Mixed spectral/hp element formulation for nonlinear elasticity

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1. Introduction

High-order finite elements, especially the p- and hp-versions [1–4], lead to exponential convergence rate for smooth solutions and for singular solutions (hp-version) [1–4]; they also enable flexibility in discretization as their accuracy does not depend strongly on the aspect ratio of the elements [1]. In the field of solid mechanics, p-FEM and hp-FEM have been studied theoretically [5–10] for linear elasticity, and numerical results have been provided for hyperelastic materials [11–13] and plastic materials [14], exhibiting locking-free behavior for both nearly-incompressible [7,12,13] and plate/shell problems [8,15–21].

From the numerical standpoint, the spectral/hp element version based on general Jacobi polynomials provides further flexibility in generating elements (e.g., hexahedral, tetrahedral, triangular prisms and even pyramids). Moreover, the polynomial basis is hierarchical and is well conditioned for enhanced computational efficiency and scalability. These aspects have led to the implementation of a displacement formulation of the method for linear elasticity and geometrically nonlinear problems in [22], where high accuracy and parallel scalability were demonstrated for several prototype problems as well as for arterial mechanics problems.

With regards to nearly incompressible problems, there are two main strategies for overcoming volumetric locking: the displacement-only formulation with higher-order elements and the modified variational forms, including the mixed formulations and reduced integration methods. Both analytical and computational results were provided for the first strategy, e.g. [7,12,13], demonstrating locking-free results can be obtained by using sufficiently high polynomial order $O_p$ for displacement. However, when the Poisson ratio is close to 0.5 (or bulk modulus for hyperelastic materials satisfies $k \geq 1$), one has to increase $O_p$ to resolve the volumetric locking problem, see, e.g. [7,13]. On the other hand, the modified variational form does not have this problem, and hence it exhibits the correct approximability for both $h$- and $p$-versions [5]. Among the methods of the second strategy there are several interesting modifications, e.g. enhanced strain formulations [23–27] and displacement/pressure formulations [5, 6, 10, 15, 28–36]. Here we will concentrate on the later. For the large deformation problems, these formulations were mainly applied to lower-order elements (usually $O_p \leq 3$), although combining the mixed formulations and high-order elements was proposed in the analysis of Stenberg and Suri [5]. Also, the violation of ellipticity condition in large deformation problems was found for the displacement/pressure formulations but has not been investigated for different combinations of high-order elements, e.g. see [36–38]. Therefore, for large deformation problems, computational results of mixed formulation on high-order elements are needed to investigate both the locking phenomena and ellipticity condition.
addition, a reduced-integration method (noted also as reduced-constraint method in [10], or pressure projection method in [33,34] and herein) for hyperelastic material was introduced by Chen and Pan [33] and Chen et al. [34], hence the displacement/pressure formulation can be reduced to displacement only without losing the similarity to mixed formulations [10].

In the current paper we extend the method presented in [22] to fully nonlinear elastic problems with large deformations following a mixed formulation and using a pressure projection scheme [33,34] unlike the displacement-only formulation employed in [22]. Unfortunately, the mixed formulation approach complicates both the numerical implementation as well as the theoretical framework of the method as compatibility of the spaces for the displacement and pressure fields should be established even for linear elasticity problems, e.g. see [28]. In addition, for large deformations the issue of ellipticity of the discrete formulation will be addressed following [36,37], where the conditions for lower order elements were studied. To this end, upon formulating and implementing in 3D the mixed formulation, we set up several numerical tests that address these issues and highlight the fast convergence, the locking-free behavior, and the stability of the high-order mixed method in large deformation analysis of nearly incompressible solids. Compared to the displacement-only formulation of p-type finite elements presented in [12,13], which also overcomes volume locking for sufficiently large interpolation order, our numerical results here demonstrate that the mixed formulation – despite the aforementioned difficulties and the added computational efforts – seems to be more efficient, especially for problems with high values of the bulk modulus.

We consider the problem in Lagrangian coordinates \( \mathbf{X} \), both for infinitesimal and finite deformations. Specifically, we aim to obtain the displacement field \( \mathbf{u}(\mathbf{X}) \) in a 3D solid domain \( \Omega \subset \mathbb{R}^3 \), with given traction and body forces, satisfying [39,40]

\[
\text{div}(\mathbf{S}) + \mathbf{f} = 0
\]

in some weak sense, where \( \mathbf{f} \) is the body force and \( \mathbf{S} \) is the appropriate stress tensor. In our implementation, we employ the hierarchical shape functions based on Jacobian polynomials for spatial discretization (see Appendix A) to obtain high-order accuracy. In the mixed formulation, we introduce the hydrostatic pressure \( p = \frac{\text{div} \mathbf{S}}{\text{vol}} \) into the equilibrium equation, and solve for \( (\mathbf{u}, p) \) [36]. However, by introducing another variable \( p \), the computational cost for inverting the linear system increases. To this end, we employ a pressure projection method [33,34], by which we only need to compute (once or iteratively via Newton–Raphson in the nonlinear case) the linear system for the \( \mathbf{u} \) part, while the pressure field is obtained from the displacement field [33].

The paper is organized as follows: in Section 2 we present briefly the linear and nonlinear material models we use, and in Section 3 we present the weak formulation and the projection method. In Section 4 we demonstrate the convergence rates of the method for linear and nonlinear elasticity problems. In Section 5 we present the main results, first focusing on the issue of stability, by investigating numerically the validity of the ellipticity and inf–sup conditions, then turning to the locking phenomena. We finish in Section 5 with the main conclusions while the appendix contains details on the Jacobian polynomial basis and the increment method employed in obtaining the nonlinear solutions.

2. Material models

We mainly consider two models represented by the following strain energy density functions:

- Linear elastic model [39,40]:

\[
\Psi(\mathbf{E}) = \frac{\lambda}{2} (\text{tr} \mathbf{E})^2 + \mu \mathbf{E} : \mathbf{E},
\]

where \( \mathbf{E} = \frac{1}{2} \left( (\mathbf{u}_0)^T + (\mathbf{u}_0) \right) \) is the strain tensor, and \( \mu, \lambda \) are the shear and Lamé modulus, respectively.

- Mooney–Rivlin model [33,34,40]:

\[
\Psi(\mathbf{E}) = A_{01}(I_1 - 2/3) + A_{02}(I_2 - 4/3) - 3/2 k (J - 1)^2,
\]

where \( \mathbf{E} = \frac{1}{2} \left( (\mathbf{u}_0)^T + (\mathbf{u}_0) \right) \) is the Green Lagrange strain tensor, \( A_{01}, A_{02} \) are constants describing the behavior of the material, and \( k \) is the bulk modulus. Also \( I_1, I_2, J \) are related with the deformation gradient \( \mathbf{F} = \mathbf{I} + \frac{\mathbf{u}_0}{\mathbf{C}} \) and the right Green deformation tensor \( \mathbf{C} = \mathbf{F}^T \mathbf{F} \) as

\[
I_1 = \text{tr}(\mathbf{C}), \quad I_2 = \frac{1}{2} ([\text{tr}(\mathbf{C}^2) - \text{tr}(\mathbf{C}^2^2)), \quad J = \det(\mathbf{F}).
\]

This model is a simple isotropic approximation for materials with rubber-like properties [41], in the region of large deformation and large strain. In this expression, we separate the distortional and dilatational deformation using Penn’s approach [42] as this facilitates the use of the pressure projection technique we will use in the next section; see also [33].

By using the hyper-elasticity relations, the second Piola–Kirchhoff stress tensor \( \mathbf{S} = \frac{\partial \Psi}{\partial \mathbf{E}} \) for these two cases can be expressed as

- Linear elastic model:

\[
\mathbf{S} = \lambda (\text{tr} \mathbf{E}) \mathbf{I} + 2 \mu \mathbf{E}.
\]

- Mooney–Rivlin model:

\[
\mathbf{S} = 2A_{01} I_1 J^{-2/3} + 4A_{02} I_2 J^{-4/3} ((\text{tr} \mathbf{E} + 1) \mathbf{I} - \mathbf{E})
\]

\[
+ \left( -\frac{2}{3} A_{01} I_1 J^{-2/3} - 4 \frac{2}{3} A_{02} I_2 J^{-4/3} + k(J - 1) \right) \mathbf{C}^{-1}.
\]

After introducing the hydrostatic pressure \( p \), the models can be rewritten as

- Linear elastic model [28]:

\[
\Psi(\mathbf{E}) = \frac{p}{2} \text{tr} \mathbf{E} + \mu \mathbf{E} : \mathbf{E},
\]

where the hydrostatic pressure \( p = \lambda \text{tr} \mathbf{E} \).

- Mooney–Rivlin model [40]:

\[
\Psi(\mathbf{E}) = A_{01}(I_1 - 2/3) + A_{02}(I_2 - 4/3) + p \left( J - 1 \right),
\]

where the hydrostatic pressure \( p = k(J - 1) \), and the last term corresponds to the “material constraint” [43].

