Efficient two-dimensional simulations of the fractional Szabo equation with different time-stepping schemes

Fangying Song\textsuperscript{a}, Fanhai Zeng\textsuperscript{a}, Wei Cai\textsuperscript{b}, Wen Chen\textsuperscript{b}, George Em Karniadakis\textsuperscript{a,}\textsuperscript{*}

\textsuperscript{a} Division of Applied Mathematics, Brown University, 182 George St, Providence RI, 02912, United States
\textsuperscript{b} State Key Laboratory of Hydrology-Water Resources and Hydraulic Engineering, Institute of Soft Matter Mechanics, College of Mechanics and Materials, Hohai University, Nanjing 211100, China

\textbf{ARTICLE INFO}

\textbf{Article history:}
Available online 4 December 2016

\textbf{Keywords:}
Direction splitting
Fractional diffusion-wave equation
Wave propagation
Viscoelastic media
Second-order convergence

\textbf{ABSTRACT}

The modified Szabo wave equation is one of the various models that have been developed to model the power law frequency-dependent attenuation phenomena in lossy media. The purpose of this study is to develop two different efficient numerical methods for the two-dimensional Szabo equation and to compare the relative merits of each method. In both methods we employ the ADI scheme to split directions, however, we use different time discretization. Specifically, in the first ADI method (ADI-I) we include a third-order correction term to achieve second-order convergence for smooth solutions, hence extending the work of Sun and Wu (2006). In the second ADI method (ADI-II), we employ the scheme in Zeng et al. (submitted for publication) to two dimensional fractional wave equation using multiple correction terms to enhance accuracy for non-smooth solutions. Our simulation results show that both methods are computationally efficient for the fractional wave equation but have different advantages in terms of accuracy. Specifically, ADI-II seems to produce more accurate results than ADI-I for non-smooth solutions. However, for smooth solutions and fractional order close to two, ADI-I seems to outperform ADI-II.

© 2016 Elsevier Ltd. All rights reserved.

1. Introduction

The attenuation of wave energy in lossy media $\gamma(\cdot)$ exhibits a frequency dependency characterized by a power law [1–4]

$$\gamma(\omega) = \gamma_0|\omega|^\alpha,$$  \hfill (1)

in which $\gamma_0$ and $\alpha$ are empirical parameters obtained by fitting measured data. It has been found that when the sound travels in lossy media, such as biological tissues and sediment, the exponent $\alpha$ falls between (0, 2), as shown in Fig. 1 (i.e., $\alpha$ is the slope). The classical damped wave equation corresponds to $\alpha = 0$, which shows frequency independence. However, in real applications there is a frequency-squared dependent attenuation, which means $\alpha = 2$ [2,3].

\textsuperscript{*} This work was supported by the OSD/ARO/MURI on “Fractional PDEs for Conservation Laws and Beyond: Theory, Numerics and Applications (W911NF-15-1-0562)”.

\textsuperscript{#} Corresponding author.
E-mail address: george_karniadakis@brown.edu (G.E. Karniadakis).

http://dx.doi.org/10.1016/j.camwa.2016.11.018
0898-1221/© 2016 Elsevier Ltd. All rights reserved.
Szabo [2,3] proposed the following equation written in the time-domain for both longitudinal and shear waves with an attenuation term of the form,

$$\frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} u + \frac{2\alpha_0}{c_0} L_{t, \alpha}(u) = \Delta u,$$

where the operator $L_{t, \alpha}(-)$ is defined as follows:

$$L_{t, \alpha}(u) = \begin{cases} 
\frac{\partial}{\partial t} u, & \alpha = 0, \\
-\frac{2 \Gamma(\alpha + 2) \cos((\alpha + 1)\pi/2)}{\pi} \int_0^t \frac{u(\tau)}{(t-\tau)^{\alpha+2}} d\tau, & 0 < \alpha < 2,
\end{cases}$$

and $c_0$ is the sound speed. However, this model contains a convolutional operator which brings in a hypersingularity. Thus, Chen and Holm [4] modified the model by incorporating a positive fractional derivative, which is formulated as

$$\frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} u + \frac{2\alpha_0}{c_0} Q_{t, \alpha}(u) = \Delta u,$$

where the fractional operator $Q_{t, \alpha}(-)$ instead of $L_{t, \alpha}(-)$ is defined as follows:

$$Q_{t, \alpha}(u) = \begin{cases} 
\frac{\partial}{\partial t} u, & \alpha = 0, \\
\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} u, & 0 < \alpha < 2,
\frac{\partial^2}{\partial t^2} u, & \alpha = 2.
\end{cases}$$

The positive fractional derivative is defined by

$$\partial_t^{\alpha+1} u(t) = \begin{cases} 
\frac{1}{(\alpha + 1)\alpha q(\alpha + 1)} \int_0^t \frac{u''(\tau)}{(t-\tau)^{\alpha}} d\tau, & 0 < \alpha \leq 1, \\
-\frac{1}{(\alpha - 1)q(\alpha + 1)} \int_0^t \frac{u'''(\tau)}{(t-\tau)^{\alpha-1}} d\tau, & 1 < \alpha < 2,
\end{cases}$$

where $q(\alpha)$ can be written as

$$q(\alpha) = \frac{\pi}{2 \Gamma(\alpha + 1) \cos((\alpha + 1)\pi/2)}.$$

Thus, the modified Szabo wave equation can also be written as

$$\frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} u + \frac{2\alpha_0}{c_0 \cos(\alpha \pi/2)} \partial_t^{\alpha+1} u = \Delta u.$$

