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Parallel Simulations of Vortex-Induced Vibrations in Turbulent Flow: Linear and Non-Linear Models

by

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B.A., University of Cambridge, 1993
Sc.M., Brown University, 1994

Thesis
Submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in the Division of Applied Mathematics at Brown University

May 1999
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• C. Evangelinos and G.E. Karniadakis *Communication in Prism 3D* presented during a visit to IBM-Poughkeepsie, May 1996.

Acknowledgments

This thesis is dedicated to my loving parents, Ioannis and Irini Evangelinou and the love of my life, Asimina Georges. None of this would have been able without their unreserved love and continuous support. I’ve been away from my family for nine years by now. My parents have been a constant source of encouragement, supporting me in any way possible. Meeting Asimina has been the best thing that happened to me while at Brown. She’s affected my life in a myriad ways, and helped me get through difficulties I couldn’t have faced alone.

A huge thanks goes to my thesis advisor, George Em Karniadakis, for his direction of this project, ideas and support over the years. Thanks are also due to Didier Lucor that is continuing the project in the CRUNCH group and helped with the data analysis and the runs in the final stages. My work started from where that of David Newman left off and a lot of the direction of the first part of the thesis is owed to him. Dave helped me get jump-started on the project and I’m grateful for that. The code base of Prisim and Nektar were written by Ron Henderson and Spencer Sherwin with Tim Warburton respectively, and many thanks are due their way.

I’d like to acknowledge my readers, Prof. Chau-Hsing Su and Martin Maxey for their help and enormous patience with the rate that I was giving them the draft manuscript before the thesis defense. I also need to stress how important Prof. Su’s contributions were towards the non-linear modeling in the second part of this thesis.

This project was supported by the Office of Naval Research, under the supervision of Dr. T.F. Swean.

The work was funded under the following grants:

- AFOSR: F49620-93-1-0090
- DOE: DE-FG02-95ER25239
- ONR: N00014-95-1-0256
- NSF: CTS-9417520
Contents

1 Introduction ................................................. 1
  1 Vortex-Induced Vibrations ..................................... 1
  2 Parallel Simulations of VIV ................................... 3
  3 Thesis Outline ........................................... 4

I Linear Models .............................................. 6

2 Models for Beams and Cables ................................. 7
  1 Introduction ........................................... 7
  2 Modeling the structure ..................................... 7
     2.1 Non-dimensionalization .................................. 8
     2.2 Linear equations for a flexible cylinder ................. 9
     2.3 Initial and Boundary conditions ......................... 12
  3 Solving for the structure .................................. 15
  4 Solving for the flow ...................................... 16
     4.1 Limitations of the mapping approach ................... 19
  5 The coupled flow-structure problem ......................... 20
  6 Summary ................................................... 22

3 Transition Regime ........................................ 23

vii
## II Non-Linear Models

### 5 Non-Linear Strings

1. Introduction ................................................. 94
2. Modeling the structure ....................................... 95
   2.1 Non-dimensionalization ................................. 95
   2.2 Non-linear equations for a string under tension ...... 96
   2.3 Modeling the tension .................................... 98
   2.4 The linear limit and first-order corrections .......... 100
   2.5 Initial and Boundary conditions ....................... 102
3. Solving for the structure .................................... 104
   3.1 Numerical scheme .................................... 104
   3.2 Validation ........................................... 105
4. Solving for the flow .......................................... 111
   4.1 Formulation ........................................... 112
5. The coupled flow-structure problem ......................... 112
   5.1 Implementation issues ................................ 114
   5.2 The problem of multiple time scales ................. 118
   5.3 Comparing the mapping and the ALE formulations ...... 119

### 6 Model Simulations

1. Introduction ................................................. 124
2. Simulation parameters ..................................... 125
3. Initial Conditions ......................................... 129
4. VIV of a non-linear string ................................ 130
   4.1 String response: Displacement and Tension ........ 130
   4.2 Hydrodynamic Forces ................................ 136
   4.3 Comparison with a simulation using a linear structural model .... 140
4.4 Flow Visualizations ........................................ 142
5 A more realistic simulation .................................. 144
6 Summary ..................................................... 151

A Parallel Algorithms and Implementation Issues 152
1 Introduction ................................................. 152
2 Spectral/hp Discretizations on Unstructured and Hybrid Grids ....................................... 154
   2.1 Local coordinate Systems ............................... 155
   2.2 Spectral Hierarchical Expansions ..................... 157
   2.3 Time Integration Algorithm ............................ 160
3 The $NekTarF$ code ........................................ 162
   3.1 Fourier Decomposition ................................ 162
   3.2 Two-Dimensional Helmholtz solves .................... 164
   3.3 Communications in $NekTarF$ .......................... 166
4 The $NekTar3D$ code ...................................... 170
   4.1 Parallelization of $NekTar3D$ ......................... 172
   4.2 Parallel validation ..................................... 176
5 Communication Timings .................................... 178
6 Perspective .................................................. 182
   6.1 Hardware ............................................... 182
   6.2 Software ............................................... 183
   6.3 Migration Problems .................................... 184
   6.4 Applications to Dynamic DNS of Turbulence ......... 185
List of Tables

3.1 Summary of cross-flow displacements at $Re = 200$: $V_c$ and $V_b$ is the reduced velocity calculated using the frequency of the 1st structural mode of vibration. The amplitude of $y/d$ is calculated after the initial transients die out. .......................... 29

4.1 Comparison of simulation results with experimental results for flow past a stationary cylinder at $Re = 1000$. Shown are the values of the Strouhal number $St$, the base pressure coefficient $-C_{pb}$, the average drag coefficient $C_D$ and the standard deviation of the value of the coefficient of drag $\bar{C}_D$ and lift $\bar{C}_L$ respectively. ........ 44

4.2 Summary of time- and span-averaged amplitude, lift and drag coefficients at lock-in.
(Zero structural damping is assumed and $Re = 1000$.) ....................... 88

6.1 Lift coefficient amplitude $C_l'$ and root mean square value $C_{l_{rms}}$; drag coefficient average $C_d$ and amplitude $C_d'$ (about the average) and Strouhal number $St$ for 2D $Re = 100$ calculations using the 110 element 2D mesh. ......................... 128

A.1 Communication patterns, corresponding message lengths and frequency per timestep
in NekTARF. ......................................................... 170
List of Figures

1  Flow past a freely vibrating flexible cylinder with pinned ends at $Re = 100$. Shown is the spanwise distribution of the amplitude of the cross-flow motion $y/d$ versus non-dimensional time. For clarity isocontours at $y/d = \pm 0.4$ are used which are far below the maximum value of 1 observed ........................................... 14

2  The coordinate system is attached to the moving flexible cylinder, producing an undeformed, stationary computational domain ...................................................... 17

3  Spectral element mesh in the $x-y$ plane; Fourier expansions are used in the periodic spanwise direction. Close up (bottom) of the discretization around the cylinder for polynomial order 8 ............................................................ 25

2  3D 110 element mesh used in some of the calculations: Deformed macro-skeleton in the $z$-direction (top) and view of the planes comprising the mesh (bottom). Only every 8th plane is shown, and the scale on the $z$-axis is $1/\pi$ times that on the other two ...................................................... 26

3  Cable cross-flow displacement ($y/d$), lift coefficient ($C_l$) and drag coefficient ($C_d$) versus non-dimensional time and spanwise distance ($z/d$) along cable for $L/d = 12.6$ and $Re = 100$. (Courtesy of D.J. Newman) ............................................. 28

4  Cross-flow displacement ($y/d$), lift coefficient ($C_l$) and drag coefficient ($C_d$) versus non-dimensional time and spanwise distance ($z/d$) at lock-in. Cable (left) and Beam (right), $Re = 200$ ................................................................. 29

xii
5 Time history and corresponding spectrum of the cable (2 top plots) and beam (2 lower plots) displacement at lock-in, Re = 200. ........................................... 30
6 Cross-flow displacement (top) and midspan displacement values and spectrum (bottom) for a vibrating beam with a reduced velocity $V_c = 7.48$, Re = 200. ....... 31
7 Cross-flow displacement (top) and midspan displacement values and spectrum (bottom) for a vibrating cable with a reduced velocity $V_c = 6.65$, Re = 200. ........ 32
8 Cross-flow displacement (top) and midspan displacement values and spectrum (bottom) for a vibrating beam at lock-in, Re = 300. ................................. 33
9 Cross-flow displacement (top) and midspan displacement values and spectrum (bottom) for a vibrating cable at lock-in, Re = 300. ................................. 34
10 Slices in the x-y plane showing spanwise vorticity at Re = 100. ................. 35
11 Cable: Slices in the x-y plane showing spanwise vorticity at Re = 200. .......... 36
12 Beam: Slices in the x-y plane showing spanwise vorticity at Re = 200. .......... 37

1 Hybrid grid in the x-y plane; Fourier expansions are used in the periodic spanwise direction. Close up of the discretization around the cylinder (upper), and variable expansion order (polynomial order + 1) per element (lower). ................................. 41
2 Fluctuating lift and drag coefficients for flow past a stationary cylinder at Re = 1000: Pressure forces (top) and viscous forces (bottom). ................................. 43
3 Two-dimensional simulations at Re = 100 (left) and Re = 1000 (right). Top: Cross-flow displacement history; Middle: Lift coefficient history; Bottom: Phase portrait. ......................................................... 46
4 Rigid cylinder: Cross-flow displacement versus time (left) and span-averaged lift coefficient versus cross-flow displacement (right). ................................. 47
5 Rigid cylinder: Top - Lift coefficient along the span versus time; Middle - Drag coefficient along the span versus time; Bottom - Energy exchange along the span versus time. ......................................................... 49
Flexible cylinder subject to forced oscillation: Standing wave pattern (top) and lift coefficient (bottom).

Cable: Cross-flow displacement along the span versus time. The initial part of the simulation is shown at the top and a later part at the bottom.

Cable - demodulation analysis: Phase difference between the cross-flow displacement and the lift coefficient (top); amplitude of the lift coefficient (bottom). The sharp discontinuities are due to the fact that individual phases are calculated modulo 180 degrees before their difference is calculated (also modulo 180 degrees).

Beam: Cross-flow displacement along the span versus time. The initial part of the simulation is shown at the top and a later part at the bottom.

Beam - demodulation analysis: Phase difference between the cross-flow displacement and the lift coefficient (top), and the amplitude of the lift coefficient (bottom).

Short beam: Cross-flow displacement along the span versus time. Top: periodic ends; Bottom: fixed ends.

Long beam: Cross-flow displacement along the span versus time. Top: periodic ends; Bottom: fixed ends.

Short beam: Lift coefficient along the span versus time. Top: periodic ends; Bottom: fixed ends.

Long beam: Lift coefficient along the span versus time. Top: periodic ends; Bottom: fixed ends.

Short beam: Drag coefficient along the span versus time. Top: periodic ends; Bottom: fixed ends.

Long beam: Drag coefficient along the span versus time. Top: periodic ends; Bottom: fixed ends.

Standard deviation of the displacement along the span for a short beam (left) and a long beam (right). The solid line is the rigid line response and the dash line is the traveling wave response.
Standard deviation of the lift coefficient along the beam for a short beam (left) and a long beam (right). The solid line is the rigid line response and the dash line is the traveling wave response. .................................................. 61

Variation of the mean drag coefficient along the beam for a short beam (left) and a long beam (right). The solid line is the rigid line response and the dash line is the traveling wave response. ......................... 61

Standard deviation of the lift coefficient along the beam for a short beam (left) and a long beam (right). ................................................................. 62

Comparison of span-averaged lift coefficient histories for stationary cylinder (top left), freely oscillating rigid cylinder (top right), short beam with free ends (middle left), long beam with free ends (middle right), short beam with fixed ends (bottom left) and long beam with fixed ends (bottom right). ..................... 63

Comparison of span-averaged drag coefficient histories for stationary cylinder (top left), freely oscillating rigid cylinder (top right), short beam with free ends (middle left), long beam with short ends (middle right), short beam with fixed ends (bottom left) and long beam with fixed ends (bottom right) ..................... 64

Mean drag coefficient along the beam for a short beam. Circles denote DNS data, stars denote the prediction by equation (2), squares denote the prediction by equation (3) for $A = 0.29$ and $B = 1.79$, and diamonds denote the prediction by equation (3) for $A = 0.35$ and $B = 1$ .... 65

Autocorrelation function for the streamwise component of velocity in the near-wake at a centerline point ($z/d = 3; y/d = 0$) (left) and an off-centerline point ($z/d = 3; y/d = 1$) (right). .................................................. 67

Autocorrelation function for the cross-flow component of velocity in the near-wake at a centerline point ($z/d = 3; y/d = 0$) (left) and an off-centerline point ($z/d = 3; y/d = 1$) (right). .................................................. 67
26 Autocorrelation function for the spanwise component of velocity in the near-wake at a centerline point \((x/d = 3; y/d = 0)\) (left) and an off-centerline point \((x/d = 3; y/d = 1)\) (right). ................................................................. 68

27 Energy spectra (streamwise velocity) for stationary \((s)\), freely oscillating rigid cylinder \((r)\), cable \((c)\) and beam \((b)\) at a point in the wake at centerline \((x/d = 3, y/d = 0)\) and off-centerline \((x/d = 3, y/d = 1)\). ................................................................. 70

28 Energy spectra (cross-flow velocity) for stationary \((s)\), freely oscillating rigid cylinder \((r)\), cable \((c)\) and beam \((b)\) at a point in the wake at centerline \((x/d = 3, y/d = 0)\) and off-centerline \((x/d = 3, y/d = 1)\). ................................................................. 70

29 Energy spectra (spanwise velocity) for stationary \((s)\), freely oscillating rigid cylinder \((r)\), cable \((c)\) and beam \((b)\) at a point in the wake at centerline \((x/d = 3, y/d = 0)\) and off-centerline \((x/d = 3, y/d = 1)\). ................................................................. 71

30 Cable (left) and Beam (right): Time-history of the (square) amplitude of the first 4 modes of cross-flow displacement. ................................................................. 72

31 Vorticity contours at \(t \approx 122\). (two-dimensional simulation, \(Re = 1000\)). Only the near-wake is shown for clarity. ................................................................. 74

32 Vorticity contours at \(t \approx 136\). (two-dimensional simulation, \(Re = 1000\)). Only the near-wake is shown for clarity. ................................................................. 74

33 Vorticity contours at \(t \approx 176\). (two-dimensional simulation, \(Re = 1000\)). Only the near-wake is shown for clarity. ................................................................. 74

34 Vorticity contours at \(t \approx 122\). (two-dimensional simulation, \(Re = 1000\)). .... 75

35 Vorticity contours at \(t \approx 136\). (two-dimensional simulation, \(Re = 1000\)). .... 75

36 Vorticity contours at \(t = 300\). (two-dimensional simulation, \(Re = 500\)). The near wake shown at the top exhibits the \(P + S\) pattern and the whole domain shown at the bottom shows the pattern persists for a while until the vortex street breaks down. ................................................................. 76
DNS of flow past a flexible beam at Reynolds number 1,000. Top left: Total vorticity magnitude at levels 3.0 and 5.5; Top right: Spanwise vorticity at −1.7 and 1.7; Bottom left: Pressure isosurfaces at −0.18 and −0.24. Bottom right: Isocontour at −1.3 of the second largest eigenvalue using the technique of Jeong & Hussain (1995). All values are non-dimensionalized with respect to the flexible cylinder diameter and the freestream velocity.

Rigidly moving cylinder at \( Re = 300 \): Pressure isocontours at values −0.3 and −0.4. View facing the outflow from the side (top) and facing the inflow from the side (bottom).

Rigidly moving cylinder at \( Re = 1000 \): Pressure isocontours at values −0.3 and −0.2. View facing the outflow from the side (top) and facing the inflow from the side (bottom).

Cable at \( Re = 200 \): Pressure isocontours at values −0.3 and −0.35 .

Beam at \( Re = 1000 \): Pressure isocontours at values −0.25 and −0.18 (\( t = 508.09 \)).

Cable at \( Re = 1000 \): Pressure isocontours at values −0.15 and −0.12 (\( t = 508.09 \)).

Cable at \( Re = 1000 \): Pressure isocontours at values −0.15 and −0.12 (\( t = 487.34 \)).

Beam at \( Re = 1000 \): Top - Pressure isocontours at values −0.24 and −0.18 (\( t = 490.09 \)). Bottom - Pressure isocontours at values −0.24 and −0.18 (\( t = 532.09 \)).

Cable at \( Re = 1000 \): Lift coefficient along the span versus time.

Standing wave response for a (short) beam with pinned endpoints at \( Re = 1000 \): Pressure isocontours at values −0.1 and −0.2. The different snapshots correspond to times one-fifth of the shedding cycle apart (top-left; top-right; bottom-left; bottom-right).

Low resolution simulation for the short beam (case As) with only two modes along the span. Amplitude along the span versus time.
Deviation from the steady state catenary position in the vertical (top) and longitudinal (bottom) direction. $\delta S = 0.01, \delta t = 0.0025, T_r = 2, EA_r = 10, m_r = 3, g = 1$.

Deviation from the steady state catenary position in the horizontal (top), vertical (middle) and longitudinal (bottom) direction. $\delta S = 0.01, \delta t = 0.0025, T_r = 2, EA_r = 10, m_r = 3, g = 1$ with a horizontal "uniform drag" force of magnitude $\frac{1}{2} m$. 

Displacement in the horizontal (top), vertical (middle) and longitudinal (bottom) direction. $\delta S = 0.01, \delta t = 0.0025, T_r = 6, EA_r = 10, m_r = 3, g = 1$. Initial condition is a parabola in the vertical direction sinusoidally perturbed in the horizontal direction. The displacement in the longitudinal direction is not a deviation from the straight stretched string position and hence varies from 0 to $L$. Small oscillations can be observed near the boundaries.

Displacement in the horizontal (top), vertical (middle) and longitudinal (bottom) direction. Initial condition is a catenary in the vertical direction sinusoidally perturbed in the horizontal direction. $\delta S = 0.01, \delta t = 0.0025, T_r = 6, EA_r = 10, m_r = 3, g = 1$.

Solution of equations 8 (top) and equation 5 (bottom) for an initial condition with no transverse component. Isocontours of the deviation from the straight stretched string reference position are shown. $\delta S = 0.01, \delta t = 0.0025, T_r = 1, EA_r = 10, m_r = 3$.

Approximate solution (expression 37) to equation 20 at the top; difference from numerical solution of the same equation at the bottom. $\delta S = 0.01, \delta t = 0.0025, T_r = 1, EA_r = 10, m_r = 3$.

460 element mesh with periodic boundary conditions (left). Vertical force coefficient for forced oscillatory motion at $Re = 100$ (right). Solid line corresponds to the ALE calculation, triangles to the calculation using the mapping. The simulation used 5th order polynomials.
490 element mesh for free (in both $x -$ and $y -$ directions) calculations. The vorticity field is shown and a closeup of the mesh is seen in the inset. The simulation used 6th order polynomials. .......................................................... 121

in-line vibrations (left) and cross-flow vibrations (right) for the freely vibrating cylinder. Solid line corresponds to the ALE calculation, triangles to the calculation using the mapping. .......................................................... 121

Drag (left) and Lift (right) coefficients for the freely vibrating cylinder. Solid line corresponds to the ALE calculation, triangles to the calculation using the mapping. 122

Base pressure for the freely vibrating cylinder. Solid line corresponds to the ALE calculation, triangles to the calculation using the mapping. ................................. 122

28 element minimalist mesh for flow past a cylinder at $Re = 100$. Cross-flow velocity isocontours are shown. The simulation was run using 5th order polynomials.125

224 element mesh for flow past a cylinder at $Re = 100$. ........................................ 126

880 element mesh for flow past a cylinder at $Re = 100$. ........................................ 127

880 element mesh for flow past a cylinder at $Re = 100$. Closeup near the cylinder surface. .......................................................... 127

Displacement from the straight stretched string reference position in the $y -$ (top) and $z -$ direction for the forced standing wave simulation leading to ICe. .... 131

Initial displacement (from the straight stretched string reference position) in the streamwise $x -$ (first from the top), cross-flow $y -$ (second from the top) and spanwise $z -$ (third from the top) direction for the simulation starting from ICv. At the bottom we plot the string tension as a function of span and time. .......... 132

Initial (top) and time asymptotic (bottom) displacement in the streamwise $z -$ direction for the simulation starting from ICe. ................................. 133

Initial (top) and time asymptotic (bottom) displacement in the cross-flow $y -$ direction for the simulation starting from ICe. ................................. 134
9 Initial (top) and time asymptotic (bottom) displacement (from the straight stretched string reference position) in the spanwise z— direction for the simulation starting from ICc. .................................................. 135

10 Initial (top) and time asymptotic (bottom) tension distribution for the simulation starting from ICc. .................................................. 135

11 Time asymptotic variation of the (locally linearized) natural frequency of vibration \( f_s \) of the string for the simulation starting from ICc. ............................... 136

12 Initial coefficient of drag \( C_d \) distribution (first from the top), with a more detailed closeup (second from the top), coefficient of lift \( C_l \) distribution (third from the top) and spanwise force coefficient \( C_x \) distribution (bottom). .................................................. 138

13 Initial (top) and time asymptotic (bottom) drag coefficient \( C_d \) distribution for the simulation starting from ICc. .................................................. 139

14 Initial (top) and time asymptotic (bottom) lift coefficient \( C_l \) distribution for the simulation starting from ICc. .................................................. 139

15 Initial (top) and time asymptotic (bottom) spanwise force coefficient \( C_x \) for the simulation starting from ICc. .................................................. 140

16 Time asymptotic energy exchange in the streamwise x— (top), cross-flow y— (middle) and spanwise z— (bottom) direction for the simulation starting from ICc. . 141

17 Streamwise displacement (first from the top), cross-flow displacement (second from the top), coefficient of drag \( C_d \) distribution (third from the top) and coefficient of lift \( C_l \) distribution (bottom) for a simulation of a linear “cable”. There is a “virtual shift” in the time axis of 100 time units. .................................................. 143

18 Cross-flow velocity isocontours at three different locations along the span (non-dimensional time \( t = 156 \)) for the simulation starting from ICc. ................. 144

19 Pressure isocontours at \(-0.08\) and \(-0.06\) as seen from above and to the side (top) and as seen from below (bottom). Non-dimensional time \( t = 156 \) for the simulation starting from ICc. .................................................. 145

xx

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20 Streamwise vorticity isocontours at ±1 as seen facing the incoming flow behind the
cylinder (top) and facing the cylinder from the front (bottom). Non-dimensional
time $t = 156$ for the simulation starting from ICc. 146

21 Cross-flow vorticity isocontours at ±0.7 as seen facing the incoming flow behind the
cylinder (top) and facing the cylinder from the front (bottom). Non-dimensional
time $t = 156$ for the simulation starting from ICc. 147

22 Initial displacement in the streamwise $x$— (first from the top), cross-flow $y$— (second
from the top) and spanwise $z$— (third from the top) direction for the simulation
employing meshB. At the bottom we plot the string tension as a function of span
and time. 149

23 Initial coefficient of drag $C_d$ distribution (top), coefficient of lift $C_l$ distribution
(middle) and spanwise force coefficient $C_z$ distribution (bottom). 150

1 Triangle to rectangle transformation 155

2 Hexahedron to tetrahedron transformation 156

3 The local coordinate systems used in each of the hybrid elements and the planes
described by fixing each local coordinate. 157

4 In the spectral/$hp$ method the solution domain is decomposed into elements of
characteristic size $h$ and then a polynomial expansion of order $N$ is used within
every element. On the left we see a cuboid decomposed into 3072 tetrahedral
elements within which we use a polynomial expansion of order 4 as indicated by
the mode shapes on the right. (Courtesy of S.J.S. Sherwin) 160

5 Solution process in NekTetF 164
Data Layout in \textit{NekTARF}. Here we have chosen to store a Fourier mode (2
"Fourier planes") per processor. This also means that we keep 2 "Physical Planes"
per processor. Because this is a Real-to-Complex FFT, \( N_z = 2^m = 2P \) "Physical
Planes" map to \( P + 1 \) independent Fourier modes (0 to \( N_z/2 = P \)) as the other
\( P - 1 \) modes (\( N_z/2 + 1 \) to \( N_z - 1 \)) are fixed by symmetry. Of these \( P + 1 \) modes,
the first and last one have vanishing imaginary parts, hence by "packing" the real
part of mode \( P \) in place of the imaginary part of mode 0 we are left with \( N_z = 2P \)
"Fourier planes" as well. .................................................. 167

A Complete Exchange ("Global Transpose") followed by a series of inverse FFTs
on the resulting "Fourier pencils" transforms the data to Physical Space arranged
along "Physical pencils". "Pencils" are contiguous vectors containing either all the
Fourier modes or all the Physical space data in the \( z \)-direction at a given \((x, y)\)
point. ................................................................. 168

Model geometry of a distal end-to-side anastomosis using a viscous tetrahedral
mesh from the Felisa package. (Courtesy of S.J.S. Sherwin) .................. 171

Partitioning of the computational mesh shown in figure 8 into \( P = 8 \) regions using
the Metis package. (Courtesy of S.J.S. Sherwin) .......................... 173

Operations in the \( i^{th} \) iteration of a preconditioned Conjugate Gradient loop. (a)
serial version, (b) parallel version on \( p^{th} \) processor. ......................... 174

Self speed up (a) and parallel efficiency (b) for a Helmholtz problem in a cubic
domain using \( n_{el} = 3072 \) and \( n_{el} = 24,576 \) elements with \((N = 6)\). ........... 176

Solver time (a) and parallel Self-Speedup (b) for Helmholtz problem in a cubic
domain decomposed in 3072 elements of polynomial order \((N = 6)\). ........... 177
Contour plot of MPI_Alltoall() performance on the Cray T3E. The black parallel lines with slope $-1$ show how $N_x T_a r F$ usually scales as the problem size scales with $P$. The grey dashed line with slope $-2$ corresponds to $N_x T_a r F$ or a 3D homogeneous turbulence code with fixed problem size and finally the dotted-dashed line with slope 1 corresponds to a 3D homogeneous turbulence code that scales the problem size with $P$.

Contour plot of MPI_Allreduce() performance on the Cray T3E. The black horizontal line corresponds to constant message size.

Contour plot of MPI_Allgather() performance on the Cray T3E. The black horizontal line corresponds to constant message size; the grey dashed one to $M_{12} \propto P$ as $N_x$ increases with $P$. 
Chapter 1

Introduction

1 Vortex-Induced Vibrations

Flow past a bluff flexible structure produces vortex shedding at a sufficiently high Reynolds number. The resulting oscillating hydrodynamic forces interact with the flexible structure to give rise to vibrations. The motion of the cylinder in turn, affects the flow around it, leading to a coupled physical system described as vortex-induced vibration (VIV).

When the frequency of a structural mode of vibration is in the neighborhood of the frequency of vortex shedding (the Strouhal number $St$), large hydrodynamic forces and vibration amplitudes are observed. This phenomenon, which is another instance of resonance in nature, is called "lock-in"; in the classical case it is characterized by a shifting of the vortex shedding frequency to the structure's natural frequency of vibration (hence the name "lock-in" - see Blevins (1990) [1], Koopmann (1967) [2]; also Gharib, Gharib, Leonard and Roshko (1998) [3] for a discussion on non-classical lock-in behavior at low mass ratios). Because of the non-linear coupling between the flow and the structure, hydrodynamic forcing function at the same time as hydrodynamic damping, limiting the lock-in vibration amplitude even in the absence of structural damping.

Fluid flows over flexible cylinders arise in many engineering situations, such as transmission lines, heat exchangers, marine cables towing instruments, flexible risers used in petroleum pro-
duction and mooring lines, and other marine applications (see Blevins (1990) [1], Naudascher and Rockwell (1994) [4], Vandiver (1991) [5], Ramberg & Griffin (1976) [6], Hover, Grosenbaugh & Triantafyllou (1994) [7], Yoerger, Grosenbaugh, Triantafyllou & Burgess (1991) [8], Furnes (1998) [9]). It is therefore important to understand the wake flow dynamics and be able to predict the hydrodynamic forces and vortex-induced motion of such structures.

Simplified models attempting to predict VIV rely on the force input as well as the added mass coefficient and correlation lengths (see Furnes (1998) [9], Vandiver & Li (1994) [10]). From the fundamental point of view, it is important to understand how the near-wake is modified by the motion of the cylinder. The latter in turn depends on the flow conditions, the structural characteristics, and the type of support of the structure. While there has been significant progress in understanding the wake of a stationary cylinder in the last few years, especially in the low Reynolds number range (see Williamson (1996) [11]), there have been very few studies of the wake of freely oscillating cylinders (see Bearman (1984) [12]) with the emphasis placed primarily on the nonlinear dynamics of the structure, while modeling the wake as a simple oscillator (see Parkinson (1989) [13]).

The prediction of VIV is currently based on semi-empirical methods, all of which depend on the drag and lift coefficients, either the sectional or the span-averaged values (see, for example, Blevins (1990) [1], Naudascher and Rockwell (1994) [4], Parkinson (1989) [13]). Despite the extensive force measurements for the rigid cylinder undergoing forced or free transverse vibrations (Sarpkaya (1997) [21], Staubli (1983) [22], Gopalkrishnan (1993) [23], Khalak & Williamson (1997) [24], Hover, Techet & Triantafyllou (1998) [25]), considerably less is known for the case of the flexible cylinder subject to VIV (Alexander (1981) [18], Yoerger et al. [8], Vandiver (1983) [26]). In particular, we are not aware of any direct measurements for lift, drag and amplitude for either the rigid or flexible cylinder with the exception of the recent work by Khalak & Williamson (1997) [24] for a freely vibrating rigid cylinder.

In addition to this lack of data from experiments, a literature survey reveals very large variations in the reported values of both the lift and drag coefficients of the order of 100% or more. For
example. Vandiver measured a drag coefficient for a cable in the range of 1.6 to 3.5 [26] whereas Kim et al. [27] obtained values of $C_d$ from 1.4 to 1.6 for cables ten times longer than in the Vandiver's experiment [26]. Yoerger et al. [8] obtained a value of $C_d$ in the range of 2.2 to 2.5 for long cables. In contrast, Alexander [18] obtained an almost constant value of about $C_d \approx 1.8$ in experiments with long flexible cylinders. Although this variation is, in most cases, due to different experimental conditions, even in cases with relatively similar conditions substantial variations in the lift, drag, and cylinder displacement have been reported. It is clear that in such a complex dynamic phenomenon as VIV, even a small variation in the many parameters of the system, i.e. mass ratio, bending stiffness, cylinder length, may lead to substantial changes in the response, with the further possibility of Reynolds number effects.

2 Parallel Simulations of VIV

In previous work (see Blackburn & Karniadakis (1993) [14], Newman & Karniadakis (1995-1997) [15], [16], [19], [17]), 2D and 3D direct numerical simulations (DNS) have been used to investigate flow-structure interactions in the case of freely vibrating rigid and flexible cylinders. Due to the great computational expense, these simulations were limited to low, mainly laminar, Reynolds numbers. Even in that case, parallel computations were required for all but the most elementary 3D simulations.

A simple wave equation was employed to model the motion of the structure, thereby neglecting the effect of bending stiffness. It was found that, in the low Reynolds number regime investigated, there are two possible states of the wake: One state that corresponds to a traveling wave response, and another one that corresponds to a standing wave response. The first state produces a vorticity field consisting of oblique "rollers" shed off the cylinder's upper and lower surface. The second state corresponds to a three-dimensional staggered pattern forming lambda-type vortices.

The objective of the work presented is the study of VIV in a simplified setting that still closely resembles experiments: A large number of our simulations deal with the case of a very long flexible
cylinder; we treat it as infinitely long, and then employ (free) periodic boundary conditions on a piece of finite length equal to the dominant wavelength of vibration we want to study. We usually choose the structure's corresponding natural frequency of vibration to be close to the stationary cylinder's Strouhal number; in this way we can expect to be in a lock-in state.

Free periodic boundary conditions seem to favor the traveling wave response, as also observed in experiments with long cables (Alexander (1981) [18]). However, for Reynolds numbers in the laminar regime the standing wave response is also possible, if the initial motion of the cable is described by a standing wave, or if the wavelength studied corresponds to a small aspect ratio (length-to-diameter ratio). In the latter case, even an initial condition of a traveling wave will break down, giving rise to a standing wave, as the oblique vortex shedding angle is too steep to be supported (Newman (1996) [19]). The existence of both states has been confirmed recently by Olinger (1996) [20], who used low-order modeling based on circle maps to represent shedding patterns behind flexible cables.

3 Thesis Outline

The research presented represents the evolution of the approach of Blackburn, Newman and Karniadakis ([14], [15], [16], [19], [17]). It focuses on:

- The extension of the direct numerical simulation (DNS) calculations employing linear structural models (for beams as well as cables) to transitional $Re = 200 - 300$ and turbulent $Re = 1000$ Reynolds numbers, presented in Part I.

