Galerkin and discontinuous Galerkin spectral/hp methods

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Received 9 February 1997; revised 1 June 1998

Abstract

We present the basic algorithms of spectral/hp methods on tetrahedral and hybrid grids in the context of Galerkin and discontinuous Galerkin projections. We consider both advection and diffusion problems and formulate algorithms for incompressible as well as compressible flows. © 1998 Elsevier Science S.A. All rights reserved.

1. Introduction

Spectral methods, although very popular in direct (DNS) and large eddy simulation (LES) of turbulent flows, have had limited use in other areas of computational fluid dynamics where non-canonical computational domains are involved. In computational aerodynamics, for example, where unstructured meshes are employed to accommodate geometric complexity, low-order finite elements and finite volumes are the prevailing discretization. However, with the interest shifted towards accurate solutions of the viscous flow equations around aerodynamic configurations instead of the Euler flow, research efforts have focused into developing high-order discretization procedures on unstructured meshes [1–3].

The standard polynomial spectral methods provide high-order accuracy but they are practically limited to simple geometries [4,5]. Spectral element methods have extended spectral discretization to more complex geometries [6,7] but they require non-standard meshes for discretization and their adaptive capability is limited. For a new numerical method to be useful it has to utilize the existing technology of mesh generators for three-dimensional geometries [8,9] in addition to providing faster convergence at cost no higher than standard finite element discretization.

The introduction of complex geometries significantly limits the Reynolds number range which is achievable in DNS and LES as compared with DNS and LES in simple homogeneous geometries. This is shown schematically in Fig. 1 which summarizes some representative geometries for which accurate DNS have been successfully completed. For periodic domains the maximum Reynolds number we can achieve in current DNS is relatively high and comparable to laboratory flows but for non-separable or multiply-connected domains the maximum Reynolds number is quite low. Progress can be made if, in addition to advances in computational speed, new flexible discretizations and hardware-aware algorithms are formulated so that dynamic DNS (dDNS) is pursued based on intelligent resolution steering.

The class of spectral methods we present here has some of the most important features required for establishing dDNS as a useful tool in simulating turbulent flows in complex geometry domains. First, it employs standard tetrahedral and hybrid grids which provide great flexibility in discretization. Second, it is based on a

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unified polynomial trial basis, which is hierarchical and can be readily used in $p$-refinement. Third, unknowns and data are expanded as tensor--products and thus three-dimensional operations are reduced to a series of inexpensive one-dimensional operations. Fourth, the advection and diffusion operators have eigenspectra similar to classical spectral operators and thus standard algorithms for time-integration can be used. Fifth, the volume/surface ratio is large, especially in high-order expansions, leading to high parallel efficiency. Sixth, discontinuous Galerkin projections can be easily formulated to provide locality and robustness.

We formulate algorithms for both incompressible and compressible flows with emphasis on high Reynolds number. In particular, for incompressible flows we employ Galerkin projections and combine a $C^0$ spectral/hp basis with a high-order splitting scheme. For compressible flows we employ discontinuous Galerkin projections and combine an $L^2$ orthogonal spectral/hp basis with an explicit multi-step time-integrator. Both algorithms are implemented in the parallel code Nek5000 which is written in layers of MPI, C++, C and Fortran.

In the following we first present the trial basis for tetrahedra, hexahedra, prisms and pyramids which are the elements in a hybrid grid, and subsequently we review the discontinuous Galerkin formulation for the advection and diffusion equations. We finally present several examples that demonstrate the spectral accuracy of the method as well as the new capability in simulating turbulent and transitional flows in non-separable and multiply-connected domains.

2. Unified hybrid expansion bases

In this section we shall develop a unified hybrid expansion basis suitable for constructing $L^2$ and $C^0$ global expansions using triangular and quadrilateral regions in two dimensions and tetrahedral, pyramidal, prismatic and hexahedral domains in three dimensions. This unified approach lends itself naturally to an object orientated implementation as originally developed in [10] using C++ in the code Nek5000. To construct these expansions we must first introduce an appropriate coordinate system as discussed in Section 2.1. Having developed the coordinate system the definition of the basis in terms of Jacobi polynomials is outlined in Section 2.3.

2.1. Coordinate systems

We define the standard quadrilateral region as

$$\mathcal{Q}^2 = \{(\xi_1, \xi_2) | -1 \leq \xi_1, \xi_2 \leq 1\},$$
within which we note that the Cartesian coordinates \((\xi_1, \xi_2)\) are bounded by constant limits. This is not, however, the case in the standard triangular region defined as

\[
\mathcal{T}^2 = \{ (\xi_1, \xi_2) \mid -1 \leq \xi_1, \xi_2; \xi_1 + \xi_2 \leq 0 \},
\]

where the bounds of the Cartesian coordinates \((\xi_1, \xi_2)\) are clearly dependent upon each other. To develop a suitable tensorial type basis within unstructured regions, such as the triangle, we need to develop a new coordinate system where the local coordinates have independent bounds. The advantage of such a system is that we can then define one-dimensional functions upon which we can construct our multi-domain tensorial basis. It also defines an appropriate system upon which we can perform important numerical operations such as integration and differentiation [11].

