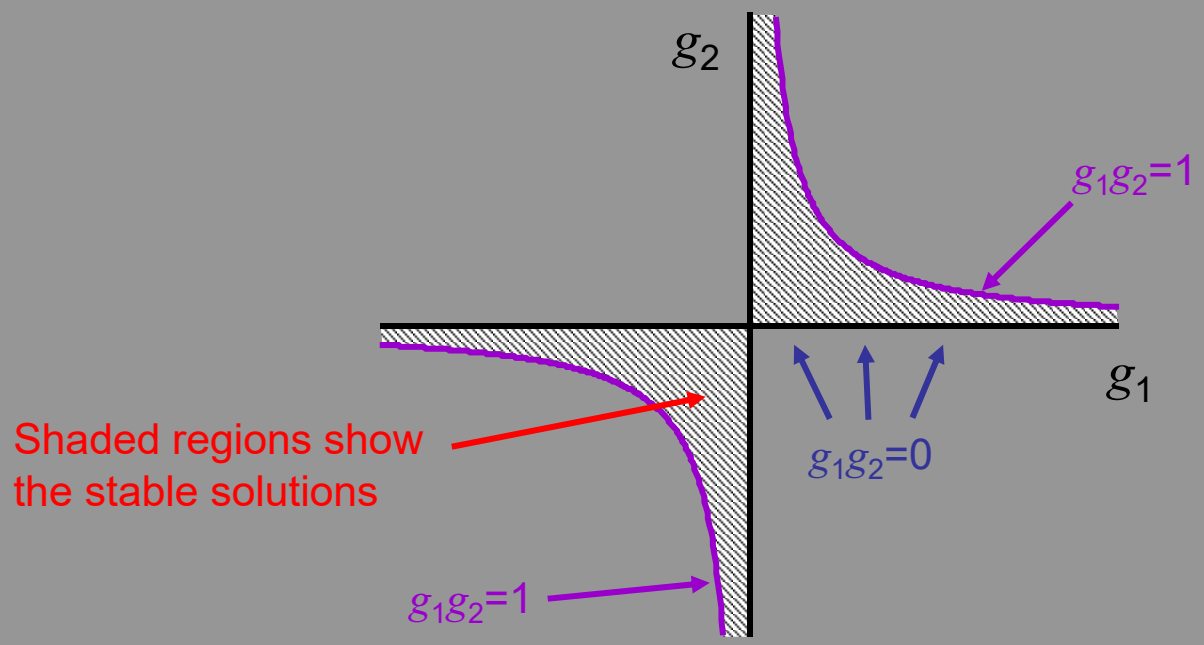
$$-1 \le 1 - \frac{2L}{R_1} - \frac{2L}{R_2} + \frac{2L^2}{R_1 R_2} \le 1 \longrightarrow \boxed{0 \le \left(1 - \frac{L}{R_1}\right)\left(1 - \frac{L}{R_2}\right) \le 1}$$

g parameters of the resonator

We define the "g parameters" of the resonator:

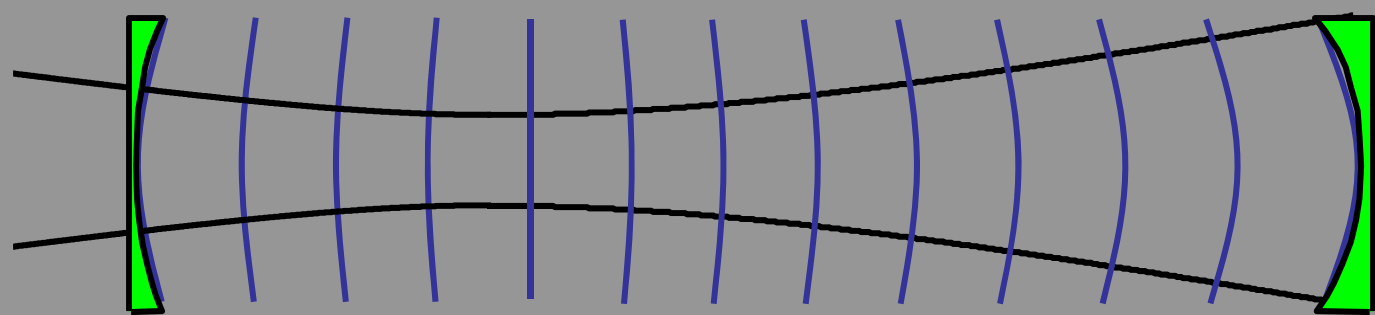
$$g_1 = \left(1 - \frac{L}{R_1}\right) \qquad g_2 = \left(1 - \frac{L}{R_2}\right)$$

Then, the stability requirement is: $\boxed{0 \le g_1 g_2 \le 1}$



Resonator analysis: Gaussian beams

Consider a Gaussian beam, focusing in empty space,
with a certain waist size and location:



Suppose we fit a pair of curved mirrors to this beam at any two points.
The radii of the mirrors should exactly match the wavefront curvature
of the Gaussian beam at each mirror location.