Correspondingly, the second Piola–Kirchhoff stress tensor \( \mathbf{S} = \frac{\partial \Psi}{\partial \mathbf{E}} \) related with the hydrostatic pressure can be expressed as

- Linear elastic model:

\[
\mathbf{S} = p \mathbf{I} + 2 \mu \mathbf{E}.
\]

- Mooney–Rivlin model:

\[
\mathbf{S} = 2A_{01} I_1 J^{-2/3} + 4A_{02} I_2 J^{-4/3} ((\text{tr} \mathbf{E} + 1) \mathbf{I} - \mathbf{E})
\]

\[
+ \left( -\frac{2}{3} A_{01} I_1 J^{-2/3} - 4 \frac{2}{3} A_{02} I_2 J^{-4/3} + p \right) \mathbf{C}^{-1}.
\]

3. Weak formulation and discretization

Next, we will derive the formulation for solving the displacement field \( \mathbf{u}(\mathbf{X}) \) in a three-dimensional object occupying domain
\[ \Omega \subset \mathbb{R}^3 \] with boundary \( \partial \Omega = \partial \Omega_d \cup \partial \Omega_n \). Here Dirichlet type boundary conditions are provided on \( \partial \Omega_d \), Neumann type boundary conditions (traction) on \( \partial \Omega_n \). \( \Omega^e \) stands for the initial configuration of each element.

### 3.1. Linear elastostatics

#### 3.1.1. Displacement and mixed formulations

First, we consider infinitesimal deformations. The weak form of the equilibrium equation can be expressed as follows [39]:

\[
\int_{\Omega} \mathbf{S} : \nabla \mathbf{\ddot{u}} \, d\Omega - \int_{\Omega} \mathbf{T} : \mathbf{v} \, d\Omega - \int_{\partial \Omega_n} \mathbf{p} \cdot \mathbf{n} \, d\Gamma = 0, \quad \forall \mathbf{v} \in \mathbf{U}_h, \tag{10}
\]

where the displacement field \( \mathbf{u}(X) \) \( \in \mathbf{U} = \{ \mathbf{u}(X) \in [H^1(\Omega)]^3 | \mathbf{w}(X) = \mathbf{u}_0(X) \} \) on \( \partial \Omega_d \), and \( \mathbf{U}_h = \{ \mathbf{w}(X) \in [H^1(\Omega)]^3 | \mathbf{w}(X) = 0 \} \) on \( \partial \Omega_n \). \( \mathbf{S} \), \( \mathbf{T} \), \( \mathbf{p} \) and \( \mathbf{\rho} \) are the stress tensor, external traction force (non-follower load) on \( \partial \Omega_n \), external body force, and mass density, respectively.

For the linear elastic model (2), we rewrite the problem as follows [28]: find the displacement field \( \mathbf{u}(X) \) \( \in \mathbf{U} \) such that

\[
\mu \int_{\Omega} \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u}) \, d\Omega + \int_{\Omega} \mathbf{p} \mathbf{v} \, d\Omega - \int_{\partial \Omega_n} \mathbf{T} : \mathbf{v} \, d\Gamma = 0, \quad \forall \mathbf{v} \in \mathbf{U}_h, \tag{11}
\]

where the operator \( \varepsilon(\cdot) = \frac{1}{2} \left( \mathbf{\nabla} \mathbf{u} + \mathbf{\nabla} \mathbf{u}^T \right) \) will refer to the above form as the displacement formulation. For the discretization of (11), one can see [22] for details.

For the mixed formulation, model (6) is considered. Problem (11) should be modified following [28]:

\[
\mu \int_{\Omega} \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u}) \, d\Omega + \int_{\Omega} \mathbf{p} \mathbf{v} \, d\Omega - \int_{\partial \Omega_n} \mathbf{T} : \mathbf{v} \, d\Gamma = 0, \quad \forall \mathbf{v} \in \mathbf{U}_h, \tag{12}
\]

along with the weak form of the definition for hydrostatic pressure \( \mathbf{p} \):

\[
\int_{\Omega} \mathbf{v} \cdot \mathbf{q} \, d\Omega = \frac{1}{\lambda} \int_{\Omega} \partial \mathbf{p} \cdot \mathbf{n} \, d\Gamma, \quad \forall \mathbf{q} \in L^2(\Omega), \tag{13}
\]

#### 3.1.2. Local discretization

Next, we consider the discretization of (12). Based on the shape functions defined in Appendix A, on each element we expand the displacement \( \mathbf{u}^e = (u^e_1, u^e_2, u^e_3) \) and the test function \( \mathbf{v}^e = (v^e_1, v^e_2, v^e_3) \):

\[
u^e_i(X) = \sum_{k=1}^{N_e} \tilde{u}^e_k \phi_k(X), \quad \mathbf{t}^e_i(X) = \sum_{k=1}^{N_e} \tilde{v}^e_k \phi_k(X), \quad i = 1, 2, 3, \tag{14}\]

where \( N_e \) is the total number of shape functions for displacement on this element, \( \phi_k(X) \) are the shape functions, and \( \tilde{u}^e_k, \tilde{v}^e_k \) are the expansion coefficients. For the local hydrostatic pressure \( p^e \), with the set of shape functions for pressure \( \phi_k(X), k = 1, \ldots, N_p^e \subseteq \{ \phi_k(X), k = 1, \ldots, N^e \} \), we can expand the pressure \( p^e \):

\[
p^e(X) = \sum_{k=1}^{N_p^e} \tilde{p}_k \phi_k(X), \tag{15}\]

where \( \tilde{p}_k \) are the corresponding expansion coefficients.

#### 3.1.3. Local pressure projection

Following the steps in [33], we carry out the projection of the hydrostatic pressure \( p^e \) at the element level onto the selected pressure function space. For the linear case, this reduced constraint method is actually equivalent to the mixed formulation [10]. Here we consider the problem of approximating \( p^e \) at the element level in a least-squares sense, by a linear combination of a sequence of shape functions \( \{ \phi_k(X), k = 1, \ldots, N_p^e \} \in L^2(\Omega^e) \) as in (15). That is, we choose

\[
\mathbf{p}^e = [\tilde{p}_1, \ldots, \tilde{p}_{N_p^e}, \tilde{p}_{k_1}, \ldots, \tilde{p}_{k_{N_p^e}}]^T. \tag{16}\]

To minimize

\[
\| p^e - \mathbf{Q}(p^e) \|^2_{L^2(\Omega^e)}, \tag{17}\]

where \( \| \cdot \|_{L^2(\Omega^e)} \) is the \( L^2 \) norm in the element domain \( \Omega^e \) and

\[
\mathbf{Q}(X) = \left[ \phi_1(X), \phi_2(X), \ldots, \phi_{N_p}^e(X) \right]. \tag{18}\]

So the minimization leads to

\[
\mathbf{M}_p \mathbf{p}^e = \mathbf{D}^e, \tag{19}\]

where the element mass matrix for the pressure is of size \( N_p^e \times N_p^e \) as follows:

\[
\mathbf{M}_p = \begin{bmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{bmatrix}, \quad m_{ij} = \int_{\Omega^e} \phi_i \phi_j \, d\Omega \tag{20}\]

and the right hand side of (19) is a vector of size \( N_p^e \):

\[
\mathbf{D}^e = \int_{\Omega^e} \phi_1 \mathbf{p}^e \, d\Omega, \int_{\Omega^e} \phi_2 \mathbf{p}^e \, d\Omega, \ldots, \int_{\Omega^e} \phi_{N_p^e} \mathbf{p}^e \, d\Omega \tag{21}\]

where the vector of unknown expansion coefficients for displacement \( \mathbf{u} \) has length \( 3N^e_c \). In order to have smaller bandwidth for the coefficient matrix [22], it is organized in a way such that the three displacement components corresponding to a particular mode are adjacent, as follows:

\[
\mathbf{U}^e = \begin{bmatrix}
(\mathbf{u}_1^e)^T, (\mathbf{u}_2^e)^T, (\mathbf{u}_3^e)^T,
(\mathbf{u}_4^e)^T, (\mathbf{u}_5^e)^T, (\mathbf{u}_6^e)^T,
(\mathbf{u}_7^e)^T, (\mathbf{u}_8^e)^T, (\mathbf{u}_9^e)^T
\end{bmatrix}^T, \tag{22}\]

\[
\mathbf{U}_e^e = [\tilde{u}_1, \tilde{u}_2, \tilde{u}_3]. \tag{23}\]

The pressure matrix \( \mathbf{K}_p^e \) with size \( 3N_p^e \times N_p^e \) is composed of \( N_p^e \times N_p^e \) numbers of 3 \times 1 submatrices:

\[
\mathbf{K}_p^e = \begin{bmatrix}
(\mathbf{K}_p^e)_{11} & \cdots & (\mathbf{K}_p^e)_{1N_p^e} \\
\vdots & \ddots & \vdots \\
(\mathbf{K}_p^e)_{N_p^e1} & \cdots & (\mathbf{K}_p^e)_{N_p^eN_p^e}
\end{bmatrix}, \tag{24}\]

where \( (\mathbf{K}_p^e)_{ij} = \int_{\Omega} \frac{\partial \phi_i}{\partial x_j} \phi_j \, d\Omega, \int_{\Omega} \frac{\partial \phi_i}{\partial y} \phi_j \, d\Omega, \int_{\Omega} \frac{\partial \phi_i}{\partial z} \phi_j \, d\Omega \) for \( i = 1, \ldots, N_p^e, j = 1, \ldots, N_p^e \).

