Generalized polynomial chaos and random oscillators

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SUMMARY

We present a new approach to obtain solutions for general random oscillators using a broad class of polynomial chaos expansions, which are more efficient than the classical Wiener–Hermite expansions. The approach is general but here we present results for linear oscillators only with random forcing or random coefficients. In this context, we are able to obtain relatively sharp error estimates in the representation of the stochastic input as well as the solution. We have also performed computational comparisons with Monte Carlo simulations which show that the new approach can be orders of magnitude faster, especially for compact distributions. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: polynomial chaos; uncertainty; stochastic modelling

1. INTRODUCTION

In the present work, we are interested in the non-stationary stochastic response of linear systems (with and without random coefficients) subject to non-stationary Gaussian or non-Gaussian random external excitation. In particular, we consider random input processes that exhibit some degree of correlation in time and study the response of the single-degree-of-freedom (SDF) oscillator. The response of more complex multi-degree-of-freedom linear systems is found by superimposing modal responses, each obtained from the study of an SDF system, see Reference [1]. Clearly, linear systems with random coefficients are similar in some ways to non-linear systems and exhibit behaviour between constant coefficient linear systems and non-linear systems, see Reference [2].

The output process of a linear system with deterministic coefficients subject to a Gaussian input is Gaussian, and the knowledge of its mean and correlation function matrix fully characterize the solution response. Therefore, it may be tempting to state, using arguments involving the central limit theorem, that responses of linear systems are approximately Gaussian even for
non-Gaussian excitations. However, this is not generally true because physical systems have a finite memory or relaxation time so that contributions of the input to the response are significant only over a relatively short period.

Closed-form solutions for the statistical moments of the non-stationary response to a generally defined external non-stationary input exist only in integral form. Only a few explicit closed-form solutions in terms of elementary functions exist for particular cases of non-stationary inputs, see Reference [3] for an exhaustive list. Another approach is to first apply a Karhunen–Loeve (K–L) spectral decomposition to the excitation process covariance matrix and then use orthogonal polynomials (such as Chebyshev polynomials) in order to obtain a compact analytical description of the data, see Reference [4]. This compact excitation data expression allows the closed-form solution of the problem. An exact closed-form solution for the transient mean-square response of a linear SDF oscillator subject to unit step modulated white noise, which is a process with constant power spectral density over the whole spectrum, was first derived in Reference [5]. The steady-state covariance matrix of the response of the SDF oscillator to a first-order Markov process has been reported in Reference [6].

Several studies have treated the case of parametric random vibration of the SDF oscillator with random time coefficients and stationary external forcing, see References [7, 8]. The stability of the response is essentially governed by the random parametric excitations regardless of the external random excitation. The stability condition for the homogeneous case with random damping and deterministic restoring force has been derived in Reference [9]. It requires that the fraction of critical damping be larger than a threshold value for the system to be stable. This value depends on the magnitude of the spectral intensity of the random coefficient at a frequency which is twice the natural frequency of the system. In another study, a simply supported column under randomly varying axial and transverse loads is modelled as a second-order oscillator with multiplicative random coefficient restoring force, deterministic damping and additive random excitation, see Reference [10]. The system is solved by stochastic averaging and the stationary probability density solution is derived with an appropriate condition on the fraction of critical damping. A stochastic averaging method is also used in Reference [11] to find the stationary probability density for the case of a SDF oscillator with parametric and additive random excitations. In this case, the random damping and restoring force coefficients and the external forcing were taken to be mutually uncorrelated, colored (which can be considered as the output of multi-dimensional linear filters to white Gaussian noise), and Gaussian noise excitations.

Conditions of moment stability were derived in Reference [12] for the case with random time damping and restoring force as well as random external forcing. The moment equation method is used in Reference [7] for solving the case of the SDF oscillator with random time damping, restoring force and random external forcing, which are independent Gaussian random processes. In other work, the stationary and non-stationary moment responses of a deterministic oscillator subjected to periodic excitation with random amplitude and random phase disturbances were modelled as uncorrelated stationary or non-stationary white noise processes, see References [13, 14].

In this paper, we use the generalized polynomial chaos approach to solve two different classes of problems. We first consider the case of stochastic forcing represented by a first-order Markov process with deterministic coefficients. In Section 4.2, we derive an exact expression for the variance of the solution to that problem over the entire time domain, i.e. including the solution initial transient. Knowing the correlation kernel of the external forcing, we represent the
random input process using a K–L decomposition and solve the problem using the generalized polynomial chaos approach. We also develop a sharp error bound for K–L representations and analyse the convergence rate of the generalized polynomial chaos expansion for different types of probability density distributions.

We then consider the mixed case of the random response of a SDF oscillator subject to both external and parametric mutually uncorrelated random excitations with fully correlated disturbances in time, which represents a random variable case. The additive noise in the system presents a deterministic time-dependent periodic function multiplied by a stationary random variable. The random input variables are represented by an appropriate generalized polynomial chaos decomposition, and the problem is then integrated in time using effectively a coupled set of deterministic equations derived by the generalized polynomial chaos approach. This approach is an extension of the polynomial chaos method, previously used to treat similar linear as well as non-linear random vibration problems [15]. However, it allows for different probability density functions for the random disturbance. The novelty of this work resides in solving non-stationary stochastic vibration problems with parametric excitation. It also provides careful theoretical validation of the numerical results and accurate second-order moment convergence rates for different types of probability density distributions.

In the following section, we first review the generalized polynomial chaos and subsequently consider the aforementioned two cases. We conclude the paper with a discussion on the relative advantages and limitations of this new method. Appendix A contains some details on the numerical integration and post-processing of the results. The nomenclature used in this paper is given in Appendix B.

2. REPRESENTATION OF RANDOM PROCESSES

In this section, we briefly review the Wiener–Askey polynomial chaos expansion along with the K–L expansion; the latter is useful for representing the input random processes. Throughout this paper, we will use the symbol $\xi$ to denote a random variable with zero mean and unit variance.

2.1. The Wiener–Askey polynomial chaos expansion

The Hermite–Chaos expansion is effective in solving stochastic differential equations with Gaussian inputs as well as certain types of non-Gaussian inputs [15–18]; theoretical justification is based on the Cameron–Martin theorem [19]. However, for general non-Gaussian random inputs, the optimal exponential convergence rate is not achieved, and in some cases the convergence rate is in fact severely deteriorated, see Reference [20].

The Wiener–Askey polynomial chaos expansion is a generalization of the original polynomial chaos, first proposed by Wiener [21]. It is well suited to represent more general random inputs. The expansion basis is the complete polynomial basis from the Askey scheme family [20, 22]. A general second-order random process $X(\theta)$ is represented by

$$X(\theta) = a_0 I_0 + \sum_{i_1=1}^{\infty} c_{i_1} I_1(\xi_{i_1}(\theta)) + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} c_{i_1i_2} I_2(\xi_{i_1}(\theta), \xi_{i_2}(\theta)) + \cdots$$

(1)
where \( I_n(\zeta_1, \ldots, \zeta_n) \) denotes the Wiener–Askey polynomial chaos of order \( n \) in terms of the random vector \( \zeta = (\zeta_1, \ldots, \zeta_n) \) and \( a_0, c_1, \ldots, c_r \) are deterministic coefficient functions that uniquely specify the process \( X(\theta) \).

For example, one possible choice for \( I_n \) are the Hermite polynomials which correspond to the original Wiener–Hermite polynomial chaos \( H_n \). The expression of the Hermite polynomials is given by

\[
I_n(\zeta_1, \ldots, \zeta_n) = H_n(\zeta_1, \ldots, \zeta_n) = e^{1/2\zeta^T\zeta} (-1)^n \frac{\zeta^n}{\partial^{\zeta_1} \cdots \partial^{\zeta_n}} e^{-1/2\zeta^T\zeta} \tag{2}
\]

where \( \zeta \) denotes the vector consisting of \( n \) Gaussian random variables \( (\zeta_1, \ldots, \zeta_n) \).