2.1. Collapsed two-dimensional coordinate system

A suitable coordinate system, which describes the triangular region between constant independent limits, is defined by the transformation

\[
\begin{align*}
\eta_1 &= \frac{1 + \xi_1}{1 - \xi_2} - 1 \\
\eta_2 &= \xi_2,
\end{align*}
\]

and has the inverse transformation

\[
\begin{align*}
\xi_1 &= \frac{(1 + \eta_1)(1 - \eta_2)}{2} - 1 \\
\xi_2 &= \eta_2.
\end{align*}
\]

These new local coordinates \((\eta_1, \eta_2)\) define the standard triangular region by

\[
\mathcal{T}^2 = \{ (\eta_1, \eta_2) \mid -1 \leq \eta_1, \eta_2 \leq 1 \}.
\]

The definition of the triangular region in terms of the coordinate system \((\eta_1, \eta_2)\) is identical to the definition of the standard quadrilateral region in terms of the Cartesian coordinates \((\xi_1, \xi_2)\). This suggests that we can interpret the transformation (1) as a mapping from the triangular region to a rectangular one as illustrated in Fig. 2. For this reason, we shall refer to the coordinate system \((\eta_1, \eta_2)\) as the collapsed coordinate system. Although this transformation introduces a multi-valued coordinate \(\eta_1\) at \((\xi_1 = -1, \xi_2 = 1)\), we note that singular points of this nature commonly occur in cylindrical and spherical coordinate systems.

2.1.2. Collapsed three-dimensional coordinate systems

The interpretation of a triangle to square mapping of the two-dimensional local coordinate system, as illustrated in Fig. 2, is helpful in the construction of a new coordinate system for three-dimensional regions. If we consider the local coordinates \((\eta_1, \eta_2)\) as independent axes (although they are not orthogonal), then the coordinate system spans a rectangular region. Therefore, if we start with a rectangular region, or hexahedral region in three dimensions, and apply the inverse transformation (2) we can derive a new local coordinate system in the triangular region \(\mathcal{T}^2\), or tetrahedron region \(\mathcal{T}^3\) in three-dimensions, where \(\mathcal{T}^3\) is defined as

![Fig. 2. Triangle to rectangle transformation.](image-url)
\[ \mathcal{F}^3 = \{ -1 \leq \xi_1, \xi_2, \xi_3; \xi_1 + \xi_2 + \xi_3 \leq -1 \} \]

To reduce the hexahedron to a tetrahedron requires repeated application of the transformation in (2) as illustrated in Fig. 3. Initially, we consider a hexahedral domain defined in terms of the local coordinate system \( (\eta_1, \eta_2, \eta_3) \) where all three coordinates are bounded by constant limits, i.e. \( -1 \leq \eta_1, \eta_2, \eta_3 \leq 1 \). Applying the rectangle to triangle transformation (2) in the \( (\eta_2, \eta_3) \) plane we obtain a new ordinate \( \eta_3 \) such that

\[ \eta_3 = \frac{(1 + \eta_2)(1 - \eta_3)}{2} - 1 \quad \eta_1 = \eta_3. \]

Treating the coordinates \( (\eta_1, \xi_2, \eta_3) \) as independent, the region which originally spanned a hexahedral domain is mapped to a rectangular prism. If we now apply transformation (2) in the \( (r_h, \eta_3) \) plane, introducing the ordinates \( r_h, \eta_3 \) defined as

\[ r_h = \frac{(1 + \eta_2)(1 - \eta_3)}{2} - 1 \quad \eta_3 = \eta_3, \]

we see that the coordinates \( (\eta_1, \xi_2, \eta_3) \) span a region of a square based pyramid. The third and final transformation to reach the tetrahedral domain is a little more complicated as to reduce the pyramidic region to a tetrahedron we need to apply the mapping in every square cross section parallel to the \( (\eta_1, \xi_2) \) plane. This means using the transformation (2) in the \( (\eta_1, \xi_2) \) plane to define the final ordinate \( \xi_1 \) as

\[ \xi_1 = \frac{(1 + \eta_1)(1 - \xi_2)}{2} - 1 \quad \xi_2 = \xi_2. \]

If we choose to define the coordinate of the tetrahedron region \( (\xi_1, \xi_2, \xi_3) \) as the orthogonal Cartesian system then, by determining the hexahedral coordinates \( (\eta_1, \eta_2, \eta_3) \) in terms of the orthogonal Cartesian system, we obtain

\[ \eta_1 = \frac{2}{-\xi_2 - \xi_3} \left( 1 + \xi_1 \right) - 1, \quad \eta_2 = \frac{2}{1 - \xi_3} \left( 1 + \xi_2 \right) - 1, \quad \eta_3 = \xi_3, \]

which is a new local coordinate system for the tetrahedral domain which is bounded by constant limits. When \( \xi_3 = -1 \) this system reduces to the two-dimensional system defined in (1).