Therefore, the element coefficients for the projected hydrostatic pressure should be

\[
\mathbf{p}^e = \lambda (\mathbf{K}_p^{-1}(\mathbf{K}_p^e))^T \mathbf{U}^e. \tag{24}\]

#### 3.1.4. Global discretization

We assemble the local displacement expansion coefficients \( \mathbf{U}^e \) of all elements into a global form \( \mathbf{U} \) and upon substitution into (12), we obtain

\[
-(\mathbf{K}_d + \mathbf{K}_p) \mathbf{U} + \mathbf{F} = 0. \tag{25}\]

\[ \text{Omega} \subset \mathbb{R}^3 \]
where \( K_e \) is the stiffness matrix for \( \int_T \epsilon(u) : \epsilon(v) \, dv, \) and \( K_e^c \) corresponds to \( \int_T \rho \text{div}(v) \, dv. \) Here, the \( 3N_e \times 3N_e \) matrix \( K_e^c \) for each element is organized into a matrix of \( N_f \times N_f \) blocks, and the \( 3 \times 3 \) submatrix \( (K_e^c)^{ij} \) is given by \( (K_e^c)^{ij} = \int_T (\mathbf{D} \phi_i \phi_j) d\Omega, \) where

\[
\mathbf{D} \phi_i = \begin{bmatrix}
\phi_{i1} \\
\phi_{i2} \\
\phi_{i3}
\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}
\mathbf{I} & \mathbf{0} \\
\mathbf{0} & \mathbf{I}
\end{bmatrix}.
\]

(26)

On the other hand, the matrix \( K_e^c \) is assembled based on local matrices from the element-wise pressure projections

\[
K_e^c = \mathbf{K} \mathbf{M}^{-1} \mathbf{K}_e^c. \]

(27)

On each element, the vector of the external forces \( \mathbf{F} \) has the same length and ordering of indices as \( \mathbf{U} \). That is, \( \mathbf{F} = [\mathbf{F}_{x}^T, \ldots, \mathbf{F}_{y}^T, \mathbf{F}_{z}^T] \), where \( \mathbf{F}_{x} = [F_{x1}^e, F_{x2}^e, F_{x3}^e] \), \( F_{x1}^e = \int_{\partial \Omega_x} T \phi_i d\Gamma^e \) and \( T \) is a component of the external traction and body forces, respectively.

In the implementation, the contribution from known expansions (due to Dirichlet boundary conditions \( \partial \mathbf{U}_D \)) is moved to the right-hand-side before the equation is solved. Although (25) is equivalent to solving the larger system \((\mathbf{U}, \mathbf{P})\), the size of unknowns is by one-fourth smaller. For details on the iterative solution of the linear system, we refer the reader to [22].

### 3.2. Nonlinear elastostatics

#### 3.2.1. Displacement and mixed formulations

Next we consider a solid object with finite deformation. Similar to the linear elastostatics, \( \mathbf{X} \) stands for the position vector of a material point in the initial configuration \( \Omega \), and \( \mathbf{X} \) denotes the position vector in the deformed configuration. The weak form of displacement formulation for the equilibrium is: find the displacement field \( \mathbf{u}(\mathbf{x}) \in \mathbf{U}(\mathbf{x}) \in [H^1(\Omega)]^3 \) such that \( \mathbf{u}(\mathbf{x}) = \mathbf{u}_D(\mathbf{x}) \) on \( \partial \Omega_D \) (where \( \partial \Omega_D \) denotes the Dirichlet boundary in the initial configuration), which satisfies

\[
P(\mathbf{u}) = \int_{\Omega} \left( \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)^T \mathbf{F}(\mathbf{u}) + \mathbf{u}(\mathbf{x})^T \left( \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right) - \int_{\partial \Omega} \mathbf{T} \cdot \mathbf{n} d\Gamma - \int_{\Omega} \rho \mathbf{f} \cdot d\Omega = 0,
\]

(28)

where \( \mathbf{U}, \mathbf{T}, \mathbf{f} \) and \( \rho \) have the same meanings as for linear elastostatics case, but with respect to the initial configuration. Note here that we assume the external \( \mathbf{T} \) to be deformation-independent (i.e., non-loadcase). Also, \( \mathbf{F}(\mathbf{u}) \) is the deformation gradient tensor defined as \( \mathbf{F}(\mathbf{u}) = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \mathbf{I} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}}. \)

To rewrite (28) into the mixed formulation framework, we consider \( S \) related with the hydrostatic pressure as in (9), and divide it into two parts: with and without the pressure term \( p \). Although it can be generalized to other cases, here we take \( S \) as in (9), i.e., \( S \) can be divided as

\[
S = 2\lambda \alpha_0 \alpha_{1/3} + 4\lambda \alpha_0 \alpha_{1/3} (\text{tr} \mathbf{E} + 1) \mathbf{I} - \mathbf{E} + \left( \frac{2}{3} \alpha_0 \alpha_{1/3} - 4 \alpha_0 \alpha_{1/3} \right) \mathbf{C}^{-1}.
\]

(29)

\[
\tilde{S} = \frac{\partial S}{\partial \mathbf{E}}.
\]

(30)

Correspondingly, the elasticity tensor \( \mathbf{C} = \frac{\partial S}{\partial \mathbf{E}} \) can also be divided into two parts (see [35,43] for the explicit expressions for these two fourth-order tensors of Mooney–Rivlin model):

\[
\tilde{C} = \frac{\partial \mathbf{S}}{\partial \mathbf{E}} \quad \text{and} \quad \mathbf{C} = \frac{\partial \mathbf{S}}{\partial \mathbf{E}}.
\]

(31)

where \( \mathbf{E} \) is the Green Lagrange strain tensor, with \( \mathbf{E}(\mathbf{u}) = \frac{1}{2}(\mathbf{F}(\mathbf{u}) - \mathbf{I}) \). Also, \( \mathbf{C} \) can be further divided into two parts

\[
\mathbf{C} = \mathbf{C}_0 + \mathbf{C}_1, \quad \mathbf{C}_0 = \frac{\partial (\mathbf{C}^{-1})}{\partial \mathbf{E}}, \quad \mathbf{C}_1 = \mathbf{J} \frac{\partial \mathbf{p}}{\partial \mathbf{E}}
\]

(32)

Following [22], we use \( p(t, \mathbf{u}) \) to denote the virtual work due to the internal stress, which is the first term in (28), and we use \( p(t, \mathbf{u}) \) to denote the last two terms, representing the virtual work due to the external forces. Separating the stress tensor and the elasticity tensor into volumetric and distorsional parts, we can write (28) as

\[
P(t, \mathbf{u}, \mathbf{p}) = \int_{\Omega} \left( \tilde{S}(\mathbf{u}) + \mathbf{S}(\mathbf{u}) \right) : \left( \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)^T \mathbf{F}(\mathbf{u}) + \mathbf{u}(\mathbf{x})^T \left( \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right) + \int_{\partial \Omega} \mathbf{T} \cdot \mathbf{n} d\Gamma - \int_{\Omega} \rho \mathbf{f} \cdot d\Omega = 0,
\]

(33)

which, along with the weak form of the definition for hydrostatic pressure \( p \):

\[
\int_{\Omega} (\mathbf{u}(\mathbf{x}) - 1) \rho \mathbf{d} \Omega - \frac{1}{K} \int_{\partial \Omega} \mathbf{q} \rho d\Omega, \quad \forall \mathbf{q} \in L^2(\Omega)
\]

(34)

forms a closed system for the mixed formulation.

### 3.2.2. Newton–Raphson procedure

Assuming that \( p \) can be approximated by a function of \( \mathbf{u} \) as in the linear case, in (33) \( p \) is nonlinear with respect to \( \mathbf{u} \), so we need to solve for \( \mathbf{u} \) iteratively. In our implementation we employed the Newton–Raphson iterative procedure. Let \( \mathbf{u}^k \) denote the solution at the \( k \)th iteration, this procedure can be written as follows.

Loop over \( k \) until convergence is reached for a pre-specified tolerance:

1. Based on the known information \( \mathbf{u}^k \), update the derivative of \( P \) with respect to \( \mathbf{u} \), which is denoted as \( \mathbf{D}P(\mathbf{u}^k) \).