In the Wiener–Askey chaos expansion, the polynomials \( I_n \) are not restricted to Hermite polynomials but rather can be all types of the orthogonal polynomials from the Askey scheme. For example, the expression of the Jacobi polynomials \( P_n^{(\alpha, \beta)} \) is given by

\[
I_n(\zeta_1, \ldots, \zeta_n) = P_n^{(\alpha, \beta)}(\zeta_1, \ldots, \zeta_n) = \frac{(1 - \zeta)^{-\alpha} (1 + \zeta)^{-\beta}}{2^n n! (-1)^n} \frac{\zeta^n}{\partial^{\zeta_1} \cdots \partial^{\zeta_n}} [(1 - \zeta)^n + (1 + \zeta)^n + \beta] \tag{3}
\]

where \( \zeta \) denotes the vector consisting of \( n \) Beta random variables \( (\zeta_1, \ldots, \zeta_n) \).

For notational convenience, we rewrite Equation (1) as

\[
X(\theta) = \sum_{j=0}^{\infty} \hat{c}_j \Phi_j(\zeta) \tag{4}
\]

where there is a one-to-one correspondence between the functions \( I_n(\zeta_1, \ldots, \zeta_n) \) and \( \Phi_j(\zeta) \). Since each type of polynomials from the Askey scheme forms a complete basis in the Hilbert space determined by their corresponding support, we can expect each type of Wiener–Askey expansion to converge to any \( L_2 \) functional in the \( L_2 \) sense in the corresponding Hilbert functional space as a generalized result of the Cameron–Martin theorem; see References [19, 23].

The orthogonality relation of the Wiener–Askey polynomial chaos takes the form

\[
\langle \Phi_i, \Phi_j \rangle = \langle \Phi_i^2 \rangle \delta_{ij} \tag{5}
\]

where \( \delta_{ij} \) is the Kronecker delta and \( \langle \cdot, \cdot \rangle \) denotes the ensemble average which is the inner product in the Hilbert space of the variables \( \zeta \). We also have

\[
\langle f(\zeta) g(\zeta) \rangle = \int f(\zeta) g(\zeta) W(\zeta) \, d\zeta \tag{6}
\]

or

\[
\langle f(\zeta) g(\zeta) \rangle = \sum_{\zeta} f(\zeta) g(\zeta) W(\zeta) \tag{7}
\]

in the discrete case. Here \( W(\zeta) \) is the weighting function corresponding to the Wiener–Askey polynomials chaos basis \( \{ \Phi_i \} \).

Most of the orthogonal polynomials from the Askey scheme have weighting functions that take the form of probability function of certain types of random distributions. We then choose
Table I. Correspondence between the type of Wiener–Askey polynomial chaos and the type of random inputs ($N \geq 0$ is a finite integer).

<table>
<thead>
<tr>
<th>Random inputs</th>
<th>Wiener–Askey chaos</th>
<th>Support</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous</td>
<td>Gaussian</td>
<td>Hermite-chaos</td>
</tr>
<tr>
<td></td>
<td>Gamma</td>
<td>Laguerre-chaos</td>
</tr>
<tr>
<td></td>
<td>Beta</td>
<td>Jacobi-chaos</td>
</tr>
<tr>
<td></td>
<td>Uniform</td>
<td>Legendre-chaos</td>
</tr>
<tr>
<td>Discrete</td>
<td>Poisson</td>
<td>Charlier-chaos</td>
</tr>
<tr>
<td></td>
<td>Binomial</td>
<td>Krawtchouk-chaos</td>
</tr>
<tr>
<td></td>
<td>Negative binomial</td>
<td>Meixner-chaos</td>
</tr>
<tr>
<td></td>
<td>Hypergeometric</td>
<td>Hahn-chaos</td>
</tr>
</tbody>
</table>

the type of independent variables $\xi$ in the polynomials $\{\Phi_i(\xi)\}$ according to the type of random distributions as shown in Table I. Legendre polynomials, which are a special case of the Jacobi polynomials with parameters $\alpha = \beta = 0$, correspond to an important distribution—the Uniform distribution.

2.2. Representation of stochastic input

The K–L expansion is another way of representing a random process [24]. It is based on the spectral expansion of the covariance function of the process. Let us denote the process as $h(t, \theta)$ and its covariance function as $R_{hh}(t_1, t_2)$, where $t_1$ and $t_2$ are the two temporal co-ordinates. By definition, the covariance function is real, symmetric and positive-definite. It has an orthogonal set of eigenfunctions which forms a complete basis. The K–L expansion then takes the following form:

$$h(t, \theta) = \bar{h}(t) + \sigma_h \sum_{i=1}^{\infty} \sqrt{\lambda_i} \phi_i(t) \xi_i(\theta)$$

(8)

where $\bar{h}(t)$ denotes the mean of the random process, $\sigma_h$ denotes the standard deviation of the process and $\xi_i(\theta)$ is a set of independent random variables with a given random distribution; they form an orthonormal random vector. Also, $\phi_i(t)$ and $\lambda_i$ are the eigenfunctions and eigenvalues of the covariance function, respectively, i.e.

$$\int R_{hh}(t_1, t_2) \phi_i(t_2) \, dt_2 = \lambda_i \phi_i(t_1)$$

(9)

Among many possible decompositions of a random process, the K–L expansion is optimal in the sense that the mean-square error resulting from a finite-term representation of the process is minimized [16]. Its use, however, is limited as the covariance function of the solution process is not known a priori. Nevertheless, the K–L expansion still provides a powerful means for representing input random processes when the covariance structure is known.

Obviously, truncated expansions will be used which brings up the issue of how many terms are needed to represent accurately the random input. Some numerical experiments regarding accuracy of truncated K–L representation for different covariance kernels have been reported in Reference [25]. Here, we derive an asymptotic expression of the relationship between the
different factors affecting the convergence of the representation in the particular case of a random first-order Markov process. We also present numerical results to validate the theory.

Let us assume that the random process is applied over the time interval \([0, T]\) and it is specified by its correlation function:

\[
R_{hh}(t_1, t_2) = \sigma^2_h e^{-|t_2 - t_1|/A}, \quad A > 0
\]

(10)

where \(A\) is the correlation length. Knowing the form of its covariance kernel, we decompose the process in its truncated K–L representation up to order \(n\). It is then possible to use this decomposition to reconstruct the covariance kernel and thus investigate how well it is represented with \(n\) terms in the expansion. To this end, we observe that the diagonal terms of the kernel should be equal to the variance of the process. For a fixed number of random dimensions, we estimated a bound on the error of the variance of the process. This bound is accurate for large values of \(i\) (see Equation (8)). In particular, in this case it becomes very sharp for the diagonal end-point (see Figure 1(b)).

The variance of the process is defined as

\[
\langle h^2(t, \theta) \rangle = \bar{h}^2(t) + \sigma^2_h \sum_{i=1}^{\infty} \lambda_i \phi_i^2(t) \quad \text{and} \quad \langle h^2_n(t, \theta) \rangle = \bar{h}^2(t) + \sigma^2_h \sum_{i=1}^{n} \lambda_i \phi_i^2(t)
\]

(11)

We define the relative error in variance as

\[
\varepsilon = \frac{\langle h^2(t, \theta) \rangle - \langle h^2_n(t, \theta) \rangle}{\langle h^2(t, \theta) \rangle} = \frac{\sigma^2_h \left( \sum_{i=n+1}^{\infty} \lambda_i \phi_i^2(t) \right)}{\sigma^2_h} = \sum_{i=n+1}^{\infty} \lambda_i \phi_i^2(t)
\]

(12)

The eigenvalues \(\lambda_i\) and eigenfunctions \(\phi_i(t)\), when \(i\) is large, take the form

\[
\lambda_i \approx \frac{2}{(i \pi)^2} \frac{T^2}{A} \quad \text{and} \quad \phi_i(t) \approx \eta \cos \left( \frac{i \pi}{T} t \right)
\]

(13)

where \(\eta\) is a normalization factor (see below, Equation (59)) and \(T\) is the length of the time domain. For large \(i\), we have \(\eta \approx \sqrt{2/T}\). Using the above expressions, we obtain the following error bound:

\[
\varepsilon \approx \sum_{i=n+1}^{\infty} \frac{4T}{\pi^2 A} \frac{\cos^2(i \pi / T t)}{i^2} \leq \frac{4}{\pi^2} \frac{1}{n A} \approx 0.4053 \frac{1}{n A}
\]

(14)

We see that it depends on the ratio \(T/A\) and it is inversely proportional to the number of retained terms.