In a similar manner, if we had chosen to define the coordinates in either the pyramidic or prismatic region as the orthogonal Cartesian system then evaluating the hexahedral coordinates in terms of these coordinates would generate a new local collapsed system for these domains. Table 1 shows the local collapsed coordinate systems.
Table 1
The local collapsed Cartesian coordinates which have constant bounds within the standard region may be expressed in terms of the Cartesian coordinates $\xi_1, \xi_2, \xi_3$. Each region may be defined in terms of the local coordinates since having a lower bound of $-1 \leq \xi_1, \xi_2, \xi_3$ and upper bound as indicated in the table. Each region and the planes of constant local coordinate are shown in Fig. 4.

<table>
<thead>
<tr>
<th>Region</th>
<th>$\xi_1, \xi_2, \xi_3 \leq 1$</th>
<th>$\xi_1$</th>
<th>$\xi_2$</th>
<th>$\xi_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hexahedron</td>
<td>$\xi_1, \xi_2 + \xi_3 \leq 1$</td>
<td>$\xi_1$</td>
<td>$\eta_2 = \frac{2(1 + \xi_3)}{(1 - \xi_3)} - 1$</td>
<td>$\xi_3$</td>
</tr>
<tr>
<td>Prism</td>
<td>$\xi_1 + \xi_2, \xi_3 \leq 1$</td>
<td>$\eta_1 = \frac{2(1 + \xi_2)}{(1 - \xi_2)} - 1$</td>
<td>$\eta_2 = \frac{2(1 + \xi_3)}{(1 - \xi_3)} - 1$</td>
<td>$\eta_3 = \xi_3$</td>
</tr>
<tr>
<td>Pyramid</td>
<td>$\xi_1 + \xi_2 + \xi_3 \leq 1$</td>
<td>$\eta_1 = \frac{2(1 + \xi_3)}{(1 - \xi_3)} - 1$</td>
<td>$\eta_2 = \frac{2(1 + \xi_2)}{(1 - \xi_2)} - 1$</td>
<td>$\eta_3 = \xi_3$</td>
</tr>
<tr>
<td>Tetrahedron</td>
<td>$\xi_1 + \xi_2 + \xi_3 \leq 1$</td>
<td>$\eta_1 = \frac{2(1 + \xi_3)}{(1 - \xi_3)} - 1$</td>
<td>$\eta_2 = \frac{2(1 + \xi_2)}{(1 - \xi_2)} - 1$</td>
<td>$\eta_3 = \xi_3$</td>
</tr>
</tbody>
</table>

in all the three-dimensional regions. A diagrammatic representation of the local collapsed coordinate system is shown in Fig. 4.

2.2. $L^2$ Orthogonal expansions

Having defined the collapsed coordinate systems in Section 2.1 we are now in a position to develop computationally efficient expansions which are orthogonal in the $L^2$ inner product. Within a quadrilateral and hexahedral domain the standard procedure is to use the tensor product of two or three one-dimensional expansions. This proves to be computationally attractive since the key operations such as differentiation and integration may then be performed as a series of one-dimensional expansions. Using the collapsed coordinate systems a similar extension process is possible for all the unstructured domains using a warped [12] or generalised product involving tensors of two and three dimensions. In the structured two-dimensional tensor product form the expansion is constructed from the same one-dimensional basis; for example the $L^2$ orthogonal expansion has the form

$$\phi_{pq}(\xi_1, \xi_2) = L_p(\xi_1)L_q(\xi_2),$$

where $L_p(x)$ is the Legendre polynomial. However, a more general product can be used in unstructured expansions combining a one-dimensional tensor $\psi_{ij}(z)$ with a two-dimensional tensor of the form $\tilde{\psi}_{ij}(z)$, i.e.

$$\phi_{pq}(\xi_1, \xi_2) = \tilde{\psi}_{ri}(\eta_1)\tilde{\psi}_{pq}(\eta_2).$$

We shall refer to the functions $\tilde{\psi}_{ij}(z)$ and $\tilde{\psi}_{ij}(z)$ as well as a third function $\tilde{\psi}_{ijk}(z)$ as the orthogonal principal functions, where $\tilde{\psi}_{ijk}(z)$ is required for the three-dimensional expansions. The following expansion was proposed in two dimensions by Dubiner [12] and extended to three dimensions in [13,14]. More recently, Owens [15] has shown how an identical basis can be derived from developing a separable two-dimensional Sturm–Liouville problem on a triangle.

2.2.1. Orthogonal principal functions

Recalling that the function $P_i^{\alpha,\beta}(z)$ denotes the $p$th-order Jacobi polynomial the principal functions, $\tilde{\psi}_{ij}(z)$, $\tilde{\psi}_{ij}(z)$, $\tilde{\psi}_{ijk}(z)$, for orthogonal expansions are

$$\tilde{\psi}_{ij}(z) = P_i^{\alpha,0}(z), \quad \tilde{\psi}_{ij}(z) = \left(\frac{1 - z}{2}\right)^{i+j} P_i^{2i+1,0}(z),$$

$$\tilde{\psi}_{ijk}(z) = \left(\frac{1 - z}{2}\right)^{i+j+k} P_i^{2i+2j+2,0}(z).$$

2.2.2. $L^2$ Orthogonal hybrid expansions

The two-dimensional expansions in terms of the principal functions are defined as
Fig. 4. Planes of constant value of the local collapsed Cartesian coordinate systems in the hexahedral, prismatic, pyramidal and tetrahedral domains. In all but the hexahedral domain, the standard Cartesian coordinates $\xi_1$, $\xi_2$, $\xi_3$ describing the region have an upper bound which couples the coordinate system as shown in Table 1. The local collapsed Cartesian coordinate system $\eta_1$, $\eta_2$, $\eta_3$, $\eta_4$ represents a system of non-orthogonal coordinates which are bounded by a constant value within the region.