2. Solve the following system of equations for the increment \( \mathbf{u} \) with a linear solver:

\[
\mathbf{D}P(\mathbf{u}^k) \Delta \mathbf{u} = -\mathbf{p}(\mathbf{u}^k), \quad \forall \mathbf{u} \in \mathbf{U}_0.
\]

3. Update the solution

\[
\mathbf{u}^{k+1} = \mathbf{u}^k + \Delta \mathbf{u}.
\]

(35)

Before spatial discretization, we write out the explicit form for the derivative:

\[
\mathbf{D}P(\mathbf{u}^k) \Delta \mathbf{u} = \left( \mathbf{D}p(t, \mathbf{u}^k, \mathbf{u}^k) \right) \Delta \mathbf{u} = \int_{\Omega} \frac{1}{2} \left( \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)^T \mathbf{F}(\mathbf{u}) + \mathbf{u}(\mathbf{x})^T \left( \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right) + \int_{\partial \Omega} \mathbf{T} \cdot \mathbf{n} d\Gamma - \int_{\Omega} \rho \mathbf{f} \cdot d\Omega, \quad \forall \mathbf{u} \in \mathbf{U}_0.
\]

(37)
where
\[
dp \frac{1}{2} k|C|^{-1} : \left( \left( \frac{\partial (\Delta u)}{\partial X} \right)^T F(u) + F(u)^T \left( \frac{\partial (\Delta u)}{\partial X} \right) \right),
\]
(38)
with \( C^{-1} \) denoting the inverse of the right Green deformation tensor \( C \).

Note that the external loads are deformation-independent under our assumption, thus \( \partial [\vec{u}|\vec{u}] = 0 \).

3.2.3. Local discretization

Similar to the linear elastostatics case, the discretization for the displacement formulation was detailed in [22]. For the mixed formulation, to discretize (35) in one element we expand \( \Delta u^e \) in terms of the shape functions as in (14):
\[
(\Delta u^e)(X) = \sum_{k=1}^{N_e} (\Delta u^e)_a \phi_k(X), \quad i = 1, 2, 3.
\]
(39)
Also, the local hydrostatic pressure \( p^e \) is expanded as in (15), while the increment of hydrostatic pressure \( dp^e \) is expanded as
\[
dp^e(X) = \sum_{k=1}^{N_e} dp^e \phi_k(X),
\]
(40)
where \( (\Delta u^e)_a \) and \( dp^e \) are the corresponding expansion coefficients.

3.2.4. Local pressure projection

Similarly as in Section 3.1.3, we carry out the projections on each element for both hydrostatic pressure \( p^e \) and its increment \( dp^e \). For the \( p^e \) part, we write the coefficients as in (16), and we minimize the function as in (17). So the minimization leads to (19) with the element mass matrix for pressure the same as in (20) and the right hand side of \( N_F \):
\[(K_u^{e-})_{h,m} = k \sum_{p=1}^{3} \sum_{q=1}^{N^e_p} \int_{\Omega^e} \left( \frac{\partial \phi^p}{\partial x^i} F_{m} + F_{m} \frac{\partial \phi^p}{\partial x^i} \right) \times J(C^{-1}) \phi^p_{a} (M^{-1}_u)^T_{a,m} \, d\Omega^e \]
\[= k \sum_{p=1}^{3} \sum_{q=1}^{N^e_p} \int_{\Omega^e} \left( \frac{\partial \phi^p}{\partial x^i} F_{m} + F_{m} \frac{\partial \phi^p}{\partial x^i} \right) \times J(C^{-1}) \phi^p_{a} (M^{-1}_u)^T_{a,m} \, d\Omega^e \]
\[= k \sum_{p=1}^{3} \sum_{q=1}^{N^e_p} \int_{\Omega^e} \left( \frac{\partial \phi^p}{\partial x^i} F_{m} + F_{m} \frac{\partial \phi^p}{\partial x^i} \right) \times J(C^{-1}) \phi^p_{a} (M^{-1}_u)^T_{a,m} \, d\Omega^e \]
\[= k \sum_{a=1}^{3} \left( K_u^e \right)_{a,m} ((M^e_u)^{-1} (K^e_u)^T)_{a,s} = k (K_u^e (M^e_u)^{-1} (K^e_u)^T)_{h,m} \]
\[= (54) \]

In the above expressions, \(F_i, S_i, S_i, C_{ijkl}, C_{ijkl}(C^{-1})_{ij}\) and \(X_i\) are the components for the deformation tensor \(F\), the distortional part of the second Piola-Kirchhoff tensor \(S\), the volumetric part \(S\), the distortional part of elasticity tensor \(C\), the volumetric part \(C\), the inverse of the right Green deformation tensor \(C\) and the position vector in initial configuration \(X\), respectively. Also, \(\delta_{ms}\) stands for the Kronecker delta. The * symbols for \(C_{ijkl}^{e-}\) and \(S_{ijkl}^{e-}\) in (53) means they are dependent on the hydrostatic pressure \(p^e\), while in these expressions the local hydrostatic pressure after projection \(p^{e-}\) is substituted in.

Regarding the residual vectors on each element, in (48) the right-hand-side \(R_u^e\) has length \(3N^e\), with the same ordering as \(\Delta U^e\). Similarly to what we did for \(p^e\), we can divide this vector into two contributions (here we denote its element as \((R_u^e)_{ij}\), \(i = 1, 2, 3\) and \(j = 1, \ldots, 3N^e\):
\[R_u^e = (R_u^{e\text{int}})^T + (R_u^{e\text{ext}})^T\]

where \((R_u^{e\text{int}})^T\) represents the contribution from the internal stress
\[(R_u^{e\text{int}})^T = \sum_{k=1}^{K-1} \int_{\Omega^e} \left( S_{ijkl} + S_{ijkl} \right) \frac{1}{2} \frac{\partial \phi^p}{\partial x^i} F_{m} + F_{m} \frac{\partial \phi^p}{\partial x^i} \, d\Omega^e\]

and \((R_u^{e\text{ext}})^T\) represents the contribution from the external loads
\[(R_u^{e\text{ext}})^T = \int_{\partial \Omega^e} T_{ijkl} \phi^p d\Gamma^s + \int_{\Omega^e} \rho \phi^p d\Omega^e\]

Here \(S\) is also marked with * in (56) because the hydrostatic pressure after projection \(p^{e-}\) is employed.

Correspondingly, the Newton–Raphson iterative procedure with pressure projection will be modified as follows.

Loop over \(k\) until the convergence tolerance is reached:

1. Based on the known information \(U^k\), following Section 3.2.4 to update for the projected hydrostatic pressure \(p^{e-}\).
2. Update the volumetric stress tensor \(S^e\) and volumetric elasticity tensor \(C^e\) based on (30) and (32).
3. Update the stiffness matrices (52)-(54) if needed. Update the internal right hand side as in (56).
4. Solve (48) for the increment \(\Delta U^e\) with a linear solver.
5. Update for displacement
\[U^{k+1} = U^k + \Delta U^e\]

4. Convergence studies

In this section, we investigate the accuracy of the pressure projection method in nearly incompressible problems by comparing simulation results against analytical solutions.

4.1. Linear elastostatics

We first investigate the convergence of the scheme for linear elastostatic problems and solve the equilibrium equations for infinitesimal deformation (12) using the pressure projection method. Three kinds of convergence \((p,h,h_p)\) are studied for solutions with and without singularities. For the material properties, we will use Young’s modulus \(E\) and the Poisson ratio \(v\), which are related with the material properties \(\lambda\) and \(\mu\) in (2) by
\[\mu = \frac{E}{2(1 + v)}, \quad \lambda = \frac{Ev}{(1 + v)(1 - 2v)}\]

4.1.1. Smooth solution: \(p\)-convergence

Consider the deformation of a cubic solid occupying the domain \(0 \leq x \leq 1, 0 \leq y \leq 1\) and \(0 \leq z \leq 1\) discretized by 8 hexahedral elements as in Fig. 1, where the thick solid lines mark the edges of the elements. For the test example, we use an analytic solution similar to [22], but with \(v = 0.49\), that is, which corresponds to nearly incompressible material. The displacements on the face \(x = 0\) are
\[u_x = 0, \quad u_y = B \sin by, \quad u_z = 0\]

where \(u_x, u_y, \) and \(u_z\) are the displacements in \(x, y,\) and \(z\) directions, respectively; also, \(B, b\) are prescribed constants. A traction force field, \(T = (T_x, T_y, T_z)\), is applied on the rest of the faces and is given by
\[T_x = n_x A, \quad T_y = n_y B \sin by + A \cos ax, \quad T_z = n_z (aA \cos ax + bB \cos by)\]

where \(n = (n_x, n_y, n_z)\) is the outward-pointing unit vector normal to the surface, and \(A, a\) are also prescribed constants. A body force field, \(\rho f = (f_x, f_y, f_z)\) is applied on the solid with components:
\[f_x = \frac{1}{v} aA \sin ax, \quad f_y = \frac{1}{v} bB \sin by, \quad f_z = 0\]

This problem has the following analytic solution for the displacement of the object:
\[\left\{ \begin{array}{l}
u_x = A \sin ax, \\
u_y = B \sin by, \\
u_z = 0 \end{array} \right.\]

so we see that the object is not totally incompressible \((p = i\lambda (aA \cos ax + bB \cos by))\).

