1. We compute the ray matrix describing propagation from the input facet of the negative lens to the output facet of the positive lens:

\[
M = \begin{bmatrix}
1 & 0 \\
-1/f_2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & d \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-1/f_1 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
-1/f_2 & 1
\end{bmatrix}
\begin{bmatrix}
1-d/f_1 & d \\
-1/f_1 & 1
\end{bmatrix}
= \begin{bmatrix}
1-d/f_1 & d \\
d/f_1f_2-1/f_1-1/f_2 & 1-d/f_2
\end{bmatrix}
\]

For a collimated input beam, we have \( \theta_{in} = 0 \). Thus, we can compute the angle of an output ray emerging from the second lens:

\[
\theta_{out} = x_{in} \left( d/f_1f_2 - 1/f_1 - 1/f_2 \right),
\]

and the position of an output ray:

\[
x_{out} = x_{in} \left( 1-d/f_1 \right).
\]

(a) The output beam converges to a focus if \( \theta_{out} < 0 \). This is equivalent to requiring that \( d/f_1f_2 < 1/f_1 + 1/f_2 \). Multiply both sides by \( f_1f_2 \) to find: \( d > f_1 + f_2 = 60 \) cm. (Note: the < sign turned into a > sign because the product \( f_1f_2 \) is negative.) The requirement is satisfied for values of \( d \) larger than 0.6 m (not smaller).

(b) Evidently, to collimate the output beam we require \( \theta_{out} = 0 \), or \( d = 60 \) cm. Using that value, we find \( x_{out} = x_{in} \left( 1 + 60/40 \right) = 2.5x_{in} \). Thus, the output beam is larger than the input beam by a factor of 2.5.

(c) To analyze Gaussian beam propagation, we can use the same matrix we found above:

\[
M = \begin{bmatrix}
1-d/f_1 & d \\
d/f_1f_2-1/f_1-1/f_2 & 1-d/f_2
\end{bmatrix}
\]

with the lens separation chosen to be \( d = 60 \) cm, so that the C parameter vanishes and \( \theta_{out} = 0 \). Now, plugging in the numbers for \( d, f_1, \) and \( f_2 \), that matrix is found to be:

\[
M = \begin{bmatrix}
2.5 & 0.6 \text{ meters} \\
0 & 0.4
\end{bmatrix}
\]

The complex beam parameter at the output, \( q_{out} \), is related to the \( q_{in} \) according to:

\[
q_{out} = \frac{Aq_{in} + B}{Cq_{in} + D} = \frac{2.5q_{in} + 0.6}{0.4} = \frac{25}{4} q_{in} + 1.5 , \text{ in units of meters.}
\]

For the input beam parameter, we have \( q_{in} = j \frac{\pi w_{in}^2}{\lambda} \), a pure imaginary quantity since the input plane is a focal plane. So the \( q \)-parameter for the output beam, at the output facet of the second lens, is \( q_{out} = \frac{25}{4} j \frac{\pi w_{in}^2}{\lambda} + 1.5 \). For shorthand, let’s define a quantity \( \xi = 25\pi w_{in}^2/4\lambda \), so that we can write \( q_{out} = j\xi + 1.5 \). We now compute the inverse of the \( q \)-parameter:

\[
\frac{1}{q_{out}} = \frac{1}{j\xi + 1.5} = -j\xi + 1.5\xi^2 + 2.25 .
\]
The beam waist at the output $w_{out}$ is found from the imaginary part of this quantity. We find:

$$\text{Im}\left\{\frac{1}{q_{out}}\right\} = -\frac{\lambda}{\pi w_{out}^2} = -\frac{\xi}{\xi^2 + 2.25}.$$  Solving for $w_{out}$, we find:  

$$w_{out} = \frac{\sqrt{\frac{\lambda}{\pi}}}{\xi + 2.25}.$$  Plugging in numbers, we find $w_{out} = 7.5$ mm, which is 2.5 times as large as $w_{in}$. Not surprisingly, the magnification factor is the same as it was last week (when we did the ray optics analysis). To find the radius of curvature, we use the real part:

$$\text{Re}\left\{\frac{1}{q_{out}}\right\} = \frac{1}{R_{out}} = \frac{1.5}{\xi^2 + 2.25}.$$  So we have $R_{out} = 5.8 \times 10^4$ meters, which is large but not infinite. The beam is not (quite) at a focus in this output plane. Instead, since $R_{out} > 0$, we find that it is diverging as it emerges from the 2nd lens.

