Coarse Resolution Turbulence Simulations With Spectral Vanishing Viscosity—Large-Eddy Simulations (SVV-LES)

We present a new implementation of the spectral vanishing viscosity method appropriate for alternative formulations of large-eddy simulations. We first review the method and subsequently present results for turbulent incompressible channel flow.

[DOI: 10.1115/1.1511321]

Introduction

Turbulence simulations using monotonicity preserving schemes initially considered homogeneous turbulence, employing both PPM and FCT-type algorithms, [1,2], but more recently emphasis has shifted to wall-bounded flows, [3,4]. Unlike other strictly monotonic discretizations (monotone schemes) of nonlinear conservation laws which are total-variation-diminishing (TVD) and thus at most first-order accurate everywhere (see theorem of LeVeque and Goodman in two dimensions, [5]), the PPM and FCT algorithms employ nonlinear limiters and guarantee monotonicity locally while preserving at least second-order accuracy both in phase and amplitude, [6,7]. These schemes honor the weaker total-variation-bounded (TVB) or other maximum principle conditions, which may allow for small amplitude oscillations. It is worth mentioning that sign-preserving schemes are monotonicity preserving and can be of high order; e.g., see [8]. The intriguing feature of the monotonically integrated LES (or MILES) approach (see [9], also [2], and references therein) is the activation of the limiter on the convective fluxes and its role in generating implicitly a tensorial form of eddy viscosity that acts to stabilize the flow and suppress oscillations. It was reported in [2] that if the resolution is fine enough to ensure that the cutoff wave number lies in the inertial range, then the simulation results seem to be independent of the generated viscosity.

In the aforementioned PPM and FCT algorithms for convection, use of nonlinear limiters or reconstruction procedures is in some form equivalent to adding diffusion to the hyperbolic conservation laws so that entropy dissipation is created, and hence a unique solution is obtained (see [10]). If the discretization lacks entropy dissipation, Gibbs oscillations are produced which eventually render the solution unstable. In convection-dominated high Reynolds number flows the situation is analogous. However, this mechanism is implicit and although the induced artificial diffusion may scale with the local resolution as \( \propto (\Delta x)^4 \), \( s > 1 \), it is an uncontrollable process that may compromise the solution accuracy. This conflict between monotonicity and accuracy, first analyzed by Godunov [11], was revisited by Tadmor [12] who has developed the first theoretical result on the convergence and stability of spectral approximations for nonlinear conservation laws. A revised formulation for polynomial spectral methods was presented more recently in [13]. Specifically, Tadmor introduced artificial dissipation via the spectral vanishing viscosity (SVV), which is sufficiently large to suppress oscillations, yet small enough not to affect the solution accuracy. In the context of spectral discretizations, for example, SVV can be viewed as a compromise between the classical TVB viscosity approximation and the exponentially accurate yet unstable spectral approximation.

The spectral vanishing viscosity approach guarantees an essentially nonoscillatory behavior although some small oscillations of bounded amplitude may be present in the solution. This theory is based on three key components:

1. a vanishing viscosity amplitude, which decreases with increasing resolution;
2. a viscosity-free spectrum for the lower, most energetic modes; and
3. an appropriate viscosity kernel for the high wave numbers.

If hierarchical discretizations are employed, the combined formulation inherits the scale dependence attempted by other authors, e.g., in the multiscale variational method of Hughes [14] or in the nonlinear Galerkin method of Temam [15]. On the other hand, monotonicity of the TVB kind is preserved, but the high-frequency regularization employed is controlled by parameters whose range is given directly by the theory.

In previous work, [16], the SVV approach was used for simulating incompressible turbulent flows using multidomain spectral methods based on the spectral/hp Galerkin approach, see [17]. The unfiltered Navier-Stokes equations enhanced on the right-hand side with a spectrally vanishing viscous operator were used. Although reasonably successful, the previous implementation was limited in two ways:

- First, the SVV filtering was accomplished on the \( C^0 \) basis, which is hierarchical but only semi-orthogonal.
- Second, the SVV implementation did not discriminate between fully resolved and unresolved regions, thereby possibly applying dissipation in regions where it was not needed.