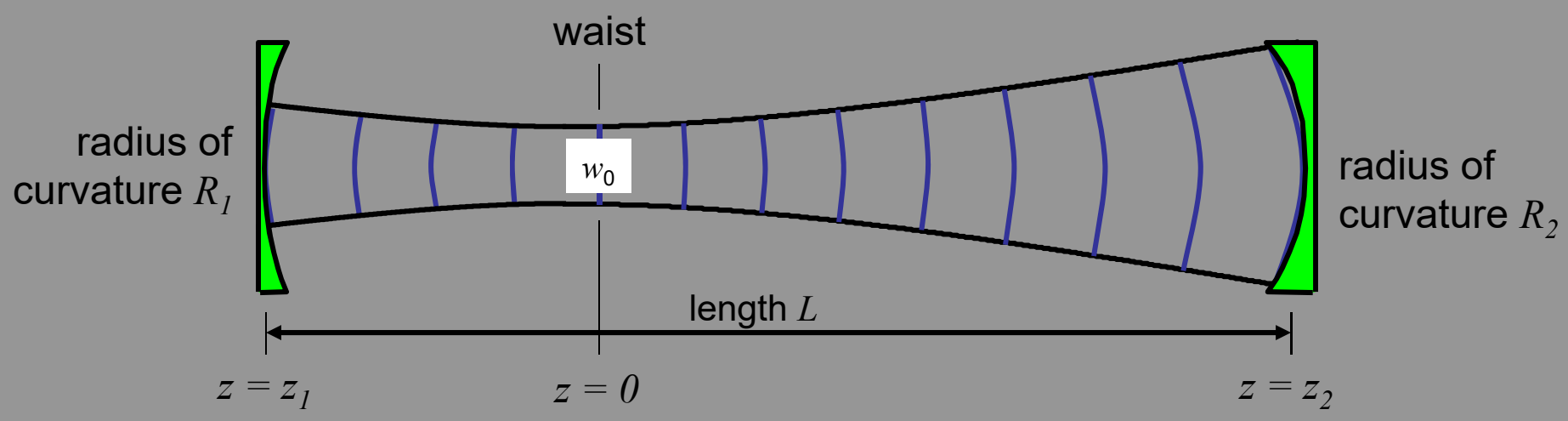
IF the mirrors are large enough so that not much of the
beam misses the mirrors, then:

Each mirror will reflect the Gaussian beam exactly back on
itself, with exactly reversed wavefront curvature and
direction. → stable mode of the cavity

Resonator analysis: Gaussian beams

More realistically, one would be given (or build) a resonator, and then need to determine the Gaussian beam solutions that it supports.

To analyze this situation, we can use the model shown here:



We assume that L, R_1 and R_2 are known. But, the spot size w_0 and the position of the waist relative to the mirrors (i.e., the values of z_1 and z_2) are unknown.

We use Gaussian beam analysis to determine these values.

Gaussian beam analysis of a resonator

Reminder: The radius of curvature of the wave front of a Gaussian beam at a distance z away from its waist is given by:

$$R(z) = \frac{z^2 + z_R^{\ 2}}{z} \qquad \text{where} \quad z_R = \frac{\pi w_0^2}{\lambda}$$

So, we have three equations:

$$-R_1 = \frac{z_1^{\ 2} + z_R^{\ 2}}{z_1} \qquad R_2 = \frac{z_2^{\ 2} + z_R^{\ 2}}{z_2} \qquad L = z_2 - z_1$$

in three unknowns: z_1, z_2, and z_R. This can be solved.

Notes on sign conventions:

- The Gaussian wave front curvature $R(z)$ is negative for a converging beam going to the right.
- Mirror curvatures R_1 and R_2 are positive for mirrors that are concave inwards (as seen looking from within the resonator).
- The distance z_1 is negative if mirror #1 is located to the left of the beam waist (so that the waist is inside the resonator).

Gaussian mode parameters in a resonator

The solutions can be written in terms of the same two g parameters defined earlier:

$$g_1 = \left(1 - \frac{L}{R_1}\right) \qquad g_2 = \left(1 - \frac{L}{R_2}\right)$$

$$z_R = \sqrt{\frac{g_1 g_2 (1 - g_1 g_2)}{(g_1 + g_2 - 2 g_1 g_2)^2}} L \qquad z_1 = \frac{-g_2 (1 - g_1)}{g_1 + g_2 - 2 g_1 g_2} L \qquad z_2 = \frac{g_1 (1 - g_2)}{g_1 + g_2 - 2 g_1 g_2} L$$

From these, we can determine the beam waist at $z = 0$ (the smallest beam spot inside the resonator):

$$w_0^{\,2} = \frac{L\lambda}{\pi} \sqrt{\frac{g_1 g_2 (1 - g_1 g_2)}{(g_1 + g_2 - 2 g_1 g_2)^2}}$$

as well as the spot sizes at the locations of the mirrors:

$$w_1^{\,2} = \frac{L\lambda}{\pi} \sqrt{\frac{g_2}{g_1 (1 - g_1 g_2)}} \qquad w_2^{\,2} = \frac{L\lambda}{\pi} \sqrt{\frac{g_1}{g_2 (1 - g_1 g_2)}}$$

These two values tell us how big the mirrors have to be.