In Figure 1 we show the diagonal terms of the kernel reconstructed from the K–L decomposition. These coefficients should be equal to the variance \(\sigma^2_h\) of the process. The plot (a) confirms the convergence of the representation when we increase the number of random dimensions \(n\). We notice that the error at the boundaries is larger compared to the middle-point. The plot (b) summarizes results from multiple cases where we estimate the minimum number of terms \(n\) needed to reach a prescribed accuracy \(\varepsilon\) as a function of \(T/A\). We present results for the middle-point of the interval \([0, T]\) (Figure 1(a): \(\sigma t = 50\)) and the right end-point (Figure 1(a): \(\sigma t = 100\)). Data are represented by crosses and are obtained for different values
of T, T/A and ε. Solid lines represent least-square approximations of the data and we indicate the corresponding slopes. The linear relationship between T/A and n for both points is verified; it is consistent with the estimate of inequality (14).

3. CONVERGENCE ISSUES

The considerable speed-up obtained by the polynomial chaos approach versus Monte–Carlo simulation (see Reference [20] and examples in Section 4), applies to stochastic input which is at least partially correlated and with a relatively low dimensionality. However, for a random process describing an input close to white noise (very short correlation length), a high dimensional chaos expansion is required and this is difficult to handle. Even if the input is a fully correlated random variable, there exist some open issues regarding the resolution properties of truncated polynomial chaos representations.

In this section, we demonstrate the dependence of the number of terms in the polynomial chaos expansion to the time domain given a prescribed level of solution accuracy. Using the particular example of a first-order linear ODE with random input, we first show that the polynomial chaos solution in this case can be found more efficiently by solving an eigenvalue problem. We then derive a theoretical expression of the number of terms \((P + 1)\) required in the expansion in order to reach a given time and a prescribed accuracy for the variance of the solution. Finally, we verify numerically the validity of the proposed analysis.

Let us consider the first-order linear ODE:

\[
\frac{dx}{dt} = -kx \quad \text{with} \quad x(t = 0) = x_0 \quad \text{and} \quad t \in [0, T]
\]  

(15)

where the decay rate coefficient \(k\) is considered to be a random variable \(k(\xi) = \sigma_\xi \xi\) (with zero mean and variance \(\sigma^2\)) and a certain probability distribution \(f(k)\). The deterministic and
**stochastic** analytic solutions are, respectively,

\[ x(t) = x_0 e^{-\bar{k} t} \quad \text{and} \quad x(t, \xi(t)) = x_0 e^{-\sigma \hat{\xi} t} \]  \hspace{1cm} (16)

where \( \bar{k} \) is the mean value of the decay rate coefficient \( k \). The polynomial chaos stochastic solution will be called \( S \) in the remaining of the section. The mean and variance of the stochastic solution are, respectively,

\[ \bar{x}(t) = \int_{\Omega} x_0 e^{-\bar{k} t} f(k) \, dk \quad \text{and} \quad \langle x^2(t) \rangle = \int_{\Omega} (x_0 e^{-\bar{k} t} - \bar{x}(t))^2 f(k) \, dk \]  \hspace{1cm} (17)

The integrations are performed over the support \( \Omega \) defined by the corresponding distribution. The initial condition is taken to be one to simplify the analysis, so we have: \( x(t = 0) = x_0 = 1 \). Here we focus on the Gaussian, Gamma and Beta continuous distribution functions, but this is still applicable to other distributions. We can express the solution \( x(t, \xi) \) as

\[ x(t, \xi) = \sum_{j=0}^{P} a_j(t) \Phi_j(\xi) \quad \text{with} \quad P = \frac{(n+p)!}{n!p!} - 1 \]  \hspace{1cm} (18)

(see Equation (4)) where the functions \( \Phi_j(\xi) \) can be Hermite (Gaussian distribution), Laguerre (Gamma–Exponential distribution) or Legendre (Beta–Uniform distribution) polynomial functionals. The variables \( n \) and \( p \) are the number of random dimensions and the highest polynomial order of the expansion, respectively. The right-hand side of Equation (15) becomes:

\[ k x(\xi, t) = \sigma \hat{\xi} x(\xi, t) = \sigma \sum_{j=0}^{P} a_j(t) \Phi_j(\xi) \]  \hspace{1cm} (19)

Using the following recurrent relation, valid for this family of orthogonal polynomials we have:

\[ k \Phi_j(\xi) = \sigma \hat{\xi} \Phi_j(\xi) = \sigma (a_j \Phi_{j-1}(\xi) + b_j \Phi_j(\xi) + c_j \Phi_{j+1}(\xi)) \]  \hspace{1cm} (20)

with \( a_0 = 0 \). Therefore, Equation (19) takes the form

\[ k x(\xi, t) = \sigma \hat{\xi} x(\xi, t) = \sigma \sum_{j=0}^{P} a_j(t) (a_j \Phi_{j-1}(\xi) + b_j \Phi_j(\xi) + c_j \Phi_{j+1}(\xi)) \]  \hspace{1cm} (21)

with \( a_0 = 0 \). The left-hand side of Equation (15) can be written as

\[ \dot{x}(\xi, t) = \sum_{j=0}^{P} \dot{a}_j(t) \Phi_j(\xi) \]  \hspace{1cm} (22)

This leads to:

\[ \dot{a}_0(t) + b_0 a_0(t) + a_1(t) = 0, \quad j = 0 \]

\[ \dot{a}_j(t) + c_{j-1} a_{j-1}(t) + b_j a_j(t) + a_{j+1} a_{j+1}(t) = 0, \quad j = 1, 2, 3, \ldots, P \]  \hspace{1cm} (23)

The recurrent relationship and corresponding set of equations for the coefficients for *Hermite polynomials* take the form

\[ k H_j(\xi) = \sigma \hat{\xi} H_j(\xi) = \sigma (j H_{j-1}(\xi) + H_{j+1}(\xi)), \quad a_j = j, \quad b_j = 0, \quad c_j = 1 \]  \hspace{1cm} (24)
and
\[ \ddot{x}_0(t) + x_1(t) = 0, \quad j = 0 \]
\[ \ddot{x}_j(t) + x_{j-1}(t) + (j + 1)x_{j+1}(t) = 0, \quad j = 1, 2, 3, \ldots, P \]
(25)

Similarly, for Laguerre polynomials, we have
\[ kL_j(\xi) = \sigma \xi L_j(\xi) = \sigma(-jL_{j-1}(\xi) + (2j + 1)L_j(\xi) - (j + 1)L_{j+1}(\xi)) \]
\[ a_j = -j, \quad b_j = 2j + 1, \quad c_j = -(j + 1) \]
(26)

and
\[ \ddot{x}_0(t) + x_0(t) - x_1(t) = 0, \quad j = 0 \]
\[ \ddot{x}_j(t) - jx_{j-1}(t) + (2j + 1)x_j(t) - (j + 1)x_{j+1}(t) = 0, \quad j = 1, 2, 3, \ldots, P \]
(27)

Finally, for Legendre polynomials we have
\[ kP_j(\xi) = \sigma \xi P_j(\xi) = \sigma \left( \frac{j}{2j + 1}P_{j-1}(\xi) + \frac{j + 1}{2j + 1}P_{j+1}(\xi) \right) \]
\[ a_j = \frac{j}{2j + 1}, \quad b_j = 0, \quad c_j = \frac{j + 1}{2j + 1} \]
(28)

\[ \ddot{x}_0(t) + x_1(t)/3 = 0, \quad j = 0 \]
\[ \ddot{x}_j(t) + \frac{j}{2j - 1}x_{j-1}(t) + \frac{j + 1}{2j + 3}x_{j+1}(t) = 0, \quad j = 1, 2, 3, \ldots, P \]
(29)