In the current paper we address these issues by presenting two new enhancements to SVV:

1. First, we present a new SVV filtering for the continuous Galerkin method in which filtering is accomplished on a fully orthogonal set of modes.
2. Second, we propose a method to compute adaptively the viscosity amplitude according to the local strain.

The Spectral Vanishing Viscosity Method

Static Implementation. Tadmor [12] first introduced the concept of spectral vanishing viscosity (SVV) using the inviscid Burgers equation

\[
\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} \left( \frac{u^2(x,t)}{2} \right) = 0, \tag{1}
\]

Contributed by the Fluids Engineering Division for publication in the JOURNAL OF FLUIDS ENGINEERING. Manuscript received by the Fluids Engineering Division March 18, 2002; revised manuscript received May 29, 2002. Associate Editor: F. F. Grinstein.

Robert M. Kirby
E-mail: kirby@cs.utah.edu

George Em Karniadakis
E-mail: gk@cfm.brown.edu

Division of Applied Mathematics,
Brown University,
Providence, RI 02912

Vol. 124, DECEMBER 2002

Copyright © 2002 by ASME

Transactions of the ASME
subject to prescribed initial and boundary conditions. The distinct feature of solutions to this problem is that spontaneous jump discontinuities (shock waves) may be developed, and hence a class of weak solutions can be admitted. Within this class, there are many possible solutions, and in order to single out the physically relevant one an additional entropy condition is applied, of the form

$$\frac{\partial}{\partial t} \left( \frac{u^2(x,t)}{2} \right) + \frac{\partial}{\partial x} \left( \frac{u^3(x,t)}{3} \right) \leq 0. \quad (2)$$

In practical applications, spectral methods are often augmented with smoothing procedures in order to reduce the Gibbs oscillations, [18], associated with discontinuities arising at the domain boundaries or due to underresolution. However, with nonlinear problems, convergence of the Fourier method, for example, may fail despite additional smoothing of the solution. Tadmor [12] introduced the spectral vanishing viscosity method, which adds a small amount of controlled dissipation that satisfies the entropy condition, yet retains spectral accuracy. It is based on viscosity solutions of nonlinear Hamilton-Jacobi equations, which have been studied systematically in [19]. Specifically, the viscosity solution for the Burgers equation has the form

$$\frac{\partial}{\partial t} u_N(x,t) + \frac{\partial}{\partial x} \left( u_N^2(x,t) \right) = \varepsilon \frac{\partial}{\partial x} \left[ Q_N \frac{\partial u_N}{\partial x} \right], \quad (3)$$

where $\varepsilon = 0$ is a viscosity amplitude and $Q_N$ is a viscosity kernel, which may be nonlinear and, in general, a function of $x$. Convergence may then be established by compactness estimates combined with entropy dissipation arguments, [12]. To respect spectral accuracy, the SVV method makes use of viscous regularization, and Eq. (3) may be rewritten in discrete form (retaining $N$ modes) as

$$\frac{\partial}{\partial t} u_N(x,t) + \frac{\partial}{\partial x} \left( u_N^2(x,t) \right) = \varepsilon \frac{\partial}{\partial x} \left[ Q_N \frac{\partial u_N}{\partial x} \right], \quad (4)$$

where the star (*) denotes convolution and $P_N$ is a projection operator. $Q_N$ is a (possibly nonlinear) viscosity kernel, which is only activated for high wave numbers. In Fourier space, this kind of spectral viscosity can be efficiently implemented as multiplication of the Fourier coefficients of $u_N$ with the Fourier coefficients of the kernel $Q_N$, i.e.,

$$\varepsilon \frac{\partial}{\partial x} \left[ Q_N \frac{\partial u_N}{\partial x} \right] = -\varepsilon \sum_{k \in [1,N]} (k^2 \hat{Q}(k) \hat{u}(k) e^{ikx},$$

where $k$ is the wave number, $N$ the number of Fourier modes, and $M$ the wave number above which the spectral vanishing viscosity is activated.