Subsequently, Chen and Holm extended their own work by including the fractional Laplacian operator in space [6]. Kelly et al. [7] also modified the aforementioned wave equation by adding a second time-fractional term to arrive at the power law wave equation. Recently, starting from the characteristic impedance and propagation coefficient, Chen et al. [8]
proposed another time-fractional wave equation for porous media, which, also obeys power law frequency-dependence and guarantees causality. Meerschaert et al. [9] gave the analytical expression for the attenuated wave in anisotropic media. In addition to the above phenomenological models, Holm et al. [10,11] also derived the fractional wave equation from viscoelastic constitutive equations. For the sake of clarity in demonstrating the proposed methods, hereafter, the formulation of the modified Szabo wave equation is formally simplified and written as

\[ \partial_t^\alpha u(x, y, t) + v \partial_x^{\alpha+1} u(x, y, t) = \mu \Delta u(x, y, t) + f(x, y, t). \]  

There are many works about the analytic solution of the fractional diffusion-wave equations. Mainardi [12,13] used the Laplace transform method to obtain the fundamental solution of the fractional diffusion-wave equation. Schneider and Wess [14] derived the corresponding Green's function for the fractional diffusion-wave equation in terms of Fox's H-functions. Agrawal [15] presented a general solution to fractional diffusion-wave equation containing fourth-order spatial derivative defined in bounded domains. To deal with absorbing and reflecting boundary problems, Metzler and Klafter [16] used Fourier–Laplace transform and separation of variables to solve the fractional diffusion equation. Meerschaert et al. [17] also developed stochastic solutions for fractional wave equations. Jiang et al. [18,19] discussed the analytical solutions for the multi-term time-fractional diffusion-wave/diffusion equations and time-space Caputo–Riesz fractional advection–diffusion equations in a finite domain.

In this paper, we mainly focus on the case for which the fractional order falls between (1,2), as it has been proved that the third-order time derivative may lead to an ill-posed problem [32]. We develop two different time discretization schemes for solving such type of equation. The first method is based on the L2 method introduced in [20]. We added a correction term to enforce that the scheme has second-order convergence. Then we employ the Newmark scheme [33] for the standard time derivative. The second method applies the second-order time discretization developed in [5]. We introduced some correction terms on the right-hand side of the scheme. The correction terms play an important role in the improvement of accuracy, and second-order accuracy can be obtained even if the exact solution is non-smooth. The Legendre spectral method is used for the spatial discretization.

The rest of the paper is organized as follows. In Section 2, the two numerical schemes for solving time-fractional wave equation are introduced. Comparisons of numerical simulations for 2D problems with different boundary conditions are presented in Section 3. The conclusions are summarized in Section 4.

2. Numerical schemes

In this section, we consider numerical discretization of the following time-fractional wave equation

\[ \partial_t^\alpha u(x, y, t) + v \partial_x^{\alpha+1} u(x, y, t) = \mu \Delta u(x, y, t) + f(x, y, t), \quad (x, y, t) \in (0, T) \times \Omega, \]

\[ U(x, y, 0) = \psi(x, y), \quad (x, y) \in \bar{\Omega}, \]

\[ \partial_t U(x, y, 0) = \phi(x, y), \quad (x, y) \in \bar{\Omega}, \]

\[ U(x, y, t) = 0, \quad (x, y, t) \in (0, T) \times \partial \Omega, \]

where \( \Omega = \Omega_1 \otimes \Omega_2, \Omega_1 = (-1, 1), \Omega_2 = (-1, 1), \Delta = \partial_x^2 + \partial_y^2, \) and \( 0 < \alpha < 1. \) Here \( \partial_t^\alpha (n - 1 < \gamma < n, n \in \mathbb{N}) \) is the Caputo fractional derivative defined as

\[ \partial_t^\alpha U(x, y, t) = \frac{1}{\Gamma(n - \gamma)} \int_0^t (t - s)^{n-\gamma-1} \partial_t^{\alpha} U(x, y, s) \, ds. \]  

Next, we propose two ADI Galerkin spectral methods [34–36] to solve (10), in which two kinds of time discretization approaches are applied.

2.1. ADI spectral method I (ADI-I)

We propose the first ADI method in this subsection. Let \( \tau \) be the time step size and \( n_\tau \) be a positive integer with \( \tau = T/n_\tau \) and \( t_n = n \tau (n = 1, \ldots, n_\tau). \) For simplicity, we also denote \( U(t) = U(\cdot, t), U^n = U(t_n) = U(\cdot, t_n) \) and assume
\( U^{-1} = U^0 - \tau \phi(-) \). Then the Caputo derivative operator with fractional order \( \alpha + 1 \) \((0 < \alpha < 1)\) can be discretized by

\[
\left[ \partial_t^{1+\alpha}U(t) \right]_{t=t_0} = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_0} \partial_s^2 U(s)(t_0 - s)^{-\alpha} \, ds \\
= \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_0 - s)^{-\alpha} \partial_s^2 U(s) \, ds \\
= \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_0 - s)^{-\alpha} \left[ \frac{U^{k+1} - 2U^k + U^{k-1}}{\tau^2} \\
- \frac{U^{k+1} - 3U^k + 3U^{k-1} - U^{k-2}}{3} \frac{t_{k+1} + t_k + t_{k-1} - 3s}{\tau^2} \right] ds + R^\alpha \\
= \frac{1}{\Gamma(3-\alpha)\tau^{\alpha+1}} \sum_{k=0}^{n-1} \left[ a_k U^{n-k+1} + b_k U^{n-k} + c_k U^{n-k-1} + d_k U^{n-k-2} \right] + R^\alpha,
\]