2. In Gaussian beam theory, the spot size of a beam grows with distance $z$ from the focus like $w_0 \sqrt{1 + z^2/z_R^2}$. In ray optics, the spot size simply increases linearly according to $\theta_{in} z$. In order to compare these two, we need to make sure that we are comparing two situations with the same divergence – that is, we require that the limiting behavior should be the same at large values of $z$. Thus, we replace $\theta_{in}$ in the ray optics expression by $w_0/z_R$. Then, the error in the ray optics prediction (relative to the Gaussian beam prediction) is given by:

$$\text{error} = \frac{\text{Gaussian beam prediction}}{\text{ray optics prediction}} - 1 = \frac{\sqrt{1 + x^2}}{x} - 1$$

where we have defined $x = \frac{z}{z_R}$ for convenience.

This function decreases with increasing $x$, and becomes equal to 0.01 when:

$$\left(\frac{\sqrt{1 + x^2}}{x}\right) = 1.01$$

which is when $x = 7.05$. You need to be a bit more than seven Rayleigh ranges away from a focal point before the ray optics prediction is within 1% of the correct (Gaussian beam) formula.

3. First we calculate the saturation intensity of this gain medium. It is given by $h\nu/\sigma\tau$, where $\nu$ is the frequency of the emission, $\nu = c/\lambda = 2.82 \times 10^{14}$ Hz. We find $I_{\text{sat}} = 9.2 \times 10^6$ watts/m$^2$. We can convert this to a saturation power because we know the spot size inside the laser gain medium:  

$$P_{\text{sat}} = I_{\text{sat}} A = I_{\text{sat}} \pi r^2 = 0.018 \text{ watts}.$$  

Next, we know that the laser produces no output for pump powers less than the threshold, given by $P_{\text{th}} = 0.04$ watts. And we know that, above threshold, the output power varies according to the saturation of the population inversion:

$$P_{\text{out}} = K \frac{\left(P_{\text{pump}} - P_{\text{th}}\right)/P_{\text{sat}}}{1 + \left(P_{\text{pump}} - P_{\text{th}}\right)/P_{\text{sat}}}$$

(expression valid only for $P_{\text{pump}} \geq P_{\text{th}}$)

where $K$ is, for the moment, an unknown proportionality constant. We have calculated $P_{\text{sat}}$, and we’re given $P_{\text{th}}$, so we know everything in this expression except for the constant $K$. We can find this value using the one piece of information that we have not yet used: the slope efficiency. At low pump powers (i.e., just barely above $P_{\text{th}}$), this expression can be approximated as a
straight line, since the denominator of the fraction is approximately equal to one. We find:

\[ P_{\text{out}} \approx K \left( P_{\text{pump}} - P_{\text{th}} \right) / P_{\text{sat}} \quad \text{(just above threshold)} \]

Obviously, the slope of this line is \( K / P_{\text{sat}} \). We require that this value be equal to 0.25 (since the slope efficiency is given as 25\%). Thus \( K = 0.25 P_{\text{sat}} = 0.0045 \) watts.

A plot of the result is shown here. In this figure, the horizontal axis is the pump power (from 0 to 1 watt), and the vertical axis is the output power (in milliwatts).

Notice that the output is zero for pump powers less than \( P_{\text{th}} \), as expected. For low pump powers (just above threshold), the curve is nearly linear, with a slope of 25\%. For higher pump powers, the curve is no longer linear – it droops below the extrapolation of the initial straight line. For the highest pump power of 1 watt, you’re able to get only about 0.045 watts out of this laser. This is what we mean by saturation.