Originally, Tadmor [12] used

$$\hat{Q}_k = \begin{cases} 0, & |k| \leq M \\ 1, & |k| > M \end{cases}, \quad (5)$$

with $\varepsilon M = 0.25$ based on the consideration of minimizing the total variation of the numerical solution. In subsequent work, however, a smooth kernel was used, since it was found that the $C^0$ smoothness of $\hat{Q}_k$ improves the resolution of the SVV method. For Legendre pseudo-spectral methods, Maday, Kaber, and Tadmor [20] used $\varepsilon = N^{-1}$, activated for modes $k > M = 5 \sqrt{N}$, with

$$\hat{Q}_k = e^{-\frac{(k-N)^2}{2(M-M^2)^2}}, \quad k > M. \quad (6)$$

In order to see the difference between the convolution operator on the right-hand side in Eq. (4) and the usual viscosity regularization, following Tadmor [21] we expand as

$$\varepsilon \frac{\partial}{\partial x} \left[ Q_N \frac{\partial u_N}{\partial x} \right] = \varepsilon \frac{\partial^2 u_N}{\partial x^2} - \varepsilon \frac{\partial}{\partial x} \left[ R_N \frac{\partial u_N}{\partial x} \right],$$

where

![Fig. 1 Normalized viscosity kernels for the spectral vanishing viscosity (dash line $C=0$ and solid line $C=5$) and the Kraichnan/Chollet-Lesieur viscosity (dashed-dot line)](image_url)

Fig. 1 shows both viscosity kernels normalized by $C = 0$ and $C = 5$, and the Kraichnan/Chollet-Lesieur viscosity (dashed-dot line). The extra term appearing in addition to the first standard viscosity term makes this method different. It measures the distance between the spectral (vanishing) viscosity and the standard viscosity. This term is bounded in the $L_2$ norm similar to the spectral projection error. We refer to the viscosity as vanishing as the theory requires that

$$\varepsilon \approx \frac{1}{N^\theta \log N}, \quad 0 < \theta < 1$$

and thus $\varepsilon = 0$ for high wave numbers. In more recent work, Tadmor and his collaborators refer to it as simply spectral viscosity but this terminology may be confused with the one used by Lesieur and his group, [22].

At this point it is also instructive to compare the spectral vanishing viscosity to the spectral eddy-viscosity introduced by Kraichnan [23] and Chollet-Lesieur [22,24]. The latter has the nondimensional form, [24],

$$\nu(k/N) = K_{0}^{-\frac{1}{2}} \left[ 0.441 + 15.2 \exp(-0.03N/k) \right],$$

where $K_{0} = 2.1$.

Comparing the Fourier analog of this eddy viscosity employed in LES, [22], to the viscosity kernel $Q_N(k,M,N)$ introduced in the SVV method, Fig. 1 shows both viscosity kernels normalized by their maximum value at $k = N$. For SVV two different values of the cutoff wave number are considered,

$$M = \sqrt{N} \quad \text{for} \quad C = 0 \quad \text{and} \quad C = 5. \quad (7)$$

This range has been used in most of the numerical experiments so far (see, for example [16,20]) and is consistent with the theoretical results in [12]. In the plot it is shown that, in general, the two
forms of viscosity have similar distributions but the SVV form does not affect the first one-third or one-half of the spectrum (viscosity-free portion), and it increases faster than the Kraichnan/Chollet-Lesieur eddy viscosity in the higher wave numbers range, e.g., in the second half of the spectrum.