- The development and coupling with the DNS flow solver of non-linear structural models for the flow-structure problem, presented in Part II.

Part I contains:

- Chapter 2: A discussion of the linear structural models and the efficient DNS approach used with them.
• Chapter 3: A study of flow past flexible cables and beams in the transition regime $Re = 200 - 300$. The structures are allowed to oscillate in both the cross-flow and the streamwise direction. Both lock-in as well as quasi-periodic states (in the boundaries of the lock-in envelope) are studied. The behavior of the different structures is compared, both in terms of displacement and force distributions as well as flow structures in the wake.

• Chapter 4: A study of flow past rigidly vibrating cylinders, cables and beams at a turbulent $Re = 1000$. We consider structures free to oscillate only in a cross-flow direction; different cylinder aspect ratios and both free and pinned endpoints are employed. Displacement and force distributions are compared, the phase relationship between lift and displacement is studied, span and time averaged forces are analyzed and spectra and autocorrelation functions for velocities in the near wake are examined. The excitation of the different structural modes is compared for cables and beams and the flow patterns in the wake are studied in order to understand their effect on the hydrodynamic forces.

Part II contains:

• Chapter 5: A presentation of a set of non-linear equations for a flexible string coupled with an Arbitrary Lagrangian-Eulerian (ALE) flow solver. The non-linear equations are derived and analyzed along with relevant tension models; a numerical discretization is proposed and validated, and the coupling with the flow solver is discussed in detail.

• Chapter 6: A set of model ALE/non-linear structure simulations at $Re = 100$. Displacement and force distributions for different initial conditions are contrasted with results from linear calculations; near wake flow structures are presented and preliminary results from a larger scale calculation are presented.
Part I

Linear Models
Chapter 2

Models for Beams and Cables

1 Introduction

We consider the problem of flow-structure interaction for incompressible flow past a long flexible cylinder. The equations that describe this problem are the coupled system of the Navier-Stokes equations and a set of equations modeling the structure. We shall initially consider linear equations for the structural part, leaving non-linear structural effects for Part II.

2 Modeling the structure

A major factor affecting the coupled flow-structure response is the structural characteristics, i.e. damping and bending stiffness. The case of zero structural damping results in the maximum response at resonant (lock-in) conditions. The elasticity and stiffness of the structure, on the other hand, determines the excited modal shape. Therefore, a tension-dominated cable with zero bending stiffness will respond differently than a beam with no tension and finite bending stiffness even at lock-in conditions. In laminar states, both structures and the resulted flow pattern are identical as the only excited mode is the first one. However, at higher Reynolds number in the transitional and turbulent flow states, a multi-modal response is obtained. Therefore, even if the
first mode dominates the response, excitation of higher modes of the structure is possible, giving rise to significant differences in the topology and dynamics of the near-wake. For long cables high modes usually dominate the response, and thus the effect of bending stiffness cannot be neglected (see Furnes (1998) [9]). In engineering practice, a simple criterion has been developed to decide on what type of modeling to be used (see Vandiver (1991) [5]). For example, a structure can be modeled as a cable if it is tension-dominated, i.e.

$$\frac{T}{EIk^2} > 30,$$  \hspace{1cm} (1)

otherwise it is modeled as a beam and its bending stiffness should be taken into account. Here $T$ is the tension, $EI$ the bending stiffness, and $k$ is the wave number describing the excitation mode.

2.1 Non-dimensionalization

We non-dimensionalize our flow problem using the fluid density $\rho$, the cylinder diameter $d$ and the free stream velocity $U$. This gives us a Reynolds number $Re = \frac{dU}{\nu}$ where $\nu = \mu/\rho$ is the kinematic viscosity of our fluid. We use the same non-dimensionalization for our structure problem. Non-dimensionalizing the structure's linear density $\rho_\text{st}(z)$ gives us the mass ratio (non-dimensional linear density)

$$m = \frac{\rho_\text{st}}{\rho d^2},$$  \hspace{1cm} (2)

This formula for calculation of the mass ratio does not include an added mass term as we will be using both the pressure and the viscous contribution to the hydrodynamic forces. For a circular cylinder composed wholly of a solid of density $\rho_s$, the linear density $\rho_s = \pi \rho_s d^2/4$ and thus the mass ratio $m = \pi/4 \rho_s/\rho$. For VIV in water, this gives us values that are not much larger than 10 for the heaviest of materials. For VIV in air the mass ratio is about 1000 times larger. Smaller values can be obtained for hollow cylinders etc. The mass ratio used in all 3D simulations is 2 (unless otherwise stated), which is a typical value for VIV in water.

Henceforth, all flow and structure variables employed can be assumed to be non-dimensional.
2.2 Linear equations for a flexible cylinder

For a flexible cylinder under tension possessing non-negligible bending stiffness and allowed to oscillate in a transverse direction only, the equation of motion is the following:

\[
m(z) \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial}{\partial z} \left( T(z) \frac{\partial \xi}{\partial z} \right) - \frac{\partial^2}{\partial z^2} \left( EI(z) \frac{\partial^2 \xi}{\partial z^2} \right) - R(z) \frac{\partial \xi}{\partial t} + F(z, t)
\]  

(3)

where \( \xi(z, t) = (\zeta(z, t), \eta(z, t)) \) is the cylinder displacement in the two transverse directions. This sets the problem for the structure as a 1D time dependent IBVP.

In the equation above the first term on the r.h.s. is due to the tension \( T(z) \) and the second one is due to the bending stiffness \( EI(z) \). The third term is a structural damping term, with a coefficient of damping \( R(z) \) and the last one is an external force (such as drag or lift or gravity). \( m(z) \) is the mass per unit length (linear density) of our flexible cylinder.

This linear equation is derived in [29] using a small angle approximation \( \sin \theta \approx \tan \theta \approx 1 \), also known as "small amplitude" approximation (dropping all \( O(\theta^2) \) terms). This basically requires that the amplitude of the oscillations is small with respect to their wavelength - the latter could very well be smaller than the length of the cylinder. This assumption also implies that the change in length of a cylinder segment is negligible \( O(\theta^2) \). This small angle approximation is one basic assumption to keep in mind insofar as the limitations of linear models are concerned.

We usually consider uniform cylinders under constant tension so that all the structural parameters are constant along the span. Hence equation 3 becomes:

\[
\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial z^2} - \gamma^2 \frac{\partial^4 \xi}{\partial z^4} - \frac{R}{m} \frac{\partial \xi}{\partial t} + \frac{1}{m} F(z, t)
\]  

(4)

where \( c = \sqrt{T/m} \) and \( \gamma = \sqrt{EI/m} \).

Although we can always consider problems with nonzero \( c \) and \( \gamma \) concurrently, for the purposes of our flow-structure interaction studies we usually want to study these two cases separately. We
model a "cable" as the structure governed by the following modified forced wave equation:

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial z^2} - \frac{R}{m} \frac{\partial \xi}{\partial t} + \frac{1}{m} F(z, t) \quad (5)$$

Similarly, we model a "beam" as the structure governed by the following modified beam equation:

$$\frac{\partial^2 \xi}{\partial t^2} = -\gamma^2 \frac{\partial^4 \xi}{\partial z^4} - \frac{R}{m} \frac{\partial \xi}{\partial t} + \frac{1}{m} F(z, t) \quad (6)$$

Applying these equation in the context of our flow-structure interaction problem, $\xi(z, t) = (\zeta(z, t), \eta(z, t))$ gives the cable/beam displacement in the streamwise and cross-flow directions. $F(z, t)$ is the fluid force on the structure (we don’t include gravity). The components of $F(z, t)$ in the streamwise and cross-flow directions are the drag and lift forces

$$F = \int (-p n + \frac{1}{Re} (\nabla u + \nabla u^T) \cdot n) ds. \quad (7)$$

including both pressure and viscous contributions. Internal damping is neglected here as we are interested in the maximum possible response of the system.

It is instructive to look at both equations, as simplified above, in their Fourier space representation:

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial z^2} + \frac{1}{m} F(z, t) \Leftrightarrow \frac{\partial^2 \hat{\xi}}{\partial t^2} = -c^2 k^2 \hat{\xi} + \frac{1}{m} \hat{F}(k, t) \quad (8)$$

and

$$\frac{\partial^2 \xi}{\partial t^2} = -\gamma^2 \frac{\partial^4 \xi}{\partial z^4} + \frac{1}{m} F(z, t) \Leftrightarrow \frac{\partial^2 \hat{\xi}}{\partial t^2} = -\gamma^2 k^4 \hat{\xi} + \frac{1}{m} \hat{F}(k, t) \quad (9)$$

The two equations in their Fourier representation can be made equal to each other for $c = \gamma k^2$.

This means that for a given wavenumber $k$ and $EI$, $T$ can be chosen such that the "cable" and the "beam" response are the same or that for given $T$ and $EI$ there exists a wavenumber $k$ for which the two equations become the same.

In the absence of an external force, the natural frequency of the oscillations is $\omega(k) = ck$ for
the cable and $\omega(k) = \gamma k^2$ for the beam. Hence the phase velocity $\omega(k)/k = c$ and the group velocity $\frac{d\epsilon}{dk} = c$ for the cable are the same. In the case of the beam $\omega(k)/k = \gamma$ but $\frac{d\epsilon}{dk} = 2\gamma k$ and dispersive waves naturally appear. So only a uni-modal response of a beam can be expected to behave like that of a cable for a suitable choice of $T$ and $EI$.

We sometimes also consider an "elastically anchored" [29] flexible cylinder where one adds a term $-\beta^2 \xi$ in equations 5 and 6. This is basically a "virtual spring" with a spring constant $K$, such that $\beta = \sqrt{K/m}$ supporting the whole length of the cylinder (and not just the endpoints). This becomes an additional $-\beta^2 \xi$ term in equations 8 and 9 and makes their dispersion relationships $D(\omega, k) = \omega^2 - c^2 k^2 - \beta^2$ and $D(\omega, k) = \omega^2 - \gamma^2 k^4 - \beta^2$ respectively. This means that the minimum frequency of propagation of any wave solutions is $\beta$ in both cases; the addition of the "virtual spring" gives a high-pass system. The cut-off frequency of propagation is $\beta$. It is therefore imperative that only small values of $K$ are employed so as not to significantly affect the dynamics of the flexible cylinder.

It is interesting to note that in the absence of external forces, for an elastically anchored cable the wave equation becomes the Klein-Gordon equation, while the addition of structural damping makes it the telegraphy equation.

Expressing equations 8 and 9 as the equations for the components of a Fourier series expansion of our cylinder variables we get:

$$\frac{\partial^2 \xi_n}{\partial t^2} = -c^2 \left(\frac{2\pi}{L}\right)^2 n^2 \xi_n + \frac{1}{m} \xi_n(t) \quad (10)$$

for the $n$-th Fourier mode of a cable, and

$$\frac{\partial^2 \xi_n}{\partial t^2} = -\gamma^2 \left(\frac{2\pi}{L}\right)^4 n^4 \xi_n + \frac{1}{m} \xi_n(t) \quad (11)$$

for the $n$-th Fourier mode of a beam, where $L$ is the length of the cylinder in the stretched equilibrium position, and $c(2\pi/L)n$ is the natural frequency for cable mode $n$ while $\gamma(2\pi/L)^2 n^2$
is the natural frequency for beam mode \( n \). The nice feature of this representation is that the homogeneous parts of the equations for the modes are decoupled (any coupling comes from the external force term). In the case of flow past a flexible cylinder that we are considering, the lift forces display oscillations of frequency \( \sim St \) (Strouhal frequency) and so resonance (lock-in) requires that \( c(2\pi/L)n = 2\pi St \) and \( \gamma(2\pi/L)^2 n^2 = 2\pi St \) respectively.

If a Fourier \( \text{sine} \) series is used instead our equations become:

\[
\frac{\partial^2 \xi_n}{\partial t^2} = -c^2 \left( \frac{\pi}{L} \right)^2 n^2 \xi_n + \frac{1}{m} \tilde{F}_n(t) \tag{12}
\]

for the \( n \)-th sine Fourier mode \( \sin \left( \frac{n\pi}{L} z \right) \) of a cable and

\[
\frac{\partial^2 \xi_n}{\partial t^2} = -\gamma^2 \left( \frac{\pi}{L} \right)^4 n^4 \xi_n + \frac{1}{m} \tilde{F}_n(t) \tag{13}
\]

for the \( n \)-th sine Fourier mode of a beam.

This is accomplished by projecting the force \( F_y \) into a Fourier \( \text{sine} \)-series that gives zero contributions at the two ends. The resonance requirements now become \( c(\pi/L)n = 2\pi St \) and \( \gamma(\pi/L)^2 n^2 = 2\pi St \) respectively.

### 2.3 Initial and Boundary conditions

For the IVP of equation 3 we need initial conditions for the displacement and the velocity of our structure. Given that in our flow-structure interaction applications the forcing term dominates the behavior of our system, the initial conditions are usually important only during an initial period of transient response.

We now examine the relevant boundary conditions for equation 3 and its derived equations of subsection 2.2. We have a choice of several boundary conditions for strings (wave equation - see \[29\]):

- pinned or fixed (zero Dirichlet: \( \xi = 0 \)),

...
• free (zero Neumann: \( \frac{\partial \xi}{\partial z} = 0 \)).

• springy \( (\frac{\partial \xi}{\partial z} + \beta^2 \xi = 0) \) and

• damped \( (T \frac{\partial^2 \xi}{\partial z^2} + R \frac{\partial \xi}{\partial t} = 0) \).

\( K, T \) and \( R \) defined in subsection 2.2. If a fourth derivative in space term exists we need additional boundary conditions on higher derivatives, for

• a simply supported (pinned) beam \( (\xi = \frac{\partial^2 \xi}{\partial z^2} = 0) \),

• a clamped beam \( (\xi = \frac{\partial \xi}{\partial z} = 0) \) and

• a free beam \( (\frac{\partial^2 \xi}{\partial z^2} = \frac{\partial^3 \xi}{\partial z^3} = 0) \).

Of course (free) periodic boundary conditions can also be used: they simplify the problem a lot enabling us to employ a Fourier series decomposition. The resulting equations 10 and 11 for the Fourier modes are decoupled. The case of pinned endpoints can also naturally be handled using a Fourier sine decomposition. A property of the decomposition itself is that \( \xi = \frac{\partial^2 \xi}{\partial z^2} = 0 \) is naturally enforced at the endpoints (that are roots of \( \sin \left( \frac{\pi \xi}{L} \right) \)).

In previous work ([15], [16], [17]) and in the simulations presented in Chapter 3 and most of the simulations in Chapter 4, (free) periodic boundary conditions were imposed. These seem to favor the traveling wave response for a flexible cylinder in VIV. This configuration resembles, in essence, a very long cable or beam that is subject to spanwise disturbances dictated by the wave length imposed by the periodic length. Although standing wave patterns are more common in experimental situations, traveling wave responses have also been realized both in laboratory and field experiments, for example in the work of Alexander (1981) [18], and in low Reynolds number experiments in the work of van Atta, Gharib & Hamache (1988) [30]. Usually, traveling wave patterns and standing wave patterns coexist, and this is also revealed in the simulations. To document such a mixed response, we have simulated a relatively long cable \( (32\pi \approx 100.5d) \), where \( d \) is the cylinder diameter) with both of its ends pinned.
Figure 1: Flow past a freely vibrating flexible cylinder with pinned ends at $Re = 100$. Shown is the spanwise distribution of the amplitude of the cross-flow motion $y/d$ versus non-dimensional time. For clarity isocontours at $y/d = \pm 0.4$ are used which are far below the maximum value of 1 observed.

In figure 1 we plot the resulting response from the motion of the flexible cylinder, which was allowed to oscillate only in the cross-flow ($y$-) direction. We see that in regions close to pinned ends a standing wave is developed, whereas in the mid-span a traveling wave is developed with a symmetry line at the mid-point. We also see that due to constructive interference large amplitudes of $y/d \approx 1$ are obtained. This value can be contrasted with the periodic ends case, which gives a maximum amplitude of $y/d \approx 0.5 - 0.6d$ at this Reynolds number ($Re = 100$) [17]. More specifically, the amplitude of about $0.5d - 0.6d$ is obtained for laminar flow irrespective of the response, i.e. traveling or standing wave, although the lift force in the former case is less than half of the lift force of the latter. This result points to the importance of the phase difference between the motion and the lift force, the selection of which is based on a subtle and yet unexplained coupling mechanism between the near-wake dynamics and the structure dynamics that is discussed further in Chapter 4.

Unlike the case of pinned endpoints, there is nothing to stop a structure with (free) periodic
ends from being moved indefinitely downstream by the drag force. To maintain a mean displacement, the structure can be lightly elastically "anchored" as discussed in subsection 2.2. The spring constant should be selected sufficiently small to have as small an effect on the structure's response as possible while at the same time maintaining a reasonably small mean displacement.

3 Solving for the structure

The numerical solution to equations 10 and 11 employs a Newmark integration scheme [31]. Since the equations for the Fourier modes are decoupled this will be the same ODE solver as the one used in the case of the 2D flow-structure interaction problem. In the standard form of a forced, damped harmonic oscillator problem, we have for the most general case,

\[ \ddot{\xi}_n + 2\epsilon\omega_n \dot{\xi}_n + \omega_n^2 \xi_n = \frac{1}{m} \ddot{F}_n \]  

(14)

where \( 2\epsilon\omega_n = \frac{R}{m} \) (\( R \) being the coefficient of structural damping discussed in subsection 2.2) and

\[ \omega_n^2 = \beta^2 + c^2 \left( \frac{2\pi}{L} \right)^2 n^2 + \gamma^2 \left( \frac{2\pi}{L} \right)^4 n^4 \]  

(15)

for the case of equations 10 and 11 combined (with \( \beta \) the "virtual spring" parameter defined in subsection 2.2). For the case of equations 12 and 13 combined we get

\[ \omega_n^2 = \beta^2 + c^2 \left( \frac{\pi}{L} \right)^2 n^2 + \gamma^2 \left( \frac{\pi}{L} \right)^4 n^4. \]  

(16)

To update the state of this system at timestep \( t \), \( (\ddot{\xi}_n, \dot{\xi}_n, \xi_n) \) using the forcing at time level \( \ddot{F}_n \), we use the

\[ \ddot{\xi}_n^{t+1} + 2\epsilon\omega_n \dot{\xi}_n^{t+1} + \omega_n^2 \xi_n^{t+1} = \frac{1}{m} \ddot{F}_n^{t+1} \]  

(17a)
\[
\begin{align*}
\dot{\xi}_{n}^{l+1} &= \dot{\xi}_{n}^{l} + \delta t \ddot{\xi}_{n}^{l} + \frac{\delta t^2}{2} \left[ \frac{\ddot{\xi}_{n}^{l} + \ddot{\xi}_{n}^{l+1}}{2} \right] \quad (17b) \\
\ddot{\xi}_{n}^{l+1} &= \ddot{\xi}_{n}^{l} + \delta t \left[ \frac{\xi_{n}^{l} + \xi_{n}^{l+1}}{2} \right], \quad (17c)
\end{align*}
\]

or more compactly

\[
A \begin{bmatrix} \ddot{\xi}_{n}^{l+1} \\ \dot{\xi}_{n}^{l+1} \\ \xi_{n}^{l+1} \end{bmatrix} = B \begin{bmatrix} \ddot{\xi}_{n}^{l} \\ \dot{\xi}_{n}^{l} \\ \xi_{n}^{l} \end{bmatrix} + \frac{1}{m} \begin{bmatrix} \ddot{F}_{n}^{l+1} \\ 0 \\ 0 \end{bmatrix}. \quad (18)
\]

The 3 × 3 matrix A is easily invertible analytically. The Newmark scheme is unconditionally stable, second order accurate. If instead of \( \ddot{F}_{n}^{l+1} \), \( \ddot{F}_{n}^{l} \) is used (because we are lagging the flow solver), an order \( O(\delta t^3) \) error is introduced for \( \ddot{\xi}_{n} \) - this is less than the order of the scheme. \( \ddot{\xi}_{n} \) incurs an extra \( O(\delta t^2) \) error and only the extra error in \( \ddot{\xi}_{n} \) \( (O(\delta t)) \) is higher than second order.

4 Solving for the flow

In a stationary, Cartesian coordinate system \((x', y', z')\) the non-dimensionalized incompressible Navier-Stokes equations (in convective form) are:

\[
\begin{align*}
\frac{\partial u'}{\partial t} + (u' \cdot \nabla) u' &= -\nabla p + Re^{-1} \nabla^2 u', \quad (19a) \\
\nabla \cdot u' &= 0, \quad (19b)
\end{align*}
\]

where \( Re = U d/\nu \) is the Reynolds number based on the free-stream velocity \( U \) and the cylinder diameter \( d \); \( \nu \) is the kinematic viscosity.

To simplify the solution of the fluid equations we consider a coordinate system attached to the moving cylinder. This maps the time-dependent and deforming problem domain to a stationary and non-deforming one as shown in figure 2. A convenient mapping is described by the following
Figure 2: The coordinate system is attached to the moving flexible cylinder, producing an undeformed, stationary computational domain.

transformation:

\[ z = x' - \zeta'(z', t'), \] (20a)
\[ y = y' - \eta'(z', t'), \] (20b)
\[ z = z', \] (20c)
\[ t = t'. \] (20d)

which changes our partial derivative operators as following:

\[ \frac{\partial}{\partial z'} = \frac{\partial}{\partial x} \] (21a)
\[ \frac{\partial}{\partial y'} = \frac{\partial}{\partial y} \] (21b)
\[ \frac{\partial}{\partial z} = \frac{\partial}{\partial z} - \frac{\partial \zeta}{\partial z} \frac{\partial}{\partial x} - \frac{\partial \eta}{\partial z} \frac{\partial}{\partial y} \] (21c)
\[ \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} - \frac{\partial \zeta}{\partial t} \frac{\partial}{\partial x} - \frac{\partial \eta}{\partial t} \frac{\partial}{\partial y}. \] (21d)

Accordingly, the velocity components and pressure are transformed as follows:

\[ u = u' - \frac{\partial \zeta}{\partial t} - w' \frac{\partial \zeta}{\partial z}, \] (22a)
\[
\begin{align*}
v &= v' - \frac{\partial \eta}{\partial t} - w \frac{\partial \eta}{\partial z}, \\
\quad (22b) \\
w &= w', \\
\quad (22c) \\
p &= p'. \\
\quad (22d)
\end{align*}
\]

In 2D flow this transformation amounts to an adjustment of \( u \) and \( v \) by the cylinder velocity:

\[
\begin{align*}
u &= u' - \frac{\partial \zeta}{\partial t'}, \\
\quad (23a) \\
v &= v' - \frac{\partial \eta}{\partial t'}, \\
\quad (23b) \\
p &= p'. \\
\quad (23c)
\end{align*}
\]

The Navier-Stokes equation and continuity equation are transformed to:

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p + Re^{-1} \nabla^2 u + A(Re, u, p, \xi), \\
\quad (24a)
\]

\[
\nabla \cdot u = 0, \\
\quad (24b)
\]

where the forcing term \( A(Re, u, p, \xi) \) is the extra acceleration term introduced by the transformation, consisting of both inviscid and viscous contributions.

In 2D flow, \( A(Re, u, p, \xi) \) has a very simple form:

\[
A = -\frac{\partial^2 \xi}{\partial t^2} \\
\quad (25)
\]

which is not \( x \) or \( y \) dependent. For 3D flow \( A(Re, u, p, \xi) \) has a more complicated form:

\[
\begin{align*}
A_x &= -\frac{d^2 \zeta}{dt^2} + \frac{1}{Re} \left[ \frac{\partial^2}{\partial z^2} \left( u + \frac{\partial c}{\partial z} w \right) - \frac{\partial^2 u}{\partial z^2} + \frac{\partial \zeta}{\partial z} \nabla^2 w + \frac{\partial \zeta}{\partial t} \frac{\partial^2 w}{\partial z^2} \right], \\
A_y &= -\frac{d^2 \eta}{dt^2} + \frac{1}{Re} \left[ \frac{\partial^2}{\partial z^2} \left( v + \frac{\partial \eta}{\partial z} w \right) - \frac{\partial^2 v}{\partial z^2} + \frac{\partial \eta}{\partial z} \nabla^2 w + \frac{\partial \eta}{\partial t} \frac{\partial^2 w}{\partial z^2} \right], \\
A_z &= \frac{\partial \zeta}{\partial z} \frac{\partial p}{\partial x} + \frac{\partial \eta}{\partial z} \frac{\partial p}{\partial y} + \nu \left[ \frac{\partial^2 w}{\partial z^2} - \frac{\partial^2 w}{\partial z^2} \right],
\quad (26a)
\end{align*}
\]
where for a more compact form of the equations we denote:

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \tag{27a}
\]

\[
\nabla_{z y}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{27b}
\]

This rather convoluted form of writing the extra terms, though not intuitive is actually the most efficient in terms of the cost of the calculations and is the actual one employed in the codes.

Note that the incompressibility condition is unchanged in both 2D and 3D:

\[
\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \tag{28}
\]

This is because the mapping produces no mesh divergence, \( \frac{\partial C}{\partial x} + \frac{\partial n}{\partial y} = 0 \). We can also view this mapping as a case of the ALE formulation, where the mesh velocity \( \hat{u} = (\frac{\partial C}{\partial t}, \frac{\partial n}{\partial t}, 0) \) throughout the domain. Tests comparing this solution approach with the ALE one are presented in subsection 5.3 of Chapter 5. Dimas and Triantafyllou [32] used this kind of mapping to study the inviscid interaction between a shear flow and a free surface, and kindly suggested its use for this problem. Its first application was in the precursor work of Newman [19] where it is also discussed in some detail.

### 4.1 Limitations of the mapping approach

This particular mapping assumes that no deformation of the mesh occurs in the \( z \) (spanwise) direction. This means that this approach can only be used if we disregard any longitudinal oscillations of our flexible cylinder. Moreover no rotation can be handled and therefore it cannot be used when any torsional effects for the structure need to be taken into account.

Because this mapping is a translation in the \( z \) and \( y \)-directions the intersection of the cylinder with the \( x - y \) plane will always be circular. As a cylinder flexes though, its circular cross-section is preserved only in a plane perpendicular to the tangent to its deformed centerline.
The corresponding intersection with the $x - y$ plane is actually an ellipse with axes of size:

\[
\begin{align*}
X_{axis} & = d \sqrt{1 + \left( \frac{\partial \xi}{\partial z} \right)^2} \\
Y_{axis} & = d \sqrt{1 + \left( \frac{\partial \eta}{\partial z} \right)^2}
\end{align*}
\] (29a) (29b)

So long as $\frac{\partial \xi}{\partial z}$ and $\frac{\partial \eta}{\partial z}$ remain small the difference between an ellipse and a circle is small enough to be negligible for our purposes. This however is the same small angle/small amplitude approximation that is required for equations 3 to hold.

Another problem of the mapping is that it is derived with an unbounded domain in mind: Boundary conditions at the far field do not enter anywhere in the derivations detailed in section 4 above, and indeed as the un-mapped computational mesh moves in space, it covers $(x, y, z)$ points that it didn’t contain at previous times and for which primitive variable information is missing. It is therefore imperative that when using this mapping one chooses a computational domain large enough that the far-field conditions are well satisfied at the boundaries and all spatial derivatives of the flow quantities are virtually zero.

5 The coupled flow-structure problem

The transformed Navier-Stokes/structure dynamics equations were discretized in space using two different spectral methods, an older code (*Prism*) based on quadrilateral elements and a new spectral code (*Nektar*, written using C++ and MPI [33]) that employs a hybrid grid in the $x - y$ plane. Both methods use Fourier complex exponentials in the $z$-direction and are covered in Appendix A.

Both codes employ the same three-step time-splitting strategy for advancing the Navier-Stokes equations in time using a stiffly stable time integration scheme ([34]):

1. First the contributions of the non-linear terms are calculated.
2. Then pressure is calculated by solving a Poisson equation that enforces the continuity constraint; the gradient of the pressure is added to the non-linear terms.

3. Finally the viscous correction is calculated.

The flow problem and the structure problem interact with each other through the boundary conditions and the hydrodynamic forces. Specifically:

- The structure's motion is affected by the calculated lift and drag forces on the cylinder. For these calculations we include both the pressure and the viscous force contribution.

- As the structure moves, the flow is affected by the change in the shape and speed of its flexible boundary. Because of the mapping that keeps the mesh constant, this translates to:
  
  - Adjustments in the velocity Dirichlet boundary conditions at the farfield. The non-slip velocity boundary conditions at the cylinder's surface remain zero because of the mapping.
  
  - Adjustments in the time accurate pressure boundary conditions employed (see [35]) for the acceleration term (which is zero for simulations with fixed boundaries).
  
  - Addition of the extra terms to the Navier-Stokes equations \( A(Re, u, p, \xi) \).

When dealing with a coupled flow-structure problem, one ideally wants to advance both the flow and the structure in time concurrently. This is, however, impractical as the coupled system is prohibitively expensive and even a predictor-corrector cycle is too costly because of the high operation count of the flow solver. Therefore we choose to lag the flow solver in a consistent manner:

1. We begin with both the flow solver and structure solver states at timestep \( n \). We have already calculated drag and lift at this timestep.

2. We calculate \( A(Re, u, p, \xi) \) along with the non-linear contributions, employing the same time integration scheme.
3. We use the structure's state (velocity, acceleration) to calculate the adjustments to the time-accurate pressure boundary conditions before solving for the pressure.

4. We use the hydrodynamic forces at timestep $n$ to advance the structure's state to timestep $n + 1$.

5. We use the structure's velocity to calculate the adjustments to the velocity Dirichlet boundary conditions before the viscous correction calculation.

6. We calculate the new forces at timestep $n + 1$.

Since $A(Re, u, p, \xi)$ contains terms that are independent of the flow variables (in 2D all of $A(\xi)$ as seen in equation 25; in 3D parts of the expressions in equation 26) it is possible to use their values at timestep $n + 1$ instead. This means adding their contribution in an "implicit" way for the time-stepping scheme: instead of calculating them in stage 2 of the list above, we compute their value after stage 4 and add them to the rest of the Navier-Stokes terms before the viscous correction. Care has to be taken to compensate the pressure boundary conditions for this change.

Use of this boundary-fitted coordinate formulation was compared against that of a different arbitrary Lagrangian-Eulerian (ALE) formulation [36] for 2D VIV in subsection 5.3 of Chapter 5. Very good agreement was obtained for both identical (periodic) as well as more standard "moving wall" boundary conditions.

6 Summary

In this chapter we formulated our flow-induced vibration problem using the incompressible Navier-Stokes equations and the forced wave equation. We proposed a coordinate transformation that mapped our moving boundary flow problem to a stationary boundary flow problem, with time dependent boundary conditions. We derived the form of the Navier-Stokes equations under this transformation, and described the temporal and spatial discretization used to solve the flow problem. Finally, we outlined the numerical method used to solve for the cable motion.
Chapter 3

Transition Regime

1 Introduction

This work extends some of the preliminary findings of [17] for the effect of flow on the amplitude of vibration as the flow undergoes transition from laminar to early turbulent state. In particular, it was found in [17] that for cable-flow interactions, the amplitude of vibration increases as the Reynolds number changes from \( Re = 100 \) to 200. As a reference point, we note that the wake of a stationary cylinder becomes three-dimensional and turbulent in the regime \( Re = 200 - 300 \) (see [37], [38] and references therein).

The question therefore arises as to what is the transition regime for a freely moving flexible cylinder, and what is the corresponding effect on the amplitude of vibration. For a rigid cylinder, cross-flow forced vibrations tend to make the wake more "two-dimensional" and therefore they tend to delay transition, [39]. However, for flexible freely moving cylinders no such studies exist nor it is known how the type of structure (i.e. its bending stiffness) influences the state of the wake.

To simplify the problem we consider long flexible cylinders, which are modeled as linear cables and beams. The complete formulation was given in Chapter 2 and a review of the numerical method is presented in appendix A. In section 2, we present the details of our simulation param-
eters, followed in section 3 by a few representative simulations of flow past cables and beams at
lock-in as well as non-lock-in states, and we conclude in section 4 with a brief summary of this
investigation.