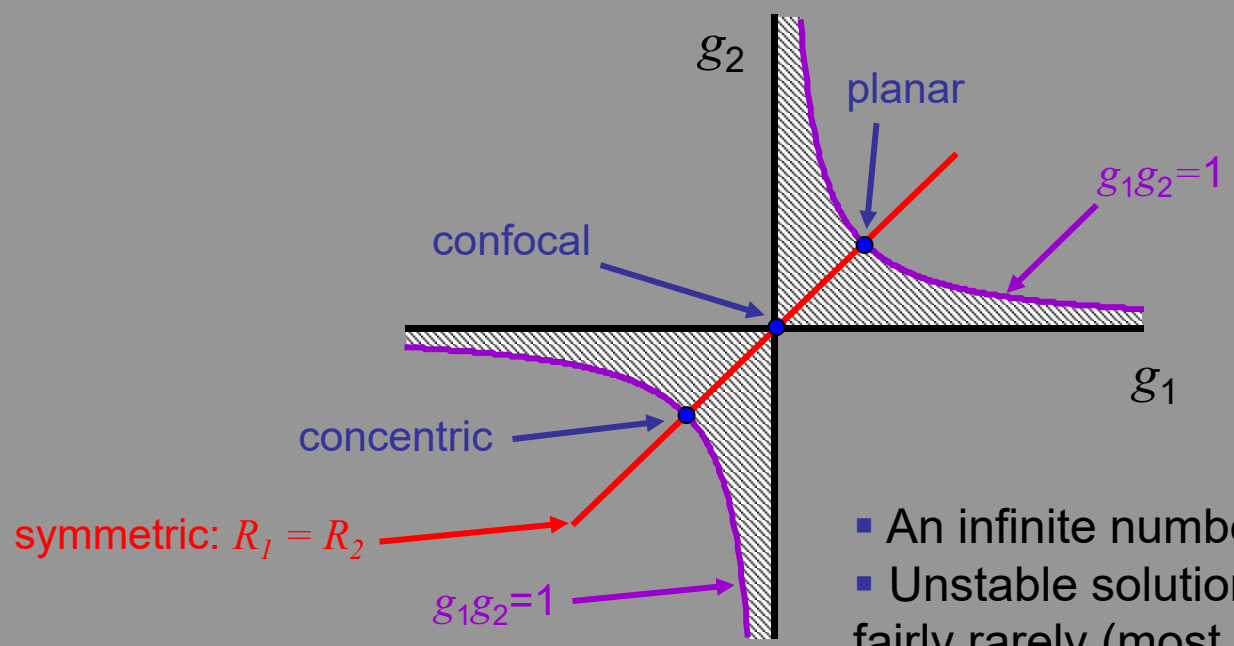
Stability of a Gaussian mode

$$z_R = \sqrt{\frac{g_1 g_2 (1 - g_1 g_2)}{(g_1 + g_2 - 2 g_1 g_2)^2}} L$$

Notice that the value of z_R is a real number if and only if: $\boxed{0 \le g_1 g_2 \le 1}$

This is also true for the beam waists.

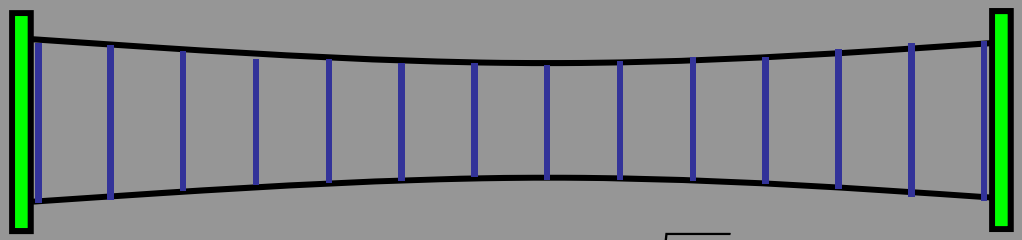
This is the same criterion found earlier using ray optics!



- An infinite number of possible solutions
- Unstable solutions are employed, although fairly rarely (most lasers are stable)
- Certain ones are particularly interesting

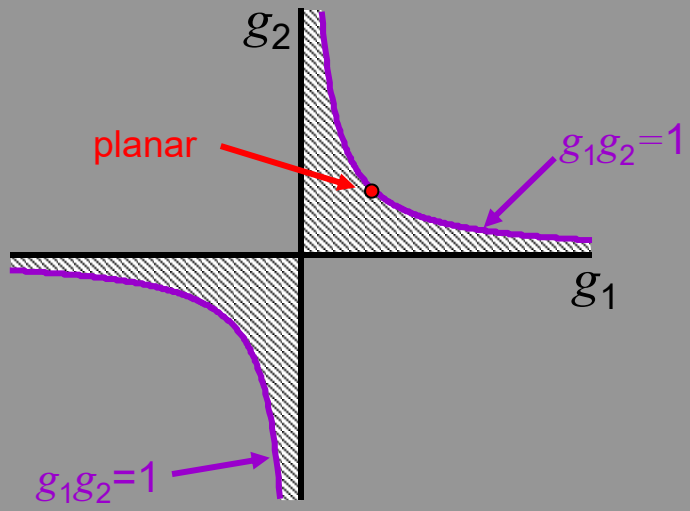
(Nearly) planar resonator

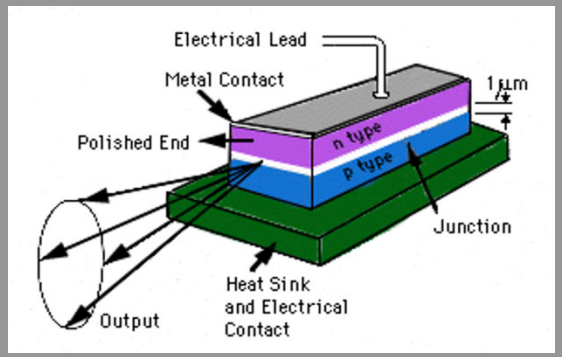
$g_1 = g_2 \approx 1$, and $R_1 = R_2 >> L$



