1. From lecture 7, we know that the absorption coefficient is proportional to the imaginary part of a complex Lorentzian:

$$\alpha(\omega) \propto \frac{\Gamma}{(\omega_0 - \omega)^2 + \Gamma^2}$$

and the refractive index is proportional to the real part:

$$n(\omega) - 1 \propto \frac{(\omega_0 - \omega)}{(\omega_0 - \omega)^2 + \Gamma^2}$$

To answer this question, we recall that the refractive index has a negative slope (i.e., anomalous dispersion) in the range between its two extrema (see lecture 7 slide 17). So, to find the frequencies at which the extrema occur, we compute the derivative of \(n - 1\) and set it equal to zero:

$$\frac{d}{d\omega} \left[ n(\omega) - 1 \right] \propto \frac{2(\omega_0 - \omega)^2}{(\omega_0 - \omega)^2 + \Gamma^2} - \frac{1}{(\omega_0 - \omega)^2 + \Gamma^2} = 0$$

This is easily solved, yielding the relation: \((\omega_0 - \omega)^2 = \Gamma^2\) which has two solutions:

$$\omega = \omega_0 \pm \Gamma$$

The dispersion is anomalous in the range from \(\Gamma\) below the resonance to \(\Gamma\) above the resonance.

2. The main point of this problem is that the maximum value of the absorption coefficient (which occurs at resonance, \(\omega = \omega_0\)) is related to the maximum value of the refractive index (which, as we saw in problem 1, occurs at \(\omega = \omega_0 - \Gamma\)). From lecture 7, slide 17, we can find these two maximum values:

$$\alpha_{\text{max}} = \alpha(\omega_0) = \frac{Ne^2 / m_e}{2\varepsilon_0 c_0} \cdot \frac{1}{\Gamma}$$

and

$$n_{\text{max}} - 1 = n(\omega_0 - \Gamma) - 1 = \frac{Ne^2 / m_e}{4\varepsilon_0 \omega_0} \cdot \frac{1}{2\Gamma}$$

(a) We observe that these two quantities are related. In fact,

$$n_{\text{max}} - 1 = \frac{\alpha_{\text{max}}}{4} \frac{c_0}{\omega_0} = \frac{\alpha_{\text{max}} \lambda_0}{8\pi}$$

where \(\lambda_0\) is the wavelength corresponding to the resonance at \(\omega_0\). For the resonance specified in this problem, we have \(\alpha_{\text{max}} = 8000 \text{ cm}^{-1}\) and \(\lambda_0 = 0.0163 \text{ cm}\). Plugging in, we determine \(n_{\text{max}} = 5.19\).

(b) Now, we look up the DC dielectric constant of CsI, and find the value 5.65 (google). Thus we predict that the refractive index has a maximum value of \(n_{\text{max}} = 5.19 + \sqrt{5.65} = 7.56\) near this resonance. This is not too far from the measured value of about 8.5 (see Optics Letters, 30, 29 (2005)).

3. Light disperses upon passing through a prism because of the dispersion of the material. Larger dispersion means greater angular separation between the colors. Thus, we want to choose the material with the largest value of |\(dn/d\lambda|\). Taking the derivative of the Cauchy formula, we find:
\[
\frac{dn}{d\lambda} = -\frac{2C}{\lambda^3}
\]
In other words, the dispersion only depends on the size of the constant \(C\), not on the other constant \(B\). The largest value of \(C\) among the six choices is SF10, for which \(C = 0.01342\). That’s the best choice.

4(a) The Drude conductivity is \(\sigma_0 = Ne^2\tau/m_{\text{eff}}\), where \(m_{\text{eff}} = 0.067m_0\). Plugging in numbers, we find that \(\sigma_0 = 4.19 \times 10^3 \, \Omega^{-1} \, \text{m}^{-1}\).
(b) In lecture we saw that the conductivity is the proportionality between the current density and the applied electric field: \(J = \sigma_0 E_{\text{applied}}\). We also saw that the current density is related to the average velocity of the electrons: \(J = NeV_{\text{ave}}\). Equate these two different expressions for \(J\), and solve for the ratio \(V_{\text{ave}}/E_{\text{applied}}\), to find the mobility. We find the result \(\mu = e\tau/m_{\text{eff}}\). Note that the mobility does not depend on the number of free conduction electrons in the semiconductor – it is an intrinsic property of the material, which is independent of how many free electrons we put into it. Using the provided values for \(\tau\) and effective mass, we find:
\[
\mu = 0.0262 \, \text{m}^2 / (\text{V sec}) = 262 \, \text{cm}^2 / (\text{V sec})
\]

5(a) The model developed in lecture predicts that the metal should become transparent for frequencies above the plasma frequency. The plot shown in lecture illustrates an abrupt drop in the reflectivity (and therefore an increase in the transmission) at a photon energy of about 15 electron volts (eV). This is a good approximation of the photon energy corresponding to the plasma frequency. Using \(h = 4.13 \times 10^{-15} \, \text{eV sec}\), we find that \(\nu = E/h = 3.63 \times 10^{15} \, \text{Hz}\).
(b) \(\lambda_0 = 500 \, \text{nm}\) corresponds to a frequency that is below the plasma frequency:
\(\nu = c/\lambda_0 = 6 \times 10^{14} \, \text{Hz}\). So the refractive index at this frequency is a pure imaginary quantity, since \(\epsilon/\epsilon_0\) is a negative number:
\[
n = \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} = 5.97 j
\]
The phase acquired by this wave as it propagates a distance \(z\) is \(\exp(jn k_0 z)\). But since \(n\) is a pure (and positive) imaginary number, this is a decaying exponential. The skin depth is therefore given by \([2 \text{ Im}\{n\} k_0]^{-1}\), which is the same as \(\lambda_0/(4\pi \times 5.97) = 6.7\) nanometers. Note how small this is, compared to the wavelength.
(c) We compute the derivative \(\frac{dn}{d\nu} = \frac{\nu_p^2/\nu^2}{\sqrt{\nu^2 - \nu_p^2}}\). For \(\nu > \nu_p\), this is always positive. Thus the material exhibits normal dispersion.