2 Simulation details

To solve the three-dimensional Navier-Stokes equations, we use a parallel spectral element-Fourier
method based on the quad based element code Prism [40]. As a reminder, spectral elements are
used to discretize the $x-y$ planes, while a Fourier expansion is used in the $z$-direction (i.e. along
the flexible cylinder length). Consequently, all flow and cylinder variables are assumed to be
periodic in the spanwise direction. The computational domain extends $35d$ (cylinder diameters)
downstream, and $15d$ above and below the cylinder. The “structure span”, i.e. wavelength of
vibrations in the flexible cylinder was $L/d = 4\pi$ for all simulations. Two meshes were used for
discretizing the $x-y$ plane: one consisting of $K = 110$ elements and one with $K = 158$ for higher
$Re$ runs. The latter mesh was a refinement of the former one, with more $h-$resolution near the
cylinder in order to better resolve the boundary layer (see figure 1). Each element contained $N = 9$
or 11 points per x,y direction corresponding to 8th or 10th polynomial order respectively. Two
different resolutions were used in the $z$-direction: 32 $z$-planes (16 independent Fourier modes)
for most cases and 64 $z$-planes for the cases far from lock-in and the higher $Re$ (see figure 2).
This gave us roughly a minimum of more than 1.1 million and a maximum of more than 4.8
million degrees of freedom over the 3 velocity fields and pressure. A non-dimensional time step of
$\Delta t U/d = 0.001$ is used giving over 5,000 time steps per shedding cycle. For each simulation, the
cylinder initial position and velocity are set from a previous run and the simulations are run for
at least 25 shedding cycles, or until the statistics are relatively stationary.

In all of the simulations the cylinder was allowed to move freely in both the $x$- and the $y$-
direction. To keep the cylinder from being dragged by drag forces indefinitely, a “virtual spring”
is used to elastically anchor the cylinder in both directions as explained in Chapter 2, where the
Figure 1: Spectral element mesh in the $x-y$ plane; Fourier expansions are used in the periodic spanwise direction. Close up (bottom) of the discretization around the cylinder for polynomial order 8.
Figure 2: 3D 110 element mesh used in some of the calculations: Deformed macro-skeleton in the z-direction (top) and view of the planes comprising the mesh (bottom). Only every 8th plane is shown, and the scale on the z-axis is $1/\pi$ times that on the other two.
dispersive wave solutions that arise because of this spring are detailed further. In all cases, the 
restoring force constant \(\omega_n\) that was used (0.105) was chosen to be small relative to the non-
dimensional \(\sqrt{\frac{E}{m}}\) (cable) or \(\sqrt{\frac{EI}{m}}\) (beam). For the purposes of our data analysis, any residual 
mean motion of our structure was subtracted out in postprocessing.

The simulations were performed mostly on 32 processors of the PSC Cray T3D - a few of the 
later calculations were performed on the PSC and NAVO Cray T3Es.

3 Simulation Results

The simplest periodic solutions to the vibrating homogeneous string equation (5) are standing 
waves and traveling waves. Both a standing wave response,

\[
y(z, t) = A \cos(\omega t) \cos(2\pi z/L),
\]

and a traveling wave response,

\[
y(z, t) = A \cos(\omega t \pm 2\pi z/L),
\]

satisfy the vibrating string equation, \(y_{tt} = c^2 y_{zz}\), when \(\omega = 2\pi c/L\), where \(c\) is the phase speed. 
In previous work [17], it was found that in flow past flexible cables the traveling wave response 
is the preferred response in the absence of Dirichlet (pinned end) boundary conditions. However, 
there are cases where standing wave responses are stable, for example in the case of very small 
spanwise wavelength.

In figure 3 we plot the response of a cylinder for laminar flow at \(Re = 100\). In particular, 
we plot the displacement \(y/d\), lift coefficient \(C_l\) and drag coefficient \(C_d\), versus time \(tU/d\), and 
position along the cylinder length \(z/d\). At this Reynolds number an identical traveling wave 
response is obtained, irrespective of whether the structure's motion is described by equation (5) 
or equation (6) in Chapter 2. This is the case for a stable, single-mode periodic spanwise motion.
Figure 3: Cable cross-flow displacement ($y/d$), lift coefficient ($C_l$) and drag coefficient ($C_d$) versus non-dimensional time and spanwise distance ($z/d$) along cable for $L/d = 12.6$ and $Re = 100$. (Courtesy of D.J. Newman)

and $T = n^2 \beta^2 EI$, where $\beta = 2\pi/L$ and $n$ is the mode number. Consequently, a beam will respond in an identical manner to a cable for single mode vibrations for an *appropriate* choice of structural parameters. This was confirmed directly for both standing and traveling wave responses.

In general, however, the coupled response of a cable is different than the response of a beam. These differences are more pronounced in the transition regime as we demonstrate next. In particular, we consider simulations of flow past cables and beams at $Re = 200$ and $Re = 300$. Both lock-in and non-lock-in cases are investigated.

In figure 4 we plot the response of a cable (left) and of a beam (right) at $Re = 200$ at lock-in. The reduced velocity $V = \frac{U}{f_d}$ is $V_c = 5.41$ for the cable and $V_b = 5.08$ for the beam. In both cases we see that a traveling wave response is the stable solution. This structure motion causes an oblique shedding at about the same angle with respect to the cylinder axis. There are two important differences between the cable and the beam response. First, for the beam we obtain a single-mode periodic response whereas for the cable we observe an aperiodic response even at synchronization state. Second, the amplitude of oscillation for the beam is approximately $0.55d$.
Figure 4: Cross-flow displacement \((y/d)\), lift coefficient \((C_l)\) and drag coefficient \((C_d)\) versus non-dimensional time and spanwise distance \((z/d)\) at lock-in. Cable (left) and Beam (right). \(Re = 200\).

whereas for the cable is \(0.8d\). Correspondingly, there are differences in lift and drag forces as shown in figure 4.

These results are typical for all lock-in states for the beam; however, no clear lock-in state for the cable was obtained presumably because of the disorganization of the flow in the wake. In figure 5 we plot the spectrum of the structure motion at a fixed spanwise location. We see that unlike the beam a subharmonic appears in the frequency spectrum of the cable, responsible for the modulated traveling wave response plotted in figure 4.

<table>
<thead>
<tr>
<th></th>
<th>(V_c)</th>
<th>(y/d)</th>
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</tr>
<tr>
<td></td>
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</tr>
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<td></td>
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<table>
<thead>
<tr>
<th></th>
<th>(V_b)</th>
<th>(y/d)</th>
</tr>
</thead>
<tbody>
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<td>0.54</td>
</tr>
<tr>
<td></td>
<td>5.08</td>
<td>0.55</td>
</tr>
<tr>
<td></td>
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<td>0.54</td>
</tr>
<tr>
<td></td>
<td>5.98</td>
<td>0.52</td>
</tr>
<tr>
<td></td>
<td>6.65</td>
<td>0.53</td>
</tr>
<tr>
<td></td>
<td>7.48</td>
<td>0.45</td>
</tr>
</tbody>
</table>

Table 3.1: Summary of cross-flow displacements at \(Re = 200\): \(V_c\) and \(V_b\) is the reduced velocity calculated using the frequency of the 1\({\textsuperscript{st}}\) structural mode of vibration. The amplitude of \(y/d\) is calculated after the initial transients die out.

For larger values of reduced velocity modulated traveling wave responses are also observed for the beam motion. In figure 6 we plot the evolution of the amplitude of the motion and corresponding spectrum as before for \(V_b = 7.47\). We observe that an intermediate between a traveling wave and a standing wave is obtained in this case; the corresponding spectrum exhibits larger subharmonic components than other beam cases. For such a detuned case, the response of
Figure 5: Time history and corresponding spectrum of the cable (2 top plots) and beam (2 lower plots) displacement at lock-in, $Re = 200$. 
Figure 6: Cross-flow displacement (top) and midspan displacement values and spectrum (bottom) for a vibrating beam with a reduced velocity $V_b = 7.48$, $Re = 200$. 
Figure 7: Cross-flow displacement (top) and midspan displacement values and spectrum (bottom) for a vibrating cable with a reduced velocity $V_c = 6.65$, $Re = 200$. 

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the cable is very irregular as shown in figure 7, where the amplitude evolution and corresponding amplitude are plotted for a reduced velocity \( V_c = 6.65 \). For smaller values of reduced velocity, a weaker modulation of the motion is caused both for the beam and the cable. In table 1 we summarize the values of cross-flow amplitude obtained for the simulations we performed at \( Re = 200 \). The striking difference is that the amplitude of the beam motion is approximately 70% of the amplitude of the cable motion. As shown in figure 3 the amplitude of the motion at lower Reynolds number \( Re = 100 \) is approximately 0.5d, i.e. close to the amplitude of a beam at \( Re = 200 \).

Figure 8: Cross-flow displacement (top) and midspan displacement values and spectrum (bottom) for a vibrating beam at lock-in, \( Re = 300 \).
Figure 9: Cross-flow displacement (top) and midspan displacement values and spectrum (bottom) for a vibrating cable at lock-in, $Re = 300$. 
At higher Reynolds number the same picture emerges, i.e. the lock-in state for the beam is a single mode response whereas for the cable there are other frequencies involved. In figure 8 we plot the motion of a beam at $Re = 300$ and corresponding frequency spectrum as before at reduced velocity $V_b = 4.93$. A very weak modulation is present but otherwise a clear lock-in state emerges. In figure 9 we plot the evolution of the cross-flow motion of the cable at $Re = 300$ and at reduced velocity $V_c = 4.93$. Here again we see that an aperiodic motion is obtained with amplitude approximately 1.4 times as large as the amplitude of the motion of the beam.

3.1 Flow Structures

The differences in cross-flow displacement between the cable and the beam at Reynolds number 200 and 300 are dictated by the differences in the induced flow structures in the near wake. To appreciate this we consider plots of instantaneous vorticity in x-y planes at different locations along the span. In all cases oblique shedding of vortices takes place induced by the traveling wave response of the structure.

Figure 10: Slices in the x-y plane showing spanwise vorticity at $Re = 100$. 

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We first plot vorticity contours at $Re = 100$ in figure 10. For a traveling wave response the vorticity pattern is similar to a two-dimensional von Karman street formed behind a stationary cylinder. The different slices in the span show phase difference at the same instance but otherwise the same picture emerges at all z-locations. As mentioned earlier at this Reynolds number there is no difference between the cable or the beam response.

Figure 11: Cable: Slices in the x-y plane showing spanwise vorticity at $Re = 200$.

As the Reynolds number increases, the flow in the wake of the cable becomes very disorganized. In figure 11 we plot vorticity contours for the cable at $Re = 200$ at the lock-in state described in figure 4. We see that a non-coherent vortex street appears, characteristic of a three-dimensional vortex street in the transition and early turbulent regime, [37]. In contrast, corresponding plots of vorticity for the beam at $Re = 200$ at lock-in (see figure 12) show a vortex street quite similar to the laminar vortex street shown in figure 10.
4 Summary

We have used direct numerical simulation to investigate differences in the coupled response of beam-flow and of cable-flow interaction. We have considered here the transition from laminar to turbulent regime that takes place between \( Re = 200 \) and \( Re = 300 \). We have employed linear models to describe the motion of a cable and of a beam, and have assumed that the effects from the supporting ends are neglected for sufficiently long structures.

The main finding of this work is that the cross-flow amplitude observed for flow-induced vibrations of beams is approximately 70\% of that observed in cables, for cases at or near lock-in for \( Re = 200 - 300 \). This was explained in terms of the corresponding flow pattern established in the wake. At \( Re = 200 \) the cable and wake responses are aperiodic but the corresponding beam response is periodic. The conclusion from this investigation is that cables tend to enhance transition compared to stationary or moving cylinder, whereas beams tend to delay transition by providing a stiffer response. In both cases an oblique shedding process is induced by the motion
of the beam or cable that follows approximately a modulated traveling wave.
Chapter 4

Turbulent Regime

1 Introduction

In the current work, we investigate the physics of VIV in a simplified setting, one which is computationally convenient but still maintains the main engineering features. In particular, we consider long flexible cylinders, which we will model as linear cables and beams subject to resonant VIV at Reynolds number $Re = 1000$, which is one order of magnitude higher than the one considered previously (Chapter 3 and [17]). At this Reynolds number, flow past a stationary cylinder exhibits a turbulent wake (see Williamson (1996) [11]). Some experimental evidence suggests that insofar as VIV phenomena are concerned, for $Re \geq 500$ there is no significant Reynolds number dependence (Vandiver (1991) [5]). More recent experiments by Ching, So & Zhou (1998) [41] suggest otherwise - future applications of DNS techniques should help investigate the existence and possible nature of Reynolds number effects.

The specific questions we address are:

- What is the maximum amplitude response of cables, beams and rigid cylinders in this turbulent regime? How does it compare with the response in the laminar flow regime? Is the response different for different structures at $Re = 1000$ and how?
• What is the phase difference between structure motion and lift forces? What flow features determine the selection process?

• What is the spatial distribution of lift and drag forces? How is it related to the periodic length used in the simulations (that is the imposed structure wavelength in our simulations)?

• What is the statistical relationship between the magnitudes of the forces and the structure's displacement?

• What is the effect of pinned-endpoints in the magnitudes and distribution of the amplitude and fluid forces? What is the structural response obtained and the corresponding flow structures?

• What are the excited structural modes and what flow structures are responsible for a multi-modal excitation in the turbulent regime?

• What are the qualitative and quantitative differences between the turbulent wake of a stationary rigid cylinder and the wakes behind freely vibrating rigid cylinders, flexible cables and beams?

To address these questions, we use direct numerical simulation based on a new spectral element method that employs unstructured hybrid (triangular/quadrilateral) grids and hp refinement. In section 2 we discuss the simulation parameters, and we include more details of the simulation and the method in Appendix A. In section 3 we present results for the vibration amplitude, lift and drag time-histories as well as their phase difference. In section 4 we study the spanwise distribution of time-averaged forces and its relationship with the corresponding spanwise distribution of the vertical displacement. The time histories of span-averaged forces are also studied. In section 5 we present an autocorrelation function and spectral analysis for the turbulent near-wake and we also look at the energy spectrum of the various structural modes. In section 6 we present and discuss representative 2D and 3D flow visualizations. We conclude with a discussion in section 7.
2 Simulation details

Figure 1: Hybrid grid in the $x - y$ plane; Fourier expansions are used in the periodic spanwise direction. Close up of the discretization around the cylinder (upper), and variable expansion order (polynomial order + 1) per element (lower).

To solve the three-dimensional Navier-Stokes equations, we shifted from using the code *Prism* to the new code *NekTao* [33], which is based on hierarchical spectral/$hp$ expansions on hybrid subdomains, i.e. triangles and quadrilaterals as shown in figure 1. This enabled us to take advantage of the enhanced resolution capabilities of *NekTao*:

- Each element can accommodate variable spectral order so that regions of different dynamics are treated accordingly as shown in figure 1, giving us variable $p$-refinement in space.

- The hybrid discretization that *NekTao* uses is also very useful when it comes to $h$-refinement: Layers of long thin quadrilaterals are used next to the cylinder, their width such that the cylinder's boundary layer is resolved within the first layer. A regular array of
small quadrilaterals is used to cover the middle and far wake region. The aim is to resolve
the vortex street while at the same time conserve memory: elements that are translations
of each other share geometric factors, operator matrices etc. A few large quadrilaterals are
used in the inflow and the side-boundaries where the solution varies very little. Standard
automatic mesh generators are used to create an unstructured triangular mesh around the
cylinder and especially in the near wake where h-refinement is required. Finally regular
arrays of triangles are used to gradually change in a conforming manner from the smaller
element sizes in the cylinder and wake region to the larger ones in the side boundaries.

These techniques enable us to put a given number of degrees of freedom (dictated by memory and
computation time constraints) to better use by using most of the resolution where it is needed,
that is in the boundary layer and the wake region.

More details about the numerical method and the discretization employed in the current
simulations can be found in Appendix A.

Validation of the new code NekTet was accomplished by repeating laminar flow simulations
as in [17] and also by comparing with experimental results.

The computational domain for the $x-y$ plane used for the $Re = 1000$ (as well as any $Re = 500$)
calculations extends $69d$ (cylinder diameters) downstream, and $22d$ in front, above and below the
cylinder. For 3D simulations the spanwise length is $L/d = 4\pi$ for all simulations, unless otherwise
stated. This also sets the wavelength of vibrations for the flexible cylinders. A $K = 1018$ element
hybrid mesh was used for the $Re = 1000$ calculations (see figure 1), with a slightly smaller $K = 988$
element one for any $Re = 500$ ones. The polynomial order used varied across the domain as shown
in figure 1. Two different resolutions were used in the z-direction for all cases unless otherwise
stated: 32 z-planes (16 independent Fourier modes) for the stationary and rigidly moving cylinder
simulations and 64 z-planes for cables and beams; a 3/2 de-aliasing rule was always used. For the
largest cases this meant more than 19 million degrees of freedom over the 3 velocity fields and
pressure. For all but the stationary cylinder calculations (which can sustain twice the timestep
size) CFL limitations force a non-dimensional time step of $\Delta tU/d = 0.0005$ to be used giving over
Figure 2: Fluctuating lift and drag coefficients for flow past a stationary cylinder at $Re = 1000$: Pressure forces (top) and viscous forces (bottom).
10,000 time steps per shedding cycle.

A systematic validation study was performed for the stationary turbulent wake at $Re = 1000$ and comparisons were made against earlier simulations using mortar spectral element simulations [42] and also accepted experimental results (see Norberg (1994) [43] and references therein). A summary of this comparison is shown in table 4.1, where we see that the current simulation results are much closer to the experimental results than the computations reported in [42]. A breakdown of the contribution to lift and drag of pressure and viscosity generated forces is shown in figure 2.

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<th>$-C_{pb}$</th>
<th>$C_D$</th>
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<th>$\bar{C}_L$</th>
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<tr>
<td>$2D$ current</td>
<td>0.238</td>
<td>1.7138</td>
<td>1.5406</td>
<td>0.1715</td>
<td>1.0696</td>
</tr>
<tr>
<td>$3D$ [42]</td>
<td>0.211</td>
<td>1.1225</td>
<td>1.1999</td>
<td>0.0491</td>
<td>0.2052</td>
</tr>
<tr>
<td>$3D$ current</td>
<td>0.202</td>
<td>0.8429</td>
<td>1.0192</td>
<td>0.0348</td>
<td>0.0991</td>
</tr>
<tr>
<td>experiment</td>
<td>0.21</td>
<td>0.8</td>
<td>1.0</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

Table 4.1: Comparison of simulation results with experimental results for flow past a stationary cylinder at $Re = 1000$. Shown are the values of the Strouhal number $St$, the base pressure coefficient $-C_{pb}$, the average drag coefficient $C_D$ and the standard deviation of the value of the coefficient of drag $\bar{C}_D$ and lift $\bar{C}_L$ respectively.

The "virtual spring" employed in Chapter 3 was not used for these simulations for cables and beams: the transients it introduces would take very long to die off and for calculations as costly as these this was impractical. Use of enough structural damping to kill-off the transients relatively quickly was also found to diminish the overall vibration amplitude substantially and hence was undesirable. Thus for the cases of simulations with free periodic boundaries we opted instead to ignore any mean (along the span) force components in our structure solver, concentrating only on the spanwise fluctuating ones. The simulations with pinned endpoints are naturally constrained in both transverse directions; therefore no extra care is needed.

A further simplification that we employ is that we only consider here the dominant motion which is in the cross-flow direction; that is, we neglect the effect of the streamwise motion, whose amplitude is typically 10% to 15% of that in the cross-flow motion. This is done in order to study the inter-relation between lift and displacement in the vertical direction without the added
complication of coupling effects with the horizontal direction motion.

Finally, as already explained, since we are interested in the maximum possible response we neglect the effect of structural damping.

The simulations were performed on 16 and 32 processors of the NAVO Cray T3E-900 and the MHPCC IBM SP2 with 160MHz P2SC nodes and took between 7 and 12 seconds per time step, depending on the problem.

3 Spatio-Temporal Variation of Amplitude, Lift and Drag

The self-limiting behavior of VIV as the damping goes to zero has been well documented by the various experimental data compiled by Griffin (1992) [44] and also reproduced in the simulations of Newman & Karniadakis (1997) [17]. More specifically, the simulations in [17] were for low Reynolds number \((Re \leq 200)\) and underpredicted the maximum amplitude at zero damping, especially the two-dimensional simulations. First, we examine possible Reynolds number effects in the two-dimensional simulations, by simulating lock-in states at \(Re = 100\) and \(Re = 1000\) but otherwise identical conditions.

In figure 3 we plot the results from these two-dimensional simulations. We see that asymptotically (after the transients die out) approximately the same response is produced in both simulations. More specifically, a small increase of less than 10% in the amplitude is observed as we increase the Reynolds number from \(Re = 100\) to \(Re = 1000\). We also note that in the case of \(Re = 1000\) a sudden increase in the amplitude is associated with a corresponding increase of the lift coefficient and a jump in phase between those two quantities. During this transient stage the lift force and the cross-flow amplitude are in-phase, as shown in the bottom plot of figure 3, and the amplitude reaches values of 0.7\(d\). The two-dimensional simulations at higher Reynolds number underpredict significantly the maximum amplitude response at zero structural damping which according to the experiments is \(y/d \approx 1\) (see Griffin (1992) [44]). As regards the frequency response, at \(Re = 100\) the input (non-dimensional) frequency, i.e. the structure eigenfrequency
Figure 3: Two-dimensional simulations at $Re = 100$ (left) and $Re = 1000$ (right). Top: Cross-flow displacement history; Middle: Lift coefficient history; Bottom: Phase portrait.
was set to $f_n = 0.167$, which is equal to the Strouhal frequency of the stationary cylinder flow at the same Reynolds number. The resulting (non-dimensional) structure frequency and wake frequency obtained from the simulation were 0.163 and 0.160, respectively. Similarly, at $Re = 1000$ the input frequency was set to 0.238 and the structure and wake frequency obtained from the simulation was 0.238. Therefore, for these two-dimensional simulations if the structure eigenfrequency matches the Strouhal frequency of the corresponding stationary flow, the frequency of the coupled structure-system deviates very little. We will see that this is not the case for the three-dimensional simulations.

3.1 VIV of rigidly moving cylinder

![Diagram](image)

Figure 4: Rigid cylinder: Cross-flow displacement versus time (left) and span-averaged lift coefficient versus cross-flow displacement (right).

In the following we will concentrate exclusively on three-dimensional simulations. First, we present results from simulations of flow past a rigid cylinder at $Re = 1000$ subject to VIV. We see in figure 4 (left) that a slightly modulated harmonic motion is produced with maximum amplitude $y/d \approx 0.74$, which is larger than the corresponding value of the two-dimensional simulation of $y/d \approx 0.55$. This motion is in-phase with the span-averaged lift coefficient as revealed in the phase portrait shown in figure 4 (right) in agreement with the experiments of Brika & Laneville.
Note that in the two-dimensional motion (see figure 3) the same large amplitude is obtained during the transient short period when the lift coefficient is in-phase with the cross-flow motion (between times 130 to 140 non-dimensional time units). Correspondingly, the values of the two-dimensional (during that transient period) and span-averaged three-dimensional lift-coefficient are also close.

The lock-in state of the freely oscillating rigid cylinder is described by a two-branch response as it was documented in the detailed experiments of Brika & Laneville (1993) [45] and Khalak & Williamson (1996) [46]: An upper branch that corresponds to large amplitude and low values of reduced velocity, and a lower branch that corresponds to low amplitudes and large values of the reduced velocity. A similar result was also obtained by Hover, Techet & Triantafyllou (1998) [25] at a Reynolds number \( Re = 3800 \), which is lower than in [46] but at comparable (small) values of the structural damping. The classical results of Feng (1968) [47] were obtained for relatively large damping (see also Brika & Laneville (1993) [45]) but they show essentially the same response at reduced levels. By comparing the numerical results here with both sets of recent experiments, it appears that the three-dimensional simulations capture the upper branch corresponding to an oscillation in-phase with the lift coefficient. There is also agreement in the amplitude of oscillation with the experimental data, especially with the data of Hover et al. (1998) [25], which having been obtained at \( Re = 3800 \) are closer to the Reynolds number in our simulation. On the other hand, it seems that the two-dimensional simulations capture the lower branch of the amplitude response curve corresponding to an oscillation, which is not phase-synchronized with the lift force as shown in figure 3. This observation was first made by Khalak & Williamson (1996) [46] in comparing their experimental data with two-dimensional simulations reported in [15].

The rigid cylinder is allowed to oscillate only in the cross-flow direction and therefore the motion is uniform along its axis. However, the corresponding flow is strongly three-dimensional as shown by the spanwise distribution of lift coefficient in figure 5. It exhibits strong cellular structure with peaks exceeding the peaks of the span-averaged coefficients by almost 50%. The same cellular structure in the span-time domain is present in the drag coefficient (see middle plot.
Figure 5: Rigid cylinder: Top - Lift coefficient along the span versus time; Middle - Drag coefficient along the span versus time; Bottom - Energy exchange along the span versus time.
in figure 5) and the energy exchange between the cylinder and the flow (see bottom plot in 5).

As regards the frequency response, the input non-dimensional frequency was 0.238 equal to the Strouhal frequency of the two-dimensional cylinder wake at $Re = 1000$. The resulting structure frequency (obtained from the spectrum of the cross-flow motion) was 0.197 and the wake frequency was 0.201. Note that the Strouhal frequency of the corresponding three-dimensional wake is 0.202. We see therefore here that the coupled structure-flow system has a frequency response that greatly deviates from the imposed frequency of the structure (here 0.238). This non-classical result has also been reported in experimental studies recently by Gharib et al. (1998) [3] and it was found to be a function of the mass ratio: for large values of mass ratio the frequency response approaches the classical lock-in behavior.

3.2 Flow past a flexible cylinder in prescribed motion

![Figure 6: Flexible cylinder subject to forced oscillation: Standing wave pattern (top) and lift coefficient (bottom).](image)

To identify differences in forces due to the cylinder deformation only, we simulated next a flexible cylinder subject to prescribed cross-flow vibrations corresponding to a standing wave pattern.
as shown in figure 6. The amplitude of the oscillation is set to match the amplitude of the free oscillation of the rigid cylinder at $Re = 1000$, and the frequency is set to 0.197, close to the Strouhal frequency of a stationary cylinder and equal to the frequency that the rigid cylinder in subsection 3.1 freely vibrated at. The lift force is in phase with the cylinder cross-flow motion as shown in figure 6, and the maximum amplitude of the lift coefficient is lower than the free-oscillating rigid cylinder by about 20%.

3.3 VIV of a cable

![Graph of VIV of a cable](image)

Figure 7: Cable: Cross-flow displacement along the span versus time. The initial part of the simulation is shown at the top and a later part at the bottom.

For the free periodic boundaries that we use, the standing wave response is not a stable state if the cylinder motion is due to vortex shedding (VIV) and not a prescribed motion. This was also shown to be the case for the laminar wake at $Re = 100$ [17], where an initial standing wave response, typically although not always, turns into a traveling wave response. The details of the motion depend on the type of structure at high Reynolds number, and in particular its bending stiffness; this structural dependence is exhibited only in the transitional and turbulent wakes. We
examine this issue first by considering the cylinder as a flexible cable and modeling its dynamics by a wave equation (see equation 5 in Chapter 2).

The initial conditions correspond to the case of a forced standing wave pattern discussed above in subsection 3.2. In figure 7 we plot the cross-flow displacement along the span as a function of time. We see that the standing wave response turns into a modulated traveling wave response after approximately 5 shedding cycles. Therefore, the coupled cable-flow response in the early turbulent regime at $Re = 1000$ is qualitatively similar to the response at laminar conditions at an order of magnitude lower in Reynolds number [17]. However, the traveling wave in the turbulent state is subject to significant modulation unlike the laminar case. We will discuss the corresponding flow structures in section 6. The cross-flow amplitude reaches values of $1d$ in the initial stages, during the transition from standing to traveling wave, and maintains values as high as $0.9d$ and slightly higher afterwards.

![Figure 8: Cable - demodulation analysis: Phase difference between the cross-flow displacement and the lift coefficient (top); amplitude of the lift coefficient (bottom). The sharp discontinuities are due to the fact that individual phases are calculated modulo 180 degrees before their difference is calculated (also modulo 180 degrees).](image)

Unlike the previous cases where we found that the lift coefficient and the cross-flow displace-
ment are essentially in-phase, in the case of the freely vibrating cable this is not true. To quantify this phase difference we employ complex demodulation analysis, which is a more general approach than harmonic analysis in dealing with non-exact periodic time series (see Bloomfield (1976) [48]). A complex demodulation of a time series \( S(t_n) \) with a dominant frequency component \( \lambda \) will give a time varying amplitude \( R(t) \) and phase \( \Phi(t) \) such that:

\[
S(t_n) \approx R(t_n)e^{i(\lambda t_n + \Phi(t_n))}
\]  

In figure 8 we plot the amplitude \( R(t) \) of the lift coefficient and its phase difference with respect to the motion. We see that there is a very strong correlation between regions of almost zero phase difference and maximum amplitude in the lift coefficient. However, and unlike the previous cases, there are also regions with sharp jumps in the phase difference between lift forces and cable motion. As regards the frequency response, here the input (non-dimensional) frequency was set to 0.197, close to the Strouhal frequency of the corresponding stationary three-dimensional wake and equal to the frequency that the rigid cylinder in subsection 3.1 freely vibrated at. The resulting frequency of the structure and wake were 0.176 and 0.1876, respectively, again deviating from the classical lock-in response, in agreement with the findings in Gharib et al. (1998) [3].

3.4 VIV of a beam

Next, we examine the response of a beam, i.e. a structure with finite bending stiffness, under identical conditions as in the cable case discussed above, including identical initial conditions. The bending stiffness is chosen so that the same structural mode \( n \) as before is mainly excited: that is using the formula \( \beta^2 n^2 EI = T \) (see Chapter 2) with \( \beta = 2\pi/L = 0.5 \) and \( n = 1 \). We see again from the simulations that the initially prescribed standing wave response turns into a modulated traveling wave response as in the cable case, but the path of transition as well as the asymptotic state are different. Therefore, different spanwise modes are excited in the beam case compared to the cable case as can be seen by comparing equations 12) and (13 of Chapter 2 in Fourier
Figure 9: Beam: Cross-flow displacement along the span versus time. The initial part of the simulation is shown at the top and a later part at the bottom.

space. In figure 9 we plot the cross-flow displacement versus time, and we see this transition from a standing to a modulated traveling wave response. However, we also observe clear differences with the corresponding plot for the cable in figure 7. Again, the cross-flow amplitude reaches values of 1d in the initial (transitional) stages and maintains values over 0.8d afterwards. For both the cable and beam cases, the 1d amplitude reached is much larger than the 0.74d of the initial conditions to the simulations - this indicates that amplitudes of that size are possible at Re = 1000 - though not necessarily representing an asymptotic state in time.

To quantify the phase difference between the beam motion and the corresponding lift coefficient, we employ again complex demodulation analysis to construct the plot in figure 10 for the amplitude of the lift coefficient and its phase difference with respect to the motion. We see that the maximum lift coefficient is subject to very large modulation following the large variation in phase difference. For example, regions of small phase difference (less than 10 degrees) result in values of maximum lift coefficient of more than $C_l \approx 2$ but phase differences of 90 degrees or higher are also possible leading to lift coefficient amplitudes of less than $C_l \approx 0.5$. As regards
the frequency response, the input frequency was set to 0.197 as in the case of the cable, and the resulting structure and wake frequencies were 0.178 and 0.1736, respectively.

3.5 Effects of aspect ratio and pinned endpoints

We simulate four different cases of turbulent flow past a flexible beam in order to investigate both the effect of the aspect ratio of the beam as well as the effect of the boundary conditions in the spanwise direction. Specifically, we consider both periodic ends (case A) as well as fixed ends (Case B); with subscripts (s) and (l) we will denote the short beam ($L/d = 4\pi$) and long beam ($L/d = 378$), respectively. Regarding initial conditions, for the short beam for case As we start by prescribing a standing wave as the initial state. For case Bs we start from simulation results of a stationary cylinder at $Re = 1000$. For the long beam we interpolated results from the shorter beam by increasing the cylinder length gradually from $L = 4\pi$ to $L = 378$ for case Al. For the fixed ends (case Bl) we used as initial conditions results from the freely moving endpoints case (A0).
Figure 11: Short beam: Cross-flow displacement along the span versus time. Top: periodic ends; Bottom: fixed ends.

Figure 12: Long beam: Cross-flow displacement along the span versus time. Top: periodic ends; Bottom: fixed ends.
In figure 11 we plot the cross-flow displacement of the short beam for both boundary conditions versus time. For case A, a transition from the initially prescribed standing wave to a modulated traveling wave response takes place: only the asymptotic stable traveling response is plotted in the figure. For case B, we also plot the asymptotic stable standing wave response, and we see that the maximum amplitude is more than one cylinder diameter, and in fact about 20% higher than the asymptotic traveling wave response. In figure 12 we plot the cross-flow displacement of the long beam for both boundary condition versus time. We see that the response is similar as in the short beam case but the amplitude of vibration is reduced. Moreover, there seems to be a substantial motion of the middle “node” for the standing wave response, as instantaneously more than one node exist around the mid-span, in contrast to the response of the short beam. This result is in agreement with experimental results as well as field data [9].