$$w_0^2 \approx w_1^2 = w_2^2 = \frac{L\lambda}{\pi}\sqrt{\frac{R}{2L}}$$

for $R >> L$

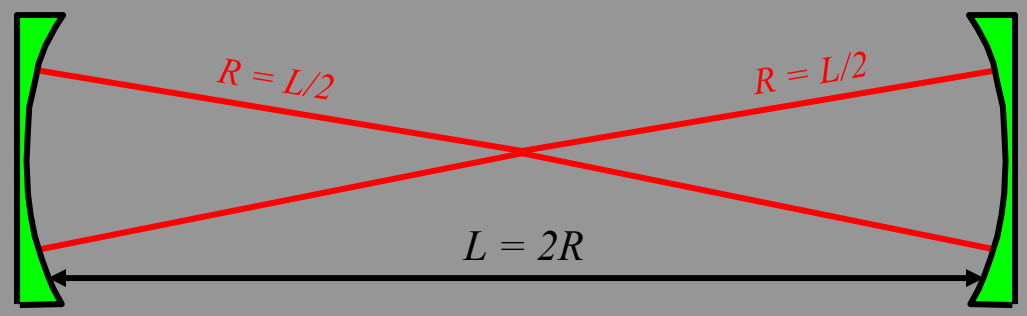


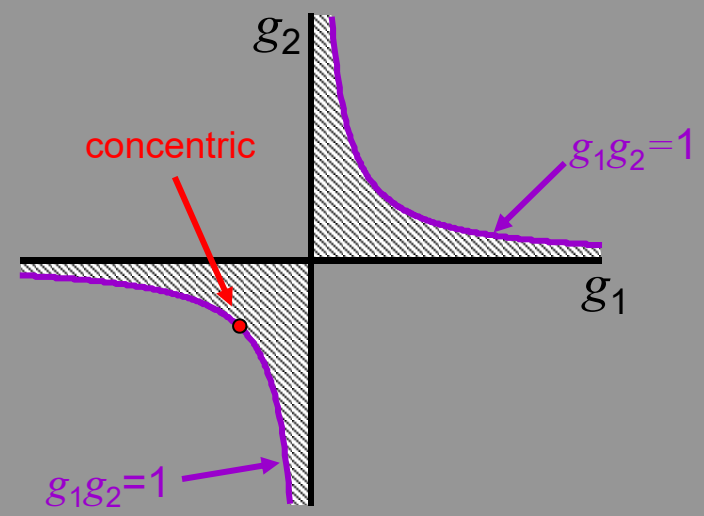


- Large mode size
- Beam waist is nearly constant inside the cavity
- For $g_1 = g_2 = 1$, waist becomes infinite: Gaussian model fails
- Very sensitive to small misalignment of the cavity, so it is very rarely used in lasers where the cavity length is more than 1 cm or so
- But it is common in small lasers where it is easy to ensure mirror parallelism: e.g., semiconductor diode lasers

Symmetric concentric resonator

$g_1 = g_2 \approx -1$, and $R_1 = R_2 = L/2$





- Small mode size in the center of the cavity
- For $g_1 = g_2 = -1$, waist becomes exactly zero: Gaussian model fails

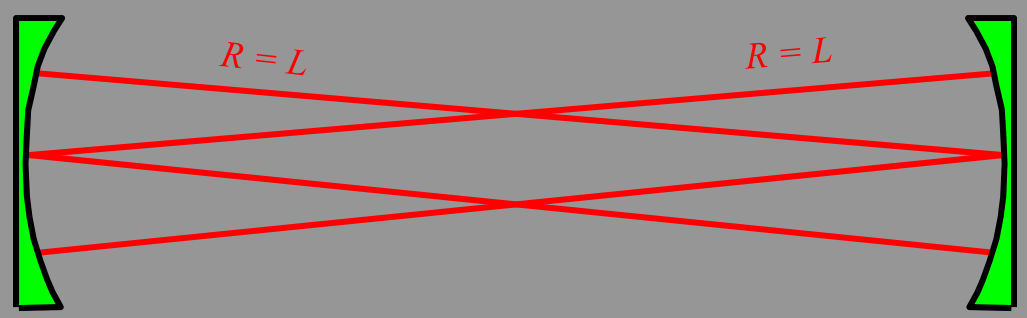
In the case of an *approximately* concentric resonator, $R_1 = R_2 = L/2 + \Delta L$ we find that the beam waists are given by:

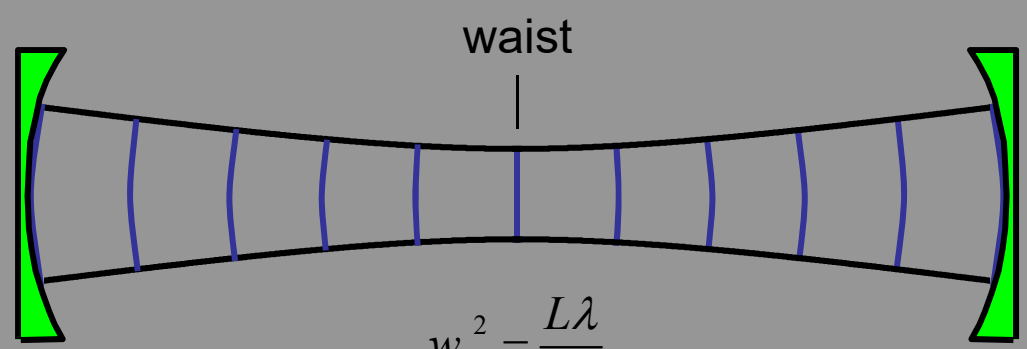
$$w_0^2 = \frac{L\lambda}{\pi}\sqrt{\frac{\Delta L}{4L}} \qquad w_1^2 = w_2^2 = \frac{L\lambda}{\pi}\sqrt{\frac{4L}{\Delta L}} \qquad \text{for } \Delta L << L$$

- Very sensitive to small misalignment of the cavity, so very rarely used

Symmetric confocal resonator

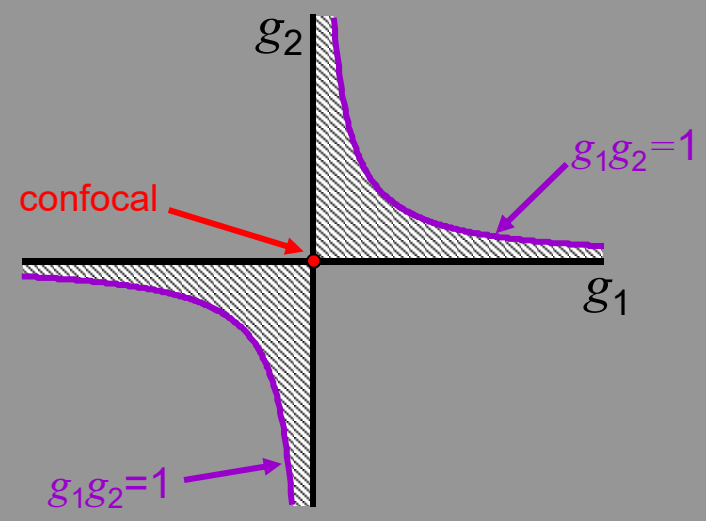
$g_1 = g_2 = 0$, and therefore $R_1 = R_2 = L$





$$w_0^2 = \frac{L\lambda}{2\pi}$$

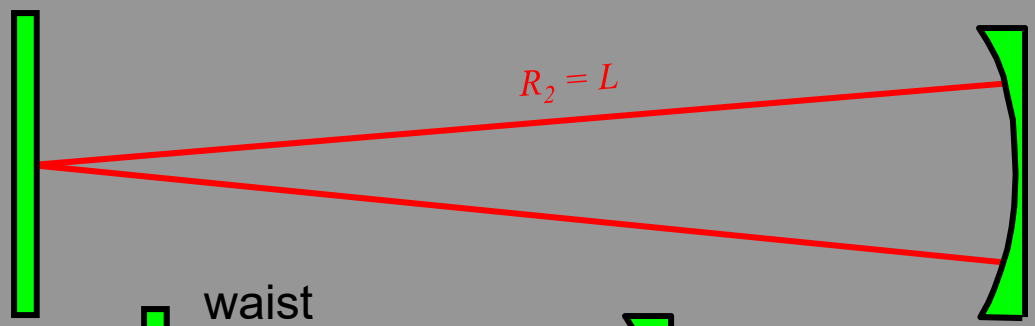
$$w_1^2 = w_2^2 = \frac{L\lambda}{\pi}$$

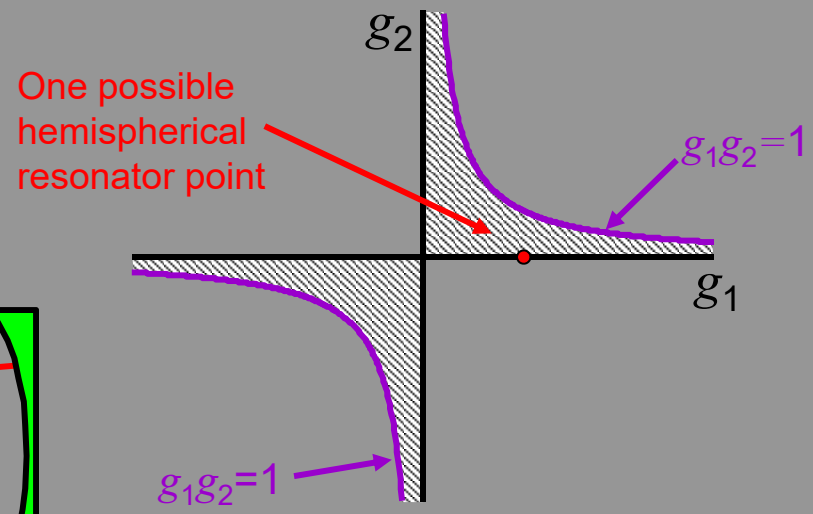


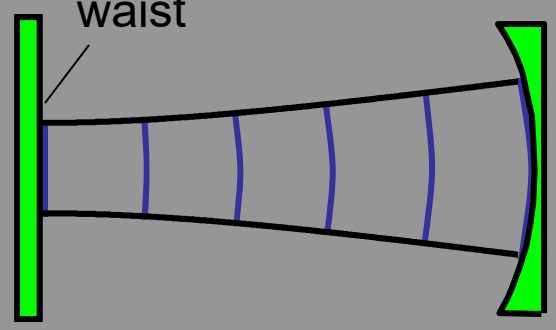
- Smallest average spot size of any resonator
- Mirror spacing = $2z_R$
- Very insensitive to small misalignment of the cavity
- a commonly used design