**Quadrilateral expansion:**

$$\phi_{pq}(\xi_1, \xi_2) = \tilde{\psi}^u_p(\xi_1) \tilde{\psi}^u_q(\xi_2)$$

**Triangular expansion:**

$$\phi_{pq}(\xi_1, \xi_2) = \tilde{\psi}^u_p(\eta_1) \tilde{\psi}^u_q(\eta_2)$$

where

$$\eta_1 = \frac{2(1 + \xi_1)}{(1 - \xi_2)} - 1, \quad \eta_2 = \xi_2$$
are the two-dimensional collapsed coordinates. The three-dimensional expansions are defined in terms of the principal functions as

**Hexahedral expansion:**

\[
\phi_{pqr}(\xi_1, \xi_2, \xi_3) = \tilde{\psi}^a_p(\xi_1) \tilde{\psi}^a_q(\xi_2) \tilde{\psi}^a_r(\xi_3)
\]

**Prismatic expansion:**

\[
\phi_{pqr}(\xi_1, \xi_2, \xi_3) = \tilde{\psi}^a_p(\xi_1) \tilde{\psi}^a_q(\eta_2) \tilde{\psi}^b_{pqr}(\xi_3)
\]

**Pyramidic expansion:**

\[
\phi_{pqr}(\xi_1, \xi_2, \xi_3) = \tilde{\psi}^a_p(\eta_1) \tilde{\psi}^a_q(\eta_2) \tilde{\psi}^c_{pqr}(\eta_3)
\]

**Tetrahedral expansion:**

\[
\phi_{pqr}(\xi_1, \xi_2, \xi_3) = \tilde{\psi}^a_p(\eta_1) \tilde{\psi}^b_q(\eta_2) \tilde{\psi}^c_{pqr}(\eta_3)
\]

where

\[
\eta_1 = \frac{2(1 + \xi_1)}{-\xi_2 - \xi_3} - 1, \quad \eta_2 = \frac{2(1 + \xi_2)}{(1 - \xi_3)} - 1, \quad \eta_3 = \xi_3.
\]

are the three-dimensional collapsed coordinates. These expansions are all polynomials in terms of their local collapsed coordinates as well as the Cartesian coordinates.

### 2.3. \(C^0\) Expansions

To construct \(C^0\) continuous bases we decompose the orthogonal expansions developed in Section 2.2 into an interior and boundary contribution as is typical of all \(hp\) finite element methods [16]. The interior modes (or bubble functions) are defined as zero on the boundary of the local domain. The completeness of the expansion is then ensured by adding boundary modes which consist of vertex, edge and face contributions. The vertex modes have unit value at one vertex and decay to zero at all other vertices; edge modes have local support along one edge and are zero on all other edges and vertices; face modes have local support on one face and are zero on all other faces, edges and vertices. Using this decomposition, \(C^0\) continuity between elements can be enforced by matching similar shaped boundary modes providing some orientation constraints are satisfied. To construct the unified hybrid expansions we shall initially define a set of principal functions in Section 2.3.1. Using these functions we then define the construction of the expansions in Section 2.3.2.

#### 2.3.1. Principal functions

Denoting by \(P_i^{\alpha,\beta}(z)\) the \(i\)th order Jacobi polynomial (as before) which satisfies the orthogonality condition

\[
\int_{-1}^1 (1-z)^\alpha(1+z)^\beta P_i^{\alpha,\beta}(z) P_j^{\alpha,\beta}(z) \, dz = C_{i,j} \quad \text{where} \quad \alpha, \beta > -1,
\]

we define three principal functions denoted by \(\psi^a_i(z), \psi^b_j(z)\) and \(\psi^c_{ik}(z)\) (\(0 \leq i \leq l, 0 \leq j \leq J, 0 \leq k \leq K\):
\[
\psi_i^\alpha (z) = \begin{cases} 
\frac{1-z}{2} & i = 0 \\
\left( \frac{1-z}{2} \right)^{i+1} \left( \frac{1+z}{2} \right) P_{i-1}^1 (z) & 1 \leq i \leq I - 1, \\
\frac{1+z}{2} & i = I
\end{cases}
\]

\[
\psi_i^\beta (z) = \begin{cases} 
\psi_j^\alpha (z) & i = 0, \\
\left( \frac{1-z}{2} \right)^{i+1} \left( \frac{1+z}{2} \right) P_{j-1}^2 (z) & 1 \leq i \leq I - 1, \\
\psi_j^\beta (z) & i = I, 
\end{cases}
\]