![Figure 13: Short beam: Lift coefficient along the span versus time. Top: periodic ends; Bottom: fixed ends.](image)

In figure 13 we plot the lift coefficient of the short beam. We see that for case A, the maximum lift coefficient is subject to very large modulation following the large variation in phase difference, unlike the freely oscillating rigid cylinder. For example, regions of small phase difference (less than
Figure 14: Long beam: Lift coefficient along the span versus time. Top: periodic ends; Bottom: fixed ends.

10 degrees) result in values of maximum lift coefficient of more than \( C_l \approx 2 \), but phase differences of 90 degrees or higher are also possible leading to lift coefficient amplitudes of less than \( C_l \approx 0.5 \). For case \( Bs \), the lift variation is also large but it follows the standing wave response. The same cellular patterns are present in the long beam but with the maximum values of the lift coefficient quite larger compared to the values for the shorter beam. Specifically, for case \( Al \) we obtained \( C_l \approx \pm 3 \) and for case \( Bl \) we obtained \( C_l \approx \pm 3.5 \).

In figure 15 we plot the drag coefficient for the short beam. Very large values of the drag coefficient are obtained locally for both boundary conditions. These values are about three times higher than the drag coefficient for a stationary cylinder. The same is true for the long beam. Specifically, the maximum value for \( C_d \) is approximately 3.4 and 4.0 for case \( Al \) and \( Bl \), respectively. Similarly, the minimum value of \( C_d \) is approximately 0.7 (case \( Al \)) and 0.55 (case \( Bl \)).
Figure 15: Short beam: Drag coefficient along the span versus time. Top: periodic ends; Bottom: fixed ends.

Figure 16: Long beam: Drag coefficient along the span versus time. Top: periodic ends; Bottom: fixed ends.
4 Time-Averaged and Span-Averaged Forces and Displacement

Figure 17: Standard deviation of the displacement along the span for a short beam (left) and a long beam (right). The solid line is the rigid line response and the dash line is the traveling wave response.

In figure 17 we plot the standard deviation of the motion of the beam with fixed ends along its axis for the short and long beam. We also include for reference the corresponding values for the rigid cylinder and the traveling wave response obtained for similar conditions. We see that for the short beam the traveling wave response is very close to the oscillating rigid cylinder, although the former corresponds to larger values of maximum amplitude. Note that the magnitude at the node is non-zero as there is some small movement of the node, which is more pronounced especially for the long beam. Also, the displacement of the long beam is lower than the short beam and similar behavior has been reported in [18]. This difference may also be due to the relatively lower resolution along the span employed in the longer beam but this could not be quantified at the present time. It is clear, however, that the smaller flow scales cannot be resolved sufficiently since for the long beam the grid spacing is about 6d in that case as compared to 0.2d in the short beam.

In figure 18 we plot the standard deviation of the lift coefficient of the beam with fixed ends along its axis for the short and long beam. We also include for reference the corresponding values for the rigid cylinder and the traveling wave response. We see that the span-averaged value is
Figure 18: Standard deviation of the lift coefficient along the beam for a short beam (left) and a long beam (right). The solid line is the rigid line response and the dash line is the traveling wave response.

about the same for cases $A_s$ and $B_s$, and similarly for the long beam. The value for the rigid cylinder is substantially larger compared to all cases simulated.

Figure 19: Variation of the mean drag coefficient along the beam for a short beam (left) and a long beam (right). The solid line is the rigid line response and the dash line is the traveling wave response.

In figures 19 and 20 we plot the mean and standard deviation of the drag coefficient of the beam with fixed ends along its axis for the short and long beam. We also include for reference the corresponding values for the rigid cylinder and the traveling wave response. We see that the mean drag coefficient for all cases is in the range of 1.6 to 1.8, in agreement with the experiments of Alexander [18], except for the rigid cylinder that corresponds to a mean drag coefficient of
Figure 20: Standard deviation of the lift coefficient along the beam for a short beam (left) and a long beam (right).

approximately 2.1. The \textit{rms} values of the drag coefficient for the oscillating rigid are more than 30 times larger than the values of the stationary cylinders, in agreement with the findings of Khalak & Williamson [24].

We now turn our attention to time variation of the span-averaged quantities. In particular, we are examining the cases where homogeneity exists along the spanwise direction, which is not true, for example, for the fixed ends cases. To summarize the results regarding the cylinder lift forces we plot in figure 21 the histories of span-averaged lift coefficient for the rigid freely oscillating cylinder, and the short and long beam. We also include for reference the corresponding values of the coefficients for a stationary cylinder subject to uniform flow at \( Re = 1000 \). We see that the lift coefficient of the freely oscillating rigid cylinder is much larger than all the other cases, and that the stationary cylinder has the smallest mean and \textit{rms} values. Measurements of the lift forces for the rigid cylinder corresponding to very \textit{small} structural damping have only been performed recently by Khalak & Williamson (1997) [24] and by Hover et al. (1998) [25]. It was found that very large values of the lift coefficient are possible at lock-in, close to the values observed in the simulations, although the experimental values are somewhat higher, especially in the experiments of [24] possibly due to the higher Reynolds number.

To summarize the results regarding the cylinder drag forces we plot in figure 22 the histories
Figure 21: Comparison of span-averaged lift coefficient histories for stationary cylinder (top left), freely oscillating rigid cylinder (top right), short beam with free ends (middle left), long beam with free ends (middle right), short beam with fixed ends (bottom left) and long beam with fixed ends (bottom right).
Figure 22: Comparison of span-averaged drag coefficient histories for stationary cylinder (top left), freely oscillating rigid cylinder (top right), short beam with free ends (middle left), long beam with short ends (middle right), short beam with fixed ends (bottom left) and long beam with fixed ends (bottom right).
of span-averaged drag coefficient for the rigid freely oscillating cylinder, and the short and long beam along with the history for a stationary cylinder. Here we can see the very large amplitudes of the drag coefficient compared to the stationary cylinder. Using the DNS data, for example the more accurate data for the short beam, we can also evaluate the empirical formula due to Skop et al. [49]

\[ C_d = C_{d0} \left[ 1 + 1.043 \left( \frac{2Y_{rms}}{d} \right)^{0.65} \right] \]  

(2)

where \( C_{d0} = 1.2 \) is the drag coefficient of a stationary cylinder, \( Y_{rms} \) is the \( rms \) amplitude of the motion, and \( d \) is the cylinder diameter. In figure 23 we plot the prediction from the above equation using the \( rms \) amplitude values for the short beam (see figure 17) against the DNS data. We see that the prediction of equation (2) is in obvious disagreement with the DNS data. Instead, a better approximation is given by

\[ C_d = C_{d1} \left[ 1 + A \left( \frac{2Y_{rms}}{D} \right)^B \right] \]  

(3)

where \( B = 1 \) and \( A = 0.35 \) (shown also in the figure marked with diamonds). Note that \( C_{d1} \)
is here the drag coefficient at the nodes which is about 1.4. Equation (3) is very similar to the
equation used for an oscillating rigid cylinder. For the DNS data presented here, if we take $A \approx 1$
and also $B = 1$ (with $C_{d1}$ the stationary cylinder value) we obtain a value for $C_d$ very close to
the mean drag coefficient for a rigid cylinder as predicted by DNS. Finally, one can improve the
formula in equation (3) by fitting the data as best as possible and select appropriate coefficients
$A$ and $B$. For example with $A = 0.29$ and $B = 1.79$ we obtain the curve marked with squares
in figure 23. In summary, the often-used in engineering analysis formula of Skop et al. seems to
lead to erroneous predictions for the standing wave responses. which are more typical in ocean
engineering applications.

5 Correlation Length and Spectra

5.1 Autocorrelation functions

In studies of VIV the value of spanwise correlation length is very important as many empirical
models rely on it (see Blevins (1990) [1]). However, related studies and experimental measurements
are relatively few, see Toebes (1969) [50], Ramberg & Griffin (1976) [6]. For stationary cylinders,
detailed measurements of correlation length based on the autocorrelation function were obtained
only recently by Mansy, Yang & Williams (1994) [51]. We define here the autocorrelation function
as follows:

$$R_{uu}(l; x, y) = \frac{\overline{u(x, y, z, t)u(x, y, z - l, t)}}{\overline{u^2(x, y, z, t)}}$$

(4)

where the bar denotes averaging over time and over $z$-planes. Also, $u(x, y, t)$ is the fluctuation
obtained after we subtract the mean quantity, i.e. averaged in time and span ($z$) at the $(x, y)$
point.

In figures 24,25 and 26 we plot the absolute value of the autocorrelation function for two
points in the near-wake for the three components of velocity. One point is on the centerline
($x/d = 3; y/d = 0$) and the other one cylinder diameter above it ($x/d = 3; y/d = 1$). We include
Figure 24: Autocorrelation function for the streamwise component of velocity in the near-wake at a centerline point ($z/d = 3; y/d = 0$) (left) and an off-centerline point ($z/d = 3; y/d = 1$) (right).

Figure 25: Autocorrelation function for the cross-flow component of velocity in the near-wake at a centerline point ($z/d = 3; y/d = 0$) (left) and an off-centerline point ($z/d = 3; y/d = 1$) (right).
Figure 26: Autocorrelation function for the spanwise component of velocity in the near-wake at a centerline point \((x/d = 3; y/d = 0)\) (left) and an off-centerline point \((x/d = 3; y/d = 1)\) (right). The results for a stationary, freely oscillating rigid cylinder, a (short) cable and (short) beam.

These results are in good agreement with the results of Toebes (1969) [50] for oscillating rigid cylinders, and of Mansy et al. (1994) [51] for stationary cylinders. In particular, the oscillatory structure in \(|R_{uu}|\) (see figure 24) for a stationary cylinder was also shown in the work of [51]. The autocorrelation function \(|R_{uv}|\) (see figure 25) indicates a high degree of correlation for the rigidly oscillating cylinder, consistent with the experimental observations ([50]; also, M.S. Triantafyllou, private communication). Finally the values for \(|R_{ww}|\) (see figure 26) are comparatively much smaller for all structures, especially in the case of the centerline point.

Looking at the behavior of the autocorrelation functions \(|R_{uu}|\), \(|R_{uv}|\) and \(|R_{ww}|\) in more detail we note:

- \(R_{uu}\) for the cable and beam at the centerline point become negative (hence the "cusp" in the \(|R_{uu}|\) graph) at approximately the mid-quarter spanwise location in agreement with the experimental measurements of Ramberg & Griffin (1976) [6]. For off-centerline locations, the rigidly oscillating cylinder displays the same behavior as the two flexible structures, with two crossings of the horizontal axis at \(L/8\) and \(3L/8\). In this case the actual value for the
cable $|R_{uu}|$ is very small, with slightly larger values for the beam.

- For the stationary cylinder $|R_{uu}|$ and $|R_{uv}|$ decay about equally slowly for the off-centerline point. In the case of the centerline point though, $|R_{uu}|$ decays rapidly while $|R_{uv}|$ decays even more slowly than in the case of the off-centerline point.

- The rigidly moving cylinder exhibits large values of $|R_{uv}|$ for both points (relatively larger in the case of the off-centerline one) - the function does not tend to zero as expected.

- The cable and the beam $R_{uv}$ values initially decay similarly but cross the horizontal axis at slightly different locations for the first time (the cable values around $L/4$, the beam at a lesser value). For larger $dz$ values, both of them cross the axis again for the centerline point but this is not so for the off-centerline one in the case of the cable. The beam values for the centerline point reach the level of those of the rigidly moving cylinder for larger $dz$-while the cable values stay small.

We note again that there is greater similarity between the $R_{uu}$ values for the cable and the beam than for the corresponding $R_{uv}$ ones; it is not clear to us at present why there is such a discrepancy.

5.2 Spectra of the velocities in the near wake

Next, we turn our attention to the pointwise statistics focusing on both the near-wake as well as the structure itself. First, we present energy spectra obtained at the points $(x/d, y/d) = (3, 0)$ (centerline) and $(x/d, y/d) = (3, 1)$ (off-centerline) for a stationary, a freely oscillating rigid cylinder, a cable and a beam.

In figure 27 we plot the results of the streamwise velocity signal by normalizing the frequency with the corresponding dominant frequency of the near-wake for each case. In figure 28 we plot the corresponding results for the cross-flow velocity (vertical component); the spanwise component graphs are in figure 29. We see that even at this relatively low Reynolds number of $Re = 1000$ an inertial subrange of about half a decade or more in wave-number has been established. For
Figure 27: Energy spectra (streamwise velocity) for stationary (s), freely oscillating rigid cylinder (r), cable (c) and beam (b) at a point in the wake at centerline ($x/d = 3, y/d = 0$) and off-centerline ($x/d = 3, y/d = 1$).

Figure 28: Energy spectra (cross-flow velocity) for stationary (s), freely oscillating rigid cylinder (r), cable (c) and beam (b) at a point in the wake at centerline ($x/d = 3, y/d = 0$) and off-centerline ($x/d = 3, y/d = 1$).
Figure 29: Energy spectra (spanwise velocity) for stationary (s), freely oscillating rigid cylinder (r), cable (c) and beam (b) at a point in the wake at centerline \((x/d = 3, y/d = 0)\) and off-centerline \((x/d = 3, y/d = 1)\).

The stationary cylinder, scales which are higher than 10 times the Strouhal frequency are rapidly decaying, however for the rigid cylinder even high frequency components are very energetic, with the cable/beam ones somewhat less energetic. We note that:

- **For the centerline point:**
  - Only the cross-flow velocity statistics exhibit a sharp peak at the Strouhal frequency.
  - The inertial range for the stationary cylinder is the widest, and for at least half a decade the spectrum for the beam and the cable is at the same level, with that of the rigidly moving cylinder slightly below.
  - In the smallest scales, the situation is reversed.

- **For the off-centerline point:**
  - All velocity statistics exhibit a sharp peak at the Strouhal frequency.
  - The inertial range for the rigidly moving cylinder, cable and beam appears to be common, except for the case of the cross-flow velocity where the rigidly moving cylinder spectrum falls off the \(-5/3\) slope first.
  - In all cases the spectrum for the stationary cylinder is below that of all the other cases past the Strouhal frequency.
- The cross-flow velocity spectrum exhibits a strong $2f$ superharmonic response for all the moving structure cases.

5.3 Excitation of the structural modes

It is clear from the results presented above that the cable-flow response is substantially different than the beam-flow response at this Reynolds number ($Re = 1000$), despite the fact that the scaled wake velocity spectra of the beam and cable wakes were shown to be almost identical. At lower Reynolds number, i.e. $Re = 100$, the responses are identical as the only excited mode is the first one ($n = 1$) [17]. We have found in our simulations described in Chapter 3 that for $100 \leq Re \leq 200$ a transition takes place for the cable-flow system and other modes, in addition to the first one, are excited although at reduced levels. However, for the beam-flow system this transition takes place at substantially higher Reynolds number ($Re_c \geq 500$) although we could not bracket exactly the critical value.

![Graphs](image)

Figure 30: Cable (left) and Beam (right): Time-history of the (square) amplitude of the first 4 modes of cross-flow displacement.

At $Re = 1000$ that we study here we have decomposed the amplitude of the vibration into Fourier modes along the spanwise direction. In figure 30 we plot the amplitude of the first 4 modes of the cable motion versus time. We see that although the first mode dominates, the other
3 modes contribute a non-negligible amount to the energy with the second and third mode almost at the same level. The same type of decomposition is shown in figure 30 for the beam, with the important difference that here all modes are clearly separated and at reduced levels compared to the cable modes. This is consistent with our aforementioned finding that the critical Reynolds number for transition is lower for the cable than for the beam.

6 Flow Visualizations

6.1 Two-Dimensional Simulations

We first examine flow structures present in the two-dimensional simulations at $Re = 1000$. As we can see from the variation of lift as a function of time in figure 3 there are two distinct states that the flow goes through before it reaches its asymptotic state. (The initial conditions at $t = 100$ correspond to a converged flow past a stationary cylinder.) In figure 31 we plot the instantaneous vorticity field at non-dimensional time $t \approx 122$, which corresponds to approximately five shedding cycles after the initial state; at that instance the lift coefficient and the motion are out of phase by 180 degrees. We see that the near-wake corresponds to a $2P$ pattern, the intermediate wake to a $P+S$ pattern, and the far-wake to a $2S$ (see 34). Here we use the terminology introduced by Williamson & Roshko (1988) [52] to characterize the shedding patterns in flow past an oscillating cylinder, with $2S$ being the standard von Karman mode, $2P$ a pattern with two pairs of vortices per shedding cycle, and $P+S$ a pattern of three vortices per shedding cycle. In figure 32 we plot vorticity contours at $t \approx 136$, which corresponds to a cylinder motion in-phase with the lift and a relatively large amplitude of oscillation (see figure 3). We see that the near-wake exhibits $P+S$ patterns and the vortex street becomes unstable beyond $x/d \approx 10$. In the far-wake we observe a mixed response downstream consisting of $2P$ pattern (at 20$d$ and 30$d$) and $P+S$ pattern (at 40$d$) (see figure 35). Finally, in figure 33 we plot vorticity contours representative of the asymptotic state at $t \approx 176$. We see that a $2P$ shedding pattern emerges in the near-wake, followed by a $2S$ pattern, before the vortex street becomes unstable farther downstream.
Figure 31: Vorticity contours at $t \approx 122$. (two-dimensional simulation, $Re = 1000$). Only the near-wake is shown for clarity.

Figure 32: Vorticity contours at $t \approx 136$. (two-dimensional simulation, $Re = 1000$). Only the near-wake is shown for clarity.

Figure 33: Vorticity contours at $t \approx 176$. (two-dimensional simulation, $Re = 1000$). Only the near-wake is shown for clarity.
The coexistence of such mixed response, revealed in our simulations, has also been discovered by Ongoren & Rockwell (1988) [39] in their systematic experimental investigation of cylinders subject to prescribed oscillations. At low Reynolds number, \( Re = 200 \), independent simulations by Newman & Karniadakis (1995) [15] and Meneghini et al. (1997) [53] have shown that the standard 2S pattern is maintained for the oscillating cylinder, similar to the stationary cylinder but with a larger lateral vortex spacing. At Reynolds number \( Re = 500 \) we found in our simulations (see figure 36) that at lock-in a \( P + S \) shedding pattern prevails, which may result in a symmetry-breaking bifurcation of the motion. However, at states somewhat far from the resonant state the 2S mode re-appears and symmetry is restored.

![Vorticity contours at \( t \approx 122 \). (two-dimensional simulation, \( Re = 1000 \)).](image1)

![Vorticity contours at \( t \approx 136 \). (two-dimensional simulation, \( Re = 1000 \)).](image2)

The amplitude of the oscillation is low for all the cases reported except from the short interval between time \( t \approx 130 \) and \( t \approx 145 \) in figure 3. These results agree with the results of Saltara, Siqueira, Meneghini & Bearman (1998) [54], who have also computed similar values for the amplitude of the motion and have observed both \( 2P \) and \( 2S \) shedding patterns depending on the exact
value of the reduced velocity. The small amplitudes observed in the two-dimensional simulations suggest that the resonant branch simulated is actually the lower one as observed in the experiments of Brika & Laneville (1993) [45] and Khalak & Williamson (1996) [46]. The simulations show that in this case a $2P$ shedding pattern prevails in the near-wake consistent with the experimental observations. However, the $P+S$ flow pattern simulated in figure 32, which is only stable for a short time interval, corresponds to the upper branch of the resonant curve, for which a $2S$ pattern was assigned in the experiments in [45] and [46]. The reason for that discrepancy between simulation and experiment is that the third vortex present in the $P+S$ pattern is so weak that it diffuses very quickly and disappears almost immediately after the first pair of vortices, so it is very hard to be captured in the experiments. At Reynolds number $Re = 500$ this $P+S$ pattern survives and is convected a large distance downstream before the vortex street breaks down at about $30d$.

![Vorticity contours](image)

Figure 36: Vorticity contours at $t = 300$. (two-dimensional simulation, $Re = 500$). The near wake shown at the top exhibits the $P+S$ pattern and the whole domain shown at the bottom shows the pattern persists for a while until the vortex street breaks down.
6.2 Three-Dimensional Simulations: Visualization Approach

![Figure 37: DNS of flow past a flexible beam at Reynolds number 1,000. Top left: Total vorticity magnitude at levels 3.0 and 5.5; Top right: Spanwise vorticity at -1.7 and 1.7; Bottom left: Pressure isosurfaces at -0.18 and -0.24. Bottom right: Isocontour at -1.3 of the second largest eigenvalue using the technique of Jeong & Hussain (1995). All values are non-dimensionalized with respect to the flexible cylinder diameter and the freestream velocity.](image)

*NekTao* uses unstructured and hybrid domains, which lead to a distribution of output data in a very non-uniform grid. Most available visualization packages suffer from severe accuracy degradation when handling data in non-uniform grids. To this end, we first interpolate the output of *NekTao* using spectral interpolations on a uniform grid by respecting the dynamic changes in the geometry, and subsequently we perform the visualizations.

We are interested in studying the near-wake vorticity dynamics and so vorticity visualization is a standard approach used. While this is useful and admits a fairly simple interpretation in two-dimensions, in three-dimensions vorticity visualizations are substantially more difficult to perform.
and to interpret. For example, if vorticity magnitude is used as a scalar quantity to visualize the near-wake in flow-structure interactions, important details of the dynamics cannot be presented. Instead, we can use isosurfaces of pressure at appropriate levels to visualize details of the wake. A more accurate approach is to find the vortex cores using the approach pioneered by Jeong and Hussain [55] based on the eigenvalues of the velocity gradient tensor, but this is rather costly.

To contrast the visualization results obtained using the different methods, we plot in figure 37 isosurfaces of vorticity magnitude at two different values, and pressure isosurfaces. In particular, we employ a solid surface at a minimum (negative value) and another grid surface at a value slightly above the previous one. To demonstrate that the iso-pressure surfaces so tracked are representative of the corresponding vorticity dynamics, we also plot in figure 37 a similar plot of the spanwise vorticity following the same two-surface tracking approach, as before. We see that indeed the pressure isosurface and the spanwise vorticity isosurfaces present the same instantaneous picture in the near-wake. We also include a plot of the isocontour of the second largest eigenvalue following the technique of Jeong & Hussain [55], which shows the resemblance with the other plots but it is not as informative as the pressure isosurface plot. In general, it is more efficient to follow pressure isosurfaces than vorticity isosurfaces, so in the following we will adopt the former approach in the case of incompressible flow, and the tracking of density isosurfaces in the case of compressible flow.

6.3 Low Re vs. high Re flow past a rigidly moving cylinder

The flow patterns in the three-dimensional simulations are substantially different than the two-dimensional ones not only for flows past a cable or a beam where the oscillation amplitude varies along the span, but also for the freely moving rigid cylinder.

To contrast low Reynolds number shedding with shedding at $Re = 1000$ we first plot in figure 38 pressure isocontours at $Re = 300$ for the freely oscillating rigid(ly moving) cylinder. We see that shedding is parallel in the near-wake and that the first strong interaction occurs at a location $x/d \approx 10$. The near-wake also appears two-dimensional up to that location. The corresponding
shifting pattern is of the type $2S$ as in the low Reynolds number two-dimensional simulations.

At $Re = 1000$ the vortex tubes shown clearly in the low Reynolds number simulations are very deformed and are only visible in the very near-wake. For the stationary cylinder (not shown here) the shedding is nominally parallel, although there is substantial three-dimensional structure in the form of small cells along the span. For the rigid freely oscillating cylinder shown in figure 39 shedding is also parallel, and by examining several slices of the flow along the span we identify a $2P$ pattern. Therefore, there is a switch from the $2S$ mode to a $2P$ mode as the Reynolds number increases, similar to the qualitative change in the two-dimensional simulations, and in accord with the visualizations of Sheridan et al. [56].

6.4 $Re = 1000$ flows past flexible cylinders

In figure 40 we plot similar pressure contours for flow past a cable at $Re = 200$. In this case an oblique shedding pattern is present, which is caused by the traveling wave along the cable as shown also in [17]. The angle of the oblique shedding with respect to the axis of the cable is approximately 18 degrees. Notice that the direction of the traveling wave, and thus, of the oblique shedding depends on the initial conditions; for example, different initial conditions may result in oblique shedding at $-18$ degrees. Although not very clear, the shedding pattern appears to be of $2S$ type.

For the beam and the cable the vortex shedding at $Re = 1000$ has changed significantly compared to the low Reynolds number states. In figure 41 we plot pressure isocontours of the flow past a beam. We see that part of the wake corresponds to parallel shedding, similar to the rigid cylinder, and part corresponds to oblique shedding at an angle of about 30 degrees. The same mixed shedding pattern is observed for cables as shown in figure 42, where we plot pressure contours at the same time instance as the beam ($t = 508.09$). The oblique shedding is at a different angle (approximately 40 degrees) compared to the beam, and the overall structure in figures 41 and 42 are quite different. This underlines the importance of the structure type, which is the only difference between these cases. We recall that spectra taken for the beam and
Figure 38: Rigidly moving cylinder at $Re = 300$: Pressure isocontours at values $-0.3$ and $-0.4$. View facing the outflow from the side (top) and facing the inflow from the side (bottom).
Figure 39: Rigidly moving cylinder at $Re = 1000$: Pressure isocontours at values $-0.3$ and $-0.2$. View facing the outflow from the side (top) and facing the inflow from the side (bottom).
cable in the near-wake are almost identical at the small scales but that difference appear the autocorrelation functions. We see from the flow visualizations here and the spectra in figures 27 and 28 that statistical differences between the cable and the beam flows are substantially smaller than instantaneous differences or real flow structures.

The parallel-oblique pattern alternates from side to side along the span. In figure 43 we plot pressure contours again for the cable at a different time instance \(t = 487.34\) for which the cable is approximately in anti-phase with the cable at time \(t = 508.09\) (see figure 42). We see that the parallel and oblique shedding patterns are on opposite sides in the two plots. This behavior is also seen clearly for the beam (see figure 44). These as well as other flow visualizations not shown here suggest that the onset of oblique shedding coincides with the node which is traveling along the span. This creates a discontinuity in phase or a vortex dislocation (see Williamson (1996) [11]), which causes vortex filaments to turn with respect to the axis of the cable resulting in oblique shedding. These vortex filaments produce patterns similar to what have been termed as phase shocks by Miller & Williamson (1994) [57] who have created such structures experimentally using variable suction at the ends of a stationary cylinder. There is, however, a difference with the
Figure 41: Beam at $Re = 1000$: Pressure isocontours at values $-0.25$ and $-0.18$ ($t = 508.09$).

Figure 42: Cable at $Re = 1000$: Pressure isocontours at values $-0.15$ and $-0.12$ ($t = 508.09$).
structures in our visualizations; unlike the experiment of Miller & Williamson (1994) [57] where the discontinuity point is fixed, here it coincides with the node, which is moving with the phase speed along the span of the cable or beam. In the experiments with the stationary cylinder, therefore, the inclined vortex filaments are located towards one side of the cylinder, whereas in the cable or beam simulations the inclined filaments alternate sides.

The form of the shedding patterns is important as it is directly related to the forces on the structure. For example, examination of the instantaneous lift force spatial distribution shows that the maximum of lift coefficient corresponds to oblique shedding downstream. This is clearly shown by plotting the lift coefficient on the cable in figure 45 and marking the locations $t = 508.09$ and $t = 487.34$ corresponding to visualizations in figures 42 and 43, respectively.

A fundamentally different pattern is produced if the endpoints of the flexible cylinder in the span are pinned. In this case, a standing wave describes approximately the motion of the flexible cylinder. In figure 46 we plot instantaneous pressure contours as before. The shedding patterns for this case are different than before, as the vortex tubes shed off the cylinder are bent towards the flow direction with fixed vortex dislocations at the two fixed nodes along the span. These
Figure 44: Beam at \( Re = 1000 \): Top - Pressure isocontours at values \(-0.24\) and \(-0.18\) \((t = 490.09)\). Bottom - Pressure isocontours at values \(-0.24\) and \(-0.18\) \((t = 532.09)\).
flow structures appear to be a higher $Re$ "relative" of the three-dimensional staggered pattern of lambda-type vortices observed in low $Re$ simulations ([15], [16], [17]).

7 Summary and Discussion

In this chapter we investigated the flow past rigidly moving and flexible cylinders subject to vortex-induced vibrations in the lock-in regime. We chose to perform all simulations at $Re = 1000$ as this corresponds to an order of magnitude increase compared to our studies in Chapter 3 and previous work in our group ([17]) and at the same time the corresponding flow exhibits a turbulent wake. In addition, with this choice we can keep the expense of the computations reasonable and still maintain very high numerical accuracy. As regards the aspect ratio value of $L/d = 4\pi$ which was used in all but the "long" beam simulations, its selection represents a reasonable choice dictated by physical reality (see Vandiver (1991) [5]), computational feasibility, and the results of our previous studies in [17], where the aspect ratio was varied up to a value of 200. Due to the periodic boundary conditions we employ, the physical system that we consider corresponds to a
Figure 46: Standing wave response for a (short) beam with pinned endpoints at $Re = 1000$: Pressure isocontours at values $-0.1$ and $-0.2$. The different snapshots correspond to times one-fifth of the shedding cycle apart (top-left; top-right; bottom-left; bottom-right).
long cylindrical structure having one of its high modes locked-in with the flow vortex shedding. The wavelength of the locked-in mode is then determined from the assumed aspect ratio. In the case of the pinned endpoint simulations, the equivalent physical system would be a very long cylinder held in place by a periodic array of supports, a distance $L$ apart.

Despite such simplifications the coupled flow-structure system exhibits quite complex dynamics, in agreement with the physical observations. In particular, we found that the amplitude of the oscillation of the structure increases to a value of about $1d$ (and asymptotically stays around $0.9d$) if the cylinder is flexible with freely moving endpoints. For a rigid cylinder the amplitude was found to be about $0.75d$. The cases with pinned endpoints gave us amplitudes even higher than $1d$. This kind of response should be contrasted with the value of $0.5-0.6d$ obtained in the laminar regime. Such Reynolds numbers effects have recently been quantified in experimental studies (see Ching et al. (1998) [41]) but at a higher Reynolds number regime. The two-dimensional simulations predict an amplitude similar to the laminar flow, i.e. of the order of $0.5-0.6d$ even at $Re = 1000$.

A correlation between the cross-flow displacement and lift forces phase difference and the magnitude of the lift forces was observed and contrasted to results from 2D simulations. We feel that this is an area that merits further study and would benefit the standard models for VIV (for an overview see Blevins (1990) [1]) which do not take into account phase variations.

<table>
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<th></th>
<th>$y_{\text{max}}/d$</th>
<th>$y_{\text{rms}}/d$</th>
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<th>$C_d$</th>
<th>$(C_d)_{\text{rms}}$</th>
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<td>2.11</td>
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<td>0.86</td>
<td>1.83</td>
<td>0.48</td>
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<tr>
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<td>0.51</td>
<td>0.83</td>
<td>1.86</td>
<td>0.48</td>
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<tr>
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<td>0.43</td>
<td>0.86</td>
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<td>0.43</td>
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<tr>
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<td>0.25</td>
<td>1.16</td>
<td>1.62</td>
<td>0.44</td>
</tr>
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Table 4.2: Summary of time- and span-averaged amplitude, lift and drag coefficients at lock-in. (Zero structural damping is assumed and $Re = 1000$.)

We observed two different types of cylinder oscillations, the first resembling a traveling (progressive) wave when the endpoints were free to move, and the second resembling a standing wave.
when the endpoints were pinned. Asymptotically, the maximum cylinder displacement is about 0.9\(d\) in the traveling wave case and 1.1\(d\) in the standing wave case. The rms lift coefficient is about 0.8 for both cases but slightly larger in the case of a the standing wave. Similarly, the drag coefficient is about the same for both cases \((C_d \approx 1.8)\) but slightly lower in the later case. Simulations with a much larger aspect ratio displayed smaller amplitude and drag, but larger rms lift values. Detailed amplitude and force distributions along the span and in time are given in the section 3, and a summary of time- and span-averaged quantities is presented in table 4.2.

We simulated our structure not only as a cable but also as a beam, i.e. a structure with non-negligible bending stiffness and compared the beam-flow response with a cable-flow response. We found that at low Reynolds number in the laminar regime the responses are identical but at \(Re = 1000\) there are significant differences both in the flow structure and the dynamics of the cylinder. It is clear that in a multi-modal response away from a lock-in state such differences will be even larger. A study of the autocorrelation function for the velocity components in the wake of a beam appears to be in-between the cases of the wake behind a rigidly moving cylinder and that behind a cable. However, despite such differences in large scales there is a remarkable statistical similarity in the small scales as exemplified by comparisons of frequency spectra between the cable-flow and the beam-flow systems.

