Ultrashort Laser Pulses I

Description of pulses

Intensity and phase

The instantaneous frequency and group delay

Zeroth and first-order phase

The linearly chirped Gaussian pulse

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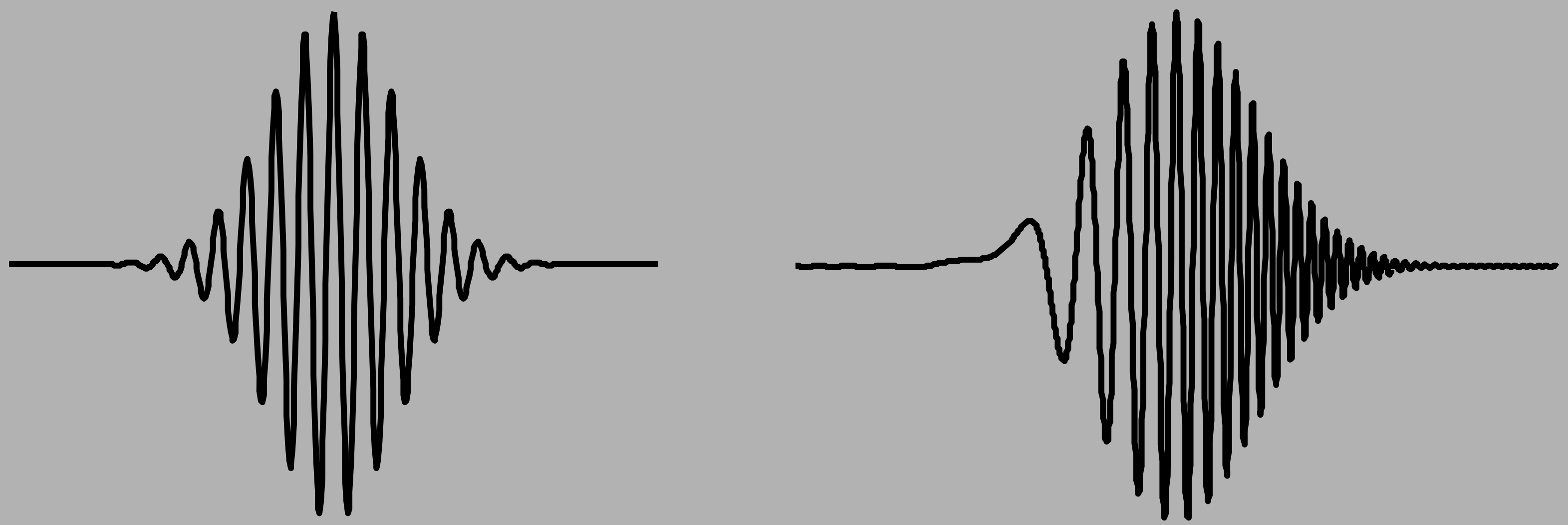
Pulses can be complicated

These two pulses have the same spectra.
But they are obviously not the same.



In optics, measuring the spectrum of a light source is easy.
But that doesn't distinguish between these two pulses.

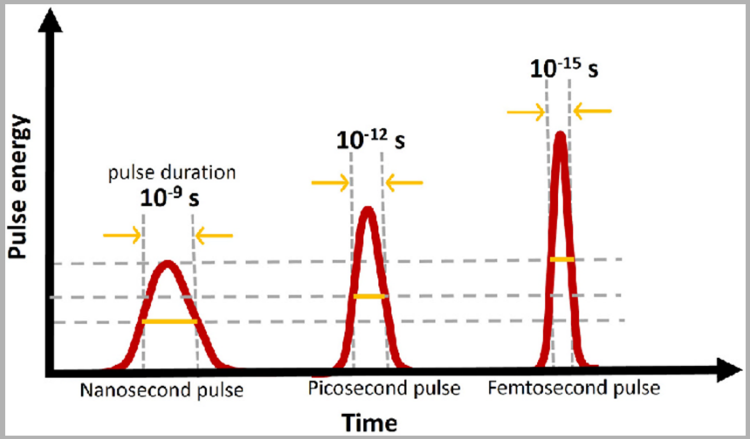
In ultrafast optics, measuring these sorts of properties of your laser pulse is often very important, and also very challenging.

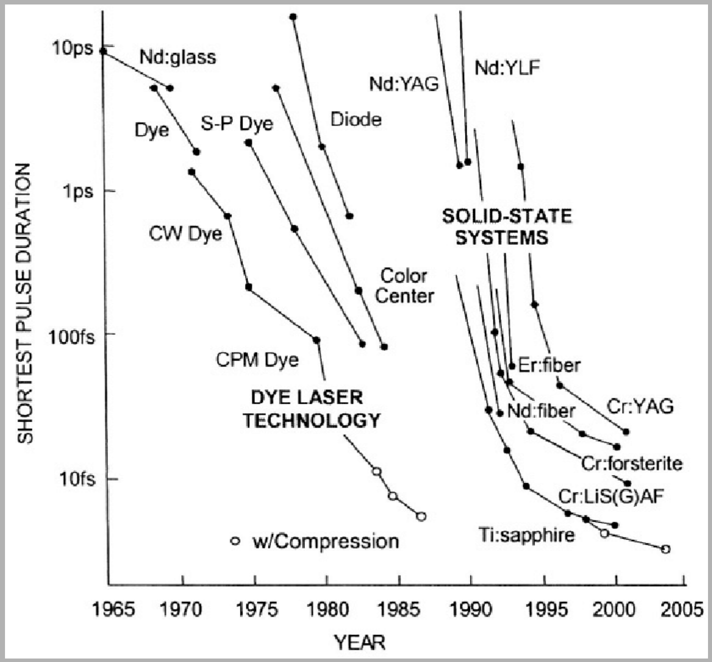
Today, we will lay the groundwork for how to understand these pulse distortions.

The duration of a short pulse

One of the two most important parameters to describe an ultrashort pulse is its duration.

(The other is its central frequency or wavelength.)



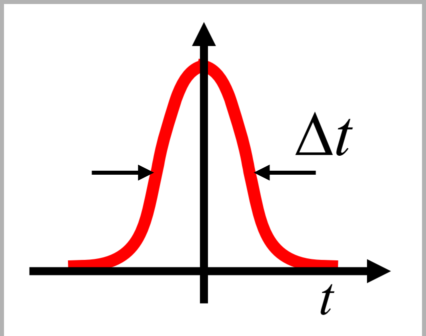


You can tell its important because it is the thing that everybody makes plots of.

But how do we define it?

The pulse length

There are many definitions of the width or length of a wave or pulse.

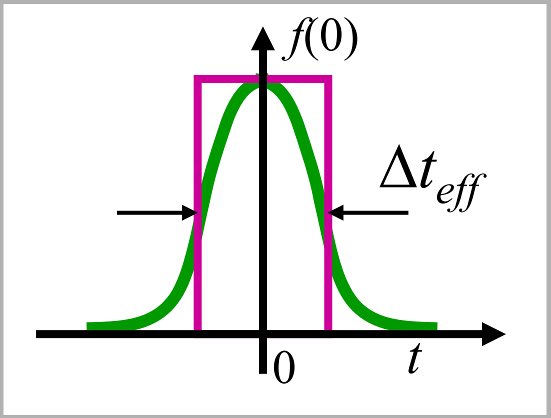


The **effective width** is the width of a rectangle whose ***height*** and ***area*** are the same as those of the pulse.

Effective width ≡ Area / height:

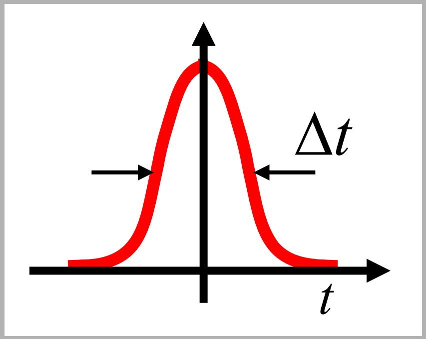
$$\Delta t_{eff} \equiv \frac{1}{E(0)} \int_{-\infty}^{\infty} \left| E(t) \right| dt$$

(Abs value is unnecessary for intensity.)



