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Stability and accuracy of periodic flow solutions obtained by a POD-penalty method

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Abstract

We develop a new penalty method to derive low-dimensional Galerkin models for fluid flows with time-dependent boundary conditions. We then outline a procedure based on bifurcation analysis in selecting the proper values of the penalty parameter(s) that yield asymptotically stable periodic solutions of the highest possible accuracy. We illustrate this new approach by studying flow past a circular cylinder using direct numerical simulation (DNS) data, and a wave-structure interaction problem using particle image velocimetry (PIV) data.

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1. Introduction

Low-dimensional systems for unsteady fluid flows, based on the proper orthogonal decomposition (POD), have had mixed success in predicting the correct dynamics even at exactly the same set of parameters for which the POD modes were obtained. Specifically, an erroneous state may be obtained after long-time integration even if the correct periodic state is set to initialize the simulation—the numerical solution eventually drifts to a new erroneous state. This has also been observed in other systems, for example in the Kuramoto–Sivashinsky equations [1]. Empirical fixes based on artificial dissipation, e.g. [2], can only correct the dynamics in short-term integration, and more rigorous procedures need to be followed to guarantee asymptotic stability, e.g. see [3].
A distinction, however, should be made between autonomous low-dimensional dynamical flow systems and non-autonomous ones. We have observed, for example in many POD studies with diverse flow systems [4,5], that autonomous systems are more susceptible to this “drifting” while non-autonomous systems may reach asymptotically stable and accurate states without incorporating any special treatment. An example of a non-autonomous system is oscillatory flow past a circular cylinder. We have observed through accurate numerical integration of the corresponding POD system (for millions of time steps) that asymptotic stable and accurate states can be reached, at least for an external frequency close to the natural frequency of the system, i.e. the vortex shedding frequency in this case.

The implementation of complicated boundary conditions in Galerkin systems has historically been a matter of some controversy [6]; see also [7–9]. An in-depth study of boundary conditions for Galerkin POD systems was performed in [10]. Herein we introduce a penalty method, similar in spirit with the “tau” method in spectral methods [11], but more flexible in many aspects as we will see in this study. In particular, we will study two different systems based on data obtained from direct numerical simulations (DNS) and experimental results using particle image velocimetry (PIV) [12]. A new aspect of the current work is the use of the penalty parameter(s) as bifurcation parameter(s) in order to perform stability analysis using a standard package, e.g. AUTO [13].

Penalty methods have been used in the past successfully in implementing boundary conditions for different types of numerical discretizations. For example, finite difference schemes on complex geometries have been developed in [14] using a penalty method to impose Dirichlet boundary conditions. Also, a penalty method was developed in [15] to enforce boundary conditions for shock-free compressible Navier-Stokes simulations. A similar penalty method was used in imposing boundary vorticity constraints in [16]. In general, the penalty approach enforces the boundary conditions but also accounts for the governing equation at the boundary in a continuous manner, thus relaxing some of the numerical stiffness associated with very steep gradients at Dirichlet boundaries.

From the fundamental point of view, we pose the following question:

- Is there a range in the penalty parameter \( \tau \) for which the periodic solutions of the flow system are asymptotically stable, and are there any particular values of \( \tau \) for which the solution is most accurate.

For full numerical discretizations of any type, the accuracy of the solution scales as the inverse of the penalty parameter in well-resolved simulations. However, efficiency considerations require that a finite value for \( \tau \) must be used. As \( \tau \rightarrow \infty \) we have a strong imposition of the boundary condition while for small values of \( \tau \) we have a weak imposition of the boundary conditions. The question then is how do low-dimensional discrete systems differ from their “full-blown” counterparts in this respect? More importantly, should we impose the boundary conditions for low-dimensional systems in a strong form or in a weak form. Intuitively, we expect to have stability above a threshold value in \( \tau \) but that does not imply good accuracy in predicting the flow dynamics.

The above are some of the issues we address in the current work. The outline of the paper is as follows: in the next section, we describe the data sets based on which the POD-penalty system is derived. Then, the POD-penalty formulation is given in some detail. Subsequently, the results of the bifurcation study using the penalty parameter are presented for the two cases corresponding to DNS and experimental data. Finally, a summary and a brief discussion is included in the last section.