It has been suggested by Khalak & Williamson (1996) [46] that the two-dimensional simulations capture the lower branch of the resonant curve, which corresponds to a 2\(P\) type shedding pattern according to their observations and in agreement with the visualizations of Brika & Laneville (1993) [45]. We re-examined this hypothesis in the current investigation and we found it to be true at \(Re = 1000\), but at lower Reynolds number a different shedding pattern emerged of the standard 2\(S\) type or the more unusual \(P + S\) type. This latter type of shedding was also present in the \(Re = 1000\) simulations but only for a brief transient period of about two to three shedding cycles resulting in a higher amplitude of oscillation. We found that the specific shedding pattern sustained depends not only on the Reynolds number but also on the specific value of the reduced velocity even within the lock-in region; this was also observed in independent two-dimensional
simulations by Saltara et al. (1998) [54].

The three-dimensional shedding patterns are substantially different even for the case of a rigid cylinder that experiences the same displacement along all the points in the span. At $Re = 1000$ the flow is strongly three-dimensional although nominally parallel shedding prevails in the near-wake but with a spanwise distribution of phase difference between the motion and the wake of about $\pm 10$ degrees. For flexible cylinders, their modulated traveling wave response causes oblique shedding patterns as in the low Reynolds number regime [17]. At $Re = 1000$ these patterns coexist with parallel shedding, and the two patterns alternate sides (along the span) during one shedding cycle. This periodic change from oblique to parallel shedding corresponds to a large variation of the lift forces along the span, so by knowing the near-wake shedding pattern we can infer the lift coefficient distribution on the cylinder. A similar mixed parallel/oblique shedding pattern has also been visualized at lower Reynolds number in the experiments of van Atta, Gharib & Hammache (1988) [30]. Such large scale phenomena characterized as phase shocks (see Miller & Williamson (1994) [57]) can be described by a low-dimensional model such as the Ginzburg-Landau equation (see Albarede & Monkewitz (1992) [58]).

Unfortunately, currently there are no available complete experimental data to substantiate the new findings presented here regarding both the statistical quantities we computed as well as the flow structures we observed. It remains still an open question, assuming that the two-dimensional simulations capture the lower branch of the resonant curve, if the three-dimensional simulations capture the upper branch given the larger values of the cross-flow displacement. The 2S shedding pattern assigned to the upper branch by Khalak & Williamson (1996) [46] and by Brika & Laneville (1993) [45] does not seem to be a unique feature of that state. The current simulations suggest that patterns of the type $P+S$ may also be associated with the upper branch, which however are difficult to discern experimentally due to the much weaker "third" vortex of the group and its fast dissipation.

We have assumed that is that there is no structural damping in the system and we have chosen our structural parameters so that we are likely to operate at lock-in. As the Reynolds number is
still relatively low compared to most experimental conditions, we expect some Reynolds number
effects for the quantities summarized in table 4.2. Other possible sources of error are: Boundary
conditions due to truncation of the domain, spatial and temporal discretization (primarily for the
long beam simulations), and time-averaging errors (due to low frequency modulations).

Figure 47: Low resolution simulation for the short beam (case As) with only two modes along the
span. Amplitude along the span versus time.

We also tested the applicability of the often-used formula for predicting the drag coefficient
based on the standard deviation of the motion [49], and we found it to be very inaccurate. A
similar model we proposed but with very different coefficients seemed to fit the data better. On
the other hand, DNS studies of the type presented here are currently prohibitive expensive to be
used in engineering design of VIV. The question, then remains, as to what will be a good and
efficient model to predict mean forces or force distribution in VIV applications. We believe that
the answer could be provided by dynamical systems modeling, given the low-dimensionality of
the wake.

To test this hypothesis, we have repeated the DNS for case As (short beam with periodic ends)
but with significantly reduced resolution, i.e., we reduced the number of degrees-of-freedom (dof)
from about 5 millions per field in the previous simulations to approximately 50,000 dof per field,
with only 2 complex Fourier modes (i.e. 4 physical planes) along the span of $4\pi d$. The response
is shown in figure 47 where we plot the amplitude of the motion along the span and in time. We
see that a standing wave response is obtained with somewhat reduced amplitude compared to
the high resolution simulation. The corresponding mean drag coefficient is $C_d \approx 2.0$, which is
about 10% higher than in the high resolution simulation but approximately 15% lower than the corresponding two-dimensional simulation. We recall here that with only two (complex) Fourier modes along the span we resolve basically the mean flow (with the zeroth mode) and the first excited mode. In other words, we resolve only one three-dimensional mode, which apparently is sufficient to give us a big improvement (compared to the two-dimensional predictions) in the force distribution and also the motion amplitude, given that the maximum amplitude of the two-dimensional simulation is only 0.55d. We also note that the Fourier representation along the span is the best possible representation from the approximation point of view. However, there was no attempt here to obtain the best representation of the flow by constructing an appropriate hierarchy of the most energetic modes in the planes perpendicular to the cylinder axis, using for example the Karhuen-Loève approach [59]. We expect that this more systematic approach will result in a substantially lower-dimensional representation to predict the dynamics of VIV. We are currently working on that front and we will report results in the future.
Part II

Non-Linear Models
Chapter 5

Non-Linear Strings

1 Introduction

The problem of large amplitude vibrations of a cable has been studied for more than a century. In 1876, Kirchhoff proposed a non-linear integro-differential equation for a flexible string; it was a modification of the wave equation with a tension constant in space but varying in time [60]. Applying separation of variables to Kirchhoff's equation, and assuming a sine wave mode for the spatial variation, results in the Duffing equation (which contains a cubic non-linearity) describing the time variation; this equation has a time-periodic solution. Carrier ([61], [62]) proceeded to attack the problem by considering longitudinal as well as transverse displacements and arrived at a set of equations very similar to the ones we will be using. Irvine and Caughey in 1974 [63] used a quasi-steady stretching mechanism to arrive at a set of linearized equations for a suspended cable with a slight sag. Perkins in 1992 [64] produced a sequence of integro-differential equations for the tangential, normal and bi-normal direction along the cable. This sequence of equations describe linearized, weakly non-linear and fully non-linear (with quadratic and cubic non-linearities) free vibrations with coupling between longitudinal and transverse as well as in-plane and out-of-plane modes. Other non-linear models (with specific application to towed underwater cables) are due to Hover, Grosenbaugh and Triantafyllou [7] and Handscomb [65]. An approach like that of Carrier
is used by Leissa and Saad [66] (the paper contains an excellent and concise historical overview) and Su ([67]), and it is these approaches that we base our modeling on.

2 Modeling the structure

We shall model our flexible cylinder as a string of circular cross-section under tension, with no bending stiffness and no torsional effects. Our analysis will treat the string as having zero thickness so far as the dynamics are concerned but will take into account its cross-sectional area when it comes to modeling the tension. A more involved treatise of the resulting equations, examining the tension models for materials of different Poisson ratios, looking at equilibrium "sagged" (catenary) configurations under gravity for suspension points at different heights and deriving approximate and linearized equations for the case of small sag are found in Su [67]. The derivation in subsection 2.2 is based on that of Leissa and Saad [66], adding a third dimension, with a different choice of variable along the longitudinal direction and considering an arbitrary "reference" position as opposed to the straight stretched string state.

2.1 Non-dimensionalization

Stretching an elastic cylinder to produce a strain (change in length over original length) \( \epsilon \) will result in a tension \( T = E A \epsilon \), where \( E \) is the modulus of elasticity (Young modulus) and \( A \) the cross-sectional area of the stretched cylinder. Typical values of \( E \) for metals are \( O(10^{10} - 10^{11}) \), for plastics \( O(10^9) \) and for rubber \( O(10^7) \), in SI (Kg, m, sec) units. Typical values for the area \( A \) range from \( O(10^{-2}) \) for large clusters of cables to \( O(10^{-7}) \) for a British standard gauge 48 ultra-thin wire. In order to non-dimensionalize the tension \( T \) and \( E A \), we multiply it by

\[
\frac{1}{\rho U^2 d^2} = \frac{1}{\rho Re^2 \nu^2} = \frac{\rho}{Re^2 \mu^2},
\]

(1)

A typical value of \( \mu \) for water is \( \mu = 10^{-3} \) and for air at room temperature is \( \mu = 1.810^{-5} \), while \( \rho \) for water is about \( 10^3 \) and for air is \( O(1) \), all of the above in the relevant SI units. Therefore
for flow in a laminar \( \text{Re} = 100 \) regime, the non-dimensional \( E \alpha \) can be as high as \( O(10^{14}) \) in water or air for the case of a thick bundle of cables treated as one. For a very thin metal wire at the same Reynolds number, \( E \alpha \) can be as low as \( O(10^9) \). As the Reynolds number increases, the non-dimensional value of \( E \alpha \) (and hence \( T \)) decreases. Typical strains in the elastic range are \( O(10^{-3}) - O(10^{-5}) \) giving us values of \( T \) that are 3 to 5 orders of magnitude less than those of \( E \alpha \). The linear density of the string is non-dimensionalized as discussed in subsection 2.1 of Chapter 2.

As before all variables used henceforth are non-dimensional.

### 2.2 Non-linear equations for a string under tension

Consider an infinitesimal segment of a stretched string in a "reference" position (which can but need not be an equilibrium position). Let the arc length of the segment in this reference position be \( dS \) (the whole process can be expressed rigorously in terms of limits but we will use this approach for a more concise presentation); the same segment has an unstretched length \( dS_0 \). Let us parameterize our string with respect to the arc length variable in the reference position, so that at any time \( t \), we have arc length parameter \( s(S,t) \) and an infinitesimal segment of length \( ds(S,t) \).

Let \( x, y \) and \( z \) be the two transverse and the longitudinal direction respectively, in an orthonormal coordinate frame for our string. We define the longitudinal direction as that of the stretched string in the absence of gravity. Then if one end of our infinitesimal string segment is at time \( t \) at \((x, y, z)\), the other is at \((x + dx, y + dy, z + dz)\). Then

\[
    ds^2 = dx^2 + dy^2 + dz^2 \Rightarrow \Delta = \frac{\partial s}{\partial S} = \sqrt{\left(\frac{\partial x}{\partial S}\right)^2 + \left(\frac{\partial y}{\partial S}\right)^2 + \left(\frac{\partial z}{\partial S}\right)^2}
\]

(2)

where \( \Delta \) represents the length increase or decrease with respect to the reference position.

Let the tension in the string be \( T(x, y, z) \) or \( T(S, t) \) in terms of our fundamental parameterization. \( T \) is tangential to the string at all points, and therefore the force components in the three
coordinate directions are:

\[
\begin{align*}
T \frac{\partial x}{\partial s} &= T \frac{\partial x}{\partial S} = \frac{T}{\Delta \partial S} T \frac{\partial x}{\partial S} \\
T \frac{\partial y}{\partial s} &= T \frac{\partial y}{\partial S} = \frac{T}{\Delta \partial S} T \frac{\partial y}{\partial S} \\
T \frac{\partial z}{\partial s} &= T \frac{\partial z}{\partial S} = \frac{T}{\Delta \partial S} T \frac{\partial z}{\partial S}
\end{align*}
\]  
(3)

The local strain in the string is

\[
\epsilon_v(s, t) = \frac{dS - dS_0}{dS_0} = \frac{\partial s}{\partial S_0} - 1 = \frac{\partial s}{\partial S} \frac{dS}{dS_0} - 1 = \Delta \frac{dS}{dS_0} - 1.
\]  
(4)

In the reference position, \( \Delta = \frac{\partial s}{\partial S} = 1 \) and hence the strain in the reference position

\[
\epsilon(S) = \frac{dS}{dS_0} - 1 \Rightarrow \frac{dS}{dS_0} = \epsilon(S) + 1.
\]  
(5)

Now, the linear density of the string is \( m_v(S, t) \) - conservation of mass for the line segment demands that:

\[
m_v(s, t)ds = m_0(S_0)ds_0 = m(S)ds.
\]  
(6)

So given structural damping forces proportional to the string velocity (by a constant \( R \)) and external forces \( F_{\text{e}y} \) in the three coordinate directions we have:

\[
\begin{align*}
\left[ \frac{\partial}{\partial S} \left( \frac{T}{\Delta \partial S} \frac{\partial x}{\partial S} \right) \right] dS + \left( F_x(S, t) - R \frac{\partial x}{\partial t} \right) dS &= m_v(s, t) \frac{\partial^2 x}{\partial t^2} ds,  \\
\left[ \frac{\partial}{\partial S} \left( \frac{T}{\Delta \partial S} \frac{\partial y}{\partial S} \right) \right] dS + \left( F_y(S, t) - R \frac{\partial y}{\partial t} \right) dS &= m_v(s, t) \frac{\partial^2 y}{\partial t^2} ds,  \\
\left[ \frac{\partial}{\partial S} \left( \frac{T}{\Delta \partial S} \frac{\partial z}{\partial S} \right) \right] dS + \left( F_z(S, t) - R \frac{\partial z}{\partial t} \right) dS &= m_v(s, t) \frac{\partial^2 z}{\partial t^2} ds.
\end{align*}
\]  
(7a) (7b) (7c)

Equation 6 then allows us to write the fully non-linear equations:

\[
\left[ \frac{\partial}{\partial S} \left( \frac{T}{\Delta \partial S} \frac{\partial x}{\partial S} \right) \right] + F_x(S, t) - R \frac{\partial x}{\partial t} = m(S) \frac{\partial^2 x}{\partial t^2},
\]  
(8a)
\[
\frac{\partial}{\partial S} \left( \frac{T}{\Delta \partial S} \right) + F_y(S,t) - R \frac{\partial y}{\partial t} = m(S) \frac{\partial^2 y}{\partial t^2}. \tag{8b}
\]
\[
\frac{\partial}{\partial S} \left( \frac{T}{\Delta \partial S} \right) + F_z(S,t) - R \frac{\partial z}{\partial t} = m(S) \frac{\partial^2 z}{\partial t^2}. \tag{8c}
\]

If the unstretched string was homogeneous and of constant cross-sectional area then \(m_0(S_0) = m_0\) is constant. By using equations 6 and 5 we can replace \(m(S)\) in equations 8 by \(\frac{m_0}{1 + \epsilon(S)}\); if the reference position is the straight stretched string one, then \(\epsilon(S) = \epsilon\) (constant) and there is no \(S\) dependence on the right hand side of equations 8.

In the context of flow-structure interactions, \(F_x\) is drag, \(F_y\) is lift possibly with a gravitational body force \(-m(S)g\), and \(F_z\) is the spanwise hydrodynamic force component.

### 2.3 Modeling the tension

Now for strains within the elastic range of the string material, tension

\[
T(s,t) = EA(s,t)\epsilon_v(s,t). \tag{9}
\]

Use of equation 4 and 5 give us

\[
T(s,t) = EA(s,t)(\Delta(\epsilon(S) + 1) - 1). \tag{10}
\]

Now for an incompressible material (volume density \(\rho_0\) constant) mass conservation implies volume conservation: That is

\[
A(s,t)ds = A_0dS_0 = A(S)dS. \tag{11}
\]

Here we have assumed that the unstretched string was of constant cross-sectional area \(A_0\). Any material that obeys equation 11 is said to have a Poisson ratio \(P = 1/2\). There is another limit,
that of Poisson ratio $P = 0$, where the material stretches without its cross-sectional area changing:

$$A(s, t) = A(S) = A_0.$$  \hfill (12)

In order for equations 6 and 12 to be satisfied at the same time, a string made of homogeneous material would need to have its (volume) density change as

$$\rho_s(s, t) ds = \rho_0(S_0) dS_0 = \rho_r(S) dS.$$  \hfill (13)

More involved models parametrized by the Poisson ratio can be constructed [66] but we will consider here only the aforementioned ones.

For the case of $P = 1/2$

$$A(s, t) = \frac{A(S)}{\frac{ds}{dS}} = A(S) = A_0 \frac{dS_0}{dS} = \frac{A_0}{\Delta(\epsilon(S) + 1)}$$  \hfill (14)

making use of equation 5. Substituting in equation 10 we get

$$T(s, t) = EA(S) \left( \epsilon(S) + 1 - \frac{1}{\Delta} \right) = EA_0 \left( 1 - \frac{1}{\Delta(\epsilon(S) + 1)} \right).$$  \hfill (15)

This can also be written as

$$T(s, t) = EA(S) \epsilon(S) + EA(S) \left( 1 - \frac{1}{\Delta} \right) = T(S) + EA(S) \left( 1 - \frac{1}{\Delta} \right),$$  \hfill (16)

where $T(s, t)$ is expressed in terms of the tension in the reference position $T(S) = EA(S) \epsilon(S)$ and of a stretching term $EA(S) \left( 1 - \frac{1}{\Delta} \right)$. In the case that the reference position is that of a straight stretched (uniform in radius) string, we can rewrite equation 16 as

$$T(s, t) = T_r + EA_r \left( 1 - \frac{1}{\Delta} \right),$$  \hfill (17)
where both the tension in the reference position \( T_r = E A_r \epsilon \) and the cross-sectional area in the reference position \( A_r \) are constants.

For the case of \( P = 0 \), equations 12 and 10 give us

\[
T(s,t) = EA_0 (\Delta(\epsilon(S) + 1) - 1). \tag{18}
\]

It should be noted that the presence or not of the incompressibility assumption gives us two different models for the tension (equations 15 and 18) and a dependence of a different nature on the non-linear term \( \Delta \). This is interesting given that the assumption that changes in cross-sectional area are negligible is not that uncommon.

It is also noteworthy that there is nothing in equations 15 and 18 that precludes the possibility that \( T < 0 \) for a sufficiently small \( \Delta < 1 \). Only \( \Delta \geq 1 \) (that is the string is at all times locally stretched at the reference level or more) doesn’t pose a problem. The physical meaning of \( T = 0 \) is that the string has become \textit{locally} slack, returning to its original segment length. Hooke’s law allows for compression as well as stretching and all of our modeling so far did not include the possibility of buckling - thus our string will support negative tension; this is not physically possible for most materials.

### 2.4 The linear limit and first-order corrections

Having a model for the tension, we can now write equations 8 solely in terms of partial derivatives of \( x, y \) and \( z \) with respect to \( S \) and \( t \). The expressions thus obtained (see [66]) however are cumbersome to work with and less intuitive than equations 8. However, they can be used as a starting point for deriving approximations with simpler non-linearities and the limit of the linear equations. Linearizing is done about a reference position that is also a steady-state (equilibrium) solution of 8. ([67]).

Allowing only for quadratic non-linear terms, the Poisson ratio \( P = 1/2 \) case gives us (refer to [67] for details) (in the absence of any structural damping, external forces and using as a reference
position the straight stretched string):

\[
m \frac{\partial^2 x}{\partial t^2} = \frac{\partial}{\partial S} \left[ T_r \frac{\partial x}{\partial S} + (EA_r - T_r) \frac{\partial x}{\partial S} \frac{\partial z_v}{\partial S} \right] \tag{19a}
\]

\[
m \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial S} \left[ T_r \frac{\partial y}{\partial S} + (EA_r - T_r) \frac{\partial y}{\partial S} \frac{\partial z_v}{\partial S} \right] \tag{19b}
\]

\[
m \frac{\partial^2 z_v}{\partial t^2} = \frac{\partial}{\partial S} \left\{ EA_r \left[ \frac{\partial z_v}{\partial S} - \left( \frac{\partial z_v}{\partial S} \right)^2 \right] + (EA_r - T_r) \frac{1}{2} \left[ \left( \frac{\partial x}{\partial S} \right)^2 + \left( \frac{\partial y}{\partial S} \right)^2 \right] \right\} \tag{19c}
\]

where \( z_v = z(S, t) - S \), that is the displacement from the straight stretched equilibrium position.

It is evident from equations 19 that while an initial condition \( x = y = 0, z_v \neq 0 \) will result in only longitudinal motion alone, obeying

\[
m \frac{\partial^2 z_v}{\partial t^2} = \frac{\partial}{\partial S} \left\{ EA_r \left[ \frac{\partial z_v}{\partial S} - \left( \frac{\partial z_v}{\partial S} \right)^2 \right] \right\} . \tag{20}
\]

any initial displacement or velocity in any of the transverse directions is going to induce longitudinal displacement because of the \((EA_r - T_r) \frac{1}{2} \left[ \left( \frac{\partial x}{\partial S} \right)^2 + \left( \frac{\partial y}{\partial S} \right)^2 \right]\) term. This is actually a feature of the fully non-linear equations 8 as well.

Dropping all non-linear (quadratic in amplitude) terms gives us:

\[
m \frac{\partial^2 x}{\partial t^2} = \frac{\partial}{\partial S} \left( T_r \frac{\partial x}{\partial S} \right) \tag{21a}
\]

\[
m \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial S} \left( T_r \frac{\partial y}{\partial S} \right) \tag{21b}
\]

\[
m \frac{\partial^2 z_v}{\partial t^2} = \frac{\partial}{\partial S} \left( EA_r \frac{\partial z_v}{\partial S} \right) . \tag{21c}
\]

Thus we arrive at the classical linearized equations: the one for the longitudinal displacement exhibits a phase speed \( c = \sqrt{EA_r/m} \) while the transverse component ones have the classical result \( c = \sqrt{T_r/m} \). Dropping all but the linear terms for the case of Poisson ratio \( P = 0 \) tension model [65] gives us \( c = \sqrt{(EA_r + T)/m} \) for the longitudinal direction. Since in most physically relevant cases \( EA \gg T \) the difference is practically negligible.
2.5 Initial and Boundary conditions

Equations 8 admit the same boundary conditions as the linearized equations 5 of Chapter 2 (Refer to the boundary condition descriptions in subsection 2.3 of that chapter). Care has to be taken in the case of the periodic boundary conditions - they cannot be used for the longitudinal displacement variable \( z(S, t) \) but they can be used for \( z_v(S, t) \) as defined above in subsection 2.4. The "pinned" end Dirichlet boundary condition can also cover the cases of non-zero (but constant) values: That would be the case of a string suspended from two points at different height - for a fuller treatment of the steady state solution and approximations including at most quadratic non-linear terms etc. see [67].

This IBVP also requires initial displacement and velocity conditions - the initial displacement \( x_{init}(S), y_{init}(S) \) and \( z_{init}(S) \) gives us the initial \( \Delta \) and thus tension \( T \). It can be argued that physically relevant initial conditions have an almost constant \( T \) [68]. For a reference position of a straight stretched string, both equations 15 and 18 require that \( \Delta \) be constant for \( T \) to be constant. In that case we can rewrite equation 2 in terms of \( x, y \) and \( z_v \) as

\[
\Delta = \sqrt{\left( \frac{\partial x}{\partial S} \right)^2 + \left( \frac{\partial y}{\partial S} \right)^2 + \left( 1 + \frac{\partial z_v}{\partial S} \right)^2} = \sqrt{1 + \left( \frac{\partial x}{\partial S} \right)^2 + \left( \frac{\partial y}{\partial S} \right)^2 + \left( \frac{\partial z_v}{\partial S} \right)^2 + 2 \frac{\partial z_v}{\partial S}} \tag{22}
\]

Note that while any form of the initial displacement in the \( x- \) and \( y- \) directions can only make the value of \( \Delta > 1 \) because of the squared contributions of the derivatives of \( x \) and \( y \), the term \( 2 \frac{\partial z_v}{\partial S} \) can be negative, thereby decreasing the value of \( \Delta \). Now if the initial condition consists only of small angle variations in the longitudinal direction,

\[
\left| \frac{\partial z_v}{\partial S} \right| = O(\delta \theta) \ll 1 \Rightarrow 2 \left| \frac{\partial z_v}{\partial S} \right| \gg \left| \frac{\partial z_v}{\partial S} \right|^2 = O(\delta \theta^2) \tag{23}
\]

and as a result \( \Delta \) can be less than unity; it can even be small enough to cause the problems with negative tension mentioned in subsection 2.3. This is a fundamental difference about the initial conditions in the longitudinal direction compared to the ones in the two transverse ones.
Still even for $\Delta \geq 1$ we get large tension variations: For example, for the simulations discussed in Chapter 4, the amplitude of the oscillations is about unity and for the fundamental vibration mode of spanwise wavelength $L = 4\pi$.

$$0 \leq \left| \frac{\partial y}{\partial S} \right| \leq \frac{2\pi}{L} = 0.5.$$  \hspace{1cm} (24)

Using only this transverse displacement we see that $1.00 \leq \Delta \leq 1.12$. Using equation 17 and for $T_r$, 3 orders of magnitude less than $E_A \tau$, $T_r \leq T \leq 0.1 E_A \tau \approx 100 T_r$.

For all of $\frac{\partial x}{\partial S}$, $\frac{\partial y}{\partial S}$ and $\frac{\partial z}{\partial S}$ small in magnitude ($O(\delta \theta)$), we can Taylor expand equation 22 about unity (employing $\sqrt{1 + \delta Q} = 1 + \frac{1}{2} \delta Q - \frac{1}{8} \delta Q^2 + O(\delta Q^3)$):

$$\Delta = 1 + \frac{1}{2} \left( \frac{\partial x}{\partial S} \right)^2 + \frac{1}{2} \left( \frac{\partial y}{\partial S} \right)^2 + \frac{\partial z}{\partial S} + O(\delta \theta^3).$$  \hspace{1cm} (25)

Therefore, a condition that ensures that $\Delta$ is a positive constant to third order in the small amplitude of the displacements is that

$$\left( \frac{\partial x}{\partial S} \right)^2 + \left( \frac{\partial y}{\partial S} \right)^2 + \frac{\partial z}{\partial S} = C.$$  \hspace{1cm} (26)

where $C > -1$ is some constant. We cannot require $C = 0$ to force $\Delta \approx 1$ to $O(\delta \theta^3)$ as equation 26 would then require that $\frac{\partial z}{\partial S} \leq 0$ for all $(S, t)$. But for pinned end boundaries,

$$\int_0^L \frac{\partial z}{\partial S} dS = z_v(S) - z_v(0) = 0,$$  \hspace{1cm} (27)

and therefore $\frac{\partial z}{\partial S}$ would have to be identically zero. Equation 26 is similar to an equation derived by Anand [58]. For

$$y(S, 0) = y_{max} \sin \frac{l \pi S}{L}, \hspace{1cm} (28a)$$

$$\frac{\partial y}{\partial t}(S, 0) = 0.$$  \hspace{1cm} (28b)
for integer \( l \) and

\[
\begin{align*}
z_v(S, 0) &= -y_{\text{max}}^2 \frac{l\pi}{8L} \sin \frac{2l\pi S}{L}, \quad (29a) \\
\frac{\partial z_v}{\partial t}(S, 0) &= 0, \quad (29b)
\end{align*}
\]

give us

\[
C = y_{\text{max}}^2 \frac{l^2 \pi^2}{2L^2} \left( \cos^2 \frac{l\pi}{L} - \frac{1}{2} \cos \frac{2l\pi}{L} \right) = y_{\text{max}}^2 \frac{l^2 \pi^2}{4L^2} > 0. \quad (30)
\]

3 Solving for the structure

3.1 Numerical scheme

We numerically solve equations 8 (using a tension model of the ones discussed in subsection 2.3) by employing the following explicit finite difference discretization for each of the 3 equations 8 \( u = x | y | z \), \( F = F_x | y | z \):

\[
\begin{align*}
u_{i+1}^{n+1} - 2u_i^n + u_{i-1}^{n-1} = \frac{1}{m_i} \left\{ \frac{1}{\delta S^2} \left[ \left( \frac{T}{\Delta} \right)_{i+\frac{1}{2}} u_{i+1}^n - \left( \frac{T}{\Delta} \right)_{i-\frac{1}{2}} u_i^n + \frac{T}{\Delta} \right] u_i^n \right. \\
+ \left. \left( \frac{T}{\Delta} \right)_{i-\frac{1}{2}} u_{i-1}^n \right) + F_i^n - R \frac{u_{i+1}^{n+1} - u_i^{n-1}}{2\delta t} \right\} \quad (31)
\end{align*}
\]

The discretization is \( O(\delta t^2) \) in time and \( O(\delta S^2) \) in space. The tension \( T \) and arc length ratio \( \Delta \) are evaluated at the half-way points: that is given a set of discrete \( x_i^n, y_i^n \) and \( z_i^n \), \( \Delta \) can be calculated with second-order spatial accuracy as follows:

\[
\Delta_{i+\frac{1}{2}}^n = \sqrt{\left( \frac{x_{i+1}^n - x_i^n}{\delta S} \right)^2 + \left( \frac{y_{i+1}^n - y_i^n}{\delta S} \right)^2 + \left( \frac{z_{i+1}^n - z_i^n}{\delta S} \right)^2}. \quad (32)
\]

Then \( T_{i+\frac{1}{2}}^n \) is calculated with the same accuracy using equation 15 or 18 and the value of \( \Delta_{i+\frac{1}{2}}^n \).

The initial timestep uses the values of \( u_i^0 \) and \( \dot{u}_i^0 \) provided as initial conditions, and the value of the acceleration \( \ddot{u}_i^0 \) calculated using the right hand side of equation 31: A second-order time-
accurate Taylor expansion is employed:

$$u_i^{n+1} = u_i^n + u_i^n \delta t + \frac{1}{2} \ddot{u}_i^n \delta t^2$$  \hspace{1cm} (33)

The boundary conditions are also second-order accurate in space, evaluated (when needed) using one-sided derivatives.

This explicit scheme is related to the simplest finite difference scheme for the wave equation

$$u_{tt} = c^2 u_{xx}: \quad \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\delta t^2} = c^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\delta S^2}. \hspace{1cm} (34)$$

The CFL limit for this scheme is $c \frac{\delta t}{\delta S} \leq 1$. The same stability restrictions would extend for scheme of equation 31 with constant $T/\Delta$ (for example for the solution of equation 21); the CFL limit is at worst $\sqrt{(EA_r + T)/m \delta t} \delta S \leq 1$ in the case of a Poisson ratio $P = 0$ based tension model.

The scheme 31 is more expensive than scheme 34: It is based on the evaluation of square roots and divisions which are not pipelined operations on any modern processor - tricks have to be employed where possible to reduce the number of such operations (e.g. by storing and using $1/\Delta$).

### 3.2 Validation

We present here some validation tests for our numerical scheme. In all cases we have used the Poisson ratio $P = 1/2$ tension model.

**Steady state**

The steady state solution of equations 8 under a constant body force is a catenary. We first tried the steady state solution as the initial condition and the error we got in both the transverse and longitudinal direction was of the order of our discretization error (see figure 1).

We then tried initial conditions perturbed from the steady state and the solution converged again to the catenary (figure 2):
Figure 1: Deviation from the steady state catenary position in the vertical (top) and longitudinal (bottom) direction. $\delta S = 0.01, \delta t = 0.0025, T_r = 2, EA_r = 10, m_r = 3, g = 1$.

Irvine and Caughey's transverse horizontal mode

Irvine and Caughey [63] consider suspended cables with a small sag (1/8 or less of their length). For the case of a parabolic cable shape (steady state solution under gravity for the linearized equations) they report that small out-of-plane (horizontal) perturbations do not induce any in-plane (longitudinal or vertical) vibrations to first order. In figure 3 we show the results of such an initial perturbation (of size $0.01L$) to a parabolic shape in the case where the sag is $1/16L$.

We can see that there is only small variation in the $y -$ and $z -$ directions with time for a given $S$. In figure 4 we show the results of the a simulation where the initial condition is a perturbed catenary (the steady state of the non-linear equations); there is no visible variation in the $y -$ and $z -$ directions. For both initial conditions the frequency of the horizontal vibrations is the one given by Irvine and Caughey.
Figure 2: Deviation from the steady state catenary position in the horizontal (top), vertical (middle) and longitudinal (bottom) direction. $\delta S = 0.01, \delta t = 0.0025, T_r = 2, EA_r = 10, m_r = 3, g = 1$ with a horizontal "uniform drag" force of magnitude $\frac{2}{3}m$. 
Figure 3: Displacement in the horizontal (top), vertical (middle) and longitudinal (bottom) direction. $\delta S = 0.01, \delta t = 0.0025, T_r = 6, EA_r = 10, m_r = 3, g = 1$. Initial condition is a parabola in the vertical direction sinusoidally perturbed in the horizontal direction. The displacement in the longitudinal direction is not a deviation from the straight stretched string position and hence varies from 0 to $L$. Small oscillations can be observed near the boundaries.
Figure 4: Displacement in the horizontal (top), vertical (middle) and longitudinal (bottom) direction. Initial condition is a catenary in the vertical direction sinusoidally perturbed in the horizontal direction. $\delta S = 0.01, \delta t = 0.0025, T_r = 6, EA_r = 10, m_r = 3, g = 1.$
Longitudinal vibrations

Finally we consider the case where there is no transverse component in the initial conditions; in figure 5 we show the results of our code as well as a simpler finite difference scheme

\[
\frac{u^n_{i+1} - 2u^n_i + u^n_{i-1}}{\delta t^2} = c^2 \left(1 - \frac{u^n_{i+1} - u^n_{i-1}}{2\delta S}\right) \frac{u^n_{i+1} - 2u^n_i + u^n_{i-1}}{\delta S^2}.
\] (35)

that solves equation 20 (discussed in subsection 2.4). The solutions are virtually identical until \(t\) becomes large enough that the difference in non-linearity becomes apparent.