Advantage: It's easy to understand.
Disadvantages: The Abs value is inconvenient.
We must integrate to ± ∞.

The rms pulse width



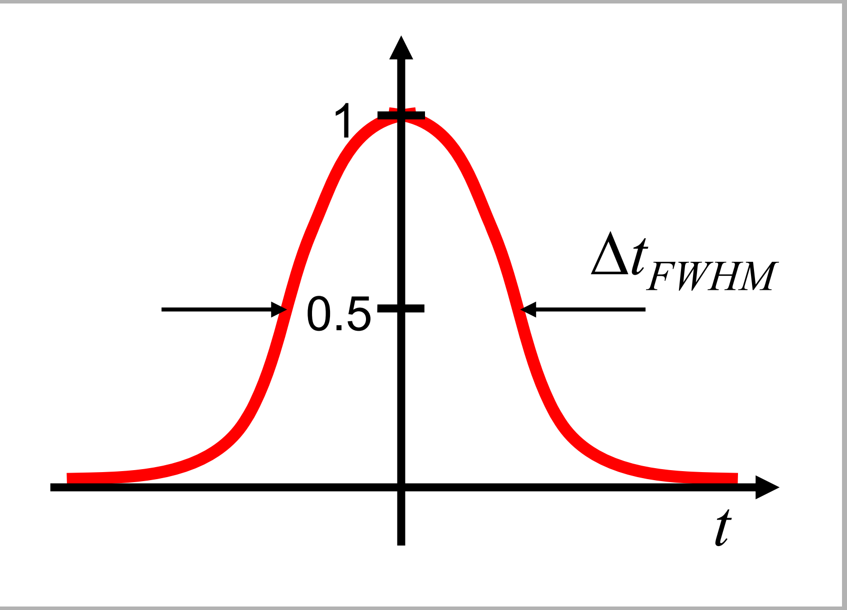
The **root-mean-squared width** or **rms width**:

$$\Delta t_{rms} \equiv \left[\frac{\int_{-\infty}^{\infty} t^2 E(t)\, dt}{\int_{-\infty}^{\infty} E(t)\, dt} \right]^{1/2}$$

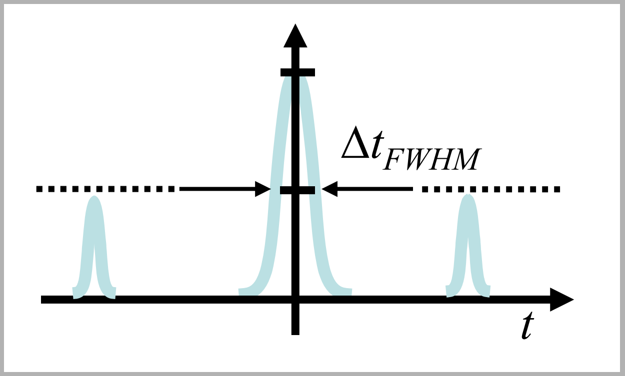
The rms width is the normalized second-order moment.

Advantages: Integrals are often easy to do analytically.
Disadvantages: It weights wings even more heavily,
so it's difficult to use for experiments, which can't scan to $\pm\infty$.

The Full-Width-Half-Maximum



Full-width-half-maximum is the distance between the half-maximum points.



Advantages: Experimentally easy.
Disadvantages: It ignores satellite pulses with heights < 49.99% of the peak!

Note: we can define these widths in terms of $E(t)$ or of its intensity, $|E(t)|^2$. Define *spectral* widths ($\Delta\omega$) similarly in the frequency domain ($t \rightarrow \omega$).

The Uncertainty Principle

The Uncertainty Principle says that the product of a function's widths in the time domain (Δt) and the frequency domain ($\Delta\omega$) has a minimum.

Use effective widths assuming $f(t)$ and $F(\omega)$ peak at 0:

$$\Delta t \equiv \frac{1}{f(0)} \int_{-\infty}^{\infty} \left| f(t) \right| dt \qquad \Delta\omega \equiv \frac{1}{F(0)} \int_{-\infty}^{\infty} \left| F(\omega) \right| d\omega$$

$$\Delta t \geq \frac{1}{f(0)} \int_{-\infty}^{\infty} f(t)\, dt = \frac{1}{f(0)} \int_{-\infty}^{\infty} f(t) \overbrace{\exp[-i(0)t]}^{1}\, dt = \frac{F(0)}{f(0)}$$

$$\Delta\omega \geq \frac{1}{F(0)} \int_{-\infty}^{\infty} F(\omega)\, d\omega = \frac{1}{F(0)} \int_{-\infty}^{\infty} F(\omega) \exp[i\omega(0)]\, d\omega = \frac{2\pi f(0)}{F(0)}$$

Combining results:

$$\Delta\omega\, \Delta t \geq 2\pi \cancel{\frac{f(0)\, F(0)}{F(0)\, f(0)}}$$

or: $\boxed{\Delta\omega\, \Delta t \geq 2\pi}$ $\boxed{\Delta\nu\, \Delta t \geq 1}$

Other width definitions yield slightly different numbers.

The Time-Bandwidth Product

For a given wave, the product of the time-domain width (Δt) and the frequency-domain width ($\Delta\nu$) is the

Time-Bandwidth Product (TBP)

$$\boxed{\Delta\nu \ \Delta t \ \equiv \ TBP}$$

sometimes people use $\Delta\omega$ instead of $\Delta\nu$.

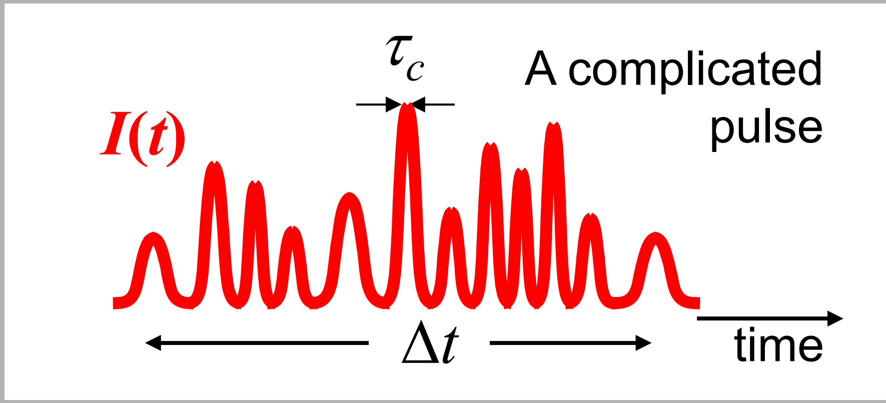
A pulse's TBP will always be greater than the theoretical minimum given by the Uncertainty Principle (for the appropriate width definition).

The TBP is a measure of how complex a wave or pulse is.

Even though every pulse's time-domain and frequency-domain functions are related by the Fourier Transform, a wave whose TBP is the theoretical minimum is called "***Fourier-Transform Limited***."

The Time-Bandwidth Product is a measure of the pulse complexity.

The coherence time ($\tau_c = 1/\Delta\nu$) indicates the smallest temporal structure of the pulse.