\[
\psi_i^{\beta j_k}(z) = \begin{cases} 
\psi_i^\mu (z) & i = 0, \\
\left( \frac{1-z}{2} \right)^{i+j+1} & 0 \leq i \leq I, \\
\psi_i^\nu (z) & 1 \leq i \leq I - 1, \\
\left( \frac{1-z}{2} \right)^{i+j+1} \left( \frac{1+z}{2} \right) P_{k-1}^{2i+2j+1} (z) & 1 \leq i \leq I - 1, \\
\psi_i^\pi (z) & i = I, \\
\psi_i^\rho (z) & 0 \leq j \leq J, \\
\psi_i^\sigma (z) & 0 \leq k \leq K
\end{cases}
\]

Fig. 5 diagrammatically indicates the structure of the principle functions \(\psi_i^\alpha (z)\), \(\psi_i^\beta (z)\) and \(\psi_i^{\beta j_k}(z)\) as well as how the function \(\psi_j^\alpha (z)\) is incorporated into \(\psi_i^\beta (z)\), and similarly how \(\psi_j^\beta (z)\) is incorporated into \(\psi_i^{\beta j_k}(z)\). The function \(\psi_j^\alpha (z)\) has been decomposed into two linearly varying components and a function which is zero at the end points. The linearly varying components generate the vertex modes which are identical to the standard linear finite element expansion. The interior contributions of all the base functions (i.e. \(1 \leq i \leq I - 1, \ 1 \leq j \leq J - 1, 1 \leq k \leq K - 1\)) are similar in form to the orthogonal basis functions defined in [14]. However, they are now pre-multiplied by a factor of the form \((1 - z)/2)((1 + z)/2)\) which ensures that these modes are zero on the boundaries of the domain. The value of \(\alpha, \beta\) in the Jacobi polynomial \(P_{\alpha,\beta}^{\alpha,\beta}(x)\) has also been slightly modified to maintain as much orthogonality as possible in the mass and Laplacian systems.

Fig. 5. Illustration of the structure of the arrays of principal functions \(\psi_i^\alpha (z)\), \(\psi_i^\beta (z)\) and \(\psi_i^{\beta j_k}(z)\). These arrays are not globally closed packed although any edge, face or interior region of the array may be treated as such. The interior of the arrays \(\psi_i^\alpha (z)\) and \(\psi_i^\beta (z)\) have been shaded to indicate the minimum functions required for a complete triangular and tetrahedral expansion.
2.3.2. Hybrid expansions

The two-dimensional expansions are defined in terms of the principal functions as

Quadrilateral expansion:
\[ \phi_{pq}(\xi_1, \xi_2) = \psi_p^a(\xi_1) \psi_q^a(\xi_2) \]

Triangular expansion:
\[ \phi_{pq}(\xi_1, \xi_2) = \psi_p^a(\eta_1) \psi_{pq}^b(\eta_2) \]

where
\[ \eta_1 = \frac{2(1 + \xi_1)}{(1 - \xi_2)} - 1, \quad \eta_2 = \xi_2, \]

are the two-dimensional collapsed coordinates. In Fig. 6 we see all of the modified expansion modes for a fourth-order \( P = 4 \) modified triangular expansion. From this figure it is immediately evident that the interior modes have zero support on the boundary of the element. This figure also illustrates that the shape of every boundary mode along a single edge is identical to one of the modes along the other two edges and which allows the modal shapes in two regions to be globally assembled into a \( C^0 \) continuous expansion. In the three-dimensional expansion an equivalent condition is ensured by the introduction of \( \psi_p^a(z) \) into \( \psi_{ija}(z) \).

The three-dimensional expansions are defined in terms of the principal functions as

Hexahedral expansion:
\[ \phi_{pqr}(\xi_1, \xi_2, \xi_3) = \psi_p^a(\xi_1) \psi_q^a(\xi_2) \psi_r^a(\xi_3) \]

Prismatic expansion:
\[ \phi_{pqr}(\xi_1, \xi_2, \xi_3) = \psi_p^a(\xi_1) \psi_q^a(\eta_2) \psi_{qr}^b(\xi_3) \]

Pyramidal expansion:
\[ \phi_{pqr}(\xi_1, \xi_2, \xi_3) = \psi_p^a(\eta_1) \psi_q^a(\eta_2) \psi_{pqr}^c(\xi_3) \]

Tetrahedral expansion:
\[ \phi_{pqr}(\xi_1, \xi_2, \xi_3) = \psi_p^a(\eta_1) \psi_{pq}^b(\eta_2) \psi_{pqr}^c(\eta_3) \]

Fig. 6. Construction of a fourth-order \( P = 4 \) triangular expansion using the product of two principal functions \( \psi_p^a(\eta_1) \) and \( \psi_{pq}^b(\eta_2) \).
3. Advection equation

In developing algorithms for the Navier–Stokes equations we will employ the $C^0$ basis presented in the incompressible version, and the $L^2$ basis for the compressible equations; both bases were presented in the previous section.