Figure 5: Solution of equations 8 (top) and equation 20 (bottom) for an initial condition with no transverse component. Isocontours of the deviation from the straight stretched string reference position are shown. \(\delta S = 0.01, \delta t = 0.0025, T_r = 1, EA_r = 10, m_r = 3\).

Su [66] provides an initial correction to the solution of the wave equation so as to satisfy equation 20 for small times \(t\) (afterwards a second and then a third correction is needed and so forth). For a base solution of the form

\[
A \sin \left(\frac{n\pi S}{L}\right) \cos \left(\frac{n\pi ct}{L}\right)
\] (36)
where \( c = \sqrt{EA_r/m_r} \), the solution with the first correction term is

\[
A \sin \left( \frac{n\pi S}{L} \right) \cos \left( \frac{n\pi ct}{L} \right) + \frac{A^2 n\pi^2}{8L} \left[ \frac{n\pi ct}{L} \sin \left( \frac{2n\pi S}{L} \right) + \left( 1 - \cos \left( \frac{2n\pi ct}{L} \right) \right) \right].
\]  

(37)

In figure 6 we plot this approximate solution and its difference from the numerical solution of equation 20. We see that there is very good initial agreement as would be expected.

![Figure 6: Approximate solution (expression 37) to equation 20 at the top; difference from numerical solution of the same equation at the bottom. \( \delta S = 0.01, \delta t = 0.0025, T_r = 1, EA_r = 10, m_r = 3 \).](image)

4 Solving for the flow

The Arbitrary Lagrangian Eulerian (ALE) method has been developed by [69],[70],[71],[72],[73] and applied to quadrilateral spectral elements in [74] and unstructured polymorphic (hybrid) spectral elements in [33].

In the ALE formulation we consider domains that are changing in time, as they naturally arise in flow-structure interaction or free surface problems. This means that we now also have
4.1 Formulation

Consider the incompressible Navier-Stokes equations in a time-dependent domain Ω(t)

\[ u_{i,z} + u_j u_{i,j} = -(p\delta_{ij})_z + \frac{1}{Re} u_{i,jj} + f_i \quad \text{in } \Omega(t) \]  
\[ u_{j,j} = 0 \quad \text{in } \Omega(t). \]  

where \( f_i \) is a body force. Following the derivation in [74] (further discussed in [33]) we arrive at the following variational form of the momentum equation (38):

\[ \frac{d}{dt} \int_{\Omega(t)} v_i u_i dx + \int_{\Omega(t)} [v_i (u_j - w_j) u_{i,j} - v_i u_i w_{j,j}] dx = \int_{\Omega(t)} [v_{i,j} p\delta_{ij} - \nu v_i,j u_{i,u} + v_i f_i] dx \]  

This reduces to the familiar Eulerian and Lagrangian form by setting \( w_j = 0 \) and \( w_j = u_j \), respectively. However, in the the standard manner that the ALE formulation is applied, \( w_j \) is chosen so as to minimize the mesh deformation. More details about the actual temporal discretization of the ALE formulation using a splitting scheme and the implementation of the method in the context of the \textit{NekTaran} family of spectral element codes are presented in [33]. The flow problem is solved in parallel using an element based domain-decomposition method following the approach presented for \textit{NekTaran3D} in Appendix A.

5 The coupled flow-structure problem

Combining the two solution approaches presented in sections 3 and 4 for the structure and flow respectively we are faced with the same problem as in section 5 of Chapter 2: it is impractical to advance both the flow and the structure in time concurrently. Therefore, as before, we choose to lag the flow solver in a consistent manner:

1. We begin with both the flow solver and structure solver states at timestep \( n \). The drag, lift
and spanwise hydrodynamic force distributions needed by the structure solver have already been calculated.

2. We use the structure’s acceleration as the time-dependent term in the time-accurate pressure boundary conditions ([35]) before solving for the pressure.

3. We use the hydrodynamic forces at timestep \( n \) to advance the structure’s state to timestep \( n + 1 \).

4. We use the structure’s velocity at timestep \( n + 1 \) to reset the velocity (no-slip) Dirichlet boundary conditions for the moving body before the viscous correction step of the Navier-Stokes solver.

5. We use the structure’s velocity at timestep \( n + 1 \) as the mesh velocity Dirichlet boundary conditions for the moving body in solving (a Laplace equation to minimize mesh distortion) for the new mesh positions.

6. We calculate the new forces at timestep \( n + 1 \). For these calculations we include both the pressure and the viscous force contribution.

This way the interaction between the flow and the structure solver takes the form of:

- the flow solver producing force distribution information at a series of cross-sections that the structure solver requires, and

- the structure solver producing velocity and acceleration information at points that the flow solver requires.

This simple form of interaction lends itself naturally to task parallelism: If the structural problem (that could be a fully 3D elasticity one instead of the one independent space variable formulation used now) is costly enough to make it worthwhile solving it in parallel, one can have two parallel codes: one for the flow and another for the structure, that communicate only force and velocity/acceleration information between themselves.
5.1 Implementation issues

There are several implementation issues that had to be addressed and can be further improved on:

- The structure solver gives the flow solver a list of \( z \)-points along the span of the flexible cylinder: the hydrodynamic forces in the \( x \)-, \( y \)- and \( z \)- directions are required for them. Since the structural model is parametrized in space only along the arc length of the flexible cylinder, the forces at a \( z \)-point must be integrated along the perimeter of the corresponding \( x - y \) plane intersection of the cylinder. This requires the relatively costly extra calculation of new line Jacobians at every timestep, for every relevant boundary element. (Given a curve in \( 3D \) space \((x(r), y(r), z(r))\) parametrized by \( r \), the line Jacobian

\[
J_l = \sqrt{\left( \frac{\partial x}{\partial r} \right)^2 + \left( \frac{\partial y}{\partial r} \right)^2 + \left( \frac{\partial z}{\partial r} \right)^2}
\]  

is necessary for integrating a function along the curve.)

- Practically, the need for line integrated force distributions requires the following:

1. Use of a structured array of quadrilateral "surface elements" (the faces of the boundary elements touching the cylinder). Thus, one of the two directions of quadrature point distribution (say direction "\( a \)"") would be parallel to the arc segment on the element along which we integrate. This enables the line integral to be calculated using polynomial interpolation from the values of the line integrals at the quadrature points along direction "\( a \)". The more general alternative is very costly as it involves calculating the line Jacobian and quadrature weights for the line integration as well as interpolated stress tensor values for a more general curve with respect to the local "surface element" geometry.

2. The above restriction implies that the boundary elements touching the cylinder can only be hexahedra, or prisms/pyramids, with their quadrilateral faces on the cylinder
surface. Unfortunately, tetrahedra cannot be used – this means that standard 3D mesh generators cannot be used to discretize the whole domain and a different approach needs to be taken near the cylinder. Moreover, even for the element types allowed, care has to be taken to ensure the parallelism property mentioned above. This means that on the cylinder’s surface, the quadrilateral faces of the elements have to be rectangles.

3. Moreover the above “rectangular face” requirement needs to be true at all times - therefore as the mesh moves with the cylinder all mesh points on the cylinder’s surface for a given $z-$coordinate have to move in unison. This is a very similar approximation to that detailed in subsection 4.1 of Chapter 2: essentially we enforce that points along an $x-y$ plane cut of our undeformed cylinder remain perpendicular to the $z-$axis.

- Since the list of $z-$points corresponds to the time-dependent $z-$coordinates of the structural finite difference discretization, a search needs to be performed at every timestep to see whether a given element contains a given $z-$ location. For $N_{be}$ moving body boundary elements per processor and $N_z$ points used to discretize the structural problem, this amounts to a $O(N_{be}N_z)$ cost per timestep.

- If a $z-$point falls exactly on an internal element boundary care needs to be taken to average the values for the line integrated forces interpolated from each element.

- This is not the case for $z-$points lying on external (side) boundaries of the flow domain (the endpoints of the finite difference structure discretization). Extra care has to be taken for these points though, as rounding error may cause the moving mesh to drift slightly with respect to these boundaries; a tolerance criterion is used when checking whether they fall at the boundaries of an element.

- Because of the domain decomposition based flow solver, not all the arc segments needed for the line integrated forces for a given $z-$point need lie on the same processor. A global summation communication operation (implemented using an MPLAllreduce() call) is needed for this information to be properly obtained on all processors. Each processor then solves the
structure problem on its own. Parallelizing the structure solver was not deemed necessary because of its small cost compared to that of the flow solver.

- The flow solver gives the structure solver a list of \((x, y, z)\) locations that velocity or acceleration values are needed for. The structure solver as currently implemented will return velocity and acceleration values for a given \(z\)–coordinate, irrespective of the values of \((x, y)\). This type of motion preserves the area of the intersection of the cylinder with the corresponding \(x - y\) plane, as all points along that intersection move in unison. This area preservation property conflicts with the case of a Poisson ratio \(P = 1/2\) elastic material discussed above (and corresponds to a \(P = 0\) one). A workable remedy would be the (radial direction dependent) adjustment of velocity and acceleration information by a radial velocity \(\partial r_c/\partial t\) and acceleration \(\partial^2 r_c/\partial t^2\) term calculated using

\[
A(S, t) = \frac{A_0}{\Delta(S, t)(1 + \epsilon)} = 2\pi r(S, t).
\]  

(42)

where \(r(S, t)\) is the radius of the circular cross-section of the cylinder. Time derivatives of \(\Delta(S, t)\) can be approximated by extending the structure solver.

- In order to return velocity and acceleration values for a given \(z\)–coordinate \(Z\), the structure solver has to:

  1. Reconstruct discrete velocity \(\dot{u}_i\) and acceleration \(\ddot{u}_i\) values from the solution of equation 8. In our case:

    (a) We use the right-hand-side of equation 31 substituting the displacement values we have at timestep \(n + 1\) to get an estimate for the acceleration at timestep \(n + 1\).

    (b) We then use a local Taylor expansion in time

\[
\ddot{u}_i^n = \ddot{u}_i^{n+1} - \delta t \dot{u}_i^{n+1} + \frac{1}{2} \delta t^2 \ddot{u}_i^{n+1} + O(\delta t^3)
\]  

(43)

using the values of the displacement at timesteps \(n\) and \(n + 1\) and acceleration at
timestep $n + 1$ to evaluate the velocity at timestep $n + 1$.

When the (proportional to velocity) structural damping term is non-zero.

(a) an initial acceleration estimate ignoring damping is done.

(b) This is then used in a appropriately modified equation 43 to get the velocity values.

(c) We then adjust the acceleration to account for the damping.

Because the flow solver lags the structure solver in the solution scheme, the acceleration at timestep $n$ gets calculated using the external forces from timestep $n - 1$. This introduces an error of $O(\delta t)$ for the acceleration and induces an error of $O(\delta t^2)$ (same order as that of the structural solver) for the velocity. As the acceleration information is not used until the pressure boundary conditions for timestep $n + 1$ need to be computed, a correction is made after the new hydrodynamic forces are calculated at the end of timestep $n$.

2. Interpolate from a discrete set of velocity and acceleration values for the value required.

Currently a simple linear interpolation is used and an obvious improvement would be some form of spline interpolation.

3. In order to do the interpolation, the unique $S$ value that corresponds to $Z$ for time $t$ is needed. Given that $S$ is also represented by a discrete variable $S_i$ in the structure solver, this is basically

(a) a root bracketing problem (find $S_i$ such that $S_i \leq S < S_{i+1}$).

(b) Once these two points are found, a $\delta l \leq \delta S$ such that $z(S_i + \delta l, t) = Z$ needs to be estimated. The estimation algorithm must be consistent with the interpolation algorithm used.

There is a tradeoff to employing more accurate estimation and interpolation techniques: the operations involved are full of conditional expressions and hence are inherently computationally inefficient.
The ALE flow solver updates the mesh positions using the mesh velocity values and the explicit part of the same stiffly stable time integrator that is used for advancing the flow variables in time. The structure solver however solves directly for the position of the centerline of the cylinder: while of the same order in $\delta t$, the error in the two timestepping schemes is different - this may lead over time in "drift" between the mesh boundaries in the flow solver and the cylinder boundaries as the structure solver expects them to be. One can use the information of the structure solver to directly set the values of the positions of the mesh boundaries. The process involves the same estimation and interpolation issues as discussed above. It is yet undetermined whether this increases the accuracy of the solution of the coupled problem or introduces instability.

5.2 The problem of multiple time scales

As explained in subsection 2.1 in actual applications in water, with an $Re = O(10^4)$ and above, $EA = O(10^5 - 10^{10})$. For the easier to solve low $Re = 100$ flows, $EA = O(10^9 - 10^{14})$. Moreover, as shown in subsection 2.4 in the linear limit the phase speed for longitudinal vibrations varies between $c_l = \sqrt{EA/m}$ in the case of a Poisson ratio $P = 1/2$ material and $c_l = \sqrt{(EA + T)/m}$ for a $P = 0$ one. As $T \ll EA$ the difference between the two cases is usually insignificant. Typical values for $m$ in water are $m = O(1-10)$. So for $Re = 100$ VIV in water a realistic $c_l = O(10^4 - 10^7)$, with the phase speed for the transverse waves in the same linear limit $c_t = \sqrt{T/m} = O(10^2 - 10^5)$.

Both numbers are very large, leading to the following problems for realistic simulations:

- Use of an explicit finite difference approximation suffers a severe CFL limit restriction as $c_l \frac{dt}{dx} \leq 1$. This can be addressed through the development of a more advanced structural solver that would use an unconditionally stable timestepping algorithm.

- Even without the stability restrictions for the timestep though, we are still faced with three different time scales for our coupled problem:

  1. The dominant time scale of the flow dynamics.
2. The time scale of oscillations in the transverse direction.

3. The time scale of oscillations in the longitudinal direction.

In a lock-in situation the first two time scales match - the third one though remains a much faster time scale and it cannot be resolved unless one decreases the timestep to appropriate levels. Unfortunately, decreasing the timestep is not practical for anything other than toy problems, as the relative cost of the flow solver makes it prohibitively expensive.

The straight-forward way to address this issue is by introducing substructuring: a timestep for the structure solver \( \delta \tau = \frac{\delta t}{n} \) for an appropriate choice of \( n \). Computationally, the difference in cost between the two solvers is such that for reasonable values of \( n \) this is a viable solution. Because of the coupled nature of the problem however, the hydrodynamic forces are needed at every "structural" timestep; they can only be updated every \( n \) "structural" timesteps and therefore an artificially piecewise constant in time hydrodynamic forcing has to be used for the structure. The effect of this remains to be investigated.

As \( Re \) increases the non-dimensional \( EA \) (and \( T \)) decrease as \( Re^{-2} \) - therefore the aforementioned problems become less acute; at the same time DNS of the flow becomes considerably more difficult.

### 5.3 Comparing the mapping and the ALE formulations

A comparison between the mapping formulation detailed in section 4 of Chapter 2 and the ALE formulation presented in section 4 is instructive. We present comparisons for 2D flow-structure interactions, for both forced and free motion:

In figure 7 we show a 460 element square mesh with periodic boundary conditions at the outer boundaries. The periodic boundary conditions effectively force the mesh velocity to become constant throughout the domain and therefore equal to the velocity of the cylinder everywhere. This is equivalent to the mapped formulation's mesh velocity. This enables us to compare the two methods without the boundary condition effects discussed in subsection 4.1 of Chapter 2. The
Figure 7: 460 element mesh with periodic boundary conditions (left). Vertical force coefficient for forced oscillatory motion at $Re = 100$ (right). Solid line corresponds to the ALE calculation, triangles to the calculation using the mapping. The simulation used 5th order polynomials.

Flow is initially at rest and the cylinder is forced to oscillate vertically, its motion given by

\begin{align*}
    y &= \frac{t^2}{t^2 + C} A \cos(\Omega t), \\
    \dot{y} &= \frac{t^2}{t^2 + C} \left(-\Omega A \sin(\Omega t)\right) + 2C \frac{t}{(t^2 + C)^2} A \cos(\Omega t), \\
    \ddot{y} &= \frac{t^2}{t^2 + C} \left(-\Omega^2 A \cos(\Omega t)\right) - 2C \frac{3t^2 - C}{(t^2 + C)^3} A \cos(\Omega t) - 4C \frac{t}{(t^2 + C)^2} (-\Omega A \sin(\Omega t)).
\end{align*}

(44a)  
(44b)  
(44c)

to ensure a smooth startup ($C = 4.0$, $A = 0.1$, $\Omega = 1.05$ for $t = 0$, $y = \dot{y} = \ddot{y} = 0$, the larger $C$ is, the smoother the startup). As can be seen in the force plot in figure 7, the two formulations give indistinguishable results.

Next we consider the case of $2D$ flow (at $Re = 100$) past an elastically supported cylinder allowed to vibrate freely in both the in-line and cross-flow directions. Structural parameters and initial conditions are exactly the same for both simulations; the difference is in the formulation used for solving the moving mesh problem. It is also in the boundary conditions (discussed in subsection 4.1 of Chapter 2) of the mapping approach as opposed to the fixed farfield boundaries of the ALE formulation. In this case a large 490 element mesh (shown in figure 8) was used.
extending 22 cylinder diameters in front, above and below the cylinder and 69 cylinder diameters in the wake.

The size of the domain appears to successfully deal with any possible boundary derived problems. As can be seen in figures 9, 10 and 11, both the cylinder displacement and forces calculated using the two different formulations are in excellent agreement. Only in the case of the base pressure is any difference visible to the eye and that is still very small.

Figure 8: 490 element mesh for free (in both $x$- and $y$- directions) calculations. The vorticity field is shown and a closeup of the mesh is seen in the inset. The simulation used 6th order polynomials.

Figure 9: in-line vibrations (left) and cross-flow vibrations (right) for the freely vibrating cylinder. Solid line corresponds to the ALE calculation, triangles to the calculation using the mapping.

The computational advantage of the mapping approach can be seen by contrasting the 0.38 seconds per timestep the code based on the mapping formulation takes versus the 3.27 the ALE
Figure 10: Drag (left) and Lift (right) coefficients for the freely vibrating cylinder. Solid line corresponds to the ALE calculation, triangles to the calculation using the mapping.

Figure 11: Base pressure for the freely vibrating cylinder. Solid line corresponds to the ALE calculation, triangles to the calculation using the mapping.
code does on the same platform. The ALE approach is slower by an order of magnitude!
Chapter 6

Model Simulations

1 Introduction

Having analyzed our structural model and solution approach to the coupled flow and non-linear structure problem in Chapter 5 we proceed to attempt a few model DNS simulations, at a laminar state in Reynolds number of $Re = 100$. The aim here is to investigate the basic capabilities of our approach: What are the similarities with the linear structure case and what about any differences in behavior, even at this laminar flow regime, because of the non-linear structural model we are employing?

In section 2 we briefly discuss the details of the simulations (mesh, structural model and parameters etc.). In section 3 we discuss the initial conditions for our freely vibrating simulations, followed in section 4 by a presentation and discussion of the results for the freely vibrating cases. Finally we present early results from a larger (more resolved) simulation in section 5 and we summarize our findings in section 6.
2 Simulation parameters

We initially consider a minimal 28 element 2D mesh shown in figure 1. This mesh, while small in size, retains two of the primary characteristics of the meshes we would want to be using for more realistic calculations:

- a layer of rectangular elements on the cylinder's surface (to resolve the boundary layer) and
- a structured rectangular element filled medium- and far-wake region.

Of course as can be seen in figure 1 the small domain (and associated blockage effect), few elements and relatively low polynomial order given the element sizes affect the solution quality; at the same time a vortex street appear in the wake and the correct physics are captured to some degree.

![Figure 1: 28 element minimalist mesh for flow past a cylinder at Re = 100. Cross-flow velocity isocontours are shown. The simulation was run using 5th order polynomials.](image)

Using a small 2D mesh as a starting point is useful because we want to extrude it in the z- (spanwise) direction to obtain a 3D mesh for our calculations. In this case we extruded the layout of this 28 element mesh, building eight layers of hexahedral and prismatic elements to cover a cylinder with an aspect (length to diameter) ratio of $4\pi$. The resulting 3D mesh has 224 elements and is shown in figure 2. The extrusion process increases the number of the elements significantly - as 3D calculations are very costly this means that using as our basis for extrusion a nice dense 2D mesh like the one in figure 1 of Chapter 4 would result in a prohibitively expensive calculation. Even with such a minimal mesh every timestep of a calculation using 5th order polynomials takes about 5-6 seconds on 32 processors of a Cray T3E or an Origin 2000. This with a tolerance
criterion for our iterative solvers of $10^{-6}$ for the pressure solve and $10^{-6}$ for all other solves; decreasing the tolerance would result in even more expensive timesteps.

![Diagram of a 3D mesh](image)

Figure 2: 224 element mesh for flow past a cylinder at $Re = 100$.

We chose to do our initial "proof-of-concept" calculations using the mesh described above henceforth referred to as meshA; to address the issues that the choice of such a minimalist mesh generates with respect to the accuracy of our simulations we also started simulations using a larger, 880-element mesh, shown in figures 3 and 4. This mesh (meshB) was generated by extruding the 110 element mesh shown in figure 2 in section 2 of Chapter 3. Again 8 levels were extruded in the spanwise direction.

In order to choose a reasonable polynomial order for this larger simulation we did a resolution analysis for this grid; the results for the amplitude $C_l'$ and root mean square value $C_{l_{rms}}$ of the coefficient of lift, average $C_d$ and amplitude $C_d'$ (about the average) coefficient of drag, and Strouhal number $St$ are presented in table 6.1.

The results of this study suggest using polynomial order 5 in order to both minimize the computational costs and resolve the flow forces acceptably well. The structure was discretized using 250 points along the span of the cylinder. Using 88 processors of the Cray T3E at NAVO,
Figure 3: 880 element mesh for flow past a cylinder at $Re = 100$.

Figure 4: 880 element mesh for flow past a cylinder at $Re = 100$. Closeup near the cylinder surface.
<table>
<thead>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<td>0.33</td>
<td>0.34</td>
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<tr>
<td>$C_{l_{rms}}$</td>
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<td>0.26</td>
<td>0.24</td>
<td>0.24</td>
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</tr>
<tr>
<td>$C_d$</td>
<td>1.38</td>
<td>1.39</td>
<td>1.37</td>
<td>1.37</td>
<td>1.37</td>
<td>1.37</td>
</tr>
<tr>
<td>$C_d'$</td>
<td>0.012</td>
<td>0.015</td>
<td>0.009</td>
<td>0.009</td>
<td>0.010</td>
<td>0.010</td>
</tr>
<tr>
<td>$St$</td>
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<td>0.1702</td>
<td>0.1706</td>
<td>0.169</td>
<td>0.168</td>
<td>0.168</td>
</tr>
</tbody>
</table>

Table 6.1: Lift coefficient amplitude $C_l$ and root mean square value $C_{l_{rms}}$; drag coefficient average $C_d$ and amplitude $C_d'$ (about the average) and Strouhal number $St$ for 2D $Re = 100$ calculations using the 110 element 2D mesh.

A timestep of our coupled simulation takes about 9.6 seconds - on 88 processors of an Origin 2000 (195MHz) it takes 11.6 seconds.

For all the simulations presented in the following sections we used a Poisson ratio $P = 1/2$ tension model. The reference position was always chosen to be the straight stretched string one, with tension $T_r = 44.1$, a value unrealistically low for VIV in water as discussed in subsection 2.1 of Chapter 5. $E.A_r$ was chosen to be an even more unrealistically low 441.0 to avoid the problems mentioned in subsection 5.2 of Chapter 5. The mass ratio $m_r = 10$. The particular choice of $T_r$ and $m_r$ was such that the phase speed of the linearized equations 21 of Chapter 5 in the transverse directions $c = \sqrt{T_r/m_r}$ would give us a value satisfying

$$c(2\pi/L) = 2\pi St$$

(1)

as discussed in subsection 2.2 of Chapter 2. Under these conditions lock-in to the 2nd (linear) vibration mode of a cable with pinned endpoints should be promoted. The value of $T_r$ thus estimated is very low because of the relatively small aspect ratio of the cable - for larger $L$, $c$ is larger and hence more realistic values of $T_r$ can be in the range needed for lock-in. No structural damping was used for any of these simulations to achieve the maximum displacement the flow will allow.

The simulations were performed on 16, 32 and 88 processors of the NAVO Cray T3E-900, with some also performed on the NAVO SGI/Cray Origin 2000 (195MHz).
3 Initial Conditions

We tried two different initial conditions for our simulations with meshA: The first one (called ICc from now on) was the \( l = 2 \) (2nd linear structural mode in the transverse direction) case of the one derived in subsection 2.5 of Chapter 5 (equations 28 and 29 with opposite signs - a change in phase of \( \pi \)). It has the property that it results in very small tension variation along the span:

\[
y(S, 0) = -y_{max} \sin \frac{2\pi S}{L}, \tag{2a}
\]

\[
z_u(S, 0) = y_{max}^2 \frac{2\pi}{8L} \sin \frac{4\pi S}{L}, \tag{2b}
\]

\[
\frac{\partial y}{\partial t}(S, 0) = 0. \tag{2c}
\]

\[
\frac{\partial z_u}{\partial t}(S, 0) = 0. \tag{2d}
\]

The second one (ICv) is the same apart from the sign of the equation describing \( z_u(S, 0) \):

\[
y(S, 0) = -y_{max} \sin \frac{2\pi S}{L}, \tag{3a}
\]

\[
z_u(S, 0) = -y_{max}^2 \frac{2\pi}{8L} \sin \frac{4\pi S}{L}, \tag{3b}
\]

\[
\frac{\partial y}{\partial t}(S, 0) = 0, \tag{3c}
\]

\[
\frac{\partial z_u}{\partial t}(S, 0) = 0. \tag{3d}
\]

This simple change of sign introduces large tension variation for the initial condition. We chose \( y_{max} = 1 \); as a reminder \( L = 4\pi \) for these simulations as well.

Given our choice of initial condition for the structure we need to get the flow solver to reach a state geometrically (displacement-wise) and dynamically (velocity-wise) consistent with this initial condition. We accomplish that by

- starting our flow simulation from the initial condition of flow past a straight cylinder at \( Re = 100 \),

- imposing a forced standing wave motion on our cylinder boundary using the 2\( D \) motion
described in subsection 5.3 of Chapter 5.

To get the standing wave we modulate this motion in space by the initial conditions we aim for. This gives

\[
y(S, t) = y_{\text{max}} \frac{t^2}{t^2 + C} \cos(\Omega t) \sin \frac{2\pi S}{L},
\]
\[
z_v(S, t) = -y_{\text{max}}^2 \frac{t^2}{t^2 + C} \cos(\Omega t) \frac{2\pi}{8L} \sin \frac{4\pi S}{L}
\]

for ICc and

\[
y(S, t) = y_{\text{max}} \frac{t^2}{t^2 + C} \cos(\Omega t) \sin \frac{2\pi S}{L},
\]
\[
z_v(S, t) = y_{\text{max}}^2 \frac{t^2}{t^2 + C} \cos(\Omega t) \frac{2\pi}{8L} \sin \frac{4\pi S}{L}
\]

for ICv respectively. This choice of modulation of our standing wave in time by an envelope of \( \frac{t^2}{t^2 + C} \) allows us to have a much smoother startup as the initial velocity and acceleration of this forced motion are zero. We used \( \Omega = 1.05 \) and \( C = 1 \) and ran our simulations until time \( t = 12 \) so that our choice of initial conditions was within 1% of the state reached.

In figure 5 we show the displacement from the straight stretched string reference position in the vertical and longitudinal direction for the simulation with motion following equation 4. This simulation was done using 3rd order polynomials. Subsequent freely vibrations calculations employed 5th order polynomials.

4 VIV of a non-linear string

4.1 String response: Displacement and Tension

In figure 6 we plot the displacement and tension distribution as a function of time and spanwise location for the freely vibrating string that started from ICv. One notes the slow variation in the streamwise displacement as the string adjusts to the presence of an average drag force pushing it.
Figure 5: Displacement from the straight stretched string reference position in the y— (top) and z— direction for the forced standing wave simulation leading to ICc.

downstream. The cross-flow displacement exhibits a standing wave pattern with extra modulation causing distortion at the antinodes. The longitudinal displacement begins as a standing wave with half the spanwise wavelength as dictated by ICv but quickly develops sharp distortions very alike the ones seen in figure 5 of Chapter 5. These distorted standing waves quickly reduce in amplitude, again as seen in the simulations and analysis of the cases in Chapter 5. We consider this behavior to be structure rather than flow driven and a result of the wide variations in tension that our initial condition exhibited. Indeed looking at the distribution of tension in figure 6 we observe that the large spanwise variations in tension result in very large and very small local values of tension by non-dimensional time $t = 15$, 3 time units after our simulation started. The initially smooth tension distribution along the span becomes progressively more “shock-like” with regions of low (or even extremely localized negative) tension separating smoother regions. By time $t = 18$ interactions between these regions diminish the smoothness even further - after this time the accuracy of the structural solver should be in doubt; a finer mesh could be used along the span, a higher order space discretization could be employed or better still a combination of both could
Figure 6: Initial displacement (from the straight stretched string reference position) in the streamwise $x$— (first from the top), cross-flow $y$— (second from the top) and spanwise $z$— (third from the top) direction for the simulation starting from ICv. At the bottom we plot the string tension as a function of span and time.
be employed to track the solution further. It is interesting to note that even with this level of non-smoothness in the structural response the simulation is stable.

Moving on to the simulation that started at $t = 12$ from ICc we plot the streamwise displacement in figure 7. The initial adjustment of the string to the presence of drag leads to oscillations of well more than half a diameter in amplitude but eventually hydrodynamic damping generates a catenary response with small time-periodic variations, more pronounced towards the mid-span point.

![Figure 7: Initial (top) and time asymptotic (bottom) displacement in the streamwise $z$—direction for the simulation starting from ICc.](image)

In figure 8 we plot the cross-flow displacement for the same simulation. The original standing wave pattern is sustained in time but both the amplitude decreases to about $0.5 - 0.6d$ (as seen in linear structure $Re = 100$ simulations [19]); the frequency of oscillations can also be seen to decrease slightly.

We plot the spanwise displacement (from the reference straight stretched string position) in figure 9. Unlike the behavior seen in figure 6 starting from ICv, the pattern established here cannot be called a standing wave (as that of figure 5) as the regions of positive and negative
Figure 8: Initial (top) and time asymptotic (bottom) displacement in the cross-flow \( y \) - direction for the simulation starting from ICc.

displacement do not change in time. An amplitude variation with time for both positive and negative displacement regions is established instead. The time asymptotic state has half the initial amplitude near the boundaries and smaller peaks near the node of the standing wave.

The tension distribution for this simulation is plotted in figure 10. The amplitude of the initial displacements was enough to make the initial tension not nearly as constant as we would like - remember equation 26 of Chapter 5 is a condition for constant tension along the span to third order in the amplitude of the displacements but our initial amplitude is large. During the initial transient period, variations of more than 50% in time and at certain times 20% along the span were observed; in the time asymptotic state 28% variations in tension with time are sustained along with periodic variations along the span (the obvious modulations in the bottom figure 10) of about 5%. The tension variations include small amplitude and wavelength variations contributing to the somewhat grainy picture in figure 10.

These variations in tension (and \( \Delta \)) change the natural frequency of vibration \( f \) of the string.
Figure 9: Initial (top) and time asymptotic (bottom) displacement (from the straight stretched string reference position) in the spanwise $z-$ direction for the simulation starting from ICc.

Figure 10: Initial (top) and time asymptotic (bottom) tension distribution for the simulation starting from ICc.
In a local linearization around point $S$ along the span,

$$f_s(S, t) = \frac{1}{L\sqrt{m_r}}\sqrt{\frac{T(S, t)}{\Delta(S, t)}}$$  \hspace{1cm} (6)

and in figure 11 we plot the time asymptotic behavior of $f_s$. It can be seen that $f_s$ varies over a range of about 10% of its average value - this means that any lock-in behavior will be non-classical as the structural frequency is significantly variable in time and space.

Figure 11: Time asymptotic variation of the (locally linearized) natural frequency of vibration $f_s$ of the string for the simulation starting from ICc.

### 4.2 Hydrodynamic Forces

Looking at the distribution of hydrodynamic forces for the simulation with initial conditions ICv in figure 12 we note that the original smoothness of the coefficient of drag variations begins to be replaced (after about 3 non-dimensional time units) by localized regions of large and rapid variations. After another 3 time units we get a complete breakdown of large scale and long spanwise wavelength variations leading to the completely unphysical drag distribution better seen in the closeup (second from the top) in figure 12. We even see large values of negative drag (thrust) corresponding to the case of the string trying to locally overtake the flow around it. This breakdown is directly related to the structural response. Rapid (and of short spanwise wavelength) variations in the structural response are fed back into the flow solver through both the velocity and the pressure boundary conditions. The latter are adjusted by the value of the structural acceleration which in this case would be very large, leading to wild pressure oscillations - indeed
further examination of the results shows that it is the pressure derived component of the force that is the main contributor to the large and rapid variations. The small size of the mesh (and thereby proximity of the boundaries to the cylinder) makes such pressure driven instabilities even more severe.