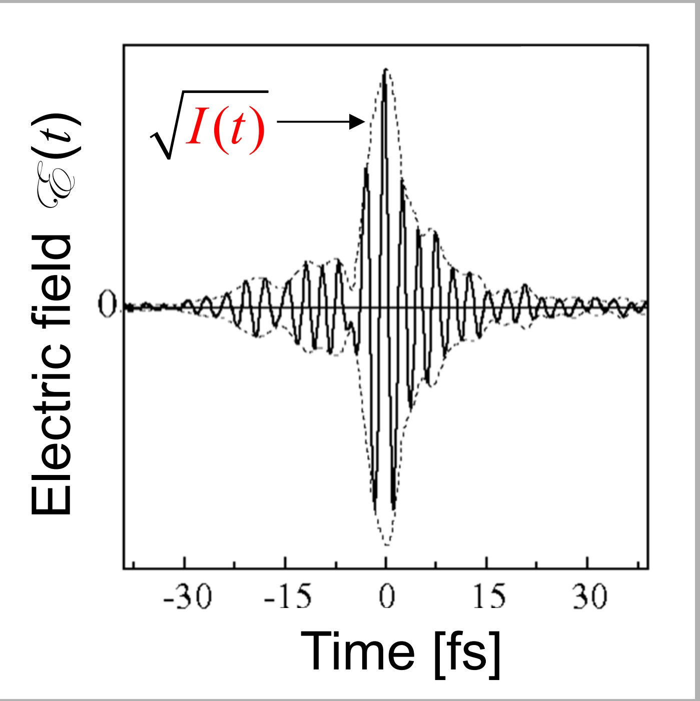
In terms of the coherence time:

$$TBP = \Delta\nu \ \Delta t = \Delta t \ / \ \tau_c$$

= about how many spikes are in the pulse

But TBP is just one number. To accurately describe a real pulse, we require a more detailed analysis which accounts for both the time-varying intensity and the time-varying phase.

An ultrashort laser pulse has an intensity and phase vs. time.



Neglecting the spatial dependence for now, the pulse electric field is given by:

$$\mathscr{E}(t) \propto \tfrac{1}{2}\sqrt{I(t)}\,\exp\{i[\omega_0 t - \phi(t)]\} + c.c.$$

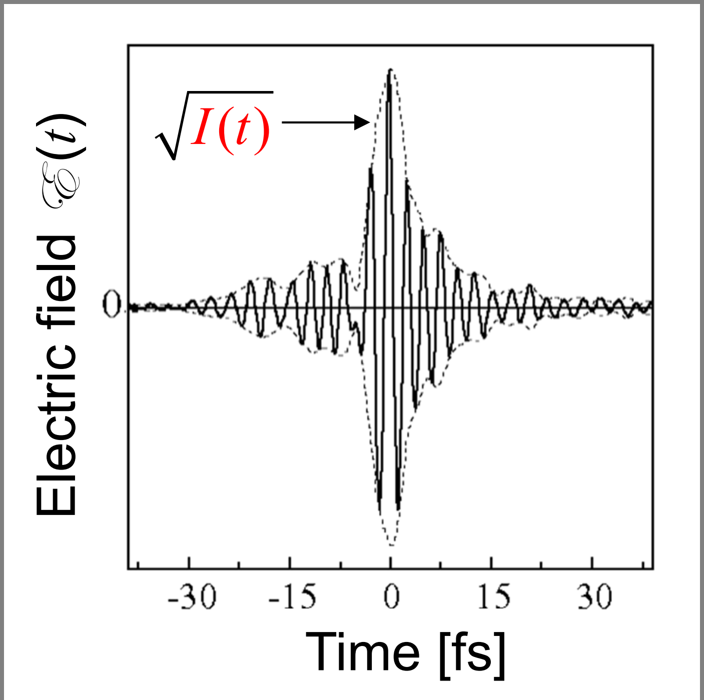
Intensity

Carrier frequency

Phase

A sharply peaked function for the intensity yields an ultrashort pulse.
The phase tells us the frequency evolution of the pulse in time.

The real and complex pulse amplitudes



Removing the 1/2, the c.c., and the exponential factor with the carrier frequency yields the **complex amplitude, $E(t)$,** of the pulse:

$$E(t) \propto \sqrt{I(t)}\,\exp\{-i\phi(t)\}$$

(notice the change of notation: $\mathscr{E}(t)$ became $E(t)$.)

This removes the rapidly varying part of the pulse electric field and yields a complex quantity, which is easier to calculate with.

$\sqrt{I(t)}$ is often called the **real amplitude, $A(t)$,** of the pulse.

The Gaussian pulse

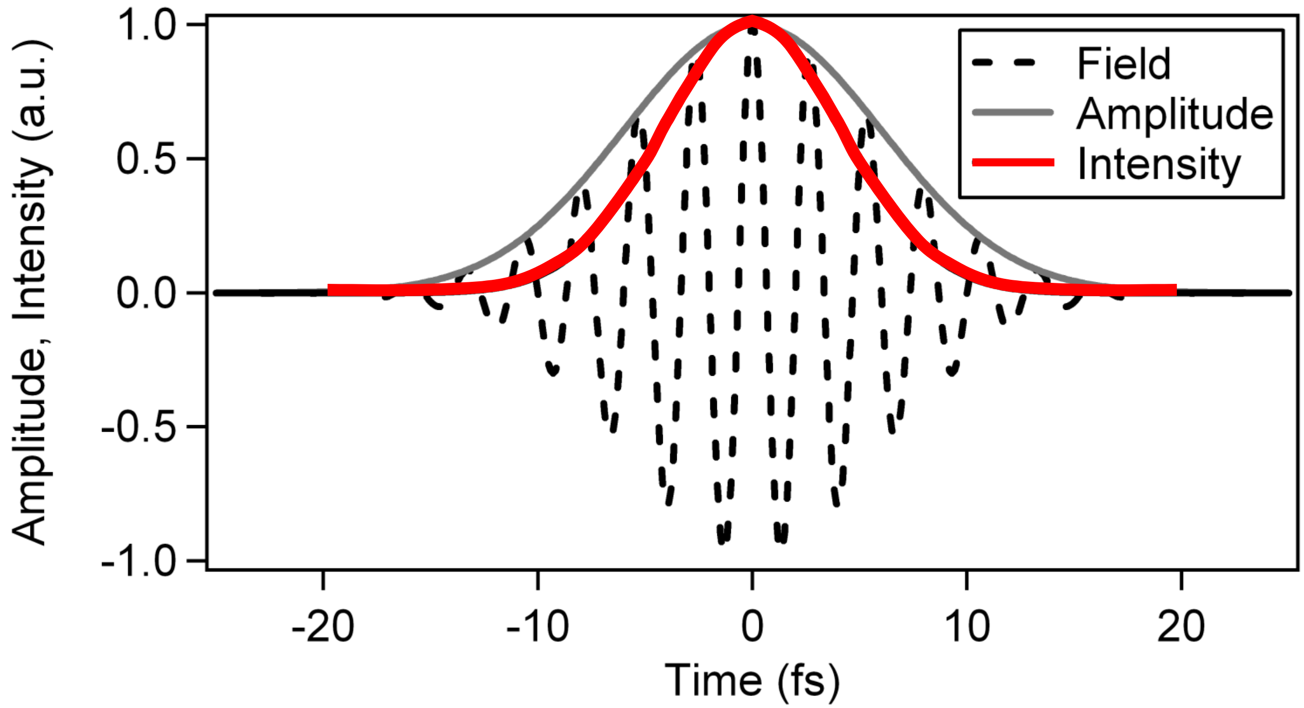
For many calculations, a nice first approximation for an ultrashort pulse is a **zero-phase Gaussian pulse**.

$$E(t) = E_0 \exp\left[-(t/\tau_{HW@1/e})^2\right]$$
$$= E_0 \exp\left[-2\ln 2\,(t/\tau_{FWHM})^2\right]$$
$$= E_0 \exp\left[-1.38\,(t/\tau_{FWHM})^2\right]$$

where $\tau_{HW@1/e}$ is the field half-width at 1/e of its maximum, and τ_{FWHM} is the intensity full-width-half-maximum.

The intensity is: $$I(t) \propto |E_0|^2 \exp\left[-4\ln 2\,(t/\tau_{FWHM})^2\right]$$
$$\propto |E_0|^2 \exp\left[-2.76\,(t/\tau_{FWHM})^2\right]$$

Intensity vs. amplitude



The phase of this pulse is constant, $\phi(t) = 0$, and is not plotted here.

The intensity of a Gaussian pulse is $\sqrt{2}$ shorter than its real amplitude. This factor of $\sqrt{2}$ is different for other pulse shapes.