In all cases, we are left with a coupled system of equations to solve that can be written as
\[ \dot{\mathbf{x}}(t) + A\mathbf{x}(t) = 0, \quad \mathbf{x}(t) = \mathbf{a}_0 \]
(30)

where \( \mathbf{a}(t) \) represents the vector of the unknown random modes at time \( t \) and \( A \) is the matrix of size \( [P + 1 \times P + 1] \) built from Equation (25), (27) or (29). The solution of this system can be expressed as
\[ \mathbf{a}(t) = \sum_{j=0}^{P} C_j \Theta^{(j)} e^{-\lambda_j t} \]
(31)

with \( \lambda \) and \( \Theta \) being the eigenvalues and corresponding eigenvectors of \( A \), respectively. The scaling of the random modes corresponding to the coefficients \( C_j \) is determined by the initial condition \( \mathbf{a}_0 \), using the relation:
\[ \mathbf{a}_0 = \sum_{j=0}^{P} C_j \Theta^{(j)} \]
(32)

This method gives us a very fast and efficient way to solve the system. It can be used for large systems with a total number of terms in the polynomial chaos expansion of the order

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of hundreds. This is particularly true for the case of the Gamma and the Beta distributions. However, in the case of the Gaussian distribution (Hermite polynomial), the matrix $A$ becomes poorly conditioned for large values of $P$ and the inversion of the matrix becomes difficult and inaccurate without any additional effort.

Because of the simple exponential form of the solution, a theoretical expression of the convergence rate of the moments of the expansion $S$ for the different distributions can be derived. In particular, we derive convergence rate estimates for the variance of the solution. We first turn our attention to the case where the decay parameter $k$ has an Exponential distribution (with zero mean and variance $\sigma^2$), which is a particular case of the general Gamma distribution (with $\alpha = 0$).

The associated polynomial used with the Exponential distribution is the Laguerre polynomial $L_j(\xi)$ and its generating function $g(z, \xi)$ is as follows:

$$g(z, \xi) = \frac{1}{1-z} e^{-\xi z/(1-z)} = \sum_{j=0}^{\infty} L_j(\xi)z^j, \quad |z| < 1$$

If we make a change of variables with $z/(1-z) = \sigma t$, where $\sigma$ is the standard deviation of $k$ and $t$ represents time, we obtain

$$S = e^{-\sigma^2 t} = \frac{1}{1+\sigma t} \sum_{j=1}^{\infty} \left( \frac{\sigma t}{1+\sigma t} \right)^j L_j(\xi)$$

Here, we do not include the mean solution in the expansion. Similarly, the truncated expansion $S_P$ becomes:

$$S_P = \frac{1}{1+\sigma t} \sum_{j=1}^{P} \left( \frac{\sigma t}{1+\sigma t} \right)^j L_j(\xi)$$

and thus,

$$S - S_P = \frac{1}{1+\sigma t} \sum_{j=P+1}^{\infty} \left( \frac{\sigma t}{1+\sigma t} \right)^j L_j(\xi)$$

We compute the norm:

$$\langle S^2 \rangle = \int_0^{\infty} S^2 e^{-\xi} d\xi = \frac{1}{1+2\sigma t} - \frac{1}{(1+\sigma t)^2}$$

And thus, we can estimate the relative error for the variance of the solution as

$$\frac{\langle (S - S_P)^2 \rangle}{\langle S^2 \rangle} = \left( \frac{\sigma t}{1+\sigma t} \right)^{2P}$$

If we call this error $\varepsilon$, we find a relation that links time $t$, $\varepsilon$, and the total number of terms $(P+1)$ in the expansion, i.e.

$$P + 1 = \frac{1}{2} \log((1+\sigma t)/(\sigma t)) + 1$$

We now consider the case where the decay parameter $k$ has a Gaussian distribution (with zero mean and variance $\sigma^2$).
The associated polynomial used with the Gaussian distribution is the Hermite polynomial \( H_j(\xi) \) and its generating function \( g(z, \xi) \) is as follows:

\[
g(z, \xi) = e^{-z^2/2 + \xi z} = \sum_{j=0}^{\infty} H_j(\xi) \frac{z^j}{j!} \quad (40)
\]

If we make a change of variables with \( z = \sigma t \), where \( \sigma \) is the standard deviation of \( k \) and \( t \) represents time, we obtain:

\[
S = e^{\sigma \xi t} = e^{(\sigma t)^2/2} \sum_{n=1}^{\infty} \frac{(\sigma t)^n}{n!} H_n(\xi) \quad (41)
\]

Here, we do not include the mean solution in the expansion. To study the rate of convergence of this expansion we consider the norm:

\[
\langle S^2 \rangle = (\sqrt{2\pi})^{-1} \int_{-\infty}^{\infty} e^{-z^2/2} S^2 d\xi = e^{2\sigma^2 t^2} - e^{\sigma^2 t^2} \quad (42)
\]

We also denote by \( S_P \) the truncation expansion in Equation (41). We note here that \( S_P \) is not exactly the polynomial chaos expansion but we will use it to obtain an approximate estimate. To this end, we compute:

\[
\langle (S - S_P)^2 \rangle = e^{(\sigma t)^2} \sum_{n=P+1}^{\infty} \frac{(\sigma t)^{2n}}{n!} \quad (43)
\]

and thus we can estimate the relative error from

\[
\frac{\langle (S - S_P)^2 \rangle}{\langle S^2 \rangle} = \frac{1}{e^{(\sigma t)^2} - 1} \sum_{n=P+1}^{\infty} \frac{(\sigma t)^{2n}}{n!} \leq \frac{1}{e^{(\sigma t)^2} - 1} \frac{(\sigma t)^{2(P+1)}}{(P+1)!((P+1) - (\sigma t)^2/(P+1))} \quad (44)
\]

where we have assumed that \( (\sigma t)^2/(P+1) < 1 \). If we call this error \( \varepsilon \), we find an error bound that links time \( t, \varepsilon \), and the number of terms \( (P+1) \), i.e.

\[
\varepsilon \leq \frac{(\sigma t)^{2(P+1)}}{e^{(\sigma t)^2} - 1} ((P+1)((1 - (\sigma t)^2/(P+1)))^{-1} \quad (45)
\]

Given a prescribed accuracy \( \varepsilon \), this non-linear equation is solved numerically for \( (P+1) \) using an iterative method, which is a combination of bisection, secant, and inverse quadratic interpolation methods. The numerical solution is used as our reference estimate for the computation using the polynomial chaos method described previously.

Numerical results for the case of a Uniform distribution (Wiener–Legendre Chaos) are also presented here. The generating function of the Legendre polynomial does not lead readily to the calculation of the coefficients of the expansion. However, a recurrent formula for the coefficients can be derived (not presented here) based on the Legendre polynomial recurrence formula in order to compute the variance of the solution.

Prescribing the error \( \varepsilon \) to a finite value, an iterative search can be employed to estimate what is the minimum number of terms \( (P+1) \) required in the truncated polynomial chaos expansion.
Figure 2 summarizes all the results for the three different distributions with $\sigma = 1$ and for an error $\varepsilon = 10^{-7}$. It compares the estimates with the results obtained by using the generalized polynomial chaos method.

Depending on the distribution, we see that for a long time integration, a very large number of modes is required to sustain the error to a prescribed level. Also, we notice that the theoretical estimates for the Exponential (Figure 2 plot (a)) and Gaussian distribution (Figure 2 plot (b)) provide a lower bound, i.e. it gives the least number of required modes to achieve the error level $\varepsilon$. As mentioned previously, in the case of the Gaussian distribution, the matrix $A$ becomes poorly conditioned for large values of $P$ and the inversion of the matrix for $P > 30$ becomes difficult and inaccurate without any additional effort. Overall, the Legendre-chaos seems to be the most robust and requires the least number of modes for the same integration time.
4. RANDOM OSCILLATORS

In this section, we determine the response of SDF mechanical systems subject to random excitations with possible randomness present in their mechanical properties. We are particularly interested in the determination of the mean and covariance function of the response.