The same breakdown is seen for the lift and spanwise forces; the extra feature in this case is that from the outset of free motion at $t = 12$, high frequency oscillations (only in time) appear for both lift and spanwise forces. These are not related to the structural response which is still at this time very smooth (see figures 7, 8 and 9). Again these are pressure driven variations at a frequency of about 13 that has no physical significance. They appear to be artifacts of the small mesh employed that did not appear in the forced motion simulations because only 3rd order polynomials were employed then. Long spanwise wavelength variations in lift and spanwise force, however, are still evident in figure 12 before the breakdown of all force distributions at later times.

In figure 13 we plot the coefficient of drag distribution for the simulation with initial conditions ICc. The pattern that emerges after the transients die out is consistent with that of a standing wave (higher drag at the antinodes, lower at the nodes). In this case, the regions right next to the string boundaries (which are nodes) actually have less drag than the boundaries themselves. This is probably an effect of the streamwise flexibility of the structure, something that was not allowed in simulations covered in Chapter 4. The drag coefficient at the midspan and boundary nodes exhibit periodic variations close to 10%; the minimum drag appears at the midspan location.

The coefficient of lift distribution seen in figure 14 exhibits from the outset the same type of rapid variations in time seen in figure 12. Similarly for the spanwise force distribution in figure 15. In both cases however this mesh artifact still allows for a large scale variation picture that is consistent with the standing wave (or "standing wave like" in the case of spanwise motion) structural response seen in figures 8 and 9 respectively.

We should note that in the case of the spanwise forces the time asymptotic magnitudes observed are $20 - 25\%$ of the corresponding lift values and are prominent at the string boundaries.

Finally, in figure 16 we plot the energy exchange (power) distribution between the flow and
Figure 12: Initial coefficient of drag $C_d$ distribution (first from the top), with a more detailed closeup (second from the top), coefficient of lift $C_l$ distribution (third from the top) and spanwise force coefficient $C_z$ distribution (bottom).
Figure 13: Initial (top) and time asymptotic (bottom) drag coefficient $C_d$ distribution for the simulation starting from ICc.

Figure 14: Initial (top) and time asymptotic (bottom) lift coefficient $C_l$ distribution for the simulation starting from ICc.
Figure 15: Initial (top) and time asymptotic (bottom) spanwise force coefficient $C_z$ for the simulation starting from ICc.

the structure as a function of spanwise location and time. Positive power means that the flow is inputing energy to the structure while negative power means that it is extracting energy from the structure. The lack of rapid fluctuations of the structure's displacement in time compared to the mesh induced fluctuations in the hydrodynamic forces make the graphs very grainy but a large scale picture still emerges: In the case of energy exchange because of drag forces a regular pattern emerges, with higher values at the antinodes and a slight time-lag at the node. The lift force picture is characteristic of a standing wave response while the energy exchange because of longitudinal motion is mainly concentrated near the boundaries, where the longitudinal displacement $z_v$ attains its largest values (see figure 9).

4.3 Comparison with a simulation using a linear structural model

For comparison purposes the methods of Chapter 2 were used to simulate VIV of a cable with pinned endpoints, and of the same structural parameters $T_r$ and $m$, at $Re = 100$. A meshA-equivalent Fourier mesh was used. The initial conditions for the free vibration were the relevant
Figure 16: Time asymptotic energy exchange in the streamwise $x-$ (top), cross-flow $y-$ (middle) and spanwise $z-$ (bottom) direction for the simulation starting from ICc.
part (the linear models do not allow for any longitudinal oscillation) of ICc, namely:

\[
\frac{\partial y(S, 0)}{\partial t} = 0. \tag{7b}
\]

Following the methodology used in section 3 to obtain consistent initial conditions for both the flow and the structure, the freely vibrating linear cable calculation was started and run for the same amount of time as the non-linear one had run. The time asymptotic results can be seen in figure 17.

One notes immediately the absence of the numerical oscillations in this case. This is not entirely surprising given the difference in the flow solvers. Moreover the mesh size is too small for boundary effects to be negligible and so good quantitative agreement should not be expected between these two cases. Never-the-less, the values of streamwise displacement are within 10% of each other. Drag coefficient values at the nodes are very close, while the values at the antinodes and the general shape differ visibly. Very large differences can be seen in the case of $C_l$, as expected, given the lack of numerical oscillations, with cross-flow displacement values appropriately affected. Moreover the frequency of the non-linear transverse oscillations is larger than that of the linear ones, as expected given the periodic increases in the value of the tension $T$.

### 4.4 Flow Visualizations

Having looked at the structural response and the forces exerted by the flow on the vibrating string, we proceed to look at the flow itself: In figure 18 we plot cross-flow velocity isocontours near the midspan node and the two antinodes. The difference in the structure of the wake is immediately evident and compares well qualitatively with the results of Newman [19].

Pressure isosurfaces in figure 19 indicate alternating shedding of vortices from the vibrating string with a clear vertical separation. These vortices again look very similar to the ones seen in
Figure 17: Streamwise displacement (first from the top), cross-flow displacement (second from the top), coefficient of drag $C_d$ distribution (third from the top) and coefficient of lift $C_l$ distribution (bottom) for a simulation of a linear "cable". There is a "virtual shift" in the time axis of 100 time units.
Figure 18: Cross-flow velocity isocontours at three different locations along the span (non-dimensional time $t = 156$) for the simulation starting from ICc.

earlier $Re = 100$ calculations (compare with interlaced spanwise vorticity visualizations in figure 4.8 of [19]).

Looking at streamwise (figure 20) and cross-flow (figure 21) vorticity isosurfaces we see prominent elongated vortical structures behind the boundary and mid-span node with pairs of curved vortices covering half the span of the cylinder each traveling downstream between them.

5 A more realistic simulation

In view of the results we have gotten with the simulations in section 4, the dependence on the initial conditions employed and the problems with the mesh used, we proceeded to use meshB for our $Re = 100$ calculations. Moreover, to avoid any problems arising from the initial conditions used we employed a "natural" initial condition exhibiting exactly constant tension: the reference straight stretched string. That is we started from $2D \ Re = 100$ flow past a fixed straight cylinder and let the flow drive the string motion thereafter. In an attempt to have a simulation closer to realistic structural parameters we chose to increase $EA_r$ by a factor of 10 to 4410. This now gives
Figure 19: Pressure isocontours at $-0.08$ and $-0.06$ as seen from above and to the side (top) and as seen from below (bottom). Non-dimensional time $t = 156$ for the simulation starting from ICc.
Figure 20: Streamwise vorticity isocontours at ±1 as seen facing the incoming flow behind the cylinder (top) and facing the cylinder from the front (bottom). Non-dimensional time $t = 156$ for the simulation starting from ICc.
Figure 21: Cross-flow vorticity isocontours at ±0.7 as seen facing the incoming flow behind the cylinder (top) and facing the cylinder from the front (bottom). Non-dimensional time $t = 156$ for the simulation starting from ICc.
a more realistic initial strain $\epsilon = 1\%$.

The disadvantage of the approach we use is that initial vibration amplitudes observed are bound to be small. Furthermore, even though we kept $T_r$ and $m_r$ the same as in the simulations in section 4 in order for the 2nd vibrational mode of the string to be the one prone to excitation, vibration in that mode would require a breaking of the symmetry about the mid-span location that the flow naturally exhibits initially. This breaking of the symmetry, seen in previous calculations with linear structural models, can take many decades of non-dimensional time units to take effect. As these computations are considerably time-consuming only preliminary results are shown here.

In figure 22 we see the initial response of the structure to the presence of the flow. Just like before there is an adjustment to the presence of drag that sets off oscillations of amplitude about $0.5d$ in the streamwise direction - hydrodynamic damping should eventually reduce their amplitude after the string assumes a shape akin a catenary in this direction. The $2D$ initial flow forces the structure to begin vibrating in the cross-flow direction in its first vibrational mode - the initial amplitude of these vibrations is one order of magnitude smaller than that of the maximum streamwise displacement; at the same time yet another order of magnitude smaller oscillations (antisymmetric about the mid-span) appear for the spanwise displacement. The tension (shown at the bottom of figure 22) exhibits very little spanwise variation but its values increase in time by more than 20%. All structural variable oscillations appear to peak at the same point in time.

In figure 23 we plot the coefficient of drag, lift and spanwise force as a function of span location and time. The effects of the pinned boundaries are more evident here, as they give rise to structures that travel towards the mid-span location and interfere with each other. The effect is more pronounced in the case of the drag coefficient where a variation of about 60% is observed at the mid-span location. This departure from the usual parallel shedding type picture that would correspond to a stationary cylinder can also be seen in the coefficient of lift variation where the effect of the endpoints is to introduce within the first 10 convective time units significant phase and amplitude differences in lift along the span. For both $C_d$ and $C_l$ (as well as streamwise and cross-flow displacement) the symmetry about the mid-span has yet to break. In the case of the
Figure 22: Initial displacement in the streamwise $x-$ (first from the top), cross-flow $y-$ (second from the top) and spanwise $z-$ (third from the top) direction for the simulation employing meshB. At the bottom we plot the string tension as a function of span and time.
Figure 23: Initial coefficient of drag $C_d$ distribution (top), coefficient of lift $C_d$ distribution (middle) and spanwise force coefficient $C_z$ distribution (bottom).
spanwise forces. Their initial magnitudes are 3 times smaller than those of the lift and they are important only near the boundaries - close to the mid-span location they are negligible.

6 Summary

In this chapter we applied the methods detailed in Chapter 5 to some model simulations of flow past non-linear strings at \( Re = 100 \).

We initially used a small mesh and investigated the effect of non-zero initial conditions with small and large variations in tension. We observed that the latter lead to a breakdown of most large features in the structural response and induce large pressure oscillations in the flow. This type of behavior had also been observed in the absence of external forces in our calculations in Chapter 5. We also observed small (time) scale oscillations in the lift and spanwise forces due to the minimalist features of the mesh. In the case of the initial condition with small variations in tension we observed persistent standing wave patterns for the displacement in the cross-flow direction and the drag coefficient. The spanwise displacement (and large scale features in the spanwise force distribution) exhibited a different type of time-periodic pattern, relate to a standing wave. The flow structures observed in the wake were qualitatively similar to the ones seen in simulations with linear structural models that had a standing wave response.

In order to address some of the problems with these calculations we started a calculation using a much large and more refined mesh, starting from zero initial conditions for our structure and letting the flow drive the motion. Boundary effects became quickly visible in the hydrodynamic force distributions and the structure exhibited longitudinal vibrations and increase in tension in the first 10 non-dimensional time units. We expect this calculations (still in progress) to exhibit a breaking of the symmetry about the mid-span it now possesses; it has been free of the pressure oscillations that appeared in the smaller calculations.
Appendix A

Parallel Algorithms and Implementation Issues

1 Introduction

The direct numerical simulation (DNS) of turbulent flows was initiated in 1972 by Orszag and Patterson, who obtained accurate simulations of wind-tunnel flows at moderate Reynolds numbers [75]. This simulation was performed on the CDC 7600 with limited memory and only 50 Mflop/s peak speed, however it opened up the possibility of simulating turbulence from first principles without any ad hoc modeling, by directly solving the Navier-Stokes equations of fluid motion.

In the last 25 years, the field has developed remarkably due to advances both in algorithms and computer hardware. In the last decade the peak performance of supercomputers sky-rocketed by a factor of 500, compared to only a factor of 10 - 15 from 1977 to 1987. At the same time, we have moved from the classical Fourier algorithms used in the first simulation of homogeneous turbulence to more sophisticated algorithms involving spectral element methods on unstructured grids handling computational domains of arbitrary geometric complexity.

A review of the state-of-the-art in DNS was presented in 1993 in [76] along with a specific
proposal for future developments in DNS: The design of a Parallel Prototype Computer (PPC), a hybrid distributed/shared memory computer of 1000 processors achieving a speed of 1 Teraflop for a load balanced turbulence simulations of $1024^3$ resolution. Today, the main concepts of that proposal have been realized as part of the developments of the Advanced Strategic Computing Initiative supported by the Department of Energy. Moreover, most of the newer architectural models such as the HP/Convex Exemplar, the SGI Origin 2000 or the new IBM SP have features similar to the PPC, thus providing high efficiency in solving the Navier-Stokes equations. With target simulations of 1 billion grid points on 100 Teraflop computer systems achievable in the next couple of years, the current focus has turned on the possibilities of simulations on the next generation of systems corresponding to once unthinkable Petaflop ($10^{15}$ flops) rate systems.

Despite such great developments on the computer hardware side, the progress on the physical modeling side has been limited both in terms of the Reynolds number range as well as the physical or geometric complexity that can be simulated. As regards to the Reynolds number, an increase by a factor of 2 would approximately require one order of magnitude increase in CPU resources [76]. The introduction of non-isothermal processes, chemical reactions or complex geometric domains in the simulation tax the computational resources in a similar manner. The computational complexity is ultimately related to the numerical algorithms used to discretize the continuous equations as well as the parallel paradigm employed. To this end, the majority of the turbulence simulations today has been performed using spectral methods either of Fourier type, as in homogeneous turbulence, or of a polynomial type as in shear turbulent flows [77, 78, 79, 80]. The parallel paradigm can vary depending on the physical situation modeled but it is typically based on domain decomposition for the most complex cases.

A review of other numerical methods for parallel simulation of fluid flows including turbulent flows has been given in [76] and [81]. Here, we will concentrate on two prototype cases that are used in simulating turbulence in non-separable and multiply-connected domains: The first one corresponds to physical situations where one of the Cartesian directions is homogeneous and thus Fourier expansions can be employed along that direction while the other two Cartesian
directions are inhomogeneous and thus general spectral methods (or finite differences [82, 83]) are applied. The second case involves computational domains where all three Cartesian directions are non-homogeneous. Specifically, we will focus on spectral/hp type methods which exhibit an intrinsic “domain-decomposition”: this leads naturally to a geometry-based distribution of work amongst processors which permits a high degree of parallelism. The key computational kernels are scalar products, matrix-vector and matrix-matrix multiplies, while the communication patterns involve pairwise exchanges, global exchanges, global reductions and gathers as well as global synchronizations.

The rest of the sections are organized as follows: We first briefly review the spectral method that we will use and the time integration algorithm of the Navier-Stokes equations as they are the basis of the parallel paradigms that we will use. We then proceed with the specifics of the spectral/Fourier solver and subsequently we discuss the second paradigm with a full unstructured solver. We analyze some of the generic communication patterns used in both these solvers and conclude with a discussion.

2 Spectral/hp Discretizations on Unstructured and Hybrid Grids

The parallel code *Nek5000* that we employ in the two aforementioned parallel paradigms is based on a new spectral basis [84, 85, 86]. It is appropriate for unstructured meshes based on triangles or tetrahedra in two- and three-dimensions, respectively. In many simulations, however, involving complex-geometry domains or external flows it is more efficient to employ hybrid discretizations, i.e. discretizations using a combination of structured and unstructured subdomains. Such an approach combines the simplicity and convenience of structured domains with the geometric flexibility of an unstructured discretization. In two-dimensions, hybrid discretization simply implies the use of triangular and rectangular subdomains, however in three-dimensions the hybrid strategy is more complex requiring the use of hexahedra, prisms, pyramids and tetrahedra.
Hexahedral domains have been used quite extensively in the $hp$ finite element field [87, 88]. More recently an unstructured $hp$ finite element approach, based upon theoretical work in two-dimensions by Dubiner [89], has been developed for unsteady problems in fluid dynamics [90, 85, 91]. In the following, we will show how these expansions can be constructed using a unified approach which incorporates all the hybrid subdomains.

This unified approach generates polynomial expansions which can be expressed in terms of a generalized product of the form $\phi_{pqr}(x, y, z) = \phi^p(x)\phi^q(y)\phi^r(z)$. Here we have used the Cartesian coordinates $x, y$ and $z$ but, in general, they can be any set of coordinates defining a specified region. The standard tensor product is simply a degenerate case of this product where the second and third functions are only dependent on one index. The primary motivation in developing an expansion of this form is computational efficiency. Such expansions can be evaluated in three-dimensions in $O(N^4)$ operations as compared to $O(N^6)$ operations necessary with non-tensor products based expansions.

2.1 Local coordinate Systems

We start by defining a convenient set of local coordinates upon which we can construct the expansions. Moving away from the use of barycentric coordinates, which are typically applied to unstructured domains, we define a set of collapsed Cartesian coordinates in non-rectangular domains. These coordinates will form the foundation of the polynomial expansions. The advantage of this system is that every domain can be bounded by constant limits of the new local coordinates; accordingly operations such as integration and differentiation can be performed using standard one-dimensional techniques.

![Figure 1: Triangle to rectangle transformation](image)

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The new coordinate systems are based upon the transformation of a triangular region to a rectangular domain (and vice versa) as shown in figure 1. The main effect of the transformation is to map the vertical lines in the rectangular domain (i.e. lines of constant $\eta_1$) onto lines radiating out of the point ($\xi_1 = -1, \xi_2 = 1$) in the triangular domain. The triangular region can now be described using the "ray" coordinate ($\eta_1$) and the standard horizontal coordinate ($\xi_2 = \eta_2$). The triangular domain is therefore defined by ($-1 \leq \eta_1, \eta_2 \leq 1$) rather than the Cartesian description ($-1 \leq \xi_1, \xi_2; \xi_1 + \xi_2 \leq 0$) where the upper bound couples the two coordinates. The "ray" coordinate ($\eta_1$) is multi-valued at ($\xi_1 = -1, \xi_2 = 1$). Nevertheless, we note that the use of singular coordinate systems is very common arising in both cylindrical and spherical coordinate systems.

![Figure 2: Hexahedron to tetrahedron transformation](image)

As illustrated in figure 2, the same transformation can be repeatedly applied to generate new coordinate systems in three-dimensions. Here, we start from the bi-unit hexahedral domain and apply the triangle to rectangle transformation in the vertical plane to generate a prismatic region. The transformation is then used in the second vertical plane to generate the pyramidal region. Finally, the rectangle to triangle transformation is applied to every square cross section parallel to the base of the pyramidal region to arrive at the tetrahedral domain.

By determining the hexahedral coordinates ($\eta_1, \eta_2, \eta_3$) in terms of the Cartesian coordinates of the tetrahedral region ($\xi_1, \xi_2, \xi_3$) we can generate a new coordinate system for the tetrahedron. This new system and the planes described by fixing the local coordinates are shown in figure 3. Also shown are the new systems for the intermediate domains which are generated in the same fashion. Here we have assumed that the local Cartesian coordinates for every domain are ($\xi_1, \xi_2, \xi_3$).
2.2 Spectral Hierarchical Expansions

For each of the hybrid domains we can develop a polynomial expansion based upon the local coordinate system derived in section 2.1. These expansions will be polynomials in terms of the local coordinates as well as the Cartesian coordinates \((\xi_1, \xi_2, \xi_3)\). This is a significant property as primary operations such as integration and differentiation can be performed with respect to the local coordinates but the expansion may still be considered as a polynomial expansion in terms of the Cartesian system.

We shall initially consider expansions which are orthogonal in the Legendre inner product. We define three principle functions \(\phi^\alpha_i(z)\), \(\phi^b_{ij}(z)\) and \(\phi^c_{ijk}(z)\), in terms of the Jacobi polynomial, \(P^\alpha_\beta(z)\), as:

\[
\phi^0_i(z) = P^0_0(z), \quad \phi^b_{ij}(z) = \left(\frac{1-z}{2}\right)^i P^2i+1,0(z), \\
\phi^c_{ijk}(z) = \left(\frac{1-z}{2}\right)^{i+j} P^{2i+2j+2,0}(z).
\]
Using these functions we can construct the orthogonal polynomial expansions:

**Hexahedral expansion:**

\[
\phi_{pq} r (\xi_1, \xi_2, \xi_3) = \phi_p^a (\xi_1) \phi_q^a (\xi_2) \phi_r^a (\xi_3)
\]

**Prismatic expansion:**

\[
\phi_{pq} r (\xi_1, \xi_2, \xi_3) = \phi_p^a (\xi_1) \phi_q^a (\eta_1) \phi_r^a (\xi_3)
\]

**Pyramidal expansion:**

\[
\phi_{pq} r (\xi_1, \xi_2, \xi_3) = \phi_p^a (\eta_1) \phi_q^a (\eta_2) \phi_r^a (\eta_3)
\]

**Tetrahedral expansion:**

\[
\phi_{pq} r (\xi_1, \xi_2, \xi_3) = \phi_p^a (\eta_1) \phi_q^b (\eta_2) \phi_r^c (\eta_3)
\]

where,

\[
\eta_1 = \frac{2(1 + \xi_1)}{(-\xi_2 - \xi_3)} - 1, \quad \eta_2 = \frac{2(1 + \xi_1)}{(1 - \xi_3)} - 1, \quad \eta_3 = \frac{2(1 + \xi_2)}{(1 - \xi_3)} - 1
\]

are the local coordinates illustrated in figure 3.

The hexahedral expansion is simply a standard tensor product of Legendre polynomials (since \( P_p^0 (z) = L_p (z) \)). In the other expansions the introduction of the degenerate local coordinate systems is linked to the use of the more unusual functions \( \phi_j^b (z) \) and \( \phi_ijk^c (z) \). These functions both contain factors of the form \( \left( \frac{1 - z}{2} \right)^p \) which is necessary to keep the expansion as a polynomial of the Cartesian coordinates \((\xi_1, \xi_2, \xi_3)\). For example, the coordinate \( \eta_2 \) in the prismatic expansion necessitates the use of the function \( \phi_r^a (\xi_3) \) which introduces a factor of \( \left( \frac{1 - \xi_3}{2} \right)^q \). The product of this factor with \( \phi_q^a (\eta_2) \) is a polynomial function in \( \xi_2 \) and \( \xi_3 \). Since the remaining part of the prismatic expansion, \( \phi_p^a (\xi_1) \), is already in terms of a Cartesian coordinate the whole expansion is a polynomial in terms of the Cartesian system.

The polynomial space, in Cartesian coordinates, for each expansion is:

\[
P = \text{Span}\{\xi_1^p, \xi_2^q, \xi_3^r\}
\]
where $pqr$ for each domain is

\begin{align*}
\text{Hexahedron} & \quad 0 \leq p \leq N_1 \quad 0 \leq q \leq N_2 \quad 0 \leq r \leq N_3 \\
\text{Prism} & \quad 0 \leq p \leq N_1 \quad 0 \leq q \leq N_2 \quad 0 \leq q + r \leq N_3 \\
\text{Pyramidal} & \quad 0 \leq p \leq N_1 \quad 0 \leq q \leq N_2 \quad 0 \leq p + q + r \leq N_3 \\
\text{Tetrahedron} & \quad 0 \leq p \leq N_1 \quad 0 \leq p + q \leq N_2 \quad 0 \leq p + q + r \leq N_3.
\end{align*}

The range of the $p$, $q$ and $r$ indices indicate how the expansions should be expanded to generate a complete polynomial space. We note that if $N_1 = N_2 = N_3$ then the tetrahedral and pyramidal expansions span the same space and are in a subspace of the prismatic expansion which is in turn a subspace of the hexahedral expansion.

To enforce $C^0$ continuity the orthogonal expansion is modified by decomposing the expansion into an interior and boundary contribution [86, 84, 92]. The interior modes (or bubble functions) are defined to be zero on the boundary of the local domain. The completeness of the expansion is then ensured by adding boundary modes which consist of vertex, edge and face contributions. The vertex modes have unit value at one vertex and decay to zero at all other vertices; edge modes have local support along one edge and are zero on all other edges, and vertices and face modes have local support on one face and are zero on all other faces, edges and vertices. Figure 4 shows the decomposition of the domain into such elements, with the vertex, edge, face, and interior modes marked for one element. $C^0$ continuity between elements can then be enforced by matching similar shaped boundary modes. The local coordinate systems do impose some restrictions on the orientation in which triangular faces may connect. However, it has been shown in [84, 93] that a $C^0$ tetrahedral expansion can be constructed for any tetrahedral mesh. A similar strategy could be applied to a hybrid discretization [33].

Finally, we note that the bases are all hierarchical, which means that increasing the polynomial order of any expansion simply adds extra modes to the existing basis. Hierarchical expansions naturally lend themselves to $p$-type adaptivity where the polynomial order of the expansion can differ within each elemental domain. This is a very attractive property as it permits the polynomial
order of the expansion to be altered in order to capture the spatial characteristics of the solution.

Jacobi polynomials of mixed weights are used for the trial basis that form tensor products of generalized type. The nonlinear products are handled using effectively a super-collocation approach followed by a Galerkin projection [34].

This is accomplished by arranging the trial basis in terms of vertex, boundary and bubble modes and ensuring matching of the boundary modes, thus satisfying the $C^0$ continuity condition required in the Galerkin formulation.

2.3 Time Integration Algorithm

A popular time-stepping algorithm for integrating the Navier-Stokes equations is the splitting or fractional scheme. Although many different versions have been developed, here we describe a particular implementation that can give high-order time accuracy [35].

Within a domain $\Omega$, the fluid velocity $u$ and the pressure $p$ can be described by the incom-
pressible Navier Stokes equations.

\[ \frac{\partial \mathbf{u}}{\partial t} = -\nabla P + \nu L(\mathbf{u}) + N(\mathbf{u}) \text{ in } \Omega \]
\[ \nabla \cdot \mathbf{u} = 0 \]  

(3)

where

\[ L(\mathbf{u}) = \nabla^2 \mathbf{u}; \quad \omega = \nabla \times \mathbf{u} \]
\[ N(\mathbf{u}) = \mathbf{u} \times \omega; \quad P = p + \frac{1}{2} \nabla (\mathbf{u} \cdot \mathbf{u}) \]  

(4)

The non-linear operator \( N(\mathbf{u}) \) has been written in rotational form to minimize the number of derivative evaluations (6 vs. 9 for the convective form). A semi-implicit time integrator is used to integrate the system (3), (4) by using a 3-substep splitting scheme [35]:

\[ \frac{\hat{\mathbf{u}} - \sum_{q=0}^{J_s-1} \alpha_q \mathbf{u}^{n-q}}{\Delta t} = \sum_{q=0}^{J_s-1} \beta_q N(\mathbf{u}^{n-q}) \]  

(5)

\[ \frac{\hat{\mathbf{u}} - \hat{\mathbf{u}}}{\Delta t} = -\nabla \hat{P}^{n+1} \]

(6)

\[ \frac{\gamma_0 \mathbf{u}^{n+1} - \hat{\mathbf{u}}}{\Delta t} = \nu \nabla^2 \mathbf{u}^{n+1} \]  

(7)

The time-stepping algorithm can then be summarized in three steps:

1. Calculate the advective terms eqn. (4) and advance the solution in time using a stiffly-stable multi-step integrator.

2. Solve a Poisson equation for the dynamic pressure \( P \) to satisfy the divergence-free condition for the solution. Consistent pressure boundary conditions are used to ensure stability and high order accuracy [35].

3. Implicitly solve the viscous terms, advancing the solution to the next timestep. This gives
rise to a Helmholtz equation for each of the velocity components.

3 The \textit{NekTARF} code

The \textit{NekTARF} code is appropriate for flows with one homogeneous direction. In this direction a Fourier expansion is used providing a natural parallel paradigm. The spectral/hp-Fourier code was a direct evolution of a previous generation code on structured domains \textit{Prism} [42, 40] which used the same decomposition in the \(x\)-\(y\) plane and \(z\)-direction. The use of hybrid elements, a hierarchical basis and variable polynomial order across each elemental domain provides greater flexibility and permits a better accuracy for a fixed number of degrees of freedom in the \(x\)–\(y\) plane than previously possible. A illustrative example of the \(x\)–\(y\) plane mesh for a 3D flow past a cylinder was shown in figure 1 of Chapter 4.

3.1 Fourier Decomposition

If we assume that the problem is periodic in the \(z\)-direction, we may use a Fourier expansion to describe the velocity and the pressure, i.e. for the velocity,

\[
\mathbf{u}(x, y, z, t) = \sum_{m=0}^{M-1} \mathbf{u}_m(x, y, t) e^{i\beta mz}
\]  

(8)

where \(\beta\) is the \(z\)-direction wave number defined as \(\beta = 2\pi/L_z\), and \(L_z\) is the length of the computational domain in the \(z\)-direction. We now take the Fourier transform of equation (3) to get the coefficient equation for each mode \(m\) of the expansion,

\[
\frac{\partial \mathbf{u}_m}{\partial t} = -\nabla \rho_m + \nu \mathbf{L}_m(\mathbf{u}_m) + \text{FFT}_m[N(\mathbf{u})] \text{ in } \Omega_m, \ m = 0 \ldots M - 1,
\]

(9)

where \(\text{FFT}_m\) is the \(m^{th}\) component of the Fourier transform of the non-linear terms and,

\[
\mathbf{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, im\beta)
\]
\[ L_m(u_m) = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta^2 m^2 \right) u_m. \] (10)

The computational domain \( \Omega_m \) is an \( x-y \) slice of the domain \( \Omega \), implying that all \( \Omega_m \) are identical copies. From equation (9) we see that the only coupling between modes is through the non-linear terms. Therefore the computation of each mode \( m \) can be treated independently of one another. The obvious parallelization is to compute the \( m^{th} \) Fourier mode on processor \( m \) for \( m = 0...M - 1 \). Therefore, the three-dimensional computation essentially becomes a set of \( N_z = 2M \) two-dimensional problems computed in parallel on \( P \) processors where \( M \) is a multiple of \( P \). We note that the factor of two comes from the real/imaginary part pairs for the Fourier modes.

To maintain computational efficiency the non-linear product is calculated in physical space while the rest of the algorithm may be calculated in transformed space. The paradigm may therefore be thought of as a two pass process as illustrated in figure 5. As mentioned previously the spectral/\( hp \) representation in the \( x-y \) plane is hierarchical and so we may also consider this representation as a set of elemental modes and corresponding coefficients. In the first pass of the paradigm we need to obtain the physical data values at the quadrature points within each elemental domain. The inverse Fourier transform and differentiation are then performed at these points. For each timestep, Pass I can be summarized by the following substeps:

1. The velocity gets transformed to Quadrature space.

2. Calculation of the vorticity.

3. To form the non-linear terms:

   (a) Global transpose of the velocity and vorticity components.

   (b) \( N_{xy} \) \( 1D \) inverse FFTs for each velocity and vorticity component,

      (where \( N_{xy} \) is the number of points in one \( x-y \) plane divided by the number of processors).

   (c) Computation of \( N(u) \) using a dealiasing \( 3/2 \) rule.
(d) $N_{xy}$ 1D FFTs for each non-linear term.

(e) Global transpose of non-linear terms.

In Pass II the explicit time-integration of the non-linear terms and then the Helmholtz solves for pressure and velocity may be performed independently on each processor.

![Diagram](image)

Figure 5: Solution process in *NeTapF*

### 3.2 Two-Dimensional Helmholtz solves

Having reduced the linear part of the three-dimensional problem to a set of two-dimensional elliptic solves, we now describe Pass II in more detail. The major operations are computation of the gradients and inner products to form the right-hand-side of the Helmholtz problem and subsequently find its solution. The operations may be either global or element-wise in nature; solution of the Helmholtz problem is a global operation, while computing gradients in the $x-$ and $y-$directions and evaluating the inner product are all element-wise operations. In the $z-$
direction, because of the Fourier decomposition, differentiation becomes an exchange of the real and imaginary parts of a mode multiplied by a constant. As both parts of a mode reside on the same processor, this operation is local to the processor and requires no communication overhead.

Furthermore, the $x$- and $y$-gradients can be calculated using matrix-matrix multiplies representing differentiation in a single direction. For example: $\frac{\partial}{\partial x} \phi_i(k) = D_x(k) \phi_i(k)$, where in this case $\phi_i(k)$ is written out as an $Q^x \times Q^b$ matrix$^1$ and $D_x(k)$ is the $x$-direction derivative matrix for element $k$. The operation count for computing these gradients is $O(N^3)$ per element. These operations involve small matrix-matrix multiplies that unfortunately do not benefit from the usual optimizations employed for large matrices. The inner product evaluation can be handled similarly.