2. Data gathering

We construct POD modes based on data obtained both from DNS as well as from experiments. We report here on two prototype cases we have studied with the penalty method.
2.1. Direct numerical simulation

We consider flow past a circular cylinder for which both two- and three-dimensional POD models have been constructed in [17] and [18], respectively. In particular, for the concepts developed here, we consider two-dimensional time-dependent inflow past a circular cylinder at Reynolds number $Re = 100$ and 500. The inflow velocity is uniform but oscillatory in time, and is given by

$$U_\infty = (1.0 + A \sin(\omega t), 0).$$

The amplitude of the forcing term $A$, is kept the same at $A = 0.1$ for both Reynolds numbers. The forcing frequency is chosen so that we have a lock-in (resonant) state; this is done by choosing the frequency to be close to the Strouhal number which is 0.1667 for $Re = 100$ and 0.22 for $Re = 500$.

The computational domain is shown in Fig. 1. A time-dependent boundary condition is imposed at the inflow boundary $\Gamma_1$; periodicity is imposed on $\Gamma_3$ and $\Gamma_4$ while on $\Gamma_2$ the zero Neumann condition on velocity is imposed and the pressure is set to be zero. On the cylinder surface $\Gamma_5$ the no-slip boundary condition is prescribed. Converged solutions were obtained using the spectral/hp element method [19]. The domain is discretized into 412 triangular elements while seventh-order Jacobi polynomial basis are used to obtain resolution independent solutions.

2.2. Particle imaging velocimetry (PIV) experiment

Here we consider wave interaction with a vertical, surface-piercing cylinder, see Fig. 2. This flow gives rise to complex forms of wake structure due to the orbital particle trajectories of the incident wave, and the sweeping of previously generated vortices past the cylinder due to the oscillatory nature of the wave. The cylinder is fixed and the flow motion is sustained by the wave action. We will employ the experimental results obtained using particle image velocimetry (PIV) by Yang and Rockwell [12]. For this flow there are two important non-dimensional parameters that need to be specified: First, the Keulegan–Carpenter number defined by

$$KC = \frac{2nA_0}{D},$$

in which $A_0$ is either the displacement amplitude of the cylinder motion or the amplitude of the oscillatory flow and $D$ is the cylinder diameter. Secondly the Stokes number

$$\beta = \frac{fD^2}{\nu},$$

in which $f$ is the frequency of the motion and $\nu$ is the kinematic viscosity.
Quantitative images were obtained using a technique of high-image-density particle velocimetry (PIV). There are 13 phased-averaged snapshots available for a time period $T = 0.89$. These images will be used to extract the POD modes. Details of the experiments and the imaging approach are described in detail in [12]. A brief summary is given next.

The vertical, rigidly suspended cylinder, which is shown in the schematic of Fig. 2, was maintained stationary during all experiments. It had a diameter of $D = 12.7$ mm and a length of $L = 876$ mm. The submerged length of the cylinder was $700$ mm. The value of the Keulegan–Carpenter number were $KC = 2\pi A_o/D = 13.9$ at the depth of the laser sheet, which is indicated in Fig. 2 as $51$ mm beneath the quiescent free-surface. The amplitude $A_o$ corresponds to the radius of the particle orbit of the wave, which was also determined at the depth of $51$ mm. Furthermore, the value of the Stokes number was $\beta = fD^2/\nu = 164$ for this experiment. The corresponding value of Reynolds number is $Re = KC \times \beta = 2280$.

3. POD-penalty systems

In order to employ time-dependent boundary conditions in low-dimensional models, we formulate a new method to construct Galerkin systems. In particular, we incorporate the boundary conditions directly into the Navier–Stokes equations as constraints, enforced via suitable penalty parameters. In the next section we will demonstrate how to select the penalty parameters through bifurcation analysis in order to achieve asymptotically stable and accurate periodic solutions.
Here we employ the hierarchical POD modes as a trial basis to represent the velocity field. In addition, we employ a Galerkin projection of the Navier–Stokes equations onto these modes to derive dynamical systems to simulate the flow. Let us decompose the total flow field \( V \) as

\[
V(x, t) = U_0(x) + u(x, t),
\]

where \( U_0 \) is the time-averaged field. We express \( u \) as the linear combination of the POD modes as written in the summation conventions:

\[
u(x, y, t) = \phi_u(x, y) a_j(t), \quad v(x, y, t) = \phi_v(x, y) a_j(t),
\]

where \( a_j(t) \) are the unknown coefficients and \( \phi = (\phi_u, \phi_v) \) defines the vector of the POD modal basis.

In the following we derive separately the low-dimensional system for the DNS data and the experimental data.