Hemispherical resonator

g_1 = 1 and g_2 = 0, and
therefore $R_1 = \infty$ and $R_2 \approx L$







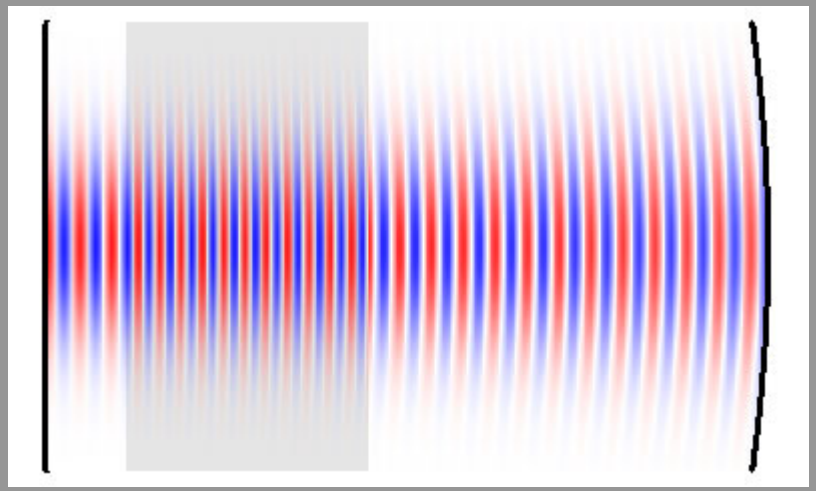
This is essentially just half of the symmetric confocal case.

- Small spot on the flat mirror, larger spot on the curved mirror
- Very insensitive to small misalignment of the cavity
- Very commonly used design

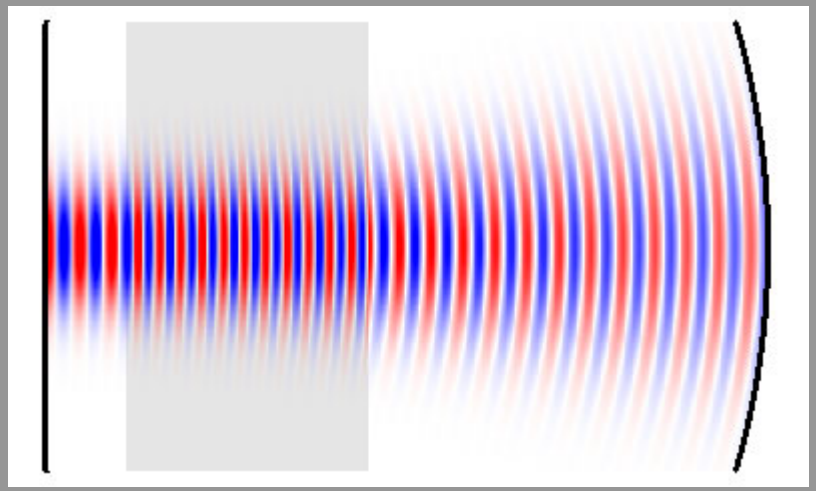
In the case of an *approximately* hemispherical radiator, $R_2 = L + \Delta L$ we find that the beam waists are given by:

$$w_0^2 = w_1^2 = \frac{L\lambda}{\pi}\sqrt{\frac{\Delta L}{L}} \qquad w_2^2 = \frac{L\lambda}{\pi}\sqrt{\frac{L}{\Delta L}} \qquad \text{for } \Delta L << L$$

Hemispherical resonator



A simple hemispherical resonator and the electric field distribution of its Gaussian mode. The wave fronts must be planar on the flat left end mirror, and the beam radius on the left mirror is so that the wave fronts also match the curvature of the right mirror.



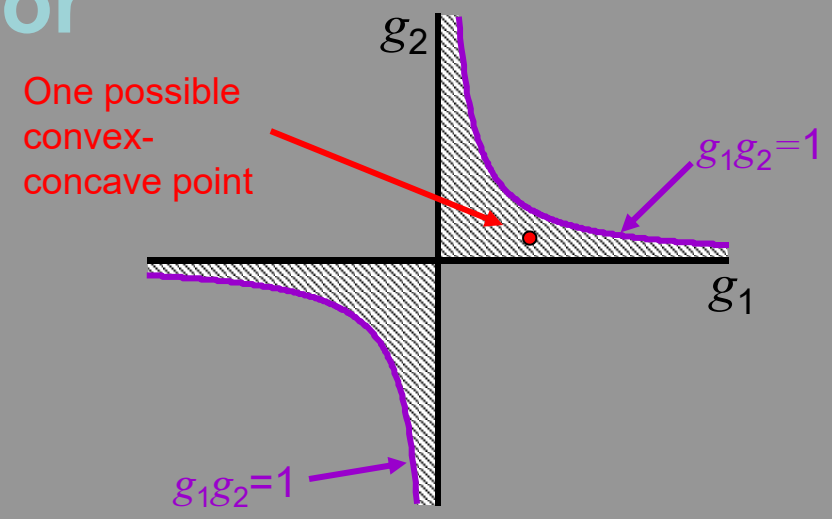
Same as above, but with a stronger curvature of the right mirror. The mode field adjusts accordingly.