Calculating the intensity and the phase

It's easy to go back and forth between the electric field and the intensity and phase:

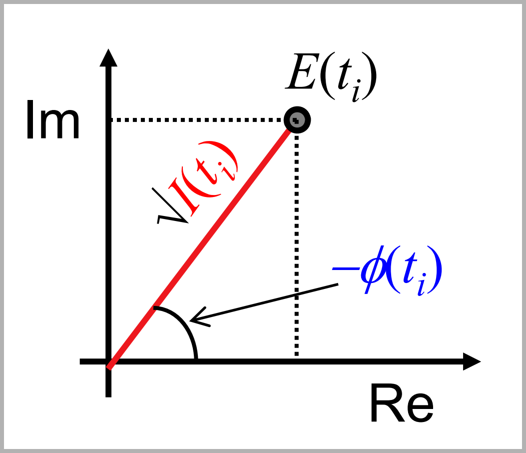
The intensity:

$$I(t) = |E(t)|^2$$

To save effort, we'll stop writing "proportional to" in these expressions and take E, $\mathscr{E}$, I, and S to be the field, intensity, and spectrum dimensionless shapes vs. time.

The phase:

$$\phi(t) = -\arctan\left\{\frac{\mathrm{Im}[E(t)]}{\mathrm{Re}[E(t)]}\right\}$$



Equivalently,

$$\phi(t) = -\,\mathrm{Im}\{\ln[E(t)]\}$$

The Fourier Transform

To think about ultrashort laser pulses, the Fourier Transform is essential.

$$\tilde{\mathscr{E}}(\omega) = \int_{-\infty}^{\infty} \mathscr{E}(t) \exp(-i\omega t)\, dt$$

$$\mathscr{E}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\mathscr{E}}(\omega) \exp(i\omega t)\, d\omega$$

We always perform Fourier transforms on the real or complex pulse electric field, and not on the intensity (unless otherwise specified).

The frequency-domain electric field

The frequency-domain equivalents of the **intensity** and **phase** are the **spectrum** and **spectral phase**.

Fourier-transforming the pulse electric field:

$$\mathcal{E}(t) \;=\; \tfrac{1}{2}\sqrt{I(t)}\,\exp\{i[\omega_0 t - \phi(t)]\} + c.c.$$

yields:

Note that ϕ and φ are different!

$$\tilde{\mathcal{E}}(\omega) \;=\; \tfrac{1}{2}\sqrt{S(\omega-\omega_0)}\,\exp\{-i[\varphi(\omega-\omega_0)]\} + \\ \tfrac{1}{2}\sqrt{S(-\omega-\omega_0)}\,\exp\{+i[\varphi(-\omega-\omega_0)]\}$$

The frequency-domain electric field has both positive- and negative-frequency components. (Recall that the Fourier transform of cos(ωt) is the sum of two delta functions at frequencies ω and $-\omega$.)

The complex frequency-domain pulse field

Since the negative-frequency component contains the same information as the positive-frequency component, we usually neglect it.

We also center the pulse on its actual frequency, not zero.

So the most commonly used expression for the complex frequency-domain pulse field is:

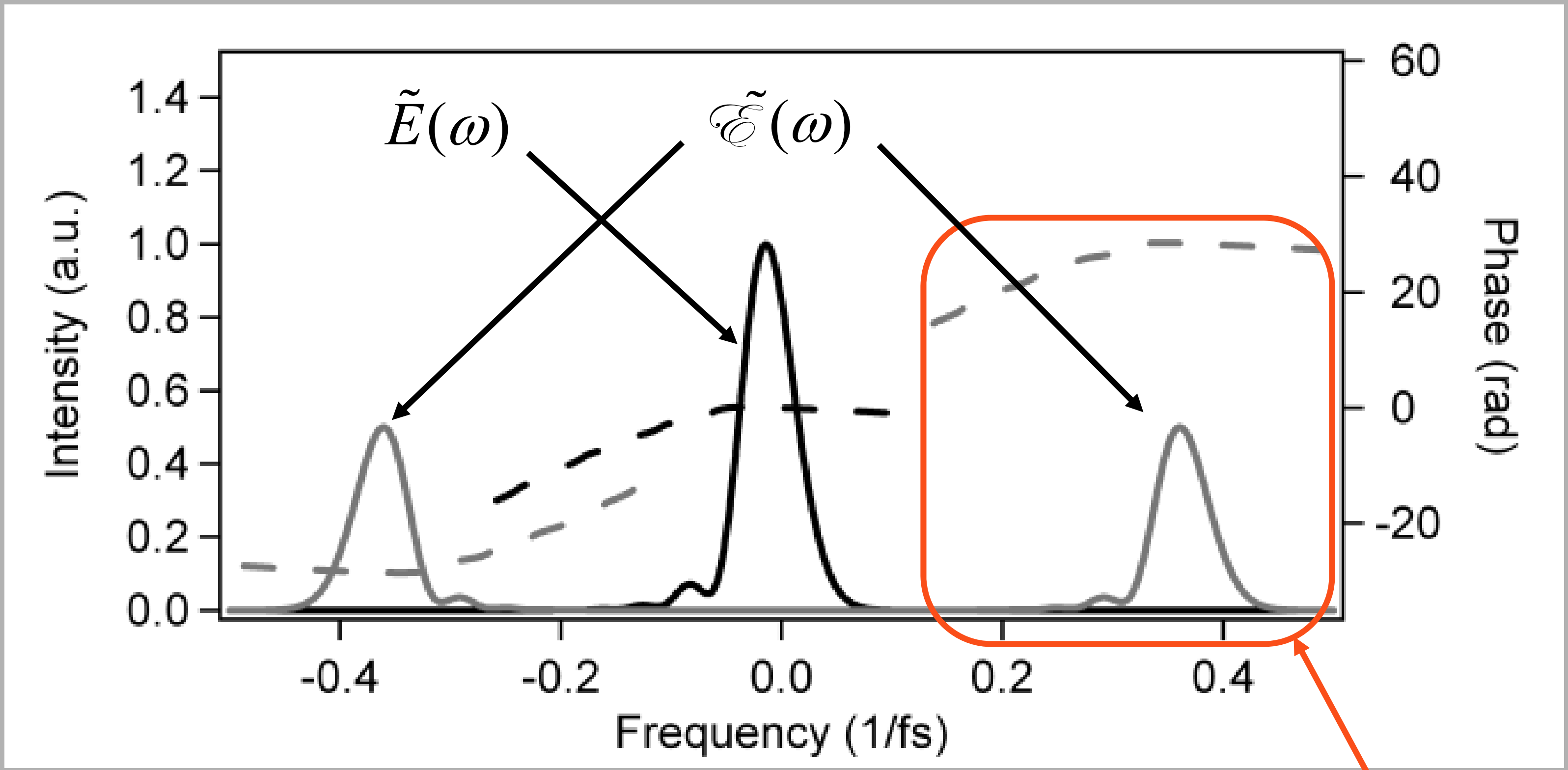
$$\tilde{\mathscr{E}}(\omega) \equiv \sqrt{S(\omega)} \exp\{-i\,\varphi(\omega)\}$$

Thus, the frequency-domain electric field also has an intensity and phase.

S is the spectrum, and φ is the spectral phase.

The spectrum with and without the carrier frequency

Fourier transforming $\mathscr{E}(t)$ and $E(t)$ yields different functions.



We usually use just this component.