where \( R^\alpha = O(\tau^2) \) when \( U(t) \) is sufficiently smooth in time and

\[
a_k = -(2 - \alpha)k^{1-\alpha} - k^{2-\alpha} + (k + 1)^{2-\alpha}, \\
b_k = 2(2 - \alpha)k^{1-\alpha} + (2 - \alpha)(k + 1)^{1-\alpha} + 3k^{2-\alpha} - 3(k + 1)^{2-\alpha}, \\
c_k = -(2 - \alpha)k^{1-\alpha} - 2(2 - \alpha)(k + 1)^{1-\alpha} - 3k^{2-\alpha} + 3(k + 1)^{2-\alpha}, \\
d_k = (2 - \alpha)(k + 1)^{1-\alpha} + k^{2-\alpha} - (k + 1)^{2-\alpha}.
\]

**Remark 2.1.** Here, \( U(s) \) is interpolated with the following function, \( \forall s \in [t_{k-1}, t_k], \) \( k = 1, \ldots, n_T \)

\[
U^k_j(s) = U^k - \frac{U^k - U^{k-1}}{\tau}(t_k - s) + \frac{U^{k+1} - 2U^k + U^{k-1}}{2\tau^2}(t_k - s)(t_{k-1} - s) - \frac{U^{k+1} - 3U^k + 3U^{k-1} - U^{k-2}}{6\tau^3}(t_k - s)(t_{k-1} - s)(t_{k-2} - s).
\]

Taking the second-order derivative of the function \( U^k_j(s), \forall s \in [t_{k-1}, t_k], \) we obtain

\[
\partial_s^2 U(s) = \partial_s^2 U^k_j(s) + \frac{\partial^4 U(\xi)}{4!\xi^4} \left( 6s^2 - 6\tau(2k - 1)s + \tau^2(6k^2 - 6k - 1) \right), \quad t_{k-1} < \xi < t_k.
\]

Then we obtain the scheme (12). In Ref. [37], the authors propose the cubic interpolation polynomial in the Hermite form for \( k = 1 \) (i.e. \( t = t_1 \))

\[
U^1_j(s) = U^0 + \phi(s - t_0) + \frac{1}{\tau}(U^1 - U^0) - \phi(s - t_0)^2 + \frac{1}{2\tau} \left[ \frac{1}{\tau}(U^1 - U^0) - \phi \right] (s - t_0)^2 - \frac{1}{2\tau} \left[ \frac{1}{\tau}(U^2 - U^1) - \frac{1}{\tau}(U^1 - U^0) \right] (s - t_0)^3.
\]

and prove that such a scheme has second-order convergence.

Letting \( t = t_n \) in (10), we obtain

\[
\partial_t^n U(t_n) + v \left[ \partial_t^{n+1} U(t) \right]_{t=t_n} = \mu \Delta U(t_n) + f(t_n).
\]

Applying the central difference method and (12) to the second-order time derivative and the time-fractional derivative in the above equation, respectively, and using \( U(t_n) = \frac{1}{4} \left( U(t_{n+1}) + 2U(t_n) + U(t_{n-1}) \right) + O(\tau^2) \), we obtain

\[
\partial_t^n U^n + \frac{v}{\Gamma(3-\alpha)\tau^{\alpha+1}} \sum_{k=0}^{n-1} \left[ a_k U^{n-k+1} + b_k U^{n-k} + c_k U^{n-k-1} + d_k U^{n-k-2} \right] \\
= \frac{\mu}{4} \left( \Delta U^{n+1} + 2\Delta U^n + \Delta U^{n-1} \right) + f^n + O(\tau^2)
\]

where \( \partial_t^n U^n = \frac{U^{n+1} - 2U^n + U^{n-1}}{\tau^2} \).
In order to obtain the ADI scheme, we let \( \delta_t U^{n+\frac{1}{2}} = \frac{U^{n+1} - U^n}{\tau} \) and add a perturbation term

\[
\frac{\mu^2 \tau^3}{16 (1 + \frac{\nu^1 - \alpha}{\tau (3 - \alpha)})} \partial_y^2 \partial_x^2 \delta_t U^{n+\frac{1}{2}} = O(\tau^3)
\]

to both sides of \((17)\) that leads to

\[
\delta_t^2 U^n + \frac{\nu}{\Gamma(3 - \alpha)\tau^{n+1}} \sum_{k=0}^{n-1} \left[ a_k U^{n-k+1} + b_k U^{n-k} + c_k U^{n-k-1} + d_k U^{n-k-2} \right]
+ \frac{\mu^2 \tau^3}{16 (1 + \frac{\nu^1 - \alpha}{\tau (3 - \alpha)})} \partial_y^2 \partial_x^2 \delta_t U^{n+\frac{1}{2}} = \frac{\mu}{4} (\Delta U^{n+1} + 2 \Delta U^n + \Delta U^{n-1}) + f^n + O(\tau^2).
\]

(18)