Fig. 1. \(p\)-convergence test: cubic object discretized with 8 hexahedral elements.
In our test, we use the following parameters:
\[ A = B = 0.2, \quad a = b = 5.0, \quad E = 1000, \quad v = 0.49 \] (64)
to study the convergence behavior. We first vary the element order \( O_p \) for displacement (i.e., highest polynomial degree in the expansions) from 1 to 10, for various pressure orders \( O_u = 0, 1, 2, 3 \), as in the upper plots of Fig. 2. Then in the lower plots, for fixed displacement orders \( O_u = 4, 6, 8, 10 \) we show the results for different choices of pressure orders \( O_p \). To compute the numerical results with respect to the exact solution, for each element order pair \((O_u, O_p)\) we calculate the relative errors for pressure in the \( L^2 \) norm, and for displacement in the energy norm as follows
\[
e_p = \frac{\|P_r - P_i\|_2}{\|P_i\|_2}, \quad e_u = \frac{\|u_r - u_i\|_2}{\|u_i\|_e},
\] (65)
where \( P_r \) and \( u_r \) are the analytic solutions for pressure and displacement, respectively, while \( P_i \) and \( u_i \) are results from computation. We plot these errors in logarithmic scale, as a function of \( O_p \) or \( O_u \) in the lower plots which is in linear scale.

From the upper plots of Fig. 2, we see that for every fixed interpolation order \( O_p \) of pressure, the computational results do not converge to the analytic solution when \( O_u \) increases. On the other hand, in the lower plots, for each fixed \( O_u \) considered approximately straight lines are obtained in the range of \( O_u < O_p \), indicating that the numerical errors decrease exponentially fast as the element order \( O_p \) for pressure increases. So for slightly compressible problems, the element order \( O_p \) for pressure is more important for obtaining accurate results. Also, as shown in the upper right plot, for each fixed \( O_p \), the two cases with \( O_u = 0 \) and \( O_u = 0 + 1 \) give the best results for displacement. Combining these results with results of the upper left plot, which shows that \( O_u = O_p \) gives the largest pressure error, we can see that in this test \( O_u = O_p + 1 \) is the best choice for accuracy.

4.1.2. Smooth solution: \( h \)-convergence

Now we focus on the \( h \)-convergence on the problem tested in Section 4.1.1 using uniform tessellations with \( i \times i \times i \) (\( i = 1, 2, \ldots, 8 \)) hexahedral elements, while \( i = 2 \) was used for all cases in Section 4.1.1 (seeing Fig. 1). The displacement order \( O_u = 5 \) is employed here while runs with four choices of pressure orders \( O_p = 1, 3, 4, 5 \) are tested. Similar to the \( p \)-convergence tests, in Fig. 3 we calculate the relative errors of pressure and displacement in appropriate norms. We plot these errors in logarithmic scale as a function of element size \( h = \frac{1}{i} \) which is in inverse logarithmic scale.

From Fig. 3, we see that in most cases, a fixed element order \( O_p \) for pressure gives a \( O_p + 1 \) order accuracy for \( h \)-convergence, except when \( O_u = 04 \) which shows an \( O_u \) order. The results, especially those for the case of \( O_u = O_p - 2 \), are consistent with the analysis in [5,10]. Also, the right plot shows a 5.5 order accuracy with the choice \( O_p = O_u - 1 = 4 \) (green curve), which is the best convergence rate we obtain from \( h \)-refinement in this test.

4.1.3. Solution with singularities: \( p \)-convergence

As in the previous two sections, we consider the deformation of a cubic solid occupying the domain \( 0 \leq x \leq 1, 0 \leq y \leq 1 \) and \( 0 \leq z \leq 1 \), but discretized by \( i = 1, 2, \ldots, 7 \) layers of prismatic elements. In Fig. 4 we show the 1- and 3-layer meshes, with the thick solid lines marking the edges of the elements.

We use a similar analytic solution as in [2,3], which is singular at the line \( x = y = 0 \). The displacements on the faces \( x = 0, y = 0 \) and \( z = 0, 1 \) are known
\[
u_x = A\sqrt{X^2 + Y^2}, \quad u_y = B\sqrt{X^2 + Y^2}, \quad u_z = 0,
\] (66)
where \( A, B \) are prescribed constants. On face \( x = 1 \), a traction force field \( \mathbf{T} = (T_x, T_y, T_z) \) is applied as
\[
T_x = \frac{\lambda (AX - BY - \frac{\alpha x}{\pi})}{\sqrt{X^2 + Y^2}}; \quad T_y = \frac{\mu (BX + AY)}{\sqrt{X^2 + Y^2}}; \quad T_z = 0
\] (67)
and on face \( y = 1 \)
\[
T_z = \frac{\mu (DX + CY)}{\sqrt{X^2 + Y^2}}; \quad T_y = \frac{\lambda (AX - BY + \frac{\alpha y}{\pi})}{\sqrt{X^2 + Y^2}}; \quad T_z = 0.
\] (68)
The body force field \( \rho \mathbf{f} = (f_x, f_y, f_z) \) applied on the solid can be described as:

![Fig. 2. \( p \)-refinement test for the smooth problem in linear case; upper: relative errors as a function of the element order \( O_u \) for displacement; lower: relative errors as a function of the element order \( O_p \) for pressure; left column: relative errors for pressure; right column: relative errors for displacement.](image-url)
\[ f_x = \frac{\lambda(AX^2(2v-1) + 2AY^2(v-1) + BXY)}{2v(X^2 + Y^2)^{3/2}}, \]
\[ f_y = \frac{\lambda(2BX^2(v-1) + BY^2(2v-1) + AXY)}{2v(X^2 + Y^2)^{3/2}}, \]
\[ f_z = 0. \]  \hfill (69)

In this problem the displacement of the object has the following analytic solution:

\[
\begin{align*}
    u_x &= A\sqrt{X^2 + Y^2}, \\
    u_y &= B\sqrt{X^2 + Y^2}, \\
    u_z &= 0.
\end{align*}
\hfill (70)

We test this problem on 1-, 2- and 3-layer meshes, with the following parameters:

\[ A = B = 1.0, \quad E = 1000, \quad v = 0.49 \]  \hfill (71)

to study the convergence behavior. On each mesh, we systematically vary the element order \( O_u \) for displacement from 1 to 16. Based on our previous results of smooth solutions, all errors presented correspond to the (best) choice \( O_p = O_u - 1 \). In Fig. 5, we give the relative errors for pressure and displacement in logarithmic scale as a function of \( O_u \) which is also in logarithmic scale. We see that all results from the 1-layer mesh are approximately on a straight line with \( O_u \) increasing. Instead of the exponential convergence for smooth solutions as in Section 4.1.1, a second-order convergence rate is obtained. Although 2- and 3-layer meshes are refined near the singularity leading to smaller errors compared with what from the 1-layer mesh, when \( O_u \) becomes sufficiently large (say, larger than 3 in the right plot), they also show a second-order convergence. These results correspond to slightly higher convergence rates compared to the estimates in \([5,10]\), because the \( r^2 \) type of singularity is at an endpoint of the element, and this type of mesh gives a 2-\( \epsilon \) order accuracy for this case according to \([3]\).

4.1.4. Solution with singularities: \( hp \)-convergence

We now study the \( hp \)-convergence on the same problem introduced in Section 4.1.3. Similar as in \([2-4]\), we refine the meshes...
gradually as the element order increases. Namely, for increasing displacement order $O_p$, we use $i = O_u$ layers mesh in the test. In Fig. 6 we calculate the relative errors of pressure and displacement in appropriate norms, for $O_p = O_u - 2$, $O_p = O_u - 1$ and $O_p = O_u$. We plot these errors in logarithmic scale, as a function of element order $O_p$ for displacement (also the number of layers) which is in linear scale. We see that approximately straight lines are obtained for all cases, which indicates the exponential convergence, as expected from the analysis in [2,3]. Also, consistent with the previous sections, we obtain larger errors when $O_p = O_u - 2$, while the results from $O_p = O_u - 1$ and $O_p = O_u$ are similar with each other.

4.2. Nonlinear elastostatics

Next we investigate the convergence of the projection spectral/$hp$ element method for nonlinear elastostatic problems. We employ the Mooney–Rivlin model (3), and solve the equilibrium Eq. (33). We consider the same domain as in the linear case shown in Fig. 1, and both $h$- and $p$-convergence tests are performed for problems with smooth solutions.

4.2.1. Smooth solution: $p$-convergence

In this test example, we choose the material property parameters as

$$A_{10} = A_{01} = 1, \quad k = 100.$$ (72)

The face $X = 0$ is clamped, and the displacements on the rest of the faces are given as follows:

$$u_x = A \sin a X, \quad u_y = 0, \quad u_z = 0;$$ (73)

where $A, a$ are prescribed constants. A body force field, $\mathbf{f} = (f_x, f_y, f_z)$ is also applied on the object:

$$f_x = \frac{4Aa^2 \sin a X}{9(a A \cos a X + 1)^{10/3}} \left( (225(a A \cos a X)^3 + 625(a A \cos a X)^2 
+ 675a A \cos a X + 225) (a A \cos a X + 1)^{1/3} 
+ a A \cos a X (a A \cos a X + 1)^{1/3} - 1) (a A \cos a X + 2) + 6 
+ 6(a A \cos a X + 1)^{1/3} \right),$$ (74)

$$f_y = f_z = 0.$$

The displacements of the solid for this problem can be expressed by the following analytic functions:

$$\begin{align*}
    u_x &= A \sin a X, \\
    u_y &= 0, \\
    u_z &= 0.
\end{align*}$$ (75)

Fig. 6. $hp$-refinement test for the singular problem in linear case: relative errors as a function of the element order $O_p$ for displacement (which is equal to the number of mesh layers). Left: relative errors for pressure. Right: relative errors for displacement.