3.1. Galerkin formulation

To explain the formulation we consider the linear two-dimensional equation for advection of a conserved quantity $u$ in a region $\Omega$

$$\frac{\partial u}{\partial t} + \nabla \cdot F(u) = 0,$$  \hspace{1cm} (4)

where $F(u) = (f(u), g(u), h(u))$ is the flux vector which defines the transport of $u(x, t)$. We start with the variational statement of the standard Galerkin formulation of (4) by multiplying by a test function $v$ and integrating by parts

$$\int_{\Omega} \frac{\partial u}{\partial t} v \, dx + \int_{\partial \Omega} v n \cdot F(u) \, ds - \int_{\Omega} \nabla v \cdot F(u) \, dx = 0. \hspace{1cm} (5)$$

The solution $u \in \mathcal{X}$ (approximation space) satisfies this equation for all $v \in \mathcal{V}$ (test space). The requirement that consists of continuous functions naturally leads to a basis consisting of functions with overlapping support, which implies Eq. (5) becomes a banded matrix equation.

3.2. Discontinuous Galerkin formulation

Continuous function spaces are not the natural place to pose the advection problem. Mathematically, hyperbolic problems of this type tend to have solutions in spaces of bounded variation. In physical problems, solutions will be piecewise continuous, that is, be smooth in regions separated by discontinuities (shocks). An additional consideration is that the formulation presented next automatically preserves conservativity in the element-wise sense. This is particularly important for transonic and supersonic flows.

These considerations suggest immediately a formulation where $\mathcal{X}$ may contain discontinuous functions. The discrete space $\mathcal{X}^d$ contains polynomials within each ‘element’, but zero outside the element. Here, the ‘element’ is, for example, an individual triangular region $T_i$ in the computational mesh applied to the problem. Thus, the computational domain $\Omega = \bigcup T_i$, and $T_i, T_j$ overlap only on edges.

Contending with the discontinuities requires a somewhat different approach to the variational formulation. Each element $(E)$ is treated separately, giving a variational statement (after integrating by parts once more):

$$\frac{\partial}{\partial t} (u, v)_E + \int_{\partial E} v (\tilde{f}(u_i, u_e) - f(u_i)) \cdot n \, ds + (\nabla \cdot f(u), v)_E = 0,$$ \hspace{1cm} (6)

where $\tilde{f}(u_i, u_e)$ is the flux of the interior values. Computations on each element are performed separately, and the connection between elements is a result of the way boundary conditions are applied. Here, boundary conditions

![Fig. 7. Interface conditions between two adjacent triangles.](image-url)
are enforced via the numerical surface flux \( \tilde{f}(u_i, u_e) \) that appears in Eq. (6). Because this value is computed at the boundary between adjacent elements, it may be computed from the value of \( u \) given at either element. These two possible values are denoted here as \( u_i \) in the interior of the element under consideration and \( u_e \) in the exterior (see Fig. 7). Upwinding considerations dictate how this flux is computed. In the more complicated case of a hyperbolic system of equations, an approximate Riemann solver should be used to compute a value of \( f, g, h \) (in three dimensions) based on \( u_i \) and \( u_e \).

Specifically, we compute the flux \( \tilde{f}(u_i, u_e) \) using upwinding, i.e.

\[
\tilde{f}(u) = RA^+ L u_i + RA^- L u_e
\]

where \( A \) (the Jacobian matrix of \( F \)) is written in terms of the left and right eigenvectors, i.e. \( A = RAL \) with \( A \) containing the corresponding eigenvalues in the diagonal; also, \( A^\pm = (A \pm |A|)/2 \). Alternatively, we can use a standard Roe flux [17]

\[
\tilde{f}(u) = \frac{1}{2} (f(u_e) + f(u_i)) - \frac{1}{2} R |A| L(u_e - u_i).
\]

This last form is what is used in the examples presented in Sections 4.1 and 4.2.

3.2.1. Eigen-spectrum

We consider the semi-discrete form of the linear advection with \( F(u) = Uu \), where \( U \) is the constant advection velocity along the propagation direction defined by the spherical polar coordinates \( \phi \) and \( \theta \). We then determine the maximum eigenvalue using the Galerkin and discontinuous Galerkin method for different values of the order \( P \). The growth of this eigenvalue with respect to the number of modes is significant since it dictates the time step restriction (CFL number) for an explicit time-integration scheme. As can be seen in Fig. 8, the maximum

Fig. 8. Left: Spectral radius of the Galerkin advection operator on a cube discretized with six pyramids. The surface is the variation of the spectral radius as a function of the spherical polar coordinates \( \phi \) and \( \theta \). Right: Expanded view of the mesh. Bottom: Spectral radius as a function of the order \( P \).
eigenvalue for the Galerkin formulation was found to grow as $P^2$ consistent with standard spectral methods on tensorial domains [5]. Similar results are valid for the discontinuous Galerkin formulation [18].