Convex-concave resonator

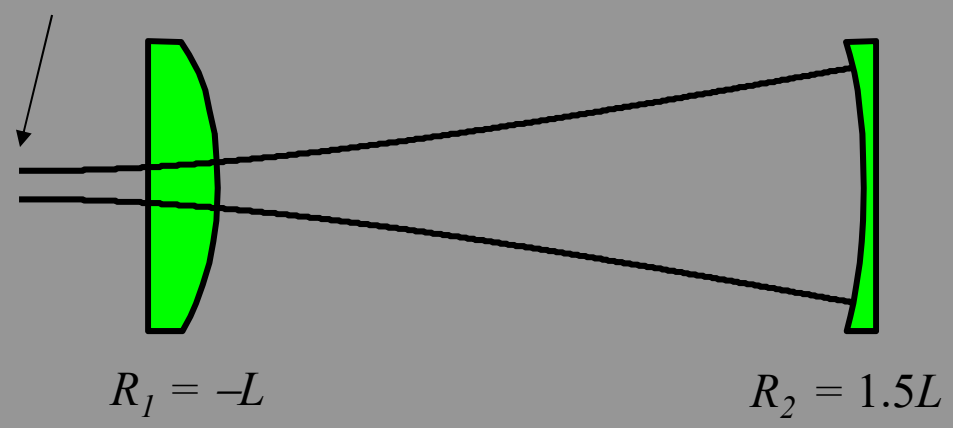
Example:

$g_1 = 2$ and $g_2 = 1/3$, and

therefore $R_1 = -L$ and $R_2 = 1.5L$



This is where the waist would be, if the beam leaked through the convex mirror.



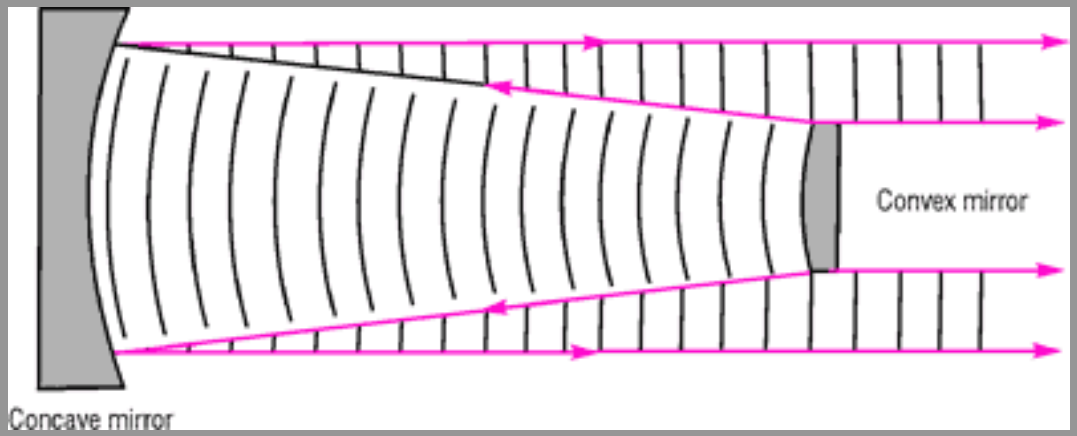
- The beam waist is outside of the laser cavity, so the beam never actually gets there.
- Common design for high-power lasers where small spots could damage mirrors.

Unstable resonators

It is also possible for lasers to operate with an unstable resonator.

unstable resonator – the beam does not reproduce itself on each round trip.

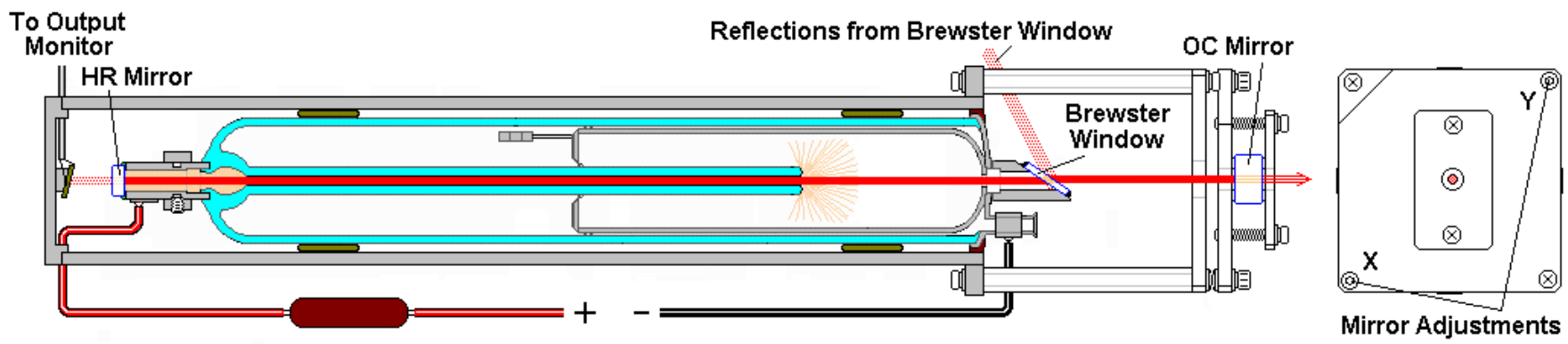
- Gaussian beam analysis is usually not useful, because the beam is usually not a Gaussian.