**Continuous Galerkin Discretization.** In previous work, [16], the SVV concept was implemented in the context of a modal spectral/hp discretization in which a C^0 formulation was used for the simulation of incompressible flows, [17]. The SVV filtering was accomplished within the context of the C^0 basis. Although the basis is hierarchical, the C^0 continuity involved in the Galerkin projection destroys partially the orthogonality of the basis. A new modification compared to previous work is now the filtering operation is applied to an orthogonal basis which results from a “rotation” of the semi-orthogonal basis. Equation (8) shows the difference in these two operations. For the SVV as implemented in [16], the Q operator which acts directly on the semi-orthogonal basis was used. In the new implementation, the Q operator is used which filters on a fully orthogonal set of basis functions. This is accomplished element-by-element by first rotating from the local nonorthogonal basis to a corresponding orthogonal basis spanning the same polynomial space; the filtering is then applied, and the resulting coefficients transformed back to the local basis.

\[ u_N = \sum_{i=1}^{N} \tilde{a}_i \phi_i \]

\[ Q u_N = \sum_{i=1}^{N} \tilde{a}_i \phi_i \]

\[ \tilde{Q} u_N = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^{-1} \tilde{a}_j \sum_{k=1}^{N} a_{jk} \phi_i \]

Technical details are provide in [25], where it is also shown that this new operator remains symmetric and semi-positive definite.

**Dynamic Implementation.** In the dynamic approach the viscosity amplitude in the SVV kernel varies as a function of space and time. We first apply this idea to the inviscid Burgers equation and subsequently we implement it in the context of Navier-Stokes equations. We thus rewrite the inviscid Burgers equation, and subsequently we implement it in the context of Navier-Stokes equations. A revised formulation of SVV for polynomial spectral methods involving application of the Q kernel at two stages was presented more recently in [13]. We adapt the proposed form and rewrite the inviscid Burgers equation in strong form with a modified SVV kernel as follows:

\[ \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u^2}{\partial x} = c(x,t) \frac{\partial^2}{\partial x^2} Q u \]

where

\[ c(x,t) = \epsilon + \frac{\kappa}{N} \frac{u_{ij}(x,t)}{u_{ij}(x,t)} \]

Here \( \kappa \) is a scalar that can be determined by optimizing the quality of the solution as we shall see below. In other words, we have incorporated the solution into determining the magnitude of the viscosity through the coefficient \( c(x,t) \). In this one-dimensional case, we have employed a normalized gradient to accomplish this. This form of \( c(x,t) \) is meant to be analogous to the adaptive coefficient for the Navier-Stokes equations \( C(x,t) = v_e(x,t)/\nu \) (where \( \nu \) is the physical viscosity and \( v_e \) is the eddy viscosity) proposed in [16]. The use of the rate of strain tensor in the computation of \( v_e \) is mimicked by the use of the magnitude of the first derivative for this one-dimensional example.

For ease of implementation, we have chosen to use the discontinuous Galerkin formulation for this investigation of Burgers equation. We first examine how the dynamic coefficient affects the quality of the solution. First, we evaluate the dynamic coefficient \( c(x,t) \) a posteriori from the numerical solution of inviscid Burgers without any viscosity treatment (Fig. 2: case A). We then compute \( c(x,t) \) using the global \( \| u_{ij}(x,t) \|_\infty \), i.e., max norm obtained across all elements, involving the inviscid Burgers equation with dynamic SVV applied to it (Fig. 2: case B). Finally, we compute \( c(x,t) \) using local \( \| u_{ij}(x,t) \|_\infty \), i.e., max norm obtained in each element, involving the inviscid Burgers equation with dynamic SVV applied to it (Fig. 2: case C). We observe in Fig. 2 that if the global \( \| u_{ij}(x,t) \|_\infty \) is used in the definition of \( c(x,t) \), the form of the \( c(x,t) \) when dynamic SVV is acting on the system is the same in shape as when \( c(x,t) \) is obtained a posteriori. When a local definition of \( \| u_{ij}(x,t) \|_\infty \) is used, however, the form of \( c(x,t) \) greatly changes. From this we conclude that

- to effectively utilize dynamic SVV, a global scaling quantity such as \( \| u_{ij}(x,t) \|_\infty \) taken over the entire domain must be used.