3.3. Diffusion equation

3.3.1. Discontinuous Galerkin formulation

We consider as a model problem the parabolic equation with variable coefficient $\nu$ to demonstrate the treatment of the viscous contributions:

$$ u_t = \nabla \cdot (\nu \nabla u) + f, \quad \text{in} \ \Omega, \quad u \in L^2(\Omega) $$

$$ u = g(x, t), \quad \text{on} \ \partial \Omega $$

We then introduce the flux variable

$$ q = - \nu \nabla u $$

with $q(x, t) \in L^2(\Omega)$, and re-write the parabolic equation

$$ u_t = - \nabla \cdot q + f, \quad \text{in} \ \Omega $$

$$ 1/\nu q = - \nabla u, \quad \text{in} \ \Omega $$

$$ u = g(x, t), \quad \text{on} \ \partial \Omega $$

The weak formulation of the problem is then as follows: Find $(q, u) \in L^2(\Omega) \times L^2(\Omega)$ such that

$$ (u_t, w)_E = (q, \nabla w)_E - (w, (q_b - q_i) \cdot n)_E + (f, w)_E, \quad \forall w \in L^2(\Omega) $$

$$ 1/\nu (q, v)_E = (u_t, \nabla v)_E - (u_i - u_b, v \cdot n)_E, \quad \forall v \in L^2(\Omega) $$

$$ u = g(x, t), \quad \text{on} \ \partial \Omega $$

where the parentheses denote standard inner product in an element $(E)$ and the angle brackets denote boundary terms on each element, with $n$ denoting the unit outwards normal. The surface terms contain weighted boundary values of $u_b, q_b$, which can be chosen as the arithmetic mean of values from the two sides of the boundary, i.e.

$$ u_b = 0.5(u_i + u_e), \quad q_b = 0.5(q_i + q_e). $$

By integrating by parts once more, we obtain an equivalent formulation which is easier to implement and it is actually used in the computer code. The new variational problem is

$$ (u_t, w)_E = (\nabla \cdot q, w)_E - (w, (q_b - q_i) \cdot n)_E + (f, w)_E, \quad \forall w \in L^2(\Omega) $$

$$ 1/\nu (q, v)_E = (-\nabla u, v)_E - (u_i - u_b, v \cdot n)_E, \quad \forall v \in L^2(\Omega) $$

$$ u = g(x, t), \quad \text{in} \ \partial \Omega $$

where the subscript $(i)$ denotes contributions evaluated at the interior side of the boundary.

4. Numerical results

In the following, we first test the accuracy of the method for smooth solutions to verify exponential convergence and subsequently present Navier–Stokes simulations. In particular, we use the $L^2$ orthogonal bases and the discontinuous Galerkin formulation to develop algorithms for the Navier–Stokes equations based on an Adams–Bashforth time integrator. For the incompressible Navier–Stokes equations, we use the $C^0$ bases and a high-order splitting formulation [19] based on a semi-implicit stiffly-stable time integrator.

4.1. Convergence tests

We first demonstrate that the Galerkin method is stable up to high polynomial orders and for hybrid discretizations. Similar results are valid for the discontinuous Galerkin formulation. In Fig. 9 we show $p$-type convergence for a specified analytical solution for the Helmholtz equation $\nabla^2 u - u = f$ and corresponding Dirichlet conditions. This example demonstrates that the method is stable to at least $P = 64$. This is much higher
Fig. 9. Galerkin formulation: Convergence test for the Helmholtz problem, \( (\nabla^2 u - \lambda u = f; \lambda = 1) \), using a triangle and a quadrilateral, with Dirichlet boundary conditions. The exact solution is \( u = \sin(\pi \cos(\pi r^2)) \) and forcing function \( f = -((\lambda + 4\pi^2 r^2 \sin(\pi r^2)) \sin(\pi(\cos(\pi r^2))) - 4\pi^2 (\pi r^2 \cos(\pi r^2) + \sin(\pi r^2))) \cos(\pi \cos(\pi r^2))) \), where \( r^2 = x^2 + y^2 \).

Fig. 10. Galerkin formulation: Convergence for the Helmholtz problem, \( (\nabla^2 u - Au = f; A = 1) \), with Dirichlet boundary conditions on a mesh of twenty-six hybrid elements. The exact solution is \( u = \sin(x) \sin(y) \sin(z) \).

order than the one used in \( hp \) finite element method [16]. In Fig. 10 we solve the Helmholtz problem on a complicated three-dimensional domain discretized using all types of elements, i.e. tetrahedra, hexahedra, prisms and pyramids. Exponential convergence is also verified for a smooth solution.

We now demonstrate exponential convergence of the discontinuous Galerkin method for smooth 3D solutions of the compressible Euler and Navier–Stokes equations. We start with a simulation of inviscid flow over a circular bump in a three-dimensional channel domain as shown in Fig. 11 consisting of 294 tetrahedra.

Fig. 11. Discontinuous Galerkin formulation: Convergence for compressible inviscid flow over a bump. The plot on the right shows convergence of entropy versus spectral order. Mach number is 0.3.
assume that all walls are adiabatic and thus no entropy is generated. To measure numerical errors we therefore consider the maximum value of the entropy in the domain, which usually occurs on the bump boundary, and perform a $p$-refinement. Although this flow is two-dimensional we perform the simulation in a three-dimensional domain for validation purposes. In Fig. 11 we see that the maximum entropy tends to zero exponentially fast as the spectral order is increased.