- The beam size is large, so one can use a wide gain medium.
- Gain per round trip must be very high.
- In this example, the output beam has a doughnut shape – dark in the middle (definitely NOT a Gaussian!)

In many cases, real lasers are actually as simple as the ones we've looked at.

For example: in many gas lasers, there are only two mirrors, and the gain medium doesn't actually affect the shape of the beam much because it is gaseous.

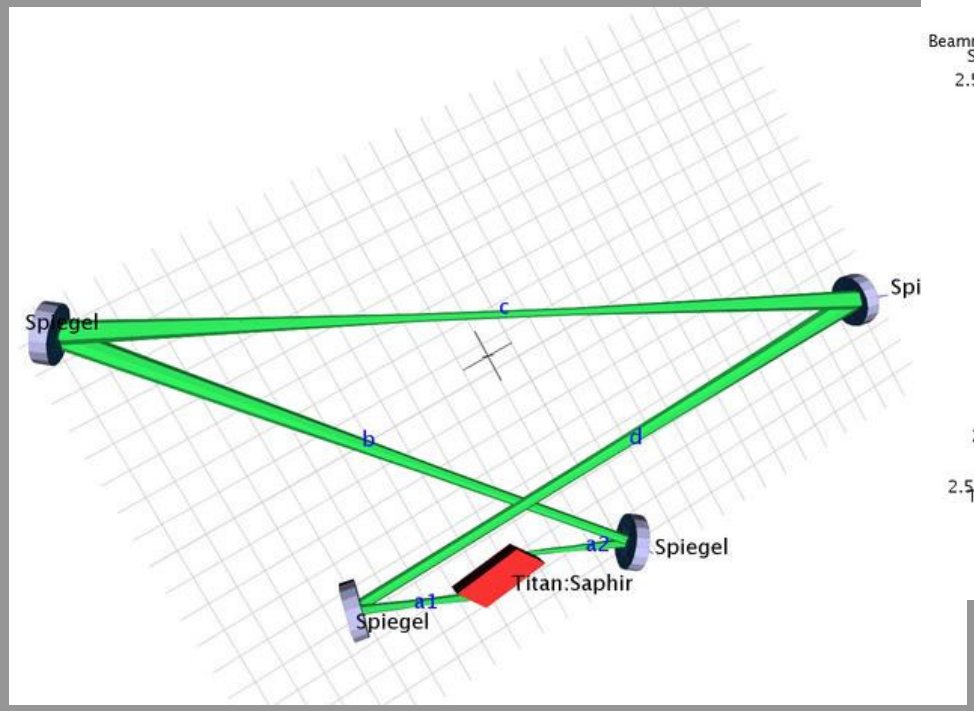


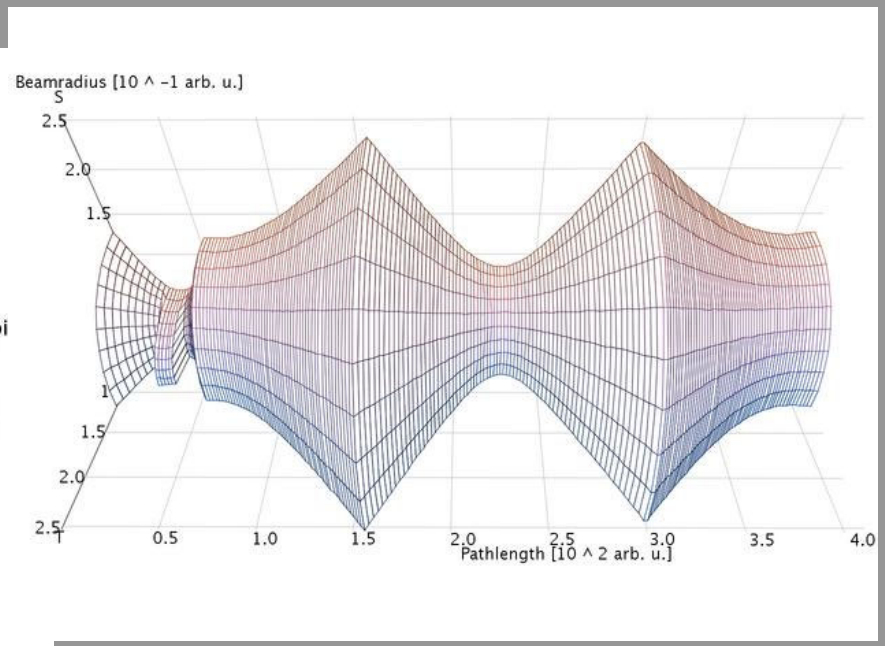
HeNe Laser Tube with Internal HR and Brewster Window with External OC

HeNe lasers most often use symmetric confocal or hemispherical cavities.

In other cases, they can be complicated

It is not unusual to use more than two mirrors. Also, there's usually something inside the laser (i.e., the gain medium) that cannot be ignored.





This shows screen shots of a Ti:sapphire laser, from a Gaussian beam resonator analysis software package called JLaserLab.

Gaussian beam analysis is the first go-to method, even in complicated laser cavities.