To understand the effect of the scaling parameter \( \kappa/N \) (where \( N \) is the number of modes on an individual element), we performed a comparison of static SVV (\( c(x,t) = \kappa/N \)) versus dynamic SVV (\( c(x,t) \) as given in Eq. (10)). The inviscid Burgers equation was solved with the added SVV term. Five equally spaced elements spanning the interval \([-1,1]\) were used, each element containing \( N = 16 \) modes. Comparisons of the \( L_2 \) error and the \( L_\infty \) error for different values of \( \kappa \) are presented in Table 1. Several observations can be made based on our studies:

Table 1 Comparison of \( L_2 \) and \( L_\infty \) errors for the inviscid Burgers equation using dynamics SVV with \( M=8; \epsilon=1/16 \)

<table>
<thead>
<tr>
<th>Form</th>
<th>( L_2 ) Error</th>
<th>( L_\infty ) Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inviscid</td>
<td>0.133395</td>
<td>0.17006</td>
</tr>
<tr>
<td>Static SVV</td>
<td>0.131833</td>
<td>0.44753</td>
</tr>
<tr>
<td>Local SVV ( \kappa = 1 )</td>
<td>0.155219</td>
<td>0.099105</td>
</tr>
<tr>
<td>Global SVV ( \kappa = 1 )</td>
<td>0.133296</td>
<td>1.15936</td>
</tr>
<tr>
<td>Global SVV ( \kappa = 5 )</td>
<td>0.131130</td>
<td>1.04961</td>
</tr>
<tr>
<td>Global SVV ( \kappa = 10 )</td>
<td>0.140145</td>
<td>0.775494</td>
</tr>
</tbody>
</table>
where

\[ |S| = \sqrt{\text{Tr}(S_jS_j)} \]

\[ S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]

and \( \rho \) is the local density. As not to affect the flow at the wall we incorporate the Panton function, [26], given by

\[ g(y^+) = \frac{2}{\pi} \tan^{-1}\left( \frac{2ky^+}{\pi} \right) \left[ 1 - \exp\left( -\frac{y^+}{C} \right) \right]^2 \]

where all quantities are expressed in viscous wall units denoted by +. This function is multiplied pointwise by the coefficient \( c(x,t) \).

In Figs. 3 and 4 we plot, respectively, a segment of the mesh around an airfoil and contours of the SVV amplitude computed using the aforementioned procedure. The spectral/hp element simulation is for a two-dimensional flow past an airfoil at 10 deg angle of attack. The mesh involved 912 quadrilateral elements with sixth-order \( N=7 \) polynomial interpolation. It is clear that the SVV is nonzero in regions of high vorticity which are the most probable candidates for underresolution.

**SVV-LES Coarse Resolution Simulations**

The effectiveness of SVV in simulations of turbulent flows using low resolution has been first demonstrated in [16]. Here, we revisit this problem using the aforementioned modification of the continuous Galerkin SVV operator to study the effect of the viscosity amplitude \( \epsilon \) and the wave number cutoff \( M \) on the solution quality. Specifically, we apply the \( \bar{Q} \) kernel as in Eq. (8) which filters on an orthogonal trial basis instead of the semi-orthogonal basis employed previously in [16].

Channel flow at \( Re=180 \) is simulated, with periodic boundary conditions in the streamwise and spanwise directions following the benchmark solutions of Kim, Moin, and Moser [27]. The mesh used here is the same as in [16], but the resolution was doubled in the streamwise direction and was substantially reduced in the crossflow plane. Specifically, the size of the computational domain was \( L_x=5, L_y=2, \) and \( L_z=2 \). In contrast to the previous simulations in [16], we increased the streamwise resolution to 32 Fourier modes (64 points) to test more carefully the effect of SVV acting only the crossflow planes and not in the streamwise direction. In the Fourier direction a 3/2 de-aliasing rule was applied for all simulations. The spectral element mesh has 25 elements \( N=7 \) polynomial interpolation. It is clear that the resolution in the wall-normal direction in the current simulations involves only 35 points!