A Helmholtz problem needs to be solved for each velocity component and the pressure at every time step. Each discrete Helmholtz problem, when discretized within a Galerkin formulation, results in a matrix problems which can be solved either directly or iteratively. However, the structure of the spectral/hp expansion basis is such that the modes may be classified in terms of boundary modes, which have support on the boundary of an elemental domains, and interior modes which do not. Therefore, when considering a matrix problems arising from a $C^0$ continuous expansion we find that the submatrices corresponding to the interior modes of a specific elemental domain are decoupled from the interior modes of another elemental domain. This fact leads to the natural decomposition where the interior modes are decoupled from the boundary system by assembling the Schur complement matrix corresponding to the elemental boundary degrees of freedom, see [90] for more details. Constructing the Schur complement means that we can employ a direct or iterative solve of a much small matrix system. If the matrix is inverted iteratively a preconditioned conjugate gradient algorithm is typically used, while the inversion of the matrix directly can be optimized by ordering the boundary degrees of freedom to reduce the matrix bandwidth. A further decomposition, similar to the interior-boundary decomposition mentioned, is possible in the boundary matrix system by identifying groups of elemental boundaries which

$^1Q^x \approx N$ and $Q^b \approx N$ are the numbers of quadrature points in each direction $\eta_1$ and $\eta_2$. $N$ is the polynomial approximation order.
do not overlap. This technique is known as substructuring [94] and can lead to a greatly reduced interior solve providing a very efficient direct solve technique with reduced memory requirement as compared to using the whole boundary matrix system.

Finally we should make the following points about $Nektar^F$:

1. A very important advantage of the spectral/$hp$-Fourier decomposition is its speed as we can usually employ direct solvers for the solution of the $2D$ Helmholtz problems.

2. Increasing the resolution in the $z$-direction (i.e. increasing the number of Fourier modes) can be accomplished without increasing the amount of memory needed per processor if the number of processors is increased accordingly.

3. Increasing the resolution in the $z$-direction while increasing the number of processors involves a longer execution time per processor but only to the extent of the increase in communication time for global exchanges as well as the time to perform a $1D$ FFT. While the operation count for an FFT is $O(n \log n)$, the speed a modern processor will perform an FFT will also increase (until a critical value of $n$) [95, 96, 97] thereby partially offsetting the increase in operation count.

4. The $x$-$y$ plane resolution on distributed memory machines is only limited by the memory available per processor. The real and imaginary parts of a mode share the same matrices which saves memory. With a slightly larger overhead due to extra communication it is also possible to store only the real or the imaginary part of a Fourier mode on each processor, thereby reducing some memory requirements.

3.3 Communications in $Nektar^F$

The main types of communication patterns used in $Nektar^F$ are:

1. Global Exchanges (All-to-All) in the non-linear substep, specifically in Pass I.

2. Global Reduction operations (addition, min, max) for any runtime flow statistics.
3. Global Gather for any possible outflow boundary conditions in the pressure substep.

4. Gather for any time-history point analysis (tracking of flow variables at some point) in the analysis substep.

5. Global Synchronization (at a few points in the code to ensure that every processor is ready to proceed to the next substep).

We will concern ourselves with the dominant communications in a NekTatF production run, namely the Global Exchange and to a lesser extent the Global Reduction and Gather operations.

Global Exchange

![Diagram of Global Exchange]

**Physical Space**

planes 0 & 1  
planes 2 & 3  
planes $2P-2$ & $2P-1$

**Fourier Space**

mode 0r & Pr  
mode 1r & 1l  
$(P-1)r$ & $(P-1)l$

Figure 6: Data Layout in NekTatF. Here we have chosen to store a Fourier mode (2 “Fourier planes”) per processor. This also means that we keep 2 “Physical Planes” per processor. Because this is a Real-to-Complex FFT, $N_z = 2^m = 2P$ “Physical Planes” map to $P + 1$ independent Fourier modes (0 to $N_z/2 = P$) as the other $P - 1$ modes ($N_z/2 + 1$ to $N_z - 1$) are fixed by symmetry. Of these $P + 1$ modes, the first and last one have vanishing imaginary parts, hence by “packing” the real part of mode $P$ in place of the imaginary part of mode 0 we are left with $N_z = 2P$ “Fourier planes” as well.

The communication upon which the code is based is the Global or Complete Exchange. This is a communication pattern of great importance to any 2D or 3D FFT-based solver, since it lies behind the transposition of a distributed matrix. For example, it is the dominant communication pattern in any spectral distributed memory homogeneous turbulence in a “box code”. It is also
used in multiplying distributed matrices when one or more of the matrices is specified to be in transposed form. In the case of $\text{NextgarF}$ it is used to move the data between Fourier and Physical space: For most of the calculation, the flow variables are in Fourier space distributed in “Fourier planes” among the processors. However when the need to form the non-linear products (Pass I) arises, the data is transferred to Physical Space (figure 6).

The rotational form of the Navier-Stokes equations requires 6 Complete Exchanges to get the 3 velocity and the 3 vorticity fields arranged along “Fourier-pencils”. Then a multiple-point inverse FFT is applied to this collection of “pencils” per processor to obtain the velocity and vorticity fields along constant $(x, y)$ lines, termed “Physical pencils”, as indicated in figure 7). The rotational form of the non-linear (convective) term is then calculated, multiple-point FFTs then transforms the data back into the form of “Fourier-pencils”, and a Complete Exchange per component of the non-linear terms is applied to bring the data back to original form of “Fourier plane” (see figure 5). In total, the rotational form of the Navier-Stokes Equations requires 9 Complete Exchanges per timestep.

Figure 7: A Complete Exchange (“Global Transpose”) followed by a series of inverse FFTs on the resulting “Fourier pencils” transforms the data to Physical Space arranged along “Physical pencils”. “Pencils” are contiguous vectors containing either all the Fourier modes or all the Physical space data in the $z-$direction at a given $(x, y)$ point.
During a Complete Exchange, each processor communicates with each other processor and so it is one of the most demanding communication patterns on network resources. In the case of $N = pT^k$, each individual message is of a size \( \left( \frac{\sum_{i=1}^{n_{el}} \sum_{j=1}^{Q_k} Q_i}{p} \times \frac{N}{p} \right) \) double precision words, where the sum is over all \( n_{el} \) elements. Therefore as the number of processors \( P \) increases the message sizes decrease, even if \( N \) remains constant, unless the number of elements \( n_{el} \) or the spectral order \( N \) increase accordingly. This means that as \( P \) increases message latency becomes ever more important.

Complete Exchanges are also needed to setup the boundary conditions or in the case of time-dependent boundary conditions.

**Global Reduction**

Global reduction operations (summation, as well as min/max) also appear in the code when runtime statistics are gathered. The values are calculated on each processor and in the end are reduced across all processors.

**Global Gather**

In the rotational form of the Navier-Stokes equations, the dynamic pressure $\Pi = p + \frac{1}{2}(u^2 + v^2 + w^2)$ needs to be evaluated at the outflow boundary. This requires the assembly of a vector consisting of the “Fourier pencils” corresponding to each of the discrete \((x, y)\) points along an outflow boundary. Thus a gather operation on all processors followed by internal rearrangements of the resulting vector of mixed “Fourier-pencils” is necessary.

**Gather**

For time-history point analysis, we again need the data in Physical space for a specific \((x, y, z)\) triad. This means that a “Fourier pencil” corresponding to the discrete \((x, y)\) pair needs to be constructed and an FFT applied to obtain the “Physical pencil” containing the required \( z \) point. Therefore, all processors send their part of the “Fourier pencil” to processor 0, the “root”
processor, where the above procedure is performed.

This gather operation needs to be done for each history point and for each field (typically 4 values - u, v, w and p) whose history is being tracked. The message size in each case is \( N_z/P \) double precision words.

**Communications Summary**

In table A.1 we summarize the frequency and message sizes of the communications in \( \texttt{NektarF} \).

<table>
<thead>
<tr>
<th>routine</th>
<th>times per timestep</th>
<th>message (Bytes) size</th>
</tr>
</thead>
<tbody>
<tr>
<td>All-to-all (rotational)</td>
<td>6 + 3</td>
<td>( 8 \sum_{k=0}^{K} a Q_k^z \times \frac{N_z}{P} )</td>
</tr>
<tr>
<td>Allgather (rotational)</td>
<td>number of outflow edges</td>
<td>( 24 N_z \sum_{k=0}^{K} a Q_k )</td>
</tr>
<tr>
<td>Gather</td>
<td>4 ( N_z ) numbers of history points</td>
<td>( 8 \frac{N_z}{P} )</td>
</tr>
</tbody>
</table>

Table A.1: Communication patterns, corresponding message lengths and frequency per timestep in \( \texttt{NektarF} \).

4  **The \( \texttt{Nektar3D} \) code**

The need to directly simulate flows past geometries of even greater complexity, which have no planes of symmetry, requires the development of fully three-dimensional high-order algorithms. Application areas for such algorithm are potentially very broad, however the need for unstructured algorithms becomes even more evident when we simply consider the complexity of meshing the solution domain. The development of three-dimensional spectral/hp unstructured and hybrid algorithms [85, 84, 86] has notably extended the original spectral element algorithms which were traditionally based on structured, hexahedral discretizations. One application area that these technique are currently being applied is the arterial haemodynamics. In this figure 8 we see...
Figure 8: Model geometry of a distal end-to-side anastomosis using a viscous tetrahedral mesh from the Felisa package. (Courtesy of S.J.S. Sherwin)

a model geometry for internal flow within a bifurcated pipe which is being used to model the distal end of an end-to-sided arterial bypass graft [98, 99]. Similar to the NekTecF code the cost of simulating large three-dimensional computations, with a high degree of accuracy, requires a parallel algorithm. We recall that parallel implementation is particularly suited to the spectral/hp type algorithms due to the close coupling of information within an element generated by the high-order polynomial representation. Typically, even in the serial algorithm many operations such as integration and differentiation are treated at a very local elemental level.

The parallelization of the three-dimensional unstructured solver was accomplished by the modification of the serial version of NekTec3D. The key components in this transition from serial to parallel were:

1. Generation of a local to global mapping of the boundary information based on an arbitrary partitioning.
2. Introduction of a communication interface to treat global operations.

3. Modification of the Preconditioned Conjugate Gradient solver to treat interprocessor communications over each iteration.

4. Implementation of a suitable I/O format for the parallel computation.

4.1 Parallelization of \( \mathcal{N}e\kappa Tar3D \)

Domain partitioning and numbering

The first modification, and from an implementation standpoint the most costly, was the development of a numbering scheme to relate the global ordering of the solution variables over the whole computational domain to the local partition of the domain. The first step to constructing this numbering scheme is to divide the solution domain across the available processors. Since, the spectral/hp method tightly couples information within each element this partitioning is performed on an elemental level. However, to maintain the generality of the implementation it is presumed that any element may be arbitrarily placed on any processor. Although it is desirable in practice to locate as many neighboring elements as possible within a specific processor, as shown in figure 9 by assuming a general distribution we are able to use any of the freely available partitioning packages such as “Metis” [100], “Chaco” [101], “Jostle” [102] and “WGPP” [103].

The spectral/hp expansion basis implicitly decouples the interior degrees of freedom of each element from the boundary degrees of freedom. This is important as it reduces the global numbering to the global degrees of freedom on the boundaries of elements. The mapping of the global boundary numbering system was then generated by constructing a global “skeleton” mesh on each processor with a unique global numbering. The scheme can be summarized in three-steps as

1. Determine the global numbering of the boundary system on the complete mesh. (This is assumed to be unique so each processor has an identical numbering scheme).

2. Construct a compact local boundary numbering using the global numbering to identify any multiply defined elemental information. (e.g. due to vertex multiplicity)
3. Determine the mapping relating the local system to the global system and then remove the global system.

Although such a scheme is conceptually easy to interpret it is rather more complex to implement. Nevertheless, determining this mapping is the most significant aspect of the work necessary to parallelize our algorithm.

At present equal weighting is attributed to every element when using the domain partitioner. However when using a variable polynomial expansion or if there is a very high multiplicity of elements at a vertex or edge a weighting system should be applied. Node and edge weighting is typically available in most graph partitioners however the relation of such a weighting system to the spectral/hp element mesh requires further investigation.

**Communication Interface**

Typically, the introduction of a communication interface would represent a major part of any parallel implementation. However, this part of the implementation was dealt with by a "Gather-Scatter" (GS) library developed by H. Tufo and P. Fischer which is currently freely available software [104]. Having obtained a global to local processor mapping, as discussed in the previous
section, this interface performs all the necessary operations in a finite element type calculation such as direct stiffness assembly and therefore greatly reduced the overhead of the implementation. We note that the package is not restricted to the spectral/hp element algorithm but could be applied to a range of potential implementations including the standard finite element method. The interface is a very versatile package, allowing the treatment of all the communications using a “binary-tree” algorithm, “pairwise” exchanges or a mix of these two approaches where pairwise exchange are used for communicating values that are shared by only a few processors and a tree-like approach is used for values shared by many processors. This later approach is basically a global reduction operation on a subset of the total number of processors.

**PCG Solver Modification**

The standard iteration of the Preconditioned Conjugate Gradient (PCG) routine to solve $Ax = b$ is shown in figure 10(a) where $\alpha_p$ and $\beta_p$ are constants and $M$ is the preconditioning matrix. It can be appreciated that in a parallel algorithm each processor may keep its local contribution to the vectors $r_i, p_i$ and $z_i$, however, to update these vectors we need to perform a global operation to determine $\alpha_i$ and $\beta_i$ which requires us to evaluate the matrix-vector product $w_i = Ap_i$ and the inner products $r_{i-1}^T z_{i-1}$ and $p_i^T w_i$.

Solve $Mz_i = r_i$.  
\[ i = i + 1 \]
\[ \beta_i = r_{i-1}^T z_{i-1} / r_{i-2}^T z_{i-2} \]
\[ p_i = z_{i-1} + \beta_i p_{i-1} \]
\[ w_i = Ap_i \]
\[ \alpha_i = r_{i-1}^T z_{i-1} / p_i^T w_i \]
\[ x_i = x_{i-1} + \alpha_i p_i \]
\[ r_i = r_{i-1} - \alpha_i w_i \]

Solve $Mz_i = r_i$.  
\[ i = i + 1 \]
\[ \beta_i = r_{i-1}^T m_{ult} z_{i-1} / r_{i-2}^T M_{ult} z_{i-2} \]
Globally sum $\beta_i^p$  
\[ p_i = z_{i-1} + \beta_i p_{i-1} \]
\[ w_i^p = Ap_i \]
\[ \alpha_i^p = r_{i-1}^T m_{ult} z_{i-1} / p_i^T w_i^p \]
Globally sum $\alpha_i^p$ and assemble $w_i^p$  
\[ x_i = x_{i-1} + \alpha_i p_i \]
\[ r_i = r_{i-1} - \alpha_i w_i \]

Figure 10: Operations in the $i^{th}$ iteration of a preconditioned Conjugate Gradient loop. (a) serial version, (b) parallel version on $p^{th}$ processor.

As shown in figure 10(b) the PCG routine can be parallelized by introducing two global communication steps. Each processor is assumed to have duplicate copies of the relevant entries of the global vectors. However, the evaluation of $\beta_i$ involves interprocessor communication. Initially,
the inner product is evaluated on the $p^{th}$ processor and then globally summed. However, since parts of the global vector are multiply stored on different processors, the local evaluation of the inner products involves the multiplicity matrix, $m_{ult}$, of the entries of the vector over all processors.

Typically the matrix $A$ can be expressed in terms of its elemental contributions and so the product $Ap$ can be performed over each element independently, which makes it trivial to distributed this operation over the processors. However, the elemental contributions need to be globally assembled. Therefore, we first determine the local product $w_p = A_p^* p$ from which we can also evaluate the contribution of the $p^{th}$ partition to $\alpha_p$. The remaining communication to globally assemble $w_p$ and sum $\alpha_p$ may then be combined, thereby reducing the overhead latency.

A further communication is generally necessary to evaluate a measure of the residual to use as a stopping criterion. This communication can be amalgamated with one of the other two existing communications if the residual calculation is lagged at the possible expense of an extra iteration.

I/O Modifications

The last modification for the parallel algorithm was the introduction of a parallel output format. At present we assume that all processors may access a single input file, however, due to the memory restriction on any given processor it is not feasible to assemble output data via a single processor. Therefore, it was necessary to replace the existing output format which produced a local "header" and "data" output file on each processors. The header files contains information about the local distribution of data on the specific processor while the data files contains all the solution data. The advantage of this format is that the a single output file may then be generate by reading the relatively small header file and concatenating the data file with a simple "cat" system call.
4.2 Parallel validation

In order to evaluate the performance of the solver we conducted tests on a scalar Helmholtz problem which forms the backbone of a splitting scheme [35] used to solve the Navier-Stokes equations. The timings were performed on up to 192 (thin) nodes of the IBM SP2 at Cornell Theory Center. The SP2 communication network has a flat logical topology, and so communications between two processors do not depend on their proximity or whether neighboring processors are concurrently communicating. Therefore, the mapping of the solution domain partitions to processors need not preserve the relative positions of the subdomains to each other.

To keep our prototype problem as simple as possible we have chosen a bi-unit box subdivided into tetrahedral elements of equal volume as illustrated in figure 4 where we see the box geometry divided up into 3072 elements. A fixed polynomial expansion order ($N = 6$) was used for all elements so as not to create load imbalance. The solution converged in all cases, giving us the same value for the $L^2$ and $H^1$ errors. The following results represent a lower limit to the performance of the main solver.

![Self-Speedup curve](image)

![Parallel Efficiency curve](image)

Figure 11: Self speed up (a) and parallel efficiency (b) for a Helmholtz problem in a cubic domain using $n_{el} = 3072$ and $n_{el} = 24,576$ elements with ($N = 6$).

Our initial tests were performed on a 3072 element mesh using a purely pairwise communication strategy. However as can be seen in figure 11 the scaling of this test case was very poor. There were two possible reasons for this bad performance. Firstly, we believed that the solver may be
saturating due to a low loading on each processor and secondly we also believed that the purely
"pairwise" communication strategy was inappropriate since it is most suited to information which
is only shared between a small number of processors. However since the inner product evaluation
of $\alpha_i$ is lumped with the direct stiffness assembly in our Conjugate Gradient algorithm there is
always at least one piece of information which is communicated over all processors.

This second fact motivated the introduction of a "binary-tree" type communication strategy
into the "GS" communication package as well as the combination of tree and pairwise commu-
nication depending on how many processors the information is distributed over. These three
approaches are labeled as "all tree", "pairwise" and "partial tree" in figure 12. From this plot
we see that the mixed "partial tree" approach is advantageous compared to purely "pairwise" or
a pure "binary-tree". The reason for this is because the majority of the communicated data is
only distributed over a limited number of processors. However we always have some data due
to the inner-product evaluation which is distributed over all processors. The pairwise approach
suffers greatly when communicating over all processors since each processor needs to send a small
message to every other processor. On the other hand, using the binary tree algorithm for all
every communicated piece of data is very wasteful as in most cases values are shared by only two
processors.

Figure 12: Solver time (a) and parallel Self-Speedup (b) for Helmholtz problem in a cubic domain
decomposed in 3072 elements of polynomial order ($N = 6$).
As can be seen in figure 12(a) the solver time for the "partial tree" runs decreased with increasing the number of processors until 64 processors where it reaches a minimum and then flattens out (allowing for the larger timing irregularities at that number of processors). This plateau is seen at a later stage for higher polynomial order calculations as Amdahl's law would suggest. In the case of the "pairwise" approach, as the number of processors increased beyond 48 the solution time actually starts increasing as previously explained. The "all tree" method is actually the worst in terms of performance, again as expected, showing that a binary tree algorithm is a very poor choice for dealing with this global reduction operation.

As a final test case we considered a $N_{el} = 24,576$ element mesh with $N = 6$ and a partial tree communication strategy. As shown in figure 11 this case demonstrates greatly improved performance. Although we were only able to scale the problem to $P = 48$ processors a parallel efficiency of 80% was achieved as compared with the $P = 8$ processor case.

5 Communication Timings

It is instructive to try and measure some of aforementioned communication primitives on their own, to see how they scale with the number of processors $P$ and the individual message sizes $M_{se}$ involved. In this manner, any limitations of algorithms or of available hardware/software can be explored in a more efficient manner.

We varied the $P$ from 1 to 256 in powers of 2, and $M_{se}$ from 8 bytes to 800,000 bytes in multiples of 8, 80, 800, etc. We timed the MPI implementations of three collective communication primitives, namely MPLAlltoall() (Global Exchange), MPLAllreduce() (Global Reduction) and MPLAllgather() (Global Gather). While timing collective communications is not very common and usually gets done for either a fixed message size or a fixed processor count, the send-receive pairs that are needed for the pairwise exchanges in the "GS" package are among the standard performance tests widely available so we will not deal with them here [105, 106, 107, 108, 109]. The timings were performed on all processors and repeated 100 times. Each time we recorded the
maximum time it took a processor to return from the MPI call. As all of these calls are blocking, the processor returns only when it has finished communicating. In figures 13.14 and 15 we plot the minimum values observed during these 100 experiments.

Figure 13: Contour plot of MPI_Alltoall() performance on the Cray T3E. The black parallel lines with slope −1 show how \texttt{NekT Volunteers} usually scales as the problem size scales with \( P \). The grey dashed line with slope −2 corresponds to \texttt{NekT Volunteers} or a 3D homogeneous turbulence code with fixed problem size and finally the dotted-dashed line with slope 1 corresponds to a 3D homogeneous turbulence code that scales the problem size with \( P \).

As can be seen in figure 13, there exists a latency dominated region for small \( M_{zz} \) where time is virtually independent of \( M_{zz} \). Above \( 64B \) messages (corresponding to 8 double precision real numbers or 4 double precision complex ones) the effect of \( M_{zz} \) becomes appreciable. Towards the upper right hand corner of the plot (the limit of large messages and many processors) the time for the communication operation increases rapidly.

In the case of \texttt{NekT Volunteers} we usually increase both \( P \) and resolution in the (homogeneous) \( z \)-direction so that their ratio remains constant. This means our message size is inversely proportional to the number of processors (\( M_{zz} \propto 1/P \)). Then for a given \( z \)-\( y \) resolution we vary along
one of the solid lines of slope $-1$ drawn in the graph. As can be readily seen, for small $P$ we cross the contour lines at a sharp angle and time increases slowly. As $P$ increases though, this angle becomes larger (we approach the latency dominated limit) and time increases faster.

Provided we start with more than one Fourier mode per processor we can increase $P$ (until that limit is reached) keeping resolution in $z$ constant. Then $M_{zz} \propto 1/P^2$ and this time we vary along the dashed line of slope $-2$. For small $P$ this line stays parallel with the contour lines but very quickly it runs into the latency dominated area and parallel efficiency declines. The same scenario holds for the case of homogeneous turbulence, where the resolution is kept constant and the number of processors is increased.

Worse still is the case of spectral homogeneous turbulence codes where we increase the resolution (in all directions) along with $P$ trying to attack bigger and bigger problems. Then we are moving along the dotted line of slope $1$ as our message sizes are proportional to the number of processors ($M_{zz} \propto P$). Time increases very rapidly making communication costs challenge computation costs that scale as $P \log P$.

Looking at figure 14 we see both a latency dominated region for small messages and a bandwidth dominated one for large ones. As $M_{zz}$ gets larger the effect of the number of $P$ on communication time becomes less pronounced. So, as long as the message size remains small (as would be in the case of a dot product during a conjugate gradient iteration for example) the communication costs remain under control.

Finally figure 15 looks similar to figure 13. This is not surprising as a Global Exchange can be interpreted as a sequence of Global Gathers. When considering $N_{\text{ex}} T_{\text{at}} F$, however, increasing the $z$-resolution with $P$, this operation has $M_{zz} \propto P$ (the dotted line) and time increases rapidly with $P$. This is a disadvantage that comes with using the rotational form of the advection operator. If, instead, only $P$ is increased keeping the same $z$-resolution, message size remains constant and time increases far less rapidly.
Figure 14: Contour plot of `MPI_Allreduce()` performance on the Cray T3E. The black horizontal line corresponds to constant message size.

Figure 15: Contour plot of `MPI_Allgather()` performance on the Cray T3E. The black horizontal line corresponds to constant message size; the grey dashed line represents $M_{zz} \propto P$ as $N_z$ increases with $P$. 

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6 Perspective

We have presented two parallel algorithms for the solution of the incompressible Navier-Stokes equations based on the MPI programming model and have timed and analyzed some collective MPI operations on the Cray T3E. This platform (and its predecessor, the T3D) was one of the first computers to incorporate "shared memory" capabilities [107] albeit in a form requiring explicit management by the user (through use of the "SMA" library). Such an architecture marks the significant transition that is currently taking place, from physically and logically distributed memory in the last decade to distributed shared memory (DSM) in the current and future designs. In the newer architectures (e.g. Convex Exemplar, SGI O2000, upcoming IBM designs) all (local and remote) memory management is handled by hardware, with no extra effort needed from the programmer to access data physically stored in a remote node.

In order to understand this trend towards DSM, the programming model it promotes, related hardware and software issues and how they can be beneficial in simulating turbulence in a Petaflop computing environment, in the following we preview the basics for upcoming computers and computational tools.

6.1 Hardware

On the hardware side, the hope is that a supercomputer based on a mass produced microprocessor (and, better still, other mass produced sub-components) will get the volume benefits with respect to price and maintenance costs. Moreover, in order to support a shared memory programming model, a Distributed Shared Memory (DSM) architecture (groups of SMP multiprocessor "hyper-nodes" connected via fast interconnects with all the memory being directly addressable from any processor in any hypernode) in some way or other appears in the roadmaps of all vendors.

At the compute engine level CMOS microprocessor technology is already prevailing. More specifically, for cost reasons even multi-chip vector supercomputer (NEC SX-4 series, Fujitsu VPP-300,700 series, CRI J90 series and future generation of CRI vector machines) as well as
mainframes (IBM, Hitachi) have moved to CMOS from expensive ECL technology without taking a significant hit in performance.

Furthermore. DRAM (and derivatives, SDRAM, SLDRAM etc.) memory is getting more inexpensive, relegating the faster but more expensive SRAM type to the role of cache, even on vector supercomputers. 64-bit processors (already most RISC microprocessors) and operating systems enable the addressing of what will be affordable Terabytes of memory within a distributed shared memory architecture. More memory of course suggests that bigger DNS problems can be attacked using the typically more reliable direct solvers.

Similarly, new disk technology with higher densities along with RAID configurations with striping (for performance) and mirroring (for data integrity) provide a parallel approach to getting very large files and filesystems, both important for large simulations involving hundreds of Gigabytes to Terabytes of data. In addition, I/O bandwidth is increasing due to both faster interfaces (eg. Fibre Channel) and parallel I/O software support.

Finally, the drive towards DSM architectures has increased the relative importance of low latencies to that of high interconnect bandwidth (of the order of 1GB/sec). While bandwidth is increasing, latencies are not going down as fast. In particular, high bandwidth is still required between hypernodes (the SMP building blocks) within a system (not visible to the user who will only see a single system image). In a message passing context the target of achieving less that 1µs latency has yet to be reached, whereas in a shared memory context, average remote memory data access even in the best system to day still reaches the 1µs mark in a system configuration of hundreds of processors [110]. Therefore data locality remains paramount and domain decomposition techniques still bear the greatest promise for DNS.

6.2 Software

The shared memory programming paradigm is seen by most as the easiest way to code in parallel and there are good reasons for it as:

- It appears to be more natural and certainly easier than the message-passing way of parti-
tioning work and manually coordinating it by exchanging messages.

- Automatically parallelizing compilers are getting better every day, although they are still a very long way off being a truly usable tool for supercomputing.


- Explicit threads based shared memory parallelism is also a possibility, and it is currently the best way to get performance. With POSIX threads it is also portable.

Some of the most popular mathematical libraries (e.g. BLAS and LAPACK) already offer *vendor optimized* parallelism for shared memory systems. As object oriented languages find their way in the High Performance Computing world, class libraries that are internally parallel will become more common, providing an easy migration to parallelism. Finally, performance tools will need to become part of everyday use for most users (as opposed to only a few today) as it is unlikely that at least initially one will be able to get decent performance otherwise.

### 6.3 Migration Problems

While the developments on both hardware and software fronts are very exciting and of direct benefit to the DNS of turbulence, there are a number of problems that even a sophisticated simulation scientist will experience during this migration to new technology.

While peak computer speeds increase with Moore’s law most applications see only a small fraction of that peak. Peak performance is achieved through high clock rates and multiple pipelined execution units but these need to be continuously fed with data. The two main problems here are bandwidth and latency of memory accesses:

*Data* bandwidth is not enough to feed these highly pipelined units when data is not streaming from cache. Once the processor needs to go to main memory for data a “memory wall” is reached and performance drops appreciably [112, 113]. Fixes such as wide memory interfaces (256 or 512 bits) or massively interleaved memory subsystems are not common due to cost reasons. Moreover,
memory interface frequency cannot be anywhere near that of the processor - a divisor of that latter frequency is used and with time this disparity increases. Specifically, SMP platforms with more than a couple of processors are forced to go to lower main memory interface frequencies.

Furthermore, memory latency [114] can play a crucial role in some cases (especially for irregular memory accesses) and its value is in the high decades if not hundreds of processor cycles. Processors now have latency tolerant and latency hiding characteristics such as multiple outstanding cache misses and prefetching, and along with compiler optimizations such as speculative code motion the pipelines can be kept working for a little longer. Such techniques do not cure the problem though, they only postpone it.

In contrast, emerging techniques such as Multi-Threaded Processors (Tera) [115] or Simultaneous MultiThreading (SMT) [116] attempt to attack the latency as well as the bandwidth problem using the notions of parallelism at the processor instruction level. However, programming efficiently such different architectures could prove to be very different [117].

While programming in object oriented languages is becoming more widespread, the corresponding current compiler technology is still unable to offer good performance. Shared memory parallel programming may be easier for most users but achieving high performance it usually needs so much more effort in ensuring good data spatial and temporal locality that the code looks very much like a message passing code! Otherwise, the message passing version of the code will be faster despite the fact that it requires a lot of redundant memory copies [118]. Finally, directives, while simple to use, do not currently offer the performance of explicit thread management. Moreover, careful management of thread creation and destruction is needed to avoid excessive overheads, especially for heavyweight threads.

6.4 Applications to Dynamic DNS of Turbulence

With the new architectural and software features realized in the near future, and particularly with the DSM architecture and shared memory programming model prevailing it is clear that algorithms such as the spectral/hp element method we presented here bear great promise for
simulating accurately complex geometry and complex physics turbulent flows while advancing DNS as a design tool. The use of multiple threads within shared memory enables the effective implementation of dynamic DNS or dDNS: The unstructured grid remains basically the same during the computation but selective $p$-refinement is performed.

General domain decomposition methods, and multi-domain high-order algorithms in particular, can exploit the hybrid nature of the DSM architecture by assigning macro-elements or subdomains to each hypernode. This assignment would follow the same logic that we have been following so far for the distributed memory model (e.g. for $NekTatF$ a Fourier plane per hypernode, for $NekTat3D$ a compact group of elements per hypernode) to preserve good data locality. The collective communications like MPI_Allto_all() could either be eliminated altogether (as they would be redundant) or kept for performance reasons but replaced by matrix transpose operations for example - these, if properly implemented, should offer equivalent or even better performance due to lower overheads. Pairwise exchanges (as needed in the “GS” library employed by $NekTat3D$) would be replaced by simple (local or remote) memory loads and stores. Moreover, extra parallelism can be used, for example the 2D solution process for each Fourier mode in $NekTatF$ can now be parallelized within each hypernode.

With the prospect of Petaflop computing not too distant in the future, we can extrapolate the range in Reynolds number we can achieve. In [76] we argued that a $1024^3$ DNS of homogeneous turbulence can be achieved with reasonable turnaround on a Teraflop DSM type system. Following similar arguments, we can foresee that DNS of turbulence in the Reynolds number range of $Re = 10^6$ will be achievable for simple geometries, and of the order of $Re = 10^5$ in complex geometries.

dDNS can be used to enhance such anticipated capabilities. Performing dDNS requires primarily on-the-fly $p$-refinement and occasional $h$-refinement, which in turn requires good load balancing. This is difficult to achieve for message passing programs although several efforts to introduce parallel frameworks that handle data migration transparently have progressed significantly [119, 120, 121]. In a shared memory environment, such dynamic refinement will be easier
with threads picking up more work as the need arises. This will allow for the easier development of more complicated solvers that can change their $h$— or $p$—resolution as the simulation progresses.

However, until the shared memory programming model can provide the performance, message passing (using MPI) should continue being the preferred model for High Performance Computing, even on shared memory machines. A mixed model making use of thread on a local hypernode level and message passing between the multithreaded processes running on each hypernode may be able to offer the best of both worlds for a while for an additional programming complexity cost. As communication bandwidth will be growing faster than latency will be decreasing, algorithms that try to minimize on the number of messages (or remote memory accesses for the shared memory model) at the expense of their size (or even extra memory copies) will be in most cases preferable. This means that even in a shared memory environment it might still be better to do a global transpose for a $3D$ FFT operation and tests such as those presented in section 5 will still be relevant.
Bibliography