4.1. Governing equations

We consider the following linear oscillator subject to an external forcing \( f(t, \theta) \):

\[
\ddot{x}(t, \theta) + c \dot{x}(t, \theta) + kx(t, \theta) = f(t, \theta), \quad x(0, \theta) = x_0 \quad \text{and} \quad \dot{x}(0, \theta) = \dot{x}_0, \quad t \in [0, T] \tag{46}
\]

The equation has been normalized with respect to the mass, so the forcing \( f(t, \theta) \) has units of acceleration. The damping factor \( c \) and spring factor \( k \) are defined as follows:

\[
c = 2\zeta \omega_0 \quad \text{and} \quad k = \omega_0^2 \tag{47}
\]

where \( \zeta \) and \( \omega_0 \) are, respectively, the damping ratio and the natural frequency of the system. This system can become stochastic if the external forcing or the input parameters or both are some random quantities. These random quantities can be evolving in time (i.e. random process) or not evolving in time (i.e. random variable).

Let us consider the case where the damping factor \( c \) and the spring constant \( k \) are random processes with unknown correlation functions and the external forcing is a random process with a given correlation function. Complementary cases with different random parametric and/or forcing inputs can be extrapolated from this case. We decompose the random process representing the forcing term in its truncated K–L expansion up to the \( n \)th random dimension to obtain:

\[
f(t, \theta) = \bar{f}(t) + \sigma_f \sum_{i=1}^{n} \sqrt{\lambda_i} \phi_i(t) \xi_i(\theta) = \sum_{i=0}^{n} f_i(t) \xi_i \tag{48}
\]

Because the correlation functions for the coefficients \( c \) and \( k \) are not known, we decompose the random input parameters in terms of their polynomial chaos expansion:

\[
c(t, \theta) = \sum_{j=0}^{P} c_j(t) \Phi_j(\xi(\theta)) \quad \text{and} \quad k(t, \theta) = \sum_{j=0}^{P} k_j(t) \Phi_j(\xi(\theta)) \tag{49}
\]

Finally, the solution of the problem is sought in the form given by its truncated Wiener–Askey polynomial chaos expansion (see Equation (18)):

\[
x(t, \theta) = \sum_{i=0}^{P} x_i(t) \Phi_i(\xi(\theta)) \tag{50}
\]

We substitute all expansions in the governing equation (see Equation (46)) to obtain:

\[
\sum_{i=0}^{P} \ddot{x}_i(t) \Phi_i + \sum_{j=0}^{P} c_j(t) \Phi_j \sum_{i=0}^{P} \dot{x}_i(t) \Phi_i + \sum_{j=0}^{P} k_j(t) \Phi_j \sum_{i=0}^{P} x_i(t) \Phi_i = \sum_{i=0}^{n} f_i(t) \xi_i \tag{51}
\]

We now project the above equation onto the random space spanned by our orthogonal polynomial basis \( \Phi_m \). To this end, we take the inner product with each basis and average, then
we use the orthogonality relation (see Equation (5)). We obtain a set of coupled deterministic differential equations:

$$\ddot{x}_m(t) + \frac{1}{\langle \Phi_m^2 \rangle} \sum_{i=0}^{P} \sum_{j=0}^{P} c_{ij}(t) \dot{x}_i(t) e_{ij}^m + \frac{1}{\langle \Phi_m^2 \rangle} \sum_{i=0}^{P} \sum_{j=0}^{P} k_{ij}(t) x_i(t) e_{ij}^m = f_m(t), \quad m = 0, 1, 2, \ldots, P$$

(52)

where $e_{ij}^m = \langle \Phi_i \Phi_j \Phi_m \rangle$. These coefficients as well as $\langle \Phi_m^2 \rangle$ can be determined analytically or numerically using multi-dimensional numerical quadratures [16]. This system of equations consists of $(P+1)$ linear equations, each equation corresponding to one random mode. More details about the numerical implementation and the temporal discretization used to solve the linear problem are given in Appendix A.

In the next section, we study different combinations and different types of random inputs. First, we consider the case of the random response of the system to a Gaussian or Uniform random forcing with correlated disturbances in time (i.e. random process case). Next, we study the case of both external and parametric Gaussian or Uniform random excitations with fully correlated disturbances in time (i.e. random variable case).

4.2. Random forcing processes

The stochastic forcing is assumed to be a weakly stationary Gaussian or Uniform random process, with zero mean and correlation function $R_{ff}(\tau)$, applied over a time interval $[0, T]$. Equation (46) becomes:

$$\ddot{x}(t) + 2\zeta \omega_0 \dot{x}(t) + \omega_0^2 x(t) = f(t, \theta), \quad x(0) = x_0 \text{ and } \dot{x}(0) = \dot{x}_0, \quad t \in [0, T]$$

(53)

In this case, we choose the random input process to be a first-order Markov process, specified by its correlation function:

$$R_{ff}(\tau) = \sigma_f^2 e^{-|\tau|/A}, \quad A > 0$$

(54)

where $A$ is the correlation length and $\sigma_f$ denotes the standard deviation of the process. It can be checked that $f(t, \theta)$ is the stationary solution of the differential equation:

$$\dot{f}(t) = -\frac{1}{A} f(t) + \sigma_f \sqrt{\frac{2}{A}} W(t)$$

(55)

in which $W(t)$ is the zero-mean stationary white noise with covariance function $\delta(t)$, see Reference [6].

Knowing the initial state of the system, there exists a theoretical solution for the asymptotic state of the response covariance matrix $\Gamma = \lim_{t \to \infty} \Gamma(t)$ with initial condition $\Gamma(0) = 0$. The time asymptotic value of the variance of the solution has been derived in Reference [6]. Here, assuming the same initial conditions, we derive a theoretical expression for the variance of the solution $\Gamma_{xx}(t)$ over the entire time domain, i.e. including the solution initial transient. This expression is independent of the type of probability distribution of the input but does depend
on the type of covariance kernel of the input. The final result is

$$\Gamma_{xx}(t) = \frac{2\sigma^2 A^2}{\pi^2((1-\gamma)^2 + \beta^2)} \left\{ \beta^2(1 + 2\gamma)(\beta^2 + (1-\gamma)^2) + \gamma - 1 \right\} e^{-2\zeta_0 A t}$$

$$+ \frac{e^{-2\zeta_0 A t}}{4(\beta^2 + \gamma^2)} \left( \beta(2\gamma - 1) \sin(2\pi t) + (\beta^2 + \gamma(1-\gamma)) \cos(2\pi t) \right)$$

$$- \frac{\beta^2 e^{-\zeta_0 (1/A) t}}{\beta^2 + (1+\gamma)^2} \left( \frac{1+\gamma}{\beta} \sin(\pi t) + \cos(\pi t) \right)$$

where

$$\alpha = \omega_0 \sqrt{1 - \zeta^2}, \quad \beta = \alpha A, \quad \gamma = \zeta_0 A$$

Knowing the correlation function of the input in the time domain, we use the K–L expansion (see Equation (8)) to decompose the random input process. The corresponding eigenvalue problem (see Equation (8)) is solved analytically. The eigenvalues and eigenfunctions are as follows:

$$\lambda_i = \frac{2/A}{(1/A)^2 + \omega_i^2} \quad \text{and} \quad \phi_i(t) = \eta \left( \cos(\omega_0 t) + \frac{1}{A \omega_i} \sin(\omega_0 t) \right), \quad i = 1, 2, \ldots, n$$

where

$$\eta = \left\{ \frac{1}{2} \left( T \left( 1 + \left( \frac{1}{A \omega_i} \right)^2 \right) + \frac{\sin(2\omega_i T)}{2\omega_i} \left( 1 - \left( \frac{1}{A \omega_i} \right)^2 \right) - \frac{1}{A \omega_i^2} (\cos(2\omega_i T) - 1) \right) \right\}^{-1/2}$$

The normalization coefficient $\eta$ ensures that $\int_0^T \phi_i(t) \, dt = 1$. Here, $A$ is the correlation length, $[0, T]$ is the size of the time domain, and $\omega_i$ are determined numerically by solving

$$\left( \omega_i^2 - \frac{1}{A} \right)^2 \tan(\omega_i T) - 2 \frac{\omega_i}{A} = 0, \quad i = 1, 2, \ldots, n$$

For a given correlation length $A$ and a standard deviation $\sigma_f$ of the random process $f(t, \theta)$, we decompose the input in its truncated K–L expansion up to the $n$th random dimension (see Equation (48)). The number of random dimensions $n$ needs to be large enough in order to resolve the scale associated with the correlation length $A$, as discussed in Section 2.2. The solution of the problem is sought in the form given by its truncated Wiener–Askey polynomial chaos expansion (Equation (50)) where $n$ is the number of random dimensions and $p$ is the highest polynomial order of the polynomial chaos expansion. We expand the right-hand side of Equation (53) in its K–L series and we expand the response process $x(t)$ in its Wiener–Askey polynomial chaos series. The system does not exhibit any non-linearity in random space.
implies that quadratic or higher order terms in the polynomial chaos expansion will not improve the accuracy of the solution. Therefore, only linear terms are used in the expansion.