From (18), we present the following fully discrete ADI-I spectral method for \((10)\) as: Find \( u^n_N \in P_N(\Omega) \cap H^1_0(\Omega) \) \((n = 2, \ldots, n_T)\) such that

\[
(\delta_t^2 u^n_N, v) + \frac{\nu}{\Gamma(3 - \alpha)\tau^{n+1}} \sum_{k=0}^{n-1} \left[ a_k (u^n_{N-k+1}, v) + b_k (u^n_{N-k}, v) + c_k (u^n_{N-k-1}, v) \right]
+ d_k (u^n_{N-k-2}, v) + \frac{\mu^2 \tau^3}{16 (1 + \frac{\nu^1 - \alpha}{\tau (3 - \alpha)})} \partial_x \partial_y (u^n_{n+\frac{1}{2}, k}) \partial_y \partial_y (u^n_{n+\frac{1}{2}, k}) = \frac{-\mu}{4} (\nabla u^{n+1}_N + 2 \nabla u^n_N + \nabla u^{n-1}_N, \nabla v) + (I_N f^n, v), \quad \forall v \in P_N(\Omega) \cap H^1_0(\Omega),
\]

(19)

where \( u^0_N = I_N \psi, u^1_N = u^0_N - \tau I_N \phi, u^1_N = u^0_N + \tau I_N \phi, P_N(\Omega) = P_{N_1}(l_1) \otimes P_{N_2}(l_2), N = (N_1, N_2), \) and \( P_k(l) \) denotes the polynomial space defined on the interval \( l \) with degree no greater than \( K \), and \( I_N \) is the Legendre–Gauss–Lobatto interpolation operator defined by

\[
(I_N u)(x_k, y_l) = u(x_k, y_l), \quad k = 0, 1, \ldots, N_1, \quad l = 0, 1, \ldots, N_2, \quad u \in C(\Omega)
\]

in which \( \{x_k\} \) and \( \{y_l\} \) are the Legendre–Gauss–Lobatto points on \( l_1 \) and \( l_2 \), respectively.

Next, we present the matrix form of the ADI-I method. Let \( \phi^1_k(x) = L_k(x) - L_{k+1}(x) \), where \( L_k(x) \) is the Legendre polynomial. Then \( \{\phi^1_k(x), k = 0, 1, \ldots, N_1 - 2\} \) are the basis of \( P_{N_1}(l_1) \otimes H^1_0(l_2) \). We can similarly define the basis of \( P_{N_2}(l_2) \otimes H^1_0(l_1) \) as \( \{\phi^2_l(y), l = 0, 1, \ldots, N_2 - 2\} \). Hence, the basis of \( P_N(\Omega) \cap H^1_0(\Omega) \) can be expressed as

\[
\phi_{k,l}(x, y) = \phi^1_k(x) \phi^2_l(y), \quad k = 0, 1, \ldots, N_1 - 2, \quad l = 0, 1, \ldots, N_2 - 2.
\]

Let \( u^n_N = \sum_{k=0}^{N_1-2} \sum_{l=0}^{N_2-2} c^n_{k,l} \phi_{k,l} \). Inserting \( u^n_N \) into (19) and letting \( v = \phi_{k,l} \) for \( k = 0, 1, \ldots, N_1 - 2, l = 0, 1, \ldots, N_2 - 2 \), we obtain

\[
\left( q_0 M_x + \frac{\mu^2 \tau^2}{q_0} S_x \right) C^{n+1} = q_0 M_y + \frac{\mu^2 \tau^2}{q_0} S_y \right)^T = RHS^n,
\]

(20)

where \( q_0 = \sqrt{1 + \frac{\nu^1 - \alpha}{\tau (3 - \alpha)}} \), \( (C^n) = c^n_{k,l} \), and \( RHS^n \) is given by

\[
RHS^n = M_k (2C^n - C^{n-1}) M_y^T - \frac{\nu^1 - \alpha}{\Gamma(3 - \alpha)} \sum_{k=2}^{n} a_k M_x C^{n-k+2} M_y^T
+ \sum_{k=1}^{n} M_x (b_k C^{n-k+1} + c_k C^{n-k+1} + d_k C^{n-k-1} M_y^T)
+ \frac{\mu^2 \tau^2}{4} \left[ 2(S_x C^n M_y^T + M_x C^n S_y^T) + (S_x C^{n-1} M_y^T + M_x C^{n-1} S_y^T) \right] + \frac{1}{16q_0^2} S_x C^n S_y^T + F^n,
\]

in which \((F^n) = (I_N f^n, \phi_{k,l})\), and the mass matrices \( M_x, M_y \) and stiff matrices \( S_x, S_y \) are given by

\[
(M_x) = (\phi^1_k, \phi^1_l), \quad (M_y) = (\phi^2_k, \phi^2_l),
(S_x) = (\partial_x \phi^1_k, \partial_x \phi^1_l), \quad (S_y) = (\partial_y \phi^2_k, \partial_y \phi^2_l).
\]

(21)
The matrix equation (20) can be solved efficiently with the following two steps:

(i) Solve $M_1C^*_n = RHS^n$ to get $C^*_n$, where $M_1 = q_0M_x + \frac{\mu^2}{q_0}S_x$;

(ii) Solve $M_2(C^{n+1})^T = (C^*_n)^T$ to get $C^{n+1}$, where $M_2 = q_0M_y + \frac{\alpha^2}{q_0}S_y$.