Fig. 7. $p$-refinement test for the smooth problem in nonlinear case: upper: relative errors as a function of the element order $O_u$ for displacement; lower: relative errors as a function of the element order $O_p$ for pressure; left column: relative errors for pressure; right column: relative errors for displacement.
and pressure \( p = k(a \cos \alpha X) \), with constants \( A = 0.04, a = 2.0 \). Similar as in the linear case, this problem is not totally incompressible.

Similar to the linear case, we study the \( p \)-convergence by two ways: first vary the element order \( O_u \) for displacement up to 7, with fixed orders of pressure \( O_p = 0, 1, 2, 3 \); then exchange the positions of \( O_u \) and \( O_p \), i.e., compare the results from different \( O_u \) (\( O_u = 7,6,5,4 \)) for increasing \( O_p \) in the range of \( 0 \leq O_p < O_u \). We calculate the relative errors of the computed displacement and pressure fields for each order element pair as in previous sections, see Fig. 7, these errors are plotted in logarithmic scale as a function of element order (in linear scale).

Consistent with the results from Section 4.1.1 for the linear case, we see that for each fixed \( O_u \), increasing \( O_p \) gives almost exponential convergence, while for each fixed \( O_p \) and sufficiently high \( O_u \), increasing \( O_u \) further does not help to decrease the errors. Also, among the cases in the upper plots, the best results are obtained by \( O_u = O_p + 1 \); this is also similar to Fig. 2 for the linear case.

### 4.2.2. Smooth solution: \( h \)-convergence

In this section, the same problem as in Section 4.2.1 is used, but with coefficients \( A = 0.01, a = 2.0 \) for stability. To study the \( h \)-convergence, we perform the tests on evenly discretized meshes with \( l \times l \times l \) \((l = 1,2,\ldots,8)\) hexahedral elements as previously for the linear case. With a fixed displacement order \( O_u = 4 \), results from different pressure orders \( O_p = 0,2,3,4 \) are studied. In Fig. 8 we show the relative errors of pressure and displacement, and plot these errors in logarithmic scale, as a function of element size \( h = \frac{1}{l} \) which is in inverse logarithmic scale. We see similar results as in Fig. 3 for the linear case: when \( O_p < O_u \), convergence rate of \( O_p + 1 \) order is obtained; when \( O_p = O_u \), only \( O_p \) order is given. Here results from \( O_p = O_u \) and \( O_p = O_u + 1 \) have similar convergence rates, while in the linear case the later gave a 1/2 order higher.

### 5. Test problems

In this section we perform several numerical tests in order to investigate the performance of the high-order projection method with respect to ellipticity condition, the \( \inf-sup \) condition, volumetric locking, and shear locking.

#### 5.1. Ellipticity condition and large deformation

As mentioned in [36,38], for the large deformation cases, sometimes the solver fails even when the element satisfies the \( \inf-sup \) condition because the ellipticity condition fails in the discrete problem (i.e., \((48)\)). Here, we will test the ellipticity condition following proposition 6 of [37].

With \( (\bar{u}, \bar{p}) \) denoting the initial conditions, and test function \( q \in L^2(\Omega) \), considering static and nearly incompressible problems for which \( k \gg 1, (34) \) can be approximated as

\[
\int_{\Omega} (J - 1)qd\Omega = 0
\]

and we linearize it, to obtain

\[
b(\Delta u, q) = \int_{\Omega} \frac{1}{2} \left( \begin{vmatrix} \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial^2 u}{\partial y^2} \end{vmatrix} - \begin{vmatrix} \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} \end{vmatrix} \right) : (C^{-1})(\bar{u})qd\Omega = 0,
\]

where \( \Delta u \) denotes the increment of \( u \) for each step in the Newton-Raphson iteration. In the continuous problem, by denoting \( \bar{U} \) and \( \bar{P} \) the admissible displacement and pressure spaces, the ellipticity in kernel is

\[
\text{inf} \frac{a(v,v)}{\|v\|^2} > 0,
\]

where

\[
K = \{ v \in \bar{U} \text{ and } b(v,q) = 0, \forall q \in \bar{P} \},
\]

\[
a(v,v) = \int_{\Omega} \frac{1}{2} \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial^2 u}{\partial y^2} \end{pmatrix} : \left( \begin{vmatrix} \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} \end{vmatrix} \right) \begin{pmatrix} \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} \end{vmatrix} d\Omega
\]

\[
+ \int_{\Omega} (S(u) + S(\bar{u}, \bar{p})) : \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial^2 u}{\partial y^2} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} \end{pmatrix} d\Omega.
\]

On the other hand, after discretization, the ellipticity in kernel is

\[
\text{inf} \frac{\bar{V}^T (K_0 + K_1) \bar{V}}{||V||^2} > 0.
\]
where
\[ \kappa_h = \{ \mathbf{V}^T K_h^2 \mathbf{V} = 0 \}. \] 

We consider two test examples, and we investigate the influence of different element orders for pressure on the ellipticity condition. From intuition, we can see that for the same element order of displacement, the smaller element order we use for pressure, the larger \( \kappa_h \) will be, and therefore the more likely is that the ellipticity condition will be violated. We will demonstrate the extent of this violation by obtaining the minimum eigenvalue of \( K_\mu(u_h, p_h) \) in \( \mathcal{K}_\lambda(u_h, p_h) \), where \( (u_h, p_h) \) is the projection of an analytic solution (see below) in the space formed by the shape functions. A similar strategy was employed in [36] for testing the ellipticity condition.

5.1.1. First numerical example

In our first example, the Mooney–Rivlin model is considered with the following material property coefficients:
\[ A_{10} = 2.0, \quad A_{01} = 0.2, \quad k = 1000 \] 
and the domain for initial configuration is as in Fig. 9. In this test, a prescribed displacement \( L \) is applied on \( X = 1: u_x = -L, \quad u_y = 0, \quad u_z = 0 \) to describe the extent of compression. For the faces \( Y = 0, X = 0, \) we clamp them in the normal direction. That is, on \( X = 0, \) we fix \( u_x = 0, \) and on \( Y = 0, \) we fix \( u_y = 0. \) To balance the system, on \( Y = 1 \) a traction force field is applied as
\[ T_x = T_z = 0, \quad T_y = 0.4(4L^2 - 16L + 27)L - 22 \] 

Here we employ a 3D code to obtain a 2D solution and hence we impose on \( Z = 0 \) and \( Z = 1 \) periodic boundary conditions, i.e., \( u_{Z=0} = u_{Z=1}. \)

Using a single hexahedral element as the mesh for this domain, we obtain the minimum eigenvalues of the stiffness matrix as in (81) which is evaluated at the projected exact solution \( (u, p) \). Next we study the minimum eigenvalues of \( K_\mu(u_h, p_h) \) in \( \mathcal{K}_\lambda(u_h, p_h) \) by systematically changing the element orders of displacement and pressure. Similar to what we did for the convergence rate tests, fixing the element order \( O_u \) of displacement \( (O_u = 2, 3, 4, 5) \), the minimum eigenvalue versus \( L \) for different element orders \( O_p \) of pressure \( (O_p = 0, \ldots, O_u) \) are plotted in Fig. 10: The horizontal axis denotes the displacement \( L \) on \( X = 1 \), which represents the extent of compression. The different lines correspond to different constant \( k \) that represents the element order combinations \( (O_u, O_p = O_u - k) \).

In Fig. 10, we see that for a fixed element order pair \( (O_u, O_p) \), as \( L \) increases, the minimum eigenvalues should decrease. That means, the larger the deformation is, the more severe the ellipticity is violated, which was also demonstrated in [37,38] for low-order elements. We also notice that for a fixed element order \( O_p \) for pressure, when \( O_u - O_p < 1 \), the minimum eigenvalues are much larger than for \( O_u \) satisfying \( O_u - O_p \geq 2 \). On the other hand, it is interesting that when \( O_u - O_p \geq 2 \), the minimum eigenvalues have almost the same magnitude independent of an increase in \( O_u \). For example, for \( L = 0.9 \), when \( O_u = 1 \) the minimum eigenvalue is \(-753.43 \) for \( O_p = 2, -814.11 \) for \( O_u = 3, -839.19 \) for \( O_u = 4, -841.80 \) for \( O_u = 5 \). Similarly, when \( O_u = 2 \) the minimum eigenvalue is \(-379.38 \) for \( O_p = 2, -358.43 \) for \( O_u = 3, -691.14 \) for \( O_u = 4, -733.14 \) for \( O_u = 5 \). These results suggest that for a fixed integer \( k \), the pair \( (O_u, O_u - k) \) should yield better performance as \( O_u \) increases. For example, for \( k = 2 \) at \( L = 0.9 \) as highlighted by the red circles, we obtain the minimum eigenvalue for \( (O_u, O_p) = (3,1), (4,2) \) and \( (5,3) \) as \(-814.11, -691.14 \) and \(-511.86 \), respectively. These results point to a particular trend, namely that for high-order \( O_u \) of displacement there are more values of the pressure order \( O_p \) that lead to simultaneous validity of the ellipticity condition and of the inf-sup condition.