The general solution of the system is expressed as an integral form of the external forcing $f(t)$:

$$x(t) = \frac{1}{2} \int_0^t f(t') e^{-\zeta_0 (t-t_1)} \sin(\alpha (t-t_1)) \, dt' = \frac{1}{2} \int_0^t f(t-t_1) e^{-\zeta_0 t_1} \sin(\alpha t_1) \, dt_1$$

(61)

where $\alpha$ is given in Equation (57). We decompose the random forcing in its K–L representation:

$$x(t, \theta) = \frac{1}{2} \int_0^t \left( \tilde{f}(t-t_1) + \sigma_f \sum_{i=1}^{\infty} \sqrt{\lambda_i} \phi_i(t-t_1) \xi_i(\theta) \right) e^{-\zeta_0 t_1} \sin(\alpha t_1) \, dt_1$$

(62)

If we assume that the mean of the forcing is zero and we use the orthogonality property of the decomposition, we can express the variance of the solution $\Gamma_{xx}(t)$ as

$$\langle x^2(t, \theta) \rangle = \left( \frac{\sigma_f}{\sigma} \right)^2 \sum_{i=n+1}^{\infty} \lambda_i \left( \int_0^t e^{-\zeta_0 t_1} \sin(\alpha t_1) \phi_i(t-t_1) \, dt_1 \right)^2 = \Gamma_{xx}(t)$$

(63)

Then, we introduce the truncated representation of the solution $x_n(t, \theta)$ and we compute the error between the exact and approximate solutions as follows:

$$\langle (x(t, \theta) - x_n(t, \theta))^2 \rangle = \langle x^2(t, \theta) \rangle - \langle x_n^2(t, \theta) \rangle$$

$$= \left( \frac{\sigma_f}{\sigma} \right)^2 \sum_{i=n+1}^{\infty} \lambda_i \left( \int_0^t e^{-\zeta_0 t_1} \sin(\alpha t_1) \phi_i(t-t_1) \, dt_1 \right)^2$$

(64)

We have already established the simplified form of the eigenvalues, eigenvectors and normalization coefficient for large $i$, which are

$$\lambda_i \approx \frac{2}{(i\pi)^2} \frac{T^2}{A}, \quad \phi_i(t) \approx \eta \cos \left( \frac{i\pi}{T} t \right) \quad \text{and} \quad \eta^2 = \frac{2}{T}$$

(65)

Therefore, the absolute error becomes:

$$\langle (x(t, \theta) - x_n(t, \theta))^2 \rangle \approx \frac{2\lambda^2}{A} \left( \frac{\sigma_f T}{\alpha \pi} \right)^2 \sum_{i=n+1}^{\infty} \frac{I_i^2(t)}{i^2}$$

(66)

we need to evaluate the following integral:

$$I_i(t) = \int_0^t e^{-\zeta_0 t_1} \sin(\alpha t_1) \cos \left( \frac{i\pi}{T} (t-t_1) \right) \, dt_1$$

(67)

This integral is computed for large $i$ and takes the value:

$$I_i(t) = \left( \frac{T}{i\pi} \right)^2 \left\{ e^{-\zeta_0 t} (\alpha \cos \alpha t - \zeta_0 \sin \alpha t) - \frac{2\zeta_0 T}{i\pi} \sin \left( \frac{i\pi}{T} t \right) + \alpha \cos \left( \frac{i\pi}{T} t \right) \right\}$$

(68)
Figure 3. Time evolution of second-order moment $\Gamma_{xx}$ (Case I) for different number of random dimensions $n$ (a). Convergence rate of second-order moment $\Gamma_{xx}$ of the output (Case I and II) and second-order moment $\Gamma_{ff}$ of the input (Case I and II) versus the number of random dimensions $n$ at $\omega_0 t = 20$ (b).

For large time $t$, it simplifies to:

$$I_i(t) \approx \left( \frac{T}{\pi} \right)^2 z \cos \left( \frac{i\pi}{T} t \right)$$

and therefore, the error can be bounded by a function of the number of random dimensions $n$. We obtain:

$$\langle (x(t, \theta) - x_n(t, \theta))^2 \rangle \approx \frac{4\sigma_f^2 T^5}{A \pi^6} \sum_{i=n+1}^{\infty} \frac{\cos^2(i\pi/T t)}{i^6} < \frac{4\sigma_f^2 T^5}{5 A \pi^6} \frac{1}{n^5}$$

Then, we normalize the error by the asymptotic value of the variance of the solution $\Gamma_{xx}(t \to \infty)$. If we call $\varepsilon$ the relative error, we have

$$\varepsilon = \frac{\langle (x(t, \theta) - x_n(t, \theta))^2 \rangle}{\langle x^2(t, \theta) \rangle} < \frac{8}{5 \pi^6} \frac{1}{n^5}$$

where

$$\kappa = \frac{\pi^2 ((1 - \gamma)^2 + \beta^2)(\gamma(\beta^2 + \gamma^2)(\beta^2 + (1 + \gamma)^2)) T^5}{(\beta^2(1 + 2\gamma)(\beta^2 + (1 - \gamma)^2)) A^3}$$

where the coefficients $z$, $\beta$ and $\gamma$ are given in Equation (57). This error bound is particularly sharp at the final time $t = T$ and does not depend on the variance $\sigma_f$ of the stochastic input.

We compute the second-order moment of the solution for different values of the random dimensions $n$ and for a correlation length $A = 1.0$. We present results for Gaussian (Case I) and Uniform (Case II) random inputs. In Figure 3(a), we present the evolution of the variance of the solution for Case I versus non-dimensional time $\tau = \omega_0 t$. As the number $n$ increases, we
see that the numerical solution converges asymptotically to the exact solution for our particular set of parameters. This is also the case for the Uniform input.

For Cases I and II, we then compare the relative $L_{\infty}$ error at the final time versus the exact solution (see Figure 3(b)) and we examine the convergence rate of this error versus the number of terms $n$ in the K–L expansion. The numerical results confirm the fact that the theoretical estimate does not depend on the input distribution but on the input covariance kernel. Indeed, we obtain exactly the same convergence rate for both distributions and the two curves overlap to form the stochastic output. The particular convergence rate of $1/n^5$ for this specific kernel and equation is perfectly validated by the numerical results for large $n$ and $t$. As one would expect from the derivation, it remains an upper bound to the polynomial chaos solution. We also plot the convergence rate of the corresponding stochastic input $f(t, \theta)$ based on the same criteria developed in Section 2.2.

In Figure 4, we plot the time evolution of the relative pointwise $L_{\infty}$ error between numerical and theoretical solution of the variance of the solution. The error is relative to the asymptotic expected value of the variance of the solution over the entire time domain. Again as $n$ increases, the accuracy of the stochastic input and output response improves, but at a different rate.

It is also worth mentioning that we do not obtain a uniform convergence rate of the solution over the time domain.