2.2. ADI spectral method II (ADI-II)

In this subsection, we apply the second-order time discretization developed in [5] to the time discretization of (10) to derive the desired ADI algorithm. The second-order difference method in [5] for the Caputo derivative operator of order $\alpha$ ($0 < \alpha < 1$) reads

$$\left[ \partial_t^\alpha U(t) \right]_{t=n} \approx A_{n,1}^{\alpha,m} U_n = \frac{1}{\tau^\alpha} \sum_{k=0}^{n} g_k^{(\alpha)} U^k + \frac{1}{\tau^\alpha} \sum_{r=1}^{m} w_{n,r}^{(\alpha)} U^r,$$

where the coefficients $\{g_k^{(\alpha)}\}$ are given by

$$g_0^{(\alpha)} = \frac{2 + \alpha}{2} w_0^{(\alpha)}, \quad g_k^{(\alpha)} = \frac{2 + \alpha}{2} w_k^{(\alpha)} - \frac{\alpha}{2} w_{k-1}^{(\alpha)}$$

in which $w_k^{(\alpha)} = (-1)^k \binom{\alpha}{k}$. The starting weights $\{w_{n,r}^{(\alpha)}\}$ are chosen such that (22) is exact for some $U(t) = t^\sigma$ ($r = 1, 2, \ldots, m$), which can be obtained by solving the following linear system

$$w_{n,1}^{(\alpha)} + 2\sigma w_{n,2}^{(\alpha)} + \cdots + m\sigma w_{n,m}^{(\alpha)} = \frac{\Gamma(\sigma_r + 1)}{\Gamma(\sigma_r + 1 - \alpha)} n^{\sigma_r - \alpha} - \sum_{k=0}^{n} g_k^{(\alpha)} (n - k)^\sigma.$$

The correction terms $\frac{1}{\tau^\alpha} \sum_{r=1}^{m} w_{n,r}^{(\alpha)} U^r$ on the right-hand side of (22) play an important role in the improvement of accuracy, and second-order accuracy can be obtained even if $U(t)$ is non-smooth, see e.g. [5].

Let $V = \partial_t U$. Then Eq. (10) can be written as

$$\partial_t V + \nu \partial_y^\alpha (V(t) - V(0)) = \mu \Delta U + f(x, y, t).$$

As in [5], the time direction of (24) together with $V = \partial_t U$ can be discretized as follows

$$\delta_t V^{n+\frac{1}{2}} + \frac{1}{\tau} \sum_{r=1}^{m_1} u_{n,r} \mu \partial_y^\alpha (V^{n+1} - V^n) = \mu \Delta U^{n+\frac{1}{2}} + f^{n+\frac{1}{2}} + R_1^n,$$

$$\delta_t U^{n+\frac{1}{2}} + \frac{1}{\tau} \sum_{r=1}^{m_1} u_{n,r} (U^r - U^n - t_r V^0) = V^{n+\frac{1}{2}} + R_2^n,$$

where $\delta_t U^{n+\frac{1}{2}} = (U^{n+1} - U^n)/\tau$, $V^{n+\frac{1}{2}} = \frac{1}{2} (V^{n+1} + V^n)$, $\mu \partial_y^\alpha (V^{n+1} - V^n)$, $\mu \partial_y^\alpha (V^n - V^0)$, and $R_1^n(i = 1, 2)$ are truncation errors from the time discretization, satisfying

$$R_1^n = O(\tau^2 t_n^{\alpha_2 + 1 - \frac{\alpha_2}{2}}) + O(\tau^2 t_n^{\alpha_3 + 1 - \frac{\alpha_3}{2}}), \quad R_2^n = O(\tau^4 t_n^{\alpha_4 + 1 - \frac{\alpha_4}{2}}).$$

In order to obtain the ADI scheme, we add a perturbation term

$$\frac{\mu^2 \tau^2}{8 \left(2 + v g_0^{(\alpha)} \tau ^{1-\alpha}\right)} \partial_y^2 \partial_y^2 (V^{n+1} - V^n) = O(\tau^4 t_n^{\alpha_5 + 1})$$

to both sides of the first equation of (25), which leads to

$$\delta_t V^{n+\frac{1}{2}} + \frac{1}{\tau} \sum_{r=1}^{m_1} u_{n,r} \mu \partial_y^\alpha (V^{n+1} - V^n) + \frac{\mu^2 \tau^3}{8 \left(2 + v g_0^{(\alpha)} \tau ^{1-\alpha}\right)} \partial_y^2 \partial_y^2 (V^{n+1} - V^n) = \mu \Delta U^{n+\frac{1}{2}} + f^{n+\frac{1}{2}} + O(\tau^4 t_n^{\alpha_5 + 1 - \frac{\alpha_5}{2}}) + O(\tau^2 t_n^{\alpha_2 + 1 - \frac{\alpha_2}{2}}) + O(\tau^2 t_n^{\alpha_3 + 1 - \frac{\alpha_3}{2}}) + O(\tau^4 t_n^{\alpha_4 + 1 - \frac{\alpha_4}{2}}).$$
From the second equation of (25) and Eq. (27), we can derive the ADI-II Legendre Galerkin spectral method for (10) as: Find $u_{N,n^1,n^2}^{n+1}, v_{N,n^1,n^2}^{n+1} \in V_N$ for $n = 0, 1, 2, \ldots, n_T - 1$, such that
\[
\left( \delta_t u_{N,n^1,n^2}^{n+1}, v \right) + \frac{1}{\tau} \sum_{r=1}^{m_2} u_{n,r}(\tilde{v}_{r}^{n+1/2}, v) + \frac{\mu^2 \tau^3}{8 \left( 2 + v_{g_0(\alpha)}^n \right) \tau^1 - \alpha} \left( \partial_\alpha \partial_y (v_{N,n}^{n+1} - v_{N,n}^{n}), \partial_\alpha \partial_y v \right) = \left( k_h f_{N,n}^{n+1/2}, v \right), \quad v \in V_N^0.
\]
(28)