The spectrum and spectral phase

The **spectrum** and **spectral phase** are obtained from the frequency-domain field the same way the **intensity** and **phase** are from the time-domain electric field.

$$S(\omega) = \left|\tilde{\mathcal{E}}(\omega)\right|^2$$

$$\varphi(\omega) = -\arctan\left\{\frac{\mathrm{Im}[\tilde{\mathcal{E}}(\omega)]}{\mathrm{Re}[\tilde{\mathcal{E}}(\omega)]}\right\}$$

or

$$\varphi(\omega) = -\mathrm{Im}\left\{\ln[\tilde{\mathcal{E}}(\omega)]\right\}$$

Intensity and phase of a Gaussian

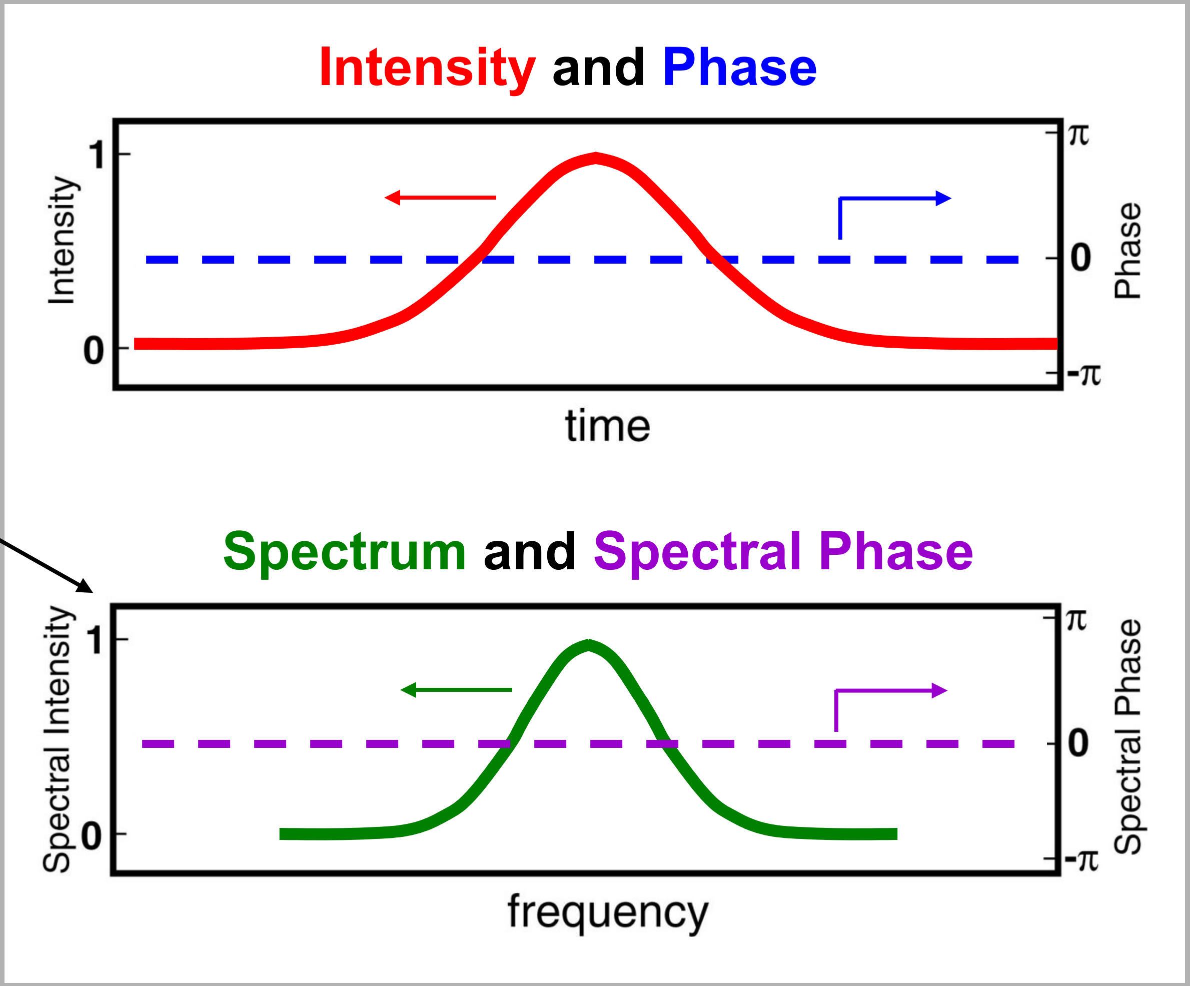
The Gaussian is real, so its phase is zero.

Time domain:

A Gaussian transforms to a Gaussian

Frequency domain:

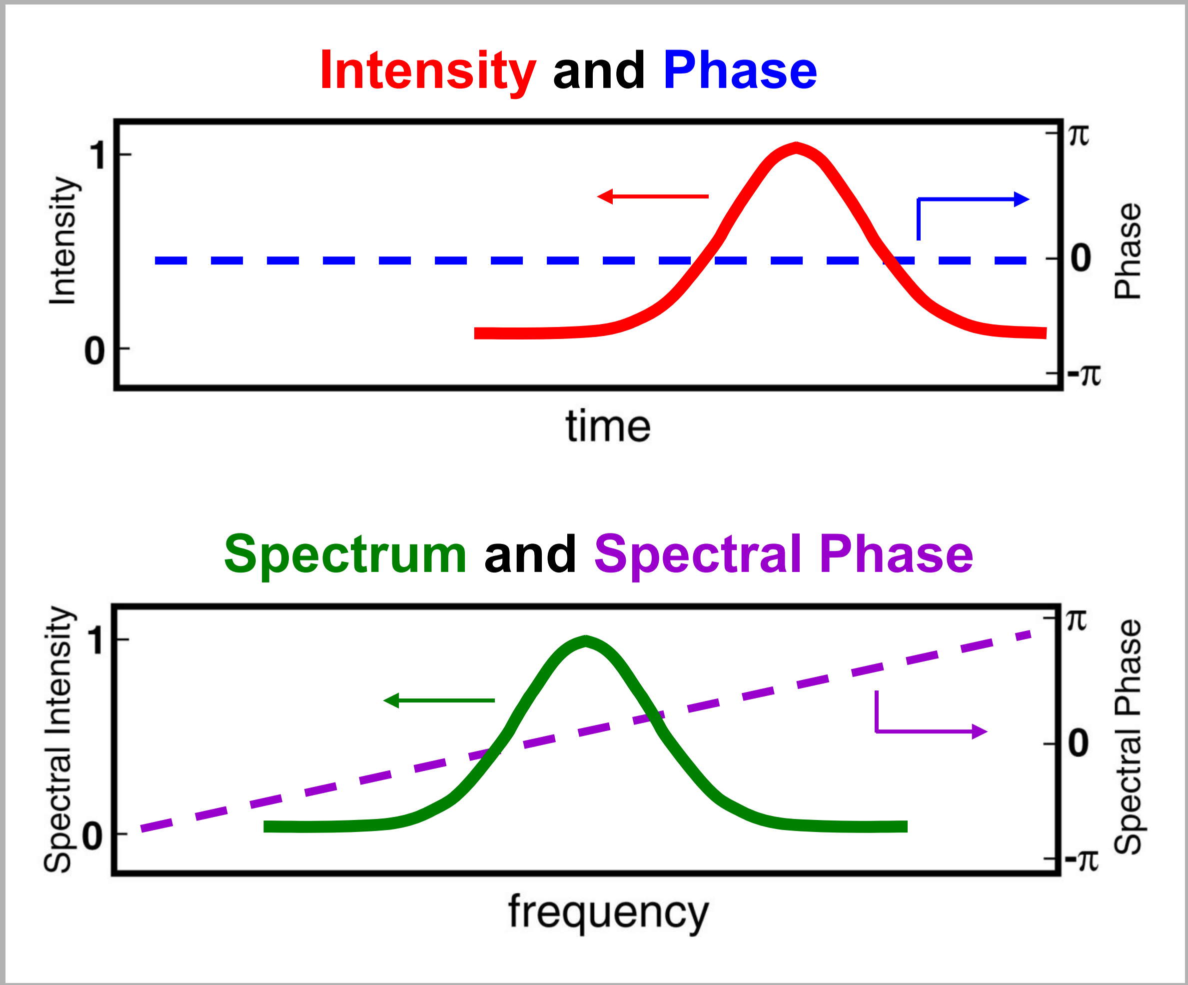
So the spectral phase is zero, too.