4.3. Random parametric and forcing variables

Here, we consider a linear oscillator subject to both random parametric and external forcing excitations. In this case, both non-stationary random forcing and random parameters are treated as random variables. We consider two types of random inputs: Gaussian input (Case I) or...
Uniform input (Case II). Equation (46) becomes:

\[
\ddot{x}(t) + 2\zeta_0 \dot{x}(t) + \omega_0^2 x(t) = f(t, \theta) = F(\theta) \cos(\omega t + \phi), \quad x(0) = x_0
\]

and \( \dot{x}(0) = \dot{x}_0, \quad t \in [0, T] \) (73)

In this case, the random quantities do not vary in time, i.e. are fully correlated in time. We assume that the probabilistic models of each of these random variables are given by

\[
2\zeta = \bar{c} + \sigma_c \xi_1 \\
\omega_0^2 = \bar{k} + \sigma_k \xi_2 \\
F = \bar{F} + \sigma_F \xi_3
\]

where \( \xi_1, \xi_2 \) and \( \xi_3 \) are three independent random variables with zero mean; \( \sigma_c, \sigma_k \) and \( \sigma_F \) are the standard deviations of \( c, k \) and \( F \), respectively.

The random inputs as well as the forcing function and the solution of the problem are represented by their Wiener–Askey polynomial chaos expansion. The number of random dimensions \( n = 3 \) is equal to the number of independent random variables \( (\xi_1, \xi_2, \xi_3) \). The system does exhibit non-linearity in random space. Therefore, quadratic or higher order terms in the polynomial chaos expansion should improve the accuracy of the solution. We use the Newmark integration scheme for this problem as described in Appendix A.

For both Case I and II, the random parameters are set to: \( (\bar{c}, \sigma_c) = (0.1, 0.01); \ (\bar{k}, \sigma_k) = (1.05, 0.105) \) and \( (\bar{F}, \sigma_F) = (0.1, 0.01) \) with a frequency \( \omega = 1.05 \) and a phase \( \phi = 0 \) for the forcing. The initial conditions \( x_0 \) and \( \dot{x}_0 \) are set to 0. We notice that there is a non-zero probability that the oscillator has a natural frequency \( \omega_0 = \sqrt{k} \) matching the forcing frequency \( \omega \).

The time evolution of the dominant modes of the solution for Case I is represented in Figure 5(a). We use a polynomial order \( p = 5 \) in the expansion \((P + 1 = 56 \text{ terms})\). Only the first four modes (mean plus Gaussian contribution to the solution) are presented in the plot. The first mode (corresponding to \( i = 0 \) in Equation (50)) is the mean solution. As expected, due to random diffusion, the amplitude of the mean solution is smaller than the deterministic solution (not presented here). The maximum amplitude of the mean solution is about 30% lower than the maximum amplitude of the deterministic solution. The higher modes, which describe the stochastic part of the solution, all start from zero and then gradually grow as the interaction of the random modes, (through the non-linearity in random space), takes place. A stationary periodic state is eventually reached by all modes (around \( t = 85 \) for the highest modes). The dominant modes of the solution for Case II follow a similar pattern [26].

We then compute the second-order moment of the solution for Case I and II for different values of the polynomial chaos order \( p \) (see Figure 5(b) for the case with \( p = 5 \)).

The numerical integration is performed up to \( T = 100 \) when the solution reaches the asymptotic periodic state. We know the exact deterministic solution of the system and the probability distribution functions of the random inputs in the system. We then integrate the solution over the support defined by the corresponding distribution to obtain the exact mean and variance of the solution. These integrations are performed numerically using a Gauss–Legendre quadrature. A sufficient number of quadrature points is used to ensure convergence to
Figure 5. (a) Fifth-order \((p = 5)\) Wiener–Askey polynomial chaos \((P + 1 = 56\) terms). Solution of the dominant random modes versus time for Case I; and (b) time evolution of the variance of the solution for Case I and II.

Figure 6. Convergence rate \(L_\infty\) error of the mean and variance of the solution versus the order of the Wiener–Askey polynomial chaos expansion \(p\): (a) Case I: Gaussian input; and (b) Case II: Uniform input.

 converged values. We also consider the convergence rate of the relative \(L_\infty\) error at the final time versus the order \(p\) of the expansion (see Figure 6). For both cases, we see in this semi-log plot that the error of the mean and variance decreases exponentially fast due to the spectral decomposition of the solution in the random space. We notice that the rates of convergence are not identical and that the mean decreases faster than the variance of the solution for the Gaussian case. The errors for the Uniform case are lower than errors for the Gaussian case.
Table II. Speed-up factors $S$ based on relative mean error $\varepsilon_{\text{mean}}$ of generalized polynomial chaos ($(P + 1)$ terms) versus Monte–Carlo simulations ($N$ events) for Gaussian and Uniform distributions.

<table>
<thead>
<tr>
<th>$\varepsilon_{\text{mean}}$ (%)</th>
<th>Monte–Carlo $N$</th>
<th>Generalized polynomial chaos $P + 1$</th>
<th>$S$</th>
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<td>6.25</td>
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<td>120</td>
<td>18</td>
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<tr>
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<td>13 000</td>
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<td>35</td>
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5. SUMMARY AND DISCUSSION

We have presented a convergence analysis and corresponding results of the generalized polynomial chaos approach, first presented in Reference [20].

An efficient way to represent stochastic input processes for differential equations is through the Karhunen–Loeve (K–L) approach. To this end, we first developed a sharp error bound for K–L representations, which is proportional to the interval of simulation $T$ normalized by the correlation length $A$ and inversely proportional to the number of terms in the expansion $n$.

We then analysed the convergence rate of the generalized polynomial chaos expansion for three different types of probability density distributions (Gamma, Beta and Gaussian). We found that the most efficient representation is accomplished with the Wiener–Legendre chaos for the Uniform distribution.

Finally, we applied the generalized polynomial chaos to linear random oscillators with stationary forcing as well as with parametric and non-stationary forcing. We derived a new analytical solution for the time-dependent covariance and demonstrated exponential convergence of the method with respect to the polynomial degree.

With regards to computational efficiency of this approach, we performed a systematic comparison with the standard Monte–Carlo approach, i.e. with no special acceleration procedures. The case we selected was the second-order random oscillator with random variables (random damping, spring and forcing), random dimensions ($n = 3$), and long time integration ($T_{\text{final}} = 100$). The results for Gaussian and Uniform distributions are summarized in Table II. The worst speed-up factor ($S \approx 6$) is for a Wiener–Hermite representation with relatively large error in the mean, while the best speed-up ($S \approx 17$ millions) is for a Wiener–Legendre representation with small error in the mean. This is not very surprising given the poor resolution properties of the Hermite polynomials in solving even deterministic problems, e.g. see Reference [27]. However, we are not aware of any previous efforts to compare the effectiveness of Wiener–Hermite expansions against other representations as we have done here.

The advantages of generalized polynomial chaos are clear in the cases of correlated input as we demonstrated. However, in the limit of very small correlation length (e.g., $T/A \to \infty$) there is a requirement for a large number of dimensions, $n$ to represent accurately the stochastic input.
process; this increases substantially the computational complexity of this approach. This is still an unresolved problem, and in this case Monte–Carlo-based approaches should be employed to deal with the high dimensionality.

APPENDIX A: NUMERICAL IMPLEMENTATION

We formulate the problem of Equation (46) in matrix form as follows:

$$\dot{X}(t) = AX(t) + \mathbf{Y}$$  \hspace{1cm} (A1)

with

$$X(t) = \begin{bmatrix} X^1(t) \\ X^2(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ k(t) & c(t) \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$ \hspace{1cm} (A2)

where $X(t)$ is the state vector, $A$ is the system matrix, and $\mathbf{Y}$ is the input distribution matrix.

Revisiting Equation (52) and writing the system in compact form gives us

$$\dot{X}^1_m(t) = X^2_m(t)$$
$$\dot{X}^2_m(t) + \frac{1}{\langle \Phi_m^2 \rangle} \sum_{i=0}^{P} \sum_{j=0}^{P} c_j X^2_i(t)e_{ijm} + \frac{1}{\langle \Phi_m^2 \rangle} \sum_{i=0}^{P} \sum_{j=0}^{P} k_j X^1_i(t)e_{ijm} = f_m(t)$$  \hspace{1cm} (A3)

with $m = 0, 1, 2, \ldots, P$.