In Fig. 6 we plot the mean-velocity profiles versus the distance from the wall, and in Fig. 7 we plot the turbulence intensities versus the distance from the wall for four different cases. The

\[ \text{Fig. 3} \quad \text{Segment of the mesh used for simulating the flow past an airfoil at 10 deg angle of attack and Re=10,000} \]

\[ \text{Fig. 4} \quad \text{Amplitude at one time instance of spectral vanishing viscosity in flow past an airfoil at 10 deg angle of attack and Re =10,000} \]
symbols denote the DNS of Kim, Moin, and Moser [27]. First, we obtained converged (in-time) statistics without SVV using the aforementioned low resolution. The corresponding results underestimate the mean velocity at the centerline ($\bar{U}/u_t = 17.67$ versus $\bar{U}/u_t = 18.2$ in [27]) as shown in Fig. 6 (solid line). Examining the turbulent intensities, the corresponding results underestimate the streamwise velocity component and overestimate the cross-flow, as shown in Fig. 7 (solid line). In our initial runs, we tried two modifications. First, we applied polynomial overintegration which effectively removes any aliasing in the crossflow direction, but the results remained effectively the same. Secondly, we applied the new SVV operator with the default parameters $(M = 5, \varepsilon = 1/8)$, which also gave results similar to the untreated case. We then experimented with several combinations of the SVV parameters $(M, \varepsilon)$. The SVV kernel is scaled with the given physical viscosity, so the actual term included in the Navier-Stokes equation is proportional to $Re^2\varepsilon$. The best results for the turbulent intensities are shown in Fig. 7 (dash-dot line) corresponding to $(M = 5, \varepsilon = 5/8)$. This set of parameters yields a mean velocity at the centerline $\bar{U}/u_t = 17.9$ as shown in Fig. 6 (dash-dot line). The other curves in the Figs. 6 and 7 correspond to $(M = 2, \varepsilon = 1/8)$ (dot) and $(M = 5, \varepsilon = 9/8)$ (dashed). The $(M = 2, \varepsilon = 1/8)$ case shows improvement in both the turbulent intensities and mean-velocity profile compared to the untreated case (solid line); however, dissipation is being added over a larger number of modes compared to the $(M = 5, \varepsilon = 5/8)$ case. Observe that for $(M = 5, \varepsilon = 9/8)$ too much dissipation has been added to the system, and hence the solution overestimates the streamwise velocity component as shown in Fig. 7. This case, however, yields the best mean-velocity profile with $\bar{U}/u_t = 18.15$ at the centerline. Although case specific, these results confirm the theoretical results that only the upper one-third of the spectrum should be treated with SVV, and that there is an optimum (but unknown) viscosity amplitude level. We propose computing dynamically the viscosity amplitude level using the methodology outlined earlier. Future work will consist of comparisons of static and dynamic SVV for high Reynolds number flow.

Acknowledgments

We acknowledge Mr. Igor Pivkin for his assistance in running the simulations. This work was supported by the Computational Mathematics program of AFOSR. Computations were performed on the IBM SP3 at MHPCC and NPACI.

Fig. 5 Mesh in the crossflow plane for turbulent channel flow at $Re_t=180$

Fig. 6 Mean-velocity profile for the turbulent channel flow. The symbols correspond to the benchmark solutions of Kim, Moin, and Moser [27]. The solid line corresponds to the underresolved DNS, the dotted line to $(M=2,\varepsilon=1/8)$, the dot-dashed line to $(M=5,\varepsilon=5/8)$, and the dashed line to $(M=5,\varepsilon=9/8)$.

Fig. 7 Turbulence intensities for the turbulent channel flow. The symbols correspond to the benchmark solutions of Kim, Moin, and Moser [27]. The solid line corresponds to the underresolved DNS, the dotted line to $(M=2,\varepsilon=1/8)$, the dot-dashed line to $(M=5,\varepsilon=5/8)$, and the dashed line to $(M=5,\varepsilon=9/8)$. 

Fig. 8
References