The spectral phase of a time-shifted pulse

Recall the Shift Theorem: $\mathscr{F}\left\{ f(t-a) \right\} = \exp(-i\omega a) F(\omega)$

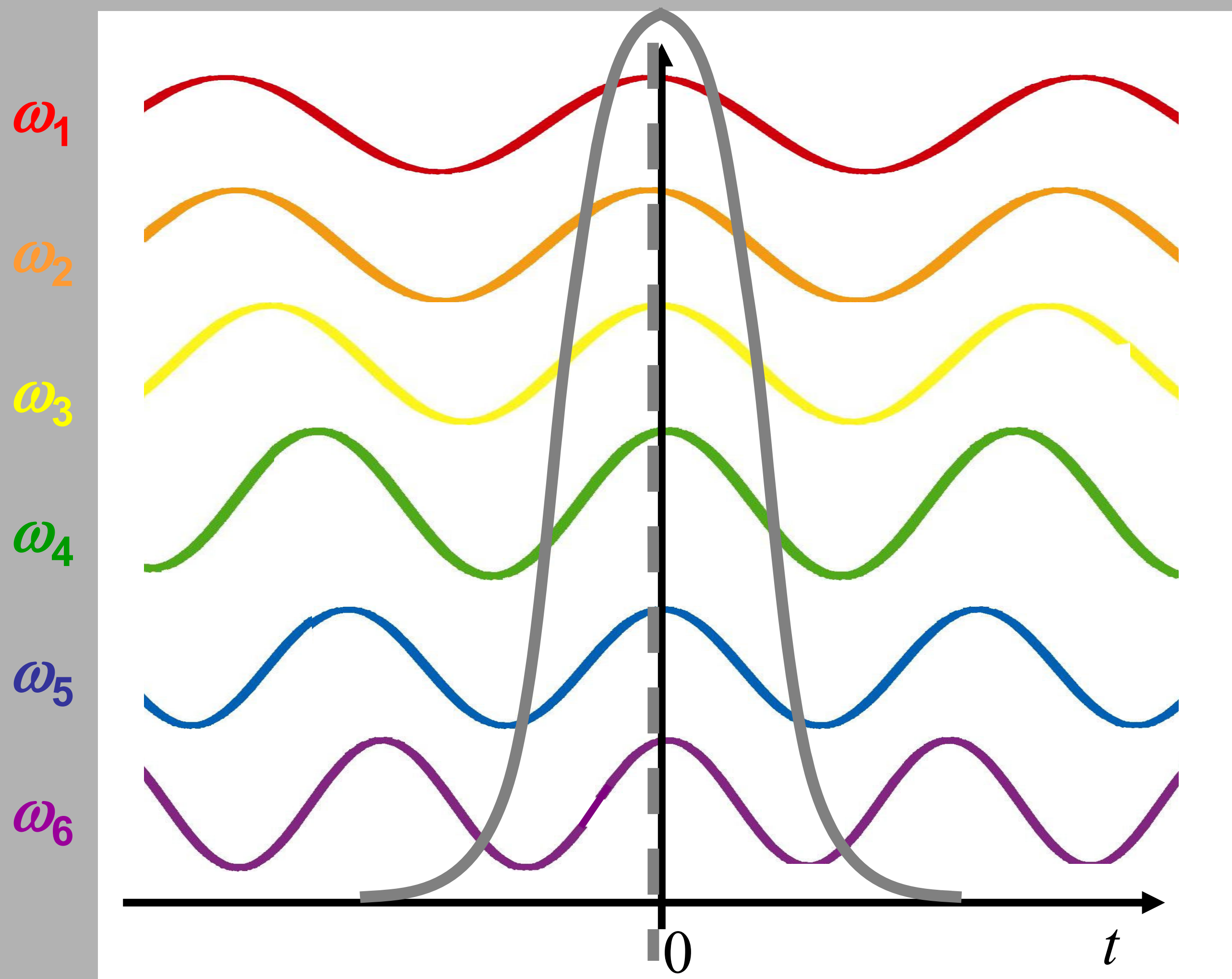
Time-shifted Gaussian pulse (with a flat phase):



So a time-shift simply adds some linear spectral phase to the pulse!

What is the spectral phase?

The spectral phase is the phase of each frequency in the wave-form.



All of these frequencies have zero phase. So this pulse has:

$$\varphi(\omega) = 0$$

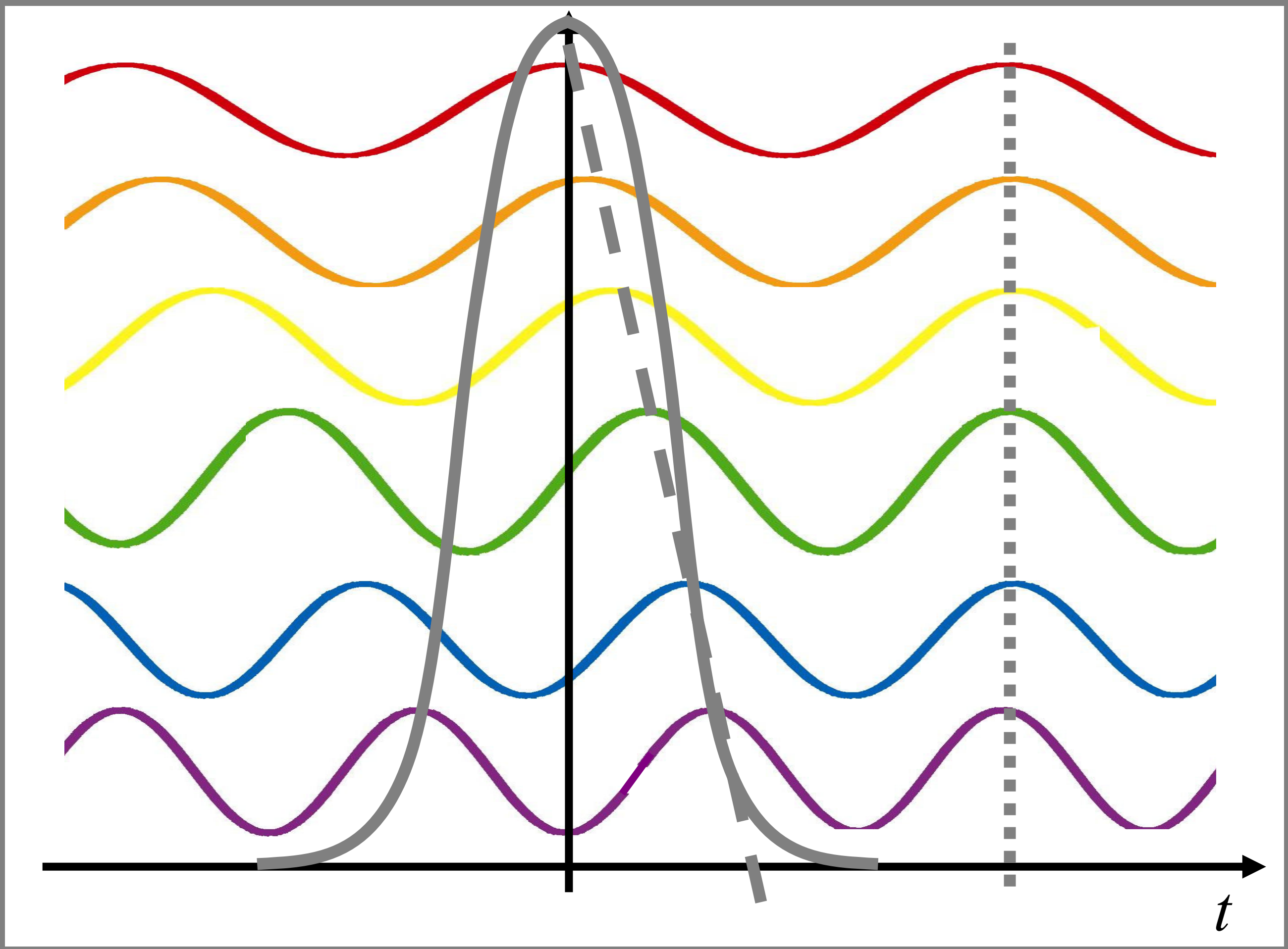
Note that this wave-form sees constructive interference, and hence peaks, at $t = 0$.