4(a) Using the expressions provided for \( P_{\text{circ}} \) and \( P_{\text{out}} \), along with the values given, we generate the following plots:

The circulating power inside the laser monotonically decreases with increasing output coupling. Thus, it is maximized for \( \delta_{\text{OC}} = 0 \), at a value near 60 Watts (see graph at left). On the other hand, with zero output coupling, we also have zero output power, since there is no output in that case. Also, if the output coupler becomes too large, then the loss is too great, and the laser won’t lase at all. Thus there is some intermediate value of \( \delta_{\text{OC}} \) which gives the maximum output power. That’s why the graph of \( P_{\text{out}} \) has a peak (see graph at right). From the graph, it appears that the maximum of \( P_{\text{out}} \) is near 0.4 W, which occurs at \( \delta_{\text{OC}} \) near 0.04. Note that these are only approximate values, obtained by squinting at the graph. In part (b) of the problem, we find more accurate values by doing math.

For both graphs, these curves are only valid for positive values of power (i.e., above the dashed lines), because negative values would correspond to the laser being below threshold, and therefore not lasing at all.
(b) We saw in (a) that the output power peaks at a particular (non-zero) value of \( \delta_{OC} \). To find an analytic expression for this value, we take the derivative of the expression for \( P_{out} \) with respect to \( \delta_{OC} \):

\[
P_{out} = \frac{\delta_{OC}}{2} \frac{P_{sat}}{\delta_{OC} + \delta_{other}} - 1 \quad \Rightarrow \quad \frac{\partial P_{out}}{\partial \delta_{OC}} = \frac{P_{sat}}{2} \left( \frac{g}{\delta_{OC} + \delta_{other}} - 1 \right) = \frac{\delta_{OC}}{2} \frac{g}{(\delta_{OC} + \delta_{other})^2}
\]

We then set this derivative equal to zero, and solve for \( \delta_{OC} \):

\[
\frac{P_{sat}}{2} \left( \frac{g}{\delta_{OC} + \delta_{other}} - 1 - \frac{g \delta_{OC}}{(\delta_{OC} + \delta_{other})^2} \right) = 0 \quad \Rightarrow \quad g \delta_{other} = (\delta_{OC} + \delta_{other})^2
\]

\[
\delta_{OC} = \sqrt{g \delta_{other}} - \delta_{other}
\]

Then, using this value for \( \delta_{OC} \), we can compute the optimum value for \( P_{out} \):

\[
P_{out} = \frac{P_{sat}}{2} \left( \sqrt{g \delta_{other} - \delta_{other}} \right) \left( \frac{g}{\delta_{other}} - 1 \right)
\]

Plugging numerical values into this expression, we find that the best value for the output coupler is \( \delta_{OC} \approx 0.035 \) (i.e., one of the laser mirrors should reflect 96.5% of the light and transmit 3.5% as our output beam). At this value, the maximum \( P_{out} \) is 0.41 Watts. In the journal article referenced in the problem set, you can see that the authors found a peak output power that is very close to this value.

An interesting aside: for this laser’s best operating conditions (optimal value of output coupling), you have more than 11 watts circulating inside the laser, and you’re getting 410 milliwatts out. This tells you that the laser mirrors and other optical components inside the laser need to be able to handle a significantly higher power than what the laser produces as its output. This is true of most lasers, since most lasers operate with relatively small values of \( \delta_{OC} \).

5. The fractional number of photons remaining in the laser is \( 1 - L \). This must be equal to the exponential decay factor defined in the problem (evaluated at \( t = T_{RT} \)), so that:

\[
\exp \left[ -\frac{T_{RT}}{\tau_c} \right] = 1 - L = 1 - \left( \delta_{OC} + \delta_{other} \right)
\]

From problem 4, the right-hand side of this expression is equal to 0.955. Solve for the cavity lifetime to find: \( \tau_c = 21.8 T_{RT} \) The typical photon makes about 22 round trips before being lost.