\[
\delta_t u_{N,n^1,n^2}^{n+1} + \frac{1}{\tau} \sum_{r=1}^{m_1} u_{n,r}(u_{N,n}^{n} - u_{0}^{n} - t_r v_{N,n}^{0}) = v_{N,n}^{n+1/2},
\]
where $\tilde{v}_{r}^{n} = v_{N,n}^{n} - v_{0}^{n}, v_{N,n}^{0} = I_h V_{0}$. 

**Theorem 2.1** ([5]). Suppose that $u_{N,n}^{n}$ and $v_{N,n}^{n}$ $(n = 0, 1, \ldots, n_T)$ are solutions to (28). If $\sigma_{m_1} \leq 3$ and $\sigma_{m_2}, \sigma_{m_3} \leq 4$, then there exists a positive constant $C$ independent of $n$, $\tau$ and $N$ such that
\[
\| u_{N,n}^{n} \|^2 + \mu \| \nabla u_{N,n}^{n} \|^2 + \frac{\mu^2 \tau^3}{8 \left( 2 + v_{g_0(\alpha)}^n \right) \tau^1 - \alpha} \| \partial_\alpha \partial_y v_{N,n}^{n} \|^2 \\
\leq C \left( \| v_{N,n}^{n} \|^2 + \mu \| \nabla u_{N,n}^{n} \|^2 + \| \nabla v_{N,n}^{n} \|^2 + \tau^3 \| \partial_\alpha \partial_y v_{N,n}^{n} \|^2 + \sum_{r=1}^{m_1} \left\| \nabla \delta_t v_{N,n}^{n} \right\|^2 \\
+ \sum_{r=1}^{m_2} \| \delta_t u_{N,n}^{n} \|^2 \right) + \tau \sum_{k=0}^{m_3} \| f^{k} \|^2.
\]
(29)

**Theorem 2.2** ([5]). Suppose that $n$, $n_T$ and $r$ are positive integers with $0 \leq n \leq n_T$ and $U(t) = U(x,y,t)$ is the solution to (10) satisfying $U(t) - U(0) - t \delta_t U(0) = \sum_{r=1}^{m_1} c_r t^{\sigma_r} + u(t) t^{\sigma_{m+1}}, \sigma_r < \sigma_{m+1}, u(t) \in C([0, T])$ for each $(x,y)$, $V(x,y,t) = \delta_t U(x,y,t), u_{N,n}^{n}$ and $v_{N,n}^{n}$ are the solutions to the scheme (28), respectively, $m_1, m_2, m_3 \leq m$ with $\sigma_{m_1} \leq 3$, $\sigma_{m_2}, \sigma_{m_3} \leq 4$. For fixed $t$, $U(t) \in H^1_{0}(\Omega) \cap H^2(\Omega)$, and $f \in C(0, T; H^1(\Omega))$. Then for small enough $\tau$, there exists a positive constant $C$ independent of $n$, $\tau$ and $h$, such that
\[
\left\| \nabla (u_{N,n}^{n} - U(t_n)) \right\| \leq C \left( \tau^{\min\{2,\sigma_{m_1}+1-1,\sigma_{m_2}+1-2,\sigma_{m_3}+1-1-\alpha\}} + N^{1-r} \right),
\]
\[
\left\| v_{N,n}^{n} - V(t_n) \right\| \leq C \left( \tau^{\min\{2,\sigma_{m_1}+1-1,\sigma_{m_2}+1-2,\sigma_{m_3}+1-1-\alpha\}} + N^{1-r} \right).
\]

3. Comparison of different time schemes for 2D problems for different boundary conditions

To demonstrate the effectiveness of the aforementioned numerical algorithms, two numerical examples for two-dimensional problems with different boundary conditions are presented here.

**Example 3.1.** Consider the following two-dimensional fractional wave equation
\[
\partial^2 U + \partial^{\alpha+1} U = (\partial_x^2 + \partial_y^2) U + f(x,y,t), \quad (x,y,t) \in \Omega \times (0, T],
\]
(30)
where $\Omega = (-1, 1) \times (-1, 1)$ and $0 < \alpha < 1$.