And it has cancellation everywhere else.

Now try a linear spectral phase: $\varphi(\omega) = a\omega$.

By the Shift Theorem, a linear spectral phase is just a delay in time.

And this cartoon shows why that occurs!



Transforming between wavelength and frequency

The spectrum and spectral phase vs. frequency aren't the same as the spectrum and spectral phase vs. wavelength.

The spectral phase is easily transformed:

$$\boxed{\varphi_\lambda(\lambda) \;=\; \varphi_\omega(2\pi c/\lambda)}$$

$$\omega = \frac{2\pi c}{\lambda}$$

To transform the spectrum, note that the energy is the same, whether we integrate the spectrum over frequency or wavelength:

$$\int_{-\infty}^{\infty} S_\lambda(\lambda)\,d\lambda \;=\; \int_{-\infty}^{\infty} S_\omega(\omega)\,d\omega$$

Changing variables:

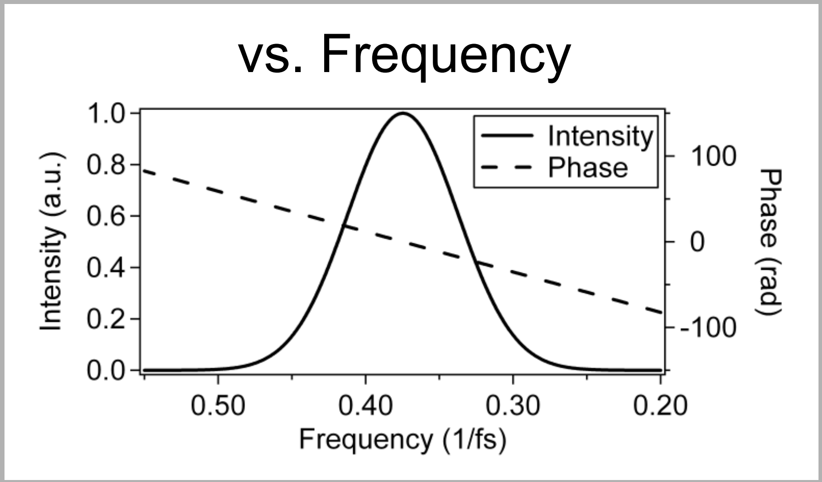
$$= \int_{\infty}^{-\infty} S_\omega(2\pi c/\lambda)\,\frac{-2\pi c}{\lambda^2}\,d\lambda \;=\; \int_{-\infty}^{\infty} S_\omega(2\pi c/\lambda)\,\frac{2\pi c}{\lambda^2}\,d\lambda$$

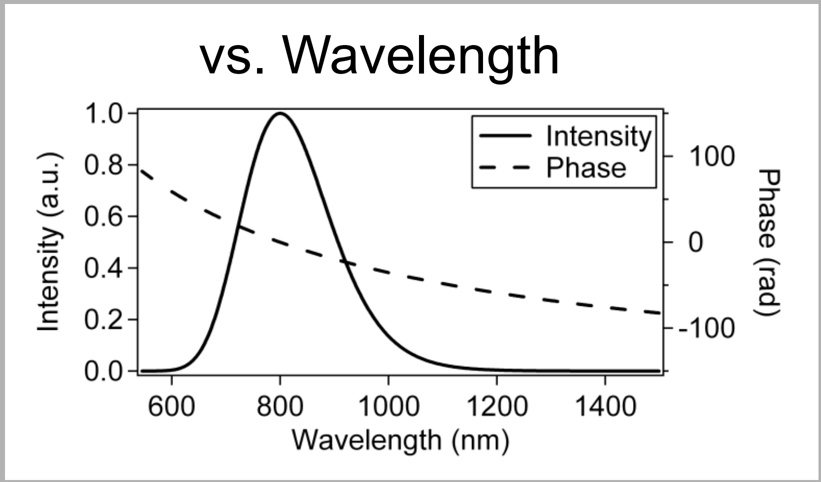
$$\frac{d\omega}{d\lambda} = \frac{-2\pi c}{\lambda^2}$$

$$\Rightarrow \quad \boxed{S_\lambda(\lambda) \;=\; S_\omega(2\pi c/\lambda)\,\frac{2\pi c}{\lambda^2}}$$

The spectrum and spectral phase vs. wavelength and frequency

Example: A Gaussian spectrum with a linear **spectral phase** vs. frequency





Note the different shapes of the **spectrum** and **spectral phase** when plotted vs. wavelength and frequency.

Bandwidth in various units

In **frequency**, by the Uncertainty Principle, a 1-ps pulse has bandwidth:

$$\boxed{\delta\nu = \sim 1/2 \text{ THz}}$$

using $\delta\nu\,\delta t \sim ½$

In **wave numbers** (cm^{-1}), we can write:

$$\nu = \frac{c}{\lambda} \quad \Rightarrow \delta\nu = c\,\delta(1/\lambda) \quad \Rightarrow \delta(1/\lambda) = \delta\nu / c$$

So $\delta(1/\lambda) = (0.5 \times 10^{12}\text{ /s}) / (3 \times 10^{10}\text{ cm/s})$ or: $\boxed{\delta(1/\lambda) = 17\text{ cm}^{-1}}$

In **wavelength**: $\delta(1/\lambda) = \left|\frac{-1}{\lambda^2}\right| \delta\lambda \quad \Rightarrow \delta\lambda = \lambda^2\,\delta(1/\lambda)$

Assuming an 800-nm wavelength:

$$\delta\lambda = (800\,\text{nm})(.8\times10^{-4}\,\text{cm})(17\text{ cm}^{-1})$$

or: $\boxed{\delta\lambda = 1\text{ nm}}$

The Instantaneous frequency

The **temporal phase, $\phi(t)$,** contains frequency-vs.-time information.

The pulse ***instantaneous angular frequency***, $\omega_{inst}(t)$, is defined as:

$$\boxed{\omega_{inst}(t) \equiv \omega_0 - \frac{d\phi}{dt}}$$

This is easy to see. At some time, t, consider the total phase of the wave. Call this quantity ϕ_0:

$$\phi_0 = \omega_0 t - \phi(t)$$

Exactly one period, T, later, the total phase will (by definition) increase to $\phi_0 + 2\pi$:

$$\phi_0 + 2\pi = \omega_0 \cdot [t+T] - \phi(t+T)$$

where $\phi(t+T)$ is the slowly varying phase at the time, $t+T$. Subtracting these two equations:

$$2\pi = \omega_0 T - [\phi(t+T) - \phi(t)]$$

Instantaneous frequency (cont'd)

Dividing by T and recognizing that $2\pi/T$ is a frequency, call it $\omega_{inst}(t)$:

$$\omega_{inst}(t) = 2\pi/T = \omega_0 - [\phi(t+T) - \phi(t)] \,/\, T$$

But T is small, so $[\phi(t+T) - \phi(t)] \,/T$ is the derivative, $d\phi\,/dt$.

So we're done!