A.1 Temporal discretization

By using the Wiener–Askey polynomial chaos expansion, the randomness of the system is transferred into the basis polynomials. Therefore the deterministic coefficients are smooth in time and any deterministic ODE solver can be employed such as explicit second- and fourth-order Runge–Kutta scheme. In this section, we focus on an implicit Newmark scheme for stochastic systems. We describe in details the implementation of the Newmark scheme for SDF stochastic differential equations with correlated random inputs in the context of polynomial chaos. Both fourth-order Runge–Kutta scheme and Newmark scheme were used in Sections 4.2 and 4.3 and gave identical results.

It is worth mentioning that in the case of stochastic systems under white-noise inputs, the validity and properties of the crude Newmark scheme are not necessarily conserved. This is because the Wiener process even though continuous may have an unbounded variation over any given time interval. Therefore, the acceleration vector does not exist mathematically [28]. If the acceleration is not smooth enough and the $C^4$-regularity of the forcing is not satisfied, a modification of the usual Newmark scheme can be done in order to show almost sure convergence of the scheme [29]. This modification is only in the stochastic part. However, in our case, the forcing term is correlated in time and the aforementioned restrictions to white-noise inputs do not apply.

We consider Equation (46) where both the external forcing $f$ and the input parameters $c$ and $k$ are random. We call $v(t) = \dot{x}(t)$ ($v$ has the dimension of a velocity) the first temporal
derivative of the polynomial chaos solution \( x(t) \) and \( a(t) = \ddot{x}(t) \) (\( a \) has the dimension of an acceleration) the second temporal derivative of the solution \( x(t) \). \( x(t) \), \( v(t) \) and \( a(t) \) are vectors which components are the modes of the polynomial chaos decomposition. To update the state of the system \( S \) at timestep \( n \), \( S^n = [a^n(t), v^n(t), x^n(t)]^T \), using the forcing at time level \( f^{n+1}(t) \), we use the following:

\[
\Gamma S^{n+1} = \Delta S^n + \Lambda^{n+1}
\]

(A4)

where \( \Gamma \) and \( \Delta \) are matrices that can be built by blocks and \( S^{n+1} \) is such as in Equation (A10), see below. The matrices take the form

\[
\Gamma = \begin{pmatrix}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\
\Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\
\Gamma_{31} & \Gamma_{32} & \Gamma_{33}
\end{pmatrix}, \quad \Delta = \begin{pmatrix}
\Delta_{11} & \Delta_{12} & \Delta_{13} \\
\Delta_{21} & \Delta_{22} & \Delta_{23} \\
\Delta_{31} & \Delta_{32} & \Delta_{33}
\end{pmatrix}
\]

(A5)

where each block submatrix \( \Gamma_{ij} \) and \( \Delta_{ij} \) is of size \((P + 1) \times (P + 1)\).

Concerning \( \Gamma \), only \( \Gamma_{12} \) and \( \Gamma_{13} \) receive a contribution from the random quantities of the left-hand side of Equation (46), the rest of the matrix is deterministic.

We have

\[
\Gamma_{12(ij)} = \sum_{l=0}^{P} c_l \frac{\langle \Phi_i \Phi_l \Phi_j \rangle}{\langle \Phi^2_l \rangle} \quad \text{and} \quad \Gamma_{13(ij)} = \sum_{l=0}^{P} k_l \frac{\langle \Phi_i \Phi_l \Phi_j \rangle}{\langle \Phi^2_l \rangle}
\]

(A6)

and \( \Gamma_{22} \) and \( \Gamma_{33} \) are zero matrices, \( \Gamma_{11}, \Gamma_{23} \) and \( \Gamma_{32} \) are identity matrices; the other blocks are diagonal matrices with constant terms on the diagonal (\( dt \) is the time step used in the temporal scheme).

\[
\Gamma_{21(ii)} = -\beta dt^2, \quad \Gamma_{31(ii)} = -\gamma dt
\]

(A7)

The matrix \( \Delta \) is fully deterministic. \( \Delta_{11}, \Delta_{12}, \Delta_{13} \) and \( \Delta_{33} \) are zero matrices, \( \Delta_{13} \) and \( \Delta_{32} \) are identity matrices, the other blocks are diagonal matrices with constant terms on the diagonal.

\[
\Delta_{21(ii)} = (1/2 - \beta) dt^2, \quad \Delta_{22(ii)} = dt, \quad \Delta_{31(ii)} = (1 - \gamma) dt
\]

(A8)

The Newmark method is unconditionally stable and second-order accurate if we choose \( \gamma = \frac{1}{2} \) and \( \beta = \frac{1}{4} \). The vector \( \Lambda \) of size \( 3(P + 1) \) is such that:

\[
\Lambda^{n+1}(t) = \begin{bmatrix}
f^{n+1}(t) \\
0 \\
0
\end{bmatrix}
\]

(A9)
Once the matrices and the forcing vector have been evaluated, the matrix \( \Gamma \) is inverted numerically and the solution of the system for the next time step is obtained by computing \textit{matrix–matrix} and \textit{matrix–vector} multiplications. We have

\[
S^{n+1} = \Gamma^{-1} \Delta S^n + \Gamma^{-1} \Lambda^{n+1}
\]  

This method is computationally very efficient if the input parameters are random \textit{variables}. In this case, the matrix \( \Gamma \) needs to be computed and inverted only once at the beginning of the computation and only the forcing term is updated at every time step. If the input parameters are random \textit{processes}, only blocks \( \Gamma_{12} \) and \( \Gamma_{13} \) need to be computed every time step in addition of the forcing term. The inversion of the matrix \( \Gamma \) at every time step should be simplified if one takes into account its structure by blocks where only two blocks are time-dependent.

\section*{APPENDIX B: NOMENCLATURE}

\begin{itemize}
  \item \( A \) \quad \text{correlation length}
  \item \( c = 2\zeta\omega_0 \) \quad \text{damping factor}
  \item \( \hat{c}_j(t) \) \quad \text{Wiener–Askey polynomial chaos coefficients}
  \item \( f \) \quad \text{external forcing}
  \item \( H_n(\xi) \) \quad \text{Hermite polynomials}
  \item \( I_n(\xi) = \Phi_j(\xi) \) \quad \text{Wiener–Askey polynomial chaos basis}
  \item \( k = \omega_0^2 \) \quad \text{spring factor}
  \item \( L_n(\xi) \) \quad \text{Legendre polynomials}
  \item \( n \) \quad \text{number of random dimensions}
  \item \( p \) \quad \text{highest polynomial order of the expansion}
  \item \( P + 1 \) \quad \text{number of terms in the Wiener–Askey polynomial chaos expansion}
  \item \( P_n^{(\alpha, \beta)}(\xi) \) \quad \text{Jacobi polynomials}
\end{itemize}

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\[ R_{XX}(t_1, t_2) \]

\( T \)

covariance function

\( \text{var}(X(\theta, t)) = \sigma_X^2 = \langle (X(\theta, t) - \bar{X}(\theta, t))^2 \rangle \)

length of the time domain

\( X(\theta, t) \)

variance of \( X(\theta, t) \)

\( \bar{X}(\theta, t) \)

second-order random process

mean value of \( X(\theta, t) \)

Greek letters

\( \varepsilon \)

relative error in variance

\( \zeta_i(\theta) \)

random variable

\( \zeta = (\zeta_{i_1}(\theta), \ldots, \zeta_{i_n}(\theta)) \)

random vector

\( \theta \)

random event

\( \lambda_i \)

eigenvalues of \( R_{XX}(t_1, t_2) \)

\( \phi_i \)\neigenfunctions of \( R_{XX}(t_1, t_2) \)

\( \langle . . . \rangle \)

ensemble average

ACKNOWLEDGEMENTS

This work is sponsored by ONR, and simulations were performed at NCSA (University of Illinois, Urbana-Champaign).

REFERENCES