5.1.2. Second numerical example

In this second example we will investigate the aforementioned finding by testing the stability of the numerical solver for different pairs of interpolation order. We consider the cubic object \( 0 < x < 1, 0 < y < 1, 0 < z < 1 \), with the mesh composed by a single hexahedral element. The Mooney–Rivlin model is specified by the following material property coefficients:
\[ A_{10} = A_{01} = 1.0, \quad k = 1000. \] 

For interpretation of color in Fig. 10, the reader is referred to the web version of this article.
For this case, the face \( X = 0 \) is clamped, and a traction force field \((T_x, T_y, T_z)\) is applied on the rest of the faces:

At \( X = 1 \):

\[
t_x = -\frac{2x^2}{25}, \quad t_y = 4x, \quad t_z = 0;
\]

at \( Y = 0 \) and \( Y = 1 \):

\[
t_x = n_y \left( \frac{4x^2}{3} + \frac{2x^3}{125} \right), \quad t_y = -n_x \frac{2x^2}{25}, \quad t_z = 0;
\]

at \( Z = 0 \) and \( Z = 1 \):

\[
t_x = t_y = t_z = 0,
\]

where \( n = (n_x, n_y, n_z) \) is the outward-pointing unit vector normal to the surface. The following body force \( \rho \mathbf{f} = (f_x, f_y, f_z) \), is also applied on the object:

\[
f_x = \frac{4x}{25}, \quad f_y = -\frac{4}{5}, \quad f_z = 0.
\]

This problem has the following analytic solution for the displacement of the object:

\[
ux = u_x = 0, \quad uy = 0.1x^2.
\]

For this test, the increment method (see Appendix B) is applied by separating the external load into \( N = 10 \) parts. In Table 1, we list the results of the stability study for different pairs of element order \((O_x, O_y)\) \((O_x \leq O_y \text{ and } O_x = 2, \ldots, 11)\); we leave the pairs with \( O_y > O_x \) as blank because they were not tested. Also, we mark with \( * \) the point where the range of choices for \( O_x \), which yield stable results, increases. We can see that for \( O_x \leq 5 \), we can only use \( O_y > O_x - 1 \) to maintain stability; when \( 6 \leq O_x \leq 10 \), the range is broadened to \( O_y \geq O_x - 2 \); and for \( O_x \geq 11 \), all \( O_y \geq O_x - 3 \) are appropriate to obtain numerical stability for this test problem.

### Table 1

<table>
<thead>
<tr>
<th>((O_x, O_y))</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\times)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
</tbody>
</table>

5.2. Inf-sup condition

In all the examples we presented so far (convergence and ellipticity tests) we obtained stable numerical results even when \( O_y = O_x \). To further investigate the inf-sup condition, we revisit here the problems defined in Sections 4.2.1 and 5.1.1 and perform further numerical tests with many non-hexahedral elements, including distorted 3D elements, and compare the accuracy of the results. In addition, we compute the inf-sup constant explicitly for different interpolation orders \((O_x, O_y)\).

For both test problems we use three types of meshes with different number of elements as in Fig. 11. Among these meshes, the prismatic elements might be especially problematic according to [44,45]. For the problem defined in Section 4.2.1 and for any pairs of \((O_x, O_y)\) we tested in the range of \( O_x \in \{1, 2, 3, 4, 5, 6, 7\} \) and \( O_y \leq O_x \) we did not observe any instabilities. The relative errors for the choices of \( O_y = O_x - 2, O_y = O_x - 1 \) and \( O_y = O_x \) are shown in Fig. 12. Although the magnitudes of these errors are different, all the lines are parallel to each other, indicating that we obtained the same convergence rate with either the \((O_x, O_y) = (O_x - 2, (O_x, O_y) = (O_x - 1) \) or the \((O_x, O_y) = (O_x)\) pairs. For the problem defined in Section 5.1.1 (see Fig. 9) and imposing a load \( L = 0.1 \), we found similar results. Also, following [46–48], we obtained the inf-sup constant

\[
\beta = \inf_{q \in P} \sup_{v \in V} \frac{b(v, q)}{||v|| ||q||} (92)
\]

for a mesh with one hexahedron, where \( b(v, q) \) is defined as in (77).

The results are shown in Fig. 13; we see that for \((O_x, O_y) = (O_x)\) cases, the constant \( \beta > 0 \) and is decreasing as \( O_x \) increases. However, for \((O_x, O_y) = (O_x - 1) \) the constant is at least an order of magnitude larger. By decreasing further the pressure interpolation order, i.e., for the pair \((O_x, O_y) = (O_x - 2)\) the results remain essentially the same as in the case \((O_x, O_y) = (O_x - 1)\).

5.3. Volumetric locking

For low-order elements the displacement-only formulation suffers from the volumetric locking behavior for problems with very high bulk modulus. This can be overcome by increasing the displacement order, i.e. typically for \( O_x \geq 4 \) [13]. Here, we compare the mixed formulation with the displacement formulation for a thick-walled tube benchmark problem, first proposed in [12], but here we are using a three-dimensional mesh. We use a neo-Hookean material model with \( A_{10} = 0.0; A_{11} = 0.5 \). Shown in Fig. 14, the thick-walled tube is under pressure load \( P_{\text{max}} = 0.5 \) MPa (applied along the radial direction); the tension on the outer surface is 0 MPa. The inner and outer radii are 10 mm and 30 mm, respectively. Along the \( Y \)-direction, the tube has length 30 mm. Therefore, on the inner and outer surfaces the boundary conditions are (here we use a follower load approach, see [40])

Fig. 11. Meshes for inf-sup condition test with example in 5.1.1: Left: skewed hexahedral elements (MESH1); Middle: prismatic elements (MESH2); Right: Many prismatic and distorted elements (MESH3).
On surface $X = 0$ and $Z = 0$, the displacements vanish along the normal directions. In other words,

$$u_r = 0 \quad \text{on} \quad X = 0; \quad u_z = 0 \quad \text{on} \quad Z = 0. \quad (94)$$

In order to describe a tube with infinite length, on surfaces $Y = 0$ and $Y = 30$, the periodic boundary condition is applied: $u_{|Y=0} = u_{|Y=30}$. All the displacements are set to 0 as the initial configuration. As in Fig. 14, the domain is decomposed into four hexahedral elements.

We test both mixed and displacement-only formulations for this problem, with displacement order $O_u = 2, 3, 4$ in the displacement formulation, and displacement/pressure order pairs $(O_u, O_p) = (2,1), (3,2), (4,3)$ in the mixed formulation; in addition we perform a simulation with $O_u = 2, O_p = 2$. The bulk modulus $k = 1, 10, 100, 1000, 10,000, 100,000$ are tested. The results for the maximum relative error of displacement along the radial $r$ direction (marked as $u_r$) at the inner boundary $r = 10$ mm are showed in Fig. 15. In this test, the increment method (see Appendix B) is applied by separating the external load into 10 parts.

From Fig. 15 we can see that using low-order elements for displacement, the displacement-only formulation suffers from volumetric locking problem, while the mixed formulation does not. When the order $O_u$ is increased to as high as 4, the displacement formulation overcomes the volumetric locking behavior. This result is in agreement with the findings of [13]. When the bulk modulus is larger than $10,000$ (MPa), the accuracy with mixed
formulation is more than an order of magnitude higher than that of the displacement formulation, reaching almost two-orders of magnitude higher accuracy for $k = 100,000$. It is interesting to also note that the mixed formulation case of $O_p = O_u = 2$ leads to volumetric locking identical to the displacement-only formulation for $O_u = 2$.

5.4. Shear locking

Next we investigate the performance of the high-order pressure projection method in shear locking with a numerical example very similar to [11]. It is known [11,21,49] that for thin plates, high-order elements can overcome the shear locking problem. Here we use a $5 \times 0.02 \times 5$ test mesh for the Mooney–Rivlin model. Based on our results for convergence rate tests, we simply use $O_p = O_u = 1$ and consider polynomial orders $O_u$ from 2 to 9.

In this example, we test the thin mesh by applying a traction $T_y = 0.00001$ at the end $X = 5$. We choose the material coefficients for the Mooney–Rivlin material according to [49], as follows: $A_{01} = 0.045$, $A_{10} = 0.177$, and since incompressibility does not effect the shear locking phenomenon, we use a smaller bulk modulus $k = 0.66666$. The numerical results are summarized in Fig. 16. We also compare the results for deformation length $L$ between our formulation and the commercial code ABAQUS [50] in Tables 2 and 3. In ABAQUS, we used 2500 incompatible-mode elements to overcome the shear locking problem. We take the results with quadratic elements in ABAQUS as the reference solution $L_s$ for the comparison of relative errors $e_R = \frac{|L - L_s|}{L_s}$.