- Case I: Choose a suitable source term $f(x,y,t)$. The boundary and initial conditions are $U(x,y,t) = \sin(\pi x) \sin(\pi y), (x,y,t) \in \partial \Omega \times (0, T]$ and $U(x,y,0) = \hat{u}(U(x,y,0) = \sin(\pi x) \sin(\pi y), (x,y) \in \hat{\Omega}$, such that the exact solution to (30) is
\[
U(x,y,t) = \left( t^5 + t^4 + t^3 + t^2 + t + 1 \right) \sin(\pi x) \sin(\pi y).
\]
- Case II: Choose a suitable source term $f$, the suitable boundary and initial conditions $U(x,y,t) = \sin(\pi x) \sin(\pi y), (x,y,t) \in \partial \Omega \times (0, T]$ and $U(x,y,0) = \hat{u}(U(x,y,0) = \sin(\pi x) \sin(\pi y), (x,y) \in \hat{\Omega}$, such that the analytical solution of (30) is given by
\[
U(x,y,t) = \left( \sum_{k=0}^{3} \frac{\sin(\pi x) \sin(\pi y)}{\Gamma(k+1-\alpha)+3} + 1 \right) \sin(\pi x) \sin(\pi y).
\]
(31)

We choose such a non-smooth solution to mimic the non-smooth solutions of real applications. For example, given a smooth input, the analytical solution is non-smooth, and the first several terms behave like (31), see [38].
In this example, we first apply the previous two algorithms to solve 3.1 for Case I, where we have the exact solution, which is smooth. We set $N = (64, 64)$ in the two numerical algorithms; the $L^2$ error $\| e_n^I \| = \| u_n^I - U(t_n) \|$ at $T = 1$ (i.e., $n_T = 1/\tau$) and the maximum $L^2$ error is defined as $\max_{0 < n \leq n_T} \| e_n \|$ are shown in Tables 1 and 2, respectively. We see that the $L^2$ error and the maximum $L^2$ error of the method ADI-II has second-accuracy in time for different fractional orders when $m = 1, 2$, while for $m = 0$, less accurate results are derived, especially for $\alpha \to 1$. It is observed that numerical results show better performance than the theoretical analysis. For method ADI-I, second-order accuracy in time is obtained for $\alpha = 0.2, 0.5, 0.8, 0.9$.

For Case II, we will show that the ADI-III method with correction terms exhibits better numerical solutions than the ADI-I method. We present the maximum $L^2$ errors and the $L^2$ errors of the two methods ADI-I and ADI-II in Tables 3 and 4, respectively. We can see that the ADI-II method with even one correction term achieves better numerical solutions than the ADI-I method. When the fractional order $\alpha$ is close to 1, i.e., $\alpha = 0.9$, we do not observe second-order accuracy from the ADI-II method with two correction terms, but the numerical accuracy is improved. In real applications, just a few correction terms are enough to obtain satisfactory accuracy; see explanation in [5].

We also present the pointwise errors of the two methods ADI-I and ADI-II for Case II in Fig. 2. One important observation is that the error of the ADI-II method at the origin $t = 0$ decreases as the number of the correction terms increases, which is important in order to obtain higher accurate numerical solutions when $t$ is far from $t = 0$. Specifically, the big errors of the numerical solutions near $t = 0$ may degrade the accuracy of the numerical solutions far from $t = 0$, which can be partly illustrated in Fig. 2(c)–(d) for $m = 0$.

Example 3.2. In this example, we extend the previous two ADI spectral methods ADI-I and ADI-II to solve the following two-dimensional fractional wave equation with mixed boundary conditions:

$$\frac{\partial^2 U}{\partial t^2} + \partial_t^{\alpha + 1} U = (\partial_x^2 + \partial_y^2)U, \quad (x, y, t) \in \Omega \times (0, T),$$

$$U(x, y, 0) = U_0, \quad \partial_t U(x, y, 0) = 0, \quad x \in \Omega,$$

$$U(x, -1, t) = \partial_t U(x, 1, t) = \partial_x U(\pm 1, x, t) = 0,$$

where $\Omega = (-1, 1) \times (-1, 1)$ and $0 < \alpha < 1$.

The above system is a simple model of the wave propagation in homogeneous media [39]. The initial condition $U_0 = U(x, y, 0)$ is chosen as

$$U_0 = \begin{cases} \frac{(y + 0.875)^2}{0.125^2}, & -0.5 \leq x \leq 0.5, \quad -0.5 \leq y \leq -0.75, \\ 0, & \text{otherwise}, \end{cases}$$

$N = (128, 128)$, and $m = 0$. Numerical solutions and the relative errors of the two methods at $t = 1.5$ with different $\alpha$ are shown in Fig. 3. We can see that consistent numerical solutions are obtained by the two methods. Fig. 3 shows that the wave speed is decreased when the fractional order $\alpha$ is increased. The equation coincides with a standard wave equation when $\alpha = 1$. In addition, both methods take the same computation cost for solving a Helmholtz equation with spectral method in space. The memory requirements of the two methods are also the same. The two schemes are efficient for the fractional
wave equation with the exact solution if it is sufficiently smooth. The second method still keeps second-order convergence in the case that the solution is non-smooth but overall ADI-I achieves lower levels of errors.

4. Summary

We mainly focus on the numerical investigation of the modified Szabo wave equation. Two different time stepping schemes are proposed to solve the fractional wave equation with space performed by the spectral method.