Usually, however, we'll think in terms of the ***instantaneous frequency***, $\nu_{inst}(t)$, so we'll need to divide by 2π:

$$\boxed{\nu_{inst}(t) = \nu_0 - (d\phi\,/dt) \,/\, 2\pi}$$

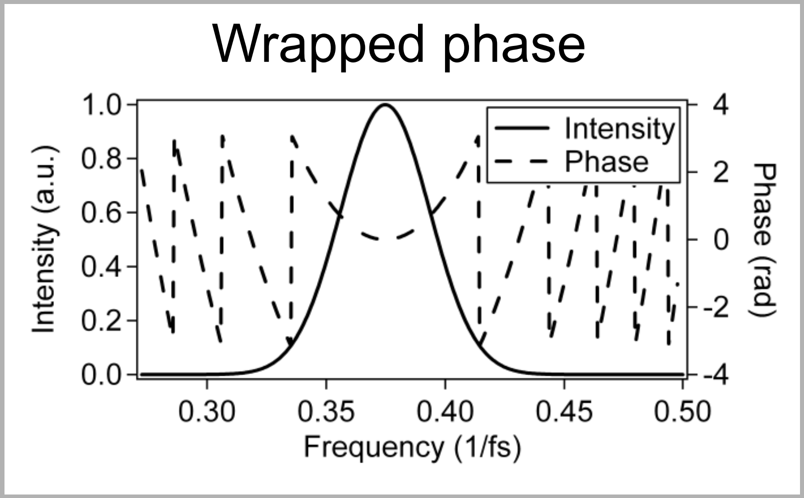
While the instantaneous frequency isn't always a rigorous quantity, it's fine for ultrashort pulses, which have broad bandwidths.

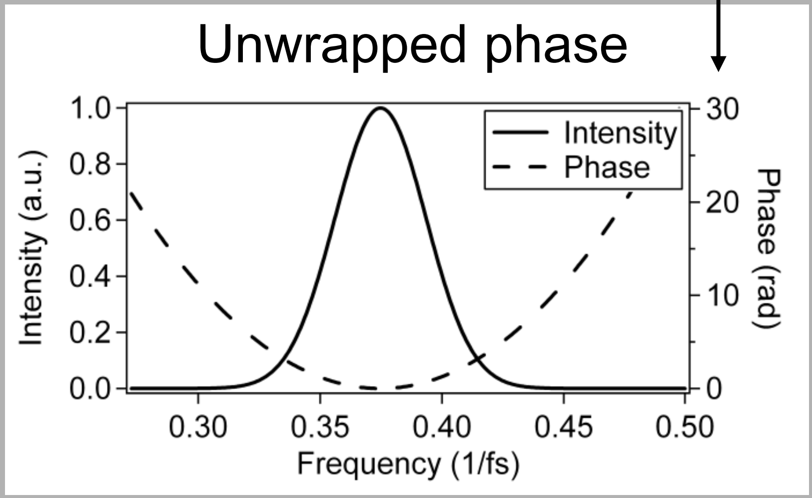
Phase wrapping and unwrapping

Technically, the phase ranges from $-\pi$ to π. But it often helps to "unwrap" it. This involves adding or subtracting 2π whenever there's a 2π phase jump.

Example: a pulse with quadratic phase

Note the scale!





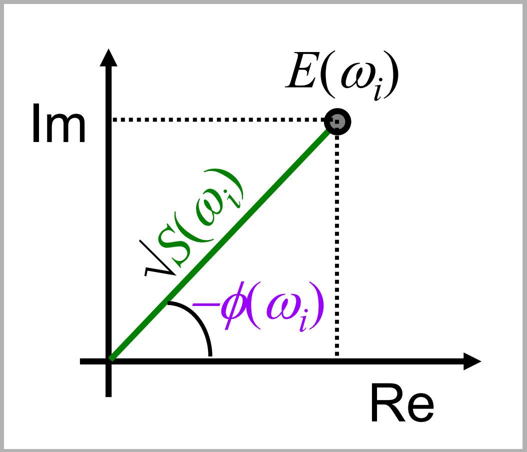
The main reason for unwrapping the phase is aesthetics.

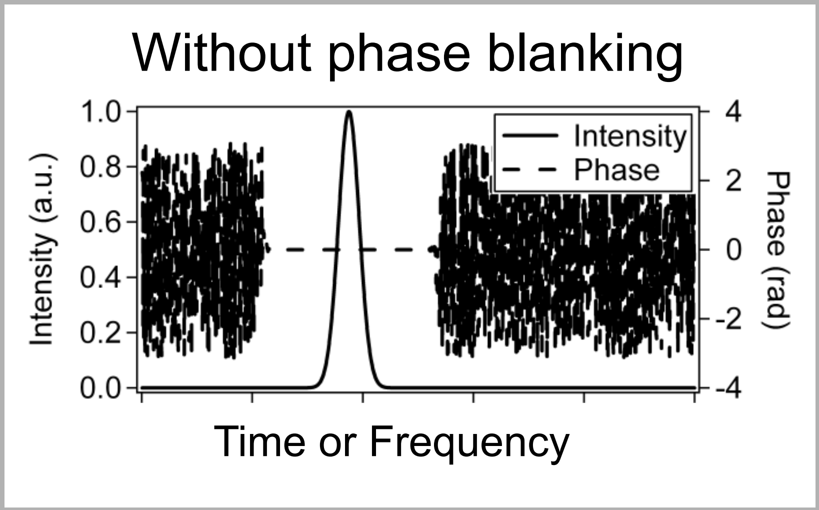
Phase-blanking

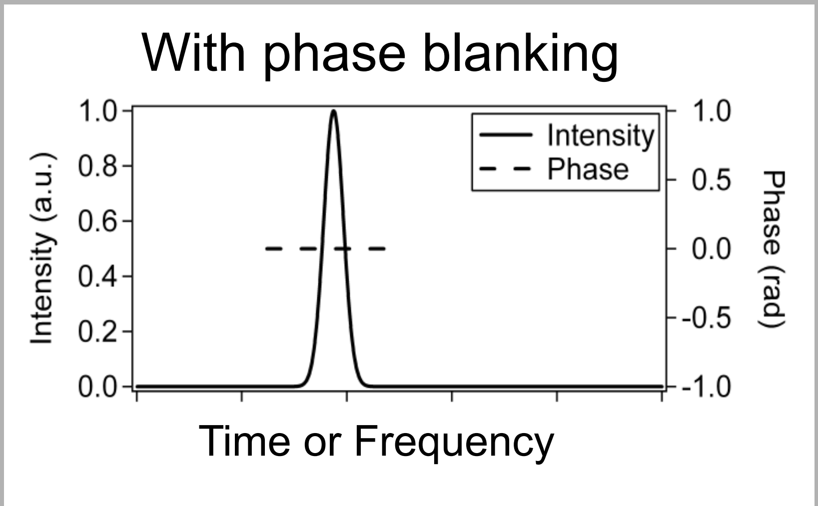
When the intensity is zero, the phase is meaningless.

When the intensity is nearly zero, the phase is nearly meaningless.

Phase-blanking involves simply not plotting the phase when the intensity is close to zero.







The only problem with phase-blanking is that you have to decide the intensity level below which the phase is meaningless.

Phase Taylor Series expansions

We can write a Taylor series for the phase, $\phi(t)$, about the time $t = 0$:

$$\phi(t) = \phi_0 + \phi_1 \frac{t}{1!} + \phi_2 \frac{t^2}{2!} + \ldots$$

where

$$\phi_1 = \left.\frac{d\phi}{dt}\right|_{t=0}$$ is related to the instantaneous frequency.

where only the first few terms are typically required to describe well-behaved pulses. Of course, sometimes you have to consider badly behaved pulses, which have higher-order terms in $\phi(t)$.

Expanding the phase in time is not common because it's hard to measure the intensity vs. time.

Frequency-domain phase expansion

It's much more common to write a Taylor series for $\varphi(\omega)$:

$$\boxed{\varphi(\omega) \;=\; \varphi_0 \;+\; \varphi_1 \frac{\omega-\omega_0}{1!} \;+\; \varphi_2 \frac{\left(\omega-\omega_0\right)^2}{2!} \;+\; \ldots}$$

where: φ_0 is called the **absolute phase**

$$\varphi_1 = \left.\frac{d\varphi}{d\omega}\right|_{\omega=\omega_0}$$ is called the **group delay**.

$$\varphi_2 = \left.\frac{d^2\varphi}{d\omega^2}\right|_{\omega=\omega_0}$$ is called the **group-delay dispersion**.

As in the time domain, only the first few terms are typically required to describe well-behaved pulses.