Next, we test the convergence of the method in a three-dimensional domain shown in Fig. 12. The analytical solution has the form

$$
\rho = A + B \sin(\omega x), \quad u = C + D \cos(\omega x) \sin(\omega y) \cos(\omega z), \quad T = E + F y + G z^2,
$$

where $\omega = \pi/2$, $A = 1$, $B = 0.1$, $C = 1$, $D = 0.04$, $E = 84$, $F = 28$, and $G = 10$. Exponential convergence is demonstrated in Fig. 12.

4.2. Simulations

4.2.1. Transition in a triangular duct

We consider incompressible flow in a duct with its cross-section being an equilateral triangle. The laminar fully developed solution is known analytically and it has been used in [20] to examine the convergence of the Galerkin spectral/hp method. Here, we are concerned with the transition of this flow from laminar to a turbulent state. We introduce some random disturbances in the flow and we integrate in time until these disturbances start decaying or growing in time.

All simulations were performed in the domain shown in Fig. 13 with the cross-section discretized using one triangular element only and 16 Fourier modes (32 collocation points) in the streamwise homogeneous direction. The Reynolds number is defined as $Re = U D_e / \nu$ where $U$ is the average velocity and $D_e$ is the equivalent (hydraulic) diameter. For $Re \ll 500$ all disturbances decay but for $Re = 1250$ the flow goes through transition and a turbulent state is sustained.

We examine here how the treatment of the nonlinear terms affects the solution, i.e. the integrated shear stress on each side walls of the duct (see Fig. 14). In particular, we perform two simulations at $Re = 1250$: the first with the number of modes $P = 16$ equal to the number of quadrature points $Q$, and the second one with $Q = 2P = 32$. In the first case we essentially under-integrate the nonlinear terms and while there is no problem in the laminar flow, in the turbulent flow there is an artificial breaking of the 3-fold symmetry as shown on the right plot of Fig. 14. In the second case, symmetry in the statistical sense is restored as it should be (plot on the left in Fig. 14). It is interesting to note that the side across the degenerate corner corresponds to the lower value of the shear stress.

![Fig. 12. Discontinuous Galerkin formulation: $L_2$ error versus the expansion order for the energy (squares), momentum (circles), and density (triangles) of an analytic solution of the steady 3D Navier–Stokes equations. The domain is shown on the left plot.](image-url)
Fig. 13. Duct flow domain: The cross-section is an equilateral triangle and the streamwise length is three times the triangle edge. Shown also are streamwise velocity contours and 2D velocity vectors at one time instance.

Fig. 14. Shear force versus time at $Re = 1250$ over the three walls of the duct. On the left plot $P = 16$ and $Q = 32$. On the right plot $P = Q = 16$.

Fig. 15. Discretization around a NACA0012 airfoil; 592 elements are used.
4.2.2. Transonic airfoil

Next, we consider a refinement study for a transonic flow past an airfoil NACA0012 at an angle of attack $\alpha = 10^\circ$, freestream Mach number $Ma = 0.8$, and Reynolds number based on the freestream velocity and the airfoil chord equal to $Re = 73$. The wall temperature is equal to the freestream total temperature. The same problem was considered in [21] and is one of the benchmark problems suggested in the GAMM (1986) workshop [22]. The mesh is shown in Fig. 15; it extends 4 chords downstream and consists of 592 elements, which is about one-fourth of the number used in [21]. Three different discretizations with $p$-refinement were used corresponding to order 2, 4 and 6. The maximum order used in [21] was 3. In Fig. 16 we plot Mach contours for the first two discretizations ($P = 2$ and 4) that show the improvement in the solution as the

![Image of Mach contour lines](image1)

Table 2

<table>
<thead>
<tr>
<th>Item</th>
<th>$P = 2$</th>
<th>$P = 4$</th>
<th>$P = 6$</th>
</tr>
</thead>
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<tr>
<td>$C_d$</td>
<td>0.68287</td>
<td>0.67858</td>
<td>0.6758</td>
</tr>
<tr>
<td>$C_l$</td>
<td>0.47625</td>
<td>0.53022</td>
<td>0.53173</td>
</tr>
</tbody>
</table>

![Image of pressure and drag coefficients](image2)

Fig. 17. Pressure (left) and drag (right) coefficients. Solid squares are data from [21] and crosses are from the current simulation for $P = 6$. 
polynomial order is increased. A more quantitative comparison is shown in Table 2 where we present the drag and lift coefficients for the three meshes; very good agreement with the results of [21] is obtained. The same is true for the distribution of the pressure and friction coefficients around the airfoil as shown in Fig. 17.

Acknowledgements

This work was supported partially by AFOSR, DOE, ONR and NSF. Computations were performed at the National Center for Supercomputing Applications, University of Illinois at Urbana-Champaign, and at Maui High Performance Computing Center in Hawaii.

References