For smooth solutions, the ADI-I method achieves second-order convergence in time for any fractional order, while the convergence of the ADI-II method without correction terms depends on the fractional order. When $\alpha$ tends to one, the ADI-I method outperforms the ADI-II method without correction terms. For non-smooth solutions, the ADI-II method with a few correction terms outperforms the ADI-I method. Theoretically, the ADI-II method can achieve second-order accuracy for both smooth and non-smooth solutions by utilizing suitable correction terms. This may cause numerical difficulties due to the stiffness that these terms introduce; however, only a few terms are necessary in real applications. Specifically, we need only several correction terms to achieve high accurate numerical solutions, see numerical results in Tables 3 and 4 and also explanations in [5]. We compare the numerical solutions of the two methods by solving the wave equation subject to the
Table 4
The $L^2$ error at $T = 2$ for Case II, $N = (64, 64)$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$1/\tau$</th>
<th>$m = 0$ Order</th>
<th>$m = 1$ Order</th>
<th>$m = 2$ Order</th>
<th>ADI-I Order</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>32</td>
<td>1.375e-4</td>
<td>1.863e-4</td>
<td>1.408e-4</td>
<td>3.979e-4</td>
<td>2.03</td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>3.333e-5</td>
<td>4.894e-5</td>
<td>3.953e-5</td>
<td>9.734e-5</td>
<td>2.02</td>
</tr>
<tr>
<td>0.5</td>
<td>128</td>
<td>7.396e-6</td>
<td>1.197e-5</td>
<td>1.023e-5</td>
<td>2.392e-5</td>
<td>2.02</td>
</tr>
<tr>
<td></td>
<td>256</td>
<td>1.542e-6</td>
<td>2.912e-6</td>
<td>2.592e-6</td>
<td>5.889e-6</td>
<td>2.02</td>
</tr>
<tr>
<td></td>
<td>512</td>
<td>2.950e-7</td>
<td>7.033e-7</td>
<td>6.520e-7</td>
<td>1.447e-6</td>
<td>2.02</td>
</tr>
<tr>
<td>0.8</td>
<td>32</td>
<td>1.547e-4</td>
<td>1.521e-4</td>
<td>7.097e-5</td>
<td>3.278e-4</td>
<td>2.14</td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>9.311e-5</td>
<td>3.742e-5</td>
<td>2.187e-5</td>
<td>7.432e-5</td>
<td>2.20</td>
</tr>
<tr>
<td>0.9</td>
<td>128</td>
<td>4.187e-5</td>
<td>8.068e-6</td>
<td>6.092e-6</td>
<td>1.616e-5</td>
<td>2.20</td>
</tr>
<tr>
<td></td>
<td>256</td>
<td>1.694e-5</td>
<td>1.464e-6</td>
<td>1.637e-6</td>
<td>3.222e-6</td>
<td>2.32</td>
</tr>
<tr>
<td></td>
<td>512</td>
<td>6.511e-6</td>
<td>1.562e-7</td>
<td>4.308e-7</td>
<td>5.900e-7</td>
<td>2.63</td>
</tr>
<tr>
<td>0.2</td>
<td>32</td>
<td>4.210e-4</td>
<td>3.493e-4</td>
<td>5.405e-5</td>
<td>1.105e-4</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>4.731e-4</td>
<td>1.006e-4</td>
<td>1.957e-5</td>
<td>5.211e-6</td>
<td>0.47</td>
</tr>
<tr>
<td>0.5</td>
<td>128</td>
<td>2.625e-4</td>
<td>2.476e-5</td>
<td>7.074e-6</td>
<td>7.397e-6</td>
<td>0.94</td>
</tr>
<tr>
<td></td>
<td>256</td>
<td>1.249e-4</td>
<td>5.044e-6</td>
<td>2.312e-6</td>
<td>5.344e-6</td>
<td>0.94</td>
</tr>
<tr>
<td></td>
<td>512</td>
<td>5.605e-5</td>
<td>6.067e-7</td>
<td>6.884e-7</td>
<td>2.776e-6</td>
<td>0.94</td>
</tr>
<tr>
<td>0.8</td>
<td>32</td>
<td>5.789e-4</td>
<td>4.713e-4</td>
<td>9.383e-5</td>
<td>5.598e-5</td>
<td>0.72</td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>4.322e-4</td>
<td>1.423e-4</td>
<td>2.300e-5</td>
<td>7.378e-6</td>
<td>0.94</td>
</tr>
<tr>
<td>0.9</td>
<td>128</td>
<td>3.448e-4</td>
<td>3.824e-5</td>
<td>7.203e-6</td>
<td>1.078e-5</td>
<td>0.72</td>
</tr>
<tr>
<td></td>
<td>256</td>
<td>1.888e-4</td>
<td>9.057e-6</td>
<td>2.316e-6</td>
<td>6.551e-6</td>
<td>0.72</td>
</tr>
<tr>
<td></td>
<td>512</td>
<td>9.270e-5</td>
<td>1.681e-6</td>
<td>7.064e-7</td>
<td>3.318e-6</td>
<td>0.98</td>
</tr>
</tbody>
</table>

Fig. 2. The pointwise errors for Case II, $N = (64, 64)$. (a)–(d) $L^2$ error versus time for time step $\tau = \frac{1}{512}$ and different fractional orders.
(a1) $\alpha = 0.2$.  

(b1) $\alpha = 0.2$.  

(a2) $\alpha = 0.5$.  

(b2) $\alpha = 0.5$.  

(a3) $\alpha = 0.8$.  

(b3) $\alpha = 0.8$.  

Fig. 3. The contours of the numerical solutions for Example 3.2 by ADI-I (left) and the corresponding difference of the two methods (right), $\tau = T/256$, $T = 1.5$, $N = (128, 128)$.  

mixed boundary conditions, see Example 3.2. We find that both methods are effective for the numerical simulation of the time-fractional wave equation.
Acknowledgment

Wei Cai was also supported by the China Scholarship Council (CSC).

References