From Fig. 16 and Tables 2 and 3, we can see that using low-order polynomial interpolation the deformation is much smaller than the accurate solution, which indicates that the shear locking phenomenon appears. However, by using higher order elements, the deformation is larger and it finally converges to the accurate solution. Hence, we obtain converged solutions for this problem for $O_u \geq 4$, which is consistent with the observations in [21]. We note that for the shear locking test we obtained similar results with either the $(O_u, O_p) = (2, 1)$ or the $(O_u, O_p) = (2, 2)$ pairs.

<table>
<thead>
<tr>
<th>$O_u$</th>
<th>$O_p$</th>
<th>$L_s$ (deformation on the end $x = 5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>3.80724</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>3.78787</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>3.79738</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>3.86840</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>3.99113</td>
</tr>
</tbody>
</table>

Table 3

Shear locking test: comparison of relative errors for $L$ between ABAQUS (ABA) with 2500 elements and our formulation (NEK) with one element.

<table>
<thead>
<tr>
<th>$O_u$</th>
<th>$O_p$</th>
<th>$L_s$ (deformation on the end $x = 5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>5.11369</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>9.00268</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>3.79738</td>
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<tr>
<td>5</td>
<td>4</td>
<td>3.86840</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>3.99113</td>
</tr>
</tbody>
</table>

Fig. 14. Volumetric locking test: mesh for thick-walled tube.

Fig. 15. Volumetric locking test: maximum relative error for $u_1$ at inner boundary $r = 10\text{ mm}$, with two formulations in thick-walled tube problem.

Fig. 16. Shear locking test: Mooney–Rivlin model.
6. Conclusion

We have presented a pressure projection method for nearly incompressible materials in solid mechanics using a general Jacobi polynomial basis that can be used to generate 3D polyomorph high-order elements. We investigated the convergence rates of the method for both smooth and singular solutions of linear and nonlinear elasticity problems, and numerically in some depth the issue of stability of the method for large deformations. We found that for linear problems the mixed formulation is stable even for elements with pressure interpolation order \( p = 0 \) equal to the displacement order \( k \), while \( p = k - 1 \) gives the best solution accuracy. On the other hand, to resolve the volumetric locking problem for hyperelastic materials, it is required that \( \frac{p}{k} = \frac{0}{k} = 1 \), which we obtained accuracies superior to displacement-only formulation, especially at high values of the bulk modulus, e.g., by almost two orders of magnitude. For large deformations we examined the stability of the method by considering the discrete ellipticity condition as has been suggested previously for low-order elements [37]. In this case, stability is enhanced by restricting the space, i.e., \( \mathcal{D}_{0} = \mathcal{D}_{0} = 0 \), where \( k \) depends on the specific value of \( \mathcal{D}_{0} \). For example, our numerical results suggest that for low values of \( \mathcal{D}_{0} \leq 5 \) we require \( k = 1 \); for intermediate values of \( \mathcal{D}_{0} = 10 \) we require \( k = 2 \), and for \( \mathcal{D}_{0} \geq 11 \) we require \( k = 3 \). These rather surprising results deserve further theoretical investigation, especially for understanding the interplay between the discrete ellipticity condition and the discrete inf-sup condition in large deformation analysis. From the practical standpoint, the present study suggests that the most robust choice of interpolation order in the mixed formulation is \( \mathcal{D}_{0} = \mathcal{D}_{0} = 1 \), leading to stable and accurate results for linear and nonlinear problems with large deformations, including resolution of the volumetric and shear locking phenomena.

Acknowledgments

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Appendix A. Shape functions

We employ the hierarchical shape functions based on Jacobi polynomials (see [1] for details). Firstly, we introduce three principal functions \( \psi_{1}^{a}(x) \), \( \psi_{1}^{b}(x) \) and \( \psi_{1}^{c}(x) \) (where the integers \( i, j, k \) are in the ranges \( 0 \leq i \leq l, 0 \leq j \leq m, 0 \leq k \leq K \), respectively), which are defined on the domain \( -1 < x < 1 \):

\[
\psi_{1}^{a}(x) = \begin{cases} \frac{1-x}{1-x^2}, & i = 0, \\ \frac{1-x}{1-x^2} P_{i-1}^{1,1}(x), & 1 \leq i \leq l, \\ \frac{1-x}{1-x^2}, & i = l, \end{cases}
\]

\[
\psi_{1}^{b}(x) = \begin{cases} \frac{1-x}{1-x^2}, & i = 0, \\ \frac{1-x}{1-x^2} P_{i-1}^{2,1}(x), & 1 \leq i \leq l, \\ \frac{1-x}{1-x^2}, & i = l, \end{cases}
\]

\[
\psi_{1}^{c}(x) = \begin{cases} \frac{1-x}{1-x^2}, & j = 0, \\ \frac{1-x}{1-x^2} P_{j-1}^{2,1}(x), & 1 \leq j \leq m, \\ \frac{1-x}{1-x^2}, & j = m, \end{cases}
\]

\[
\psi_{1}^{d}(x) = \begin{cases} \frac{1-x}{1-x^2}, & k = 0, \\ \frac{1-x}{1-x^2} P_{k-1}^{2,1}(x), & 1 \leq k \leq K, \end{cases}
\]

where \( P_{n}^{\alpha, \beta}(x) \) are the Jacobi polynomials.

Here we define the shape functions in three-dimensional space based on these three principal functions. Denote the coordinates of the standard domain as \( \xi_1, \xi_2, \xi_3 \), respectively. Then the hierarchical shape functions will be:

- For a hexahedral element \( [\xi_1, \xi_2, \xi_3] \), the shape function is defined by

\[
\psi_{ppr}(\xi_1, \xi_2, \xi_3) = \psi_{1}^{a}(\xi_1) \psi_{1}^{b}(\xi_2) \psi_{1}^{c}(\xi_3).
\]

- For a prismatic element \( [\xi_1, \xi_2, \xi_3] \), the shape function is defined by

\[
\psi_{ppr}(\xi_1, \xi_2, \xi_3) = \psi_{1}^{a}(\xi_1) \psi_{1}^{b}(\xi_2) \psi_{1}^{c}(\xi_3).
\]

where \( \eta_1 = \frac{2(1+\xi_1)}{\xi_1 - 1} - 1 \).

- For a tetrahedral element \( [\xi_1, \xi_2, \xi_3] \), the shape function is defined by

\[
\psi_{ppr}(\xi_1, \xi_2, \xi_3) = \psi_{1}^{a}(\eta_1) \psi_{1}^{b}(\eta_2) \psi_{1}^{c}(\xi_3).
\]

where \( \eta_1 = \frac{2(1+\xi_1)}{\xi_1 - 1} - 1, \eta_2 = \frac{2(1+\xi_2)}{\xi_2 - 1} - 1 \).

- For a pyramid element \( [\xi_1, \xi_2, \xi_3] \), the shape function is defined by

\[
\psi_{ppr}(\xi_1, \xi_2, \xi_3) = \psi_{1}^{a}(\eta_1) \psi_{1}^{b}(\eta_2) \psi_{1}^{c}(\xi_3).
\]

where \( \eta_1 = \frac{2(1+\xi_1)}{\xi_1 - 1} - 1, \eta_2 = \frac{2(1+\xi_2)}{\xi_2 - 1} - 1 \).

Appendix B. Increment method

The increment method is a widely used method in solving nonlinear elastic problem such as (28). By dividing the external load

\[
r^{\text{ext}} = \int_{\Delta t} \mathbf{T} \cdot \mathbf{v} \, \text{d}t + \int_{\Omega} \rho f \cdot \mathbf{v} \, \text{d}\Omega
\]

into \( N \) parts, one can enhance the stability in the Newton–Raphson iterative procedure while applying a large load directly. Take the displacement formulation, for example, (the increment method for mixed formulation can be derived similarly), this method can be rewritten as follows:

- With the initial condition of problem (28), solve

\[
P(\mathbf{w}_0) = \int_{\Omega} \mathbf{S}(\mathbf{w}_0) : \left( \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \right)^T \mathbf{F}(\mathbf{w}_0) + \mathbf{F}(\mathbf{w}_0)^T \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \right) \, \text{d}\Omega
\]

by Newton–Raphson iterative procedure, and get \( \mathbf{w}_1 \).

- For the ith step, take \( \mathbf{w}_{i-1} \), and its derivative as the initial condition, solve

\[
P(\mathbf{w}_i) = \int_{\Omega} \mathbf{S}(\mathbf{w}_i) : \left( \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \right)^T \mathbf{F}(\mathbf{w}_1) + \mathbf{F}(\mathbf{w}_1)^T \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \right) \, \text{d}\Omega
\]

by Newton–Raphson iterative procedure, and get \( \mathbf{w}_i \), until \( i = N \).

- Take \( \mathbf{w}_N \) as the final solution for problem (28).
References


[33] K.D. Papoulis, Fully nonlinear hyperelastic analysis of nearly incompressible solids: elements and material models in MSC/NASTRAN.