### 3.1. DNS: POD-penalty system

The Galerkin projection of the Navier–Stokes equations with penalty terms included onto the \( j \)th POD mode is

\[
\int_{\Omega} \phi_j \left( \frac{\partial V}{\partial t} + (V \cdot \nabla)V + \nabla p - \frac{1}{Re} \nabla^2 V + \tau_1 \Upsilon(x)(V - U_{\infty}) \right) \, dx = 0,
\]

where \( \tau_1 \) is the penalty parameter and \( U_{\infty} \) is the imposed velocity at the inflow boundary \( \Gamma_1 \) (see Fig. 1). The function \( \Upsilon(x) \) is defined as

\[
\Upsilon(x) = \begin{cases} 
1, & \text{if } x \text{ on } \Gamma_1 \\
0, & \text{otherwise}.
\end{cases}
\]

We note here that on boundary \( \Gamma_1 \) we do not impose any boundary conditions as it is now treated as part of the interior domain. The treatment of the pressure term is of particular importance, so we analyze the corresponding Galerkin projection by using the Gauss’s theorem to obtain

\[
\int_{\Omega} \phi_j \nabla p \, dx = - \int_{\Omega} \nabla \phi_j p \, dx + \int_{\partial \Omega} \phi_j \cdot n p \, ds.
\]

Obtaining the POD modes from DNS of an incompressible flow field leads to divergence-free eigenmodes, and thus the pressure term inside the domain is eliminated (first term in the above equation). On the side boundaries \( \Gamma_3 \) and \( \Gamma_4 \) we assume periodicity and hence the pressure boundary terms cancel each other. On the outflow boundary \( \Gamma_2 \) the pressure is set to zero in the corresponding DNS. The inflow boundary \( \Gamma_1 \) should not be included in the computation of the second term of Eq. (4) since we have already included it in the Navier–Stokes equations. Therefore, there is no contribution from the pressure on this boundary in the integration by parts procedure. Finally, on the cylinder boundary the test function is zero and thus there is no pressure contribution there either.

In summary, the Galerkin projection leads to the dynamical system:

\[
\frac{da_j}{dt} = f_j(a) - G_j(a)
\]
with \(a = [a_1, a_2, \ldots, a_M]\), where \(M\) is the number of POD modes. The term \(f_j(a)\) includes the convective and viscous terms and has the form:

\[
f_j(a) = -\left(\int_{\Omega_1} \phi_j \cdot (\nabla \phi_j) \, dx\right) a_i \
- \left(\frac{1}{Re} \int_{\Omega_1} \phi_j \cdot \nabla^2 \phi_j \, dx + \int_{\Omega_1} \phi_j \cdot (U_0 \cdot \nabla) \phi_j \, dx + \int_{\Omega_1} \phi_j \cdot (U_0 \cdot \nabla) U_0 \phi_j \, dx\right) a_i \
- \left(\frac{1}{Re} \int_{\Omega_1} \phi_j \cdot (U_0 \cdot \nabla) U_0 \phi_j \, dx\right).
\]

Also, \(G_j(a)\) is the boundary penalty term, which is written as follows (in summation convention):

\[
G_j(a) = \tau_j \left(\int_{\Gamma_1} \phi_j(y) \phi_j(y) \, dy - \int_{\Gamma_2} (U_{\infty} - U_0(y)) \cdot \phi_j(y) \, dy\right).
\]

where \(\phi_j(y)\) means the function of \(y\) obtained by evaluating \(\phi_{a_i}\) on \(\Gamma_y\).

Since \(U_{\infty}(t)\) is time-dependent, we obtain a non-autonomous system. More details on the derivation of the above formulation for the Navier–Stokes equation are presented in [20].

3.2. Experiment: POD-penalty system

The POD-penalty system for the experimental data is derived similarly. By referring to Fig. 3, we now employ two penalty terms \(\tau_1\) and \(\tau_2\) for the boundaries \(\Gamma_1\) and \(\Gamma_2\), respectively. The Galerkin projection of the Navier–Stokes equations with penalty terms onto the \(j\)th POD mode is now

\[
\int_{\Omega_1} \phi_j \left(\frac{\partial (V \cdot \nabla V)}{\partial t} + \nabla p - \frac{1}{Re} \nabla^2 V + \tau_1 \nabla^2 \phi_j \phi_j + \tau_2 \nabla^2 \phi_j \phi_j \right) \, dx = 0,
\]

where the projection vector \(\phi_j\) is as previously defined, and \(U_{\infty}^{\Gamma_1}, U_{\infty}^{\Gamma_2}\) are the velocity vectors at the boundaries \(\Gamma_1\) and \(\Gamma_2\), respectively (see Fig. 3). The function \(Y(x)\) is defined as

\[
Y(x) = \begin{cases} 
1, & \text{if } x \text{ on } \Gamma_1 \\
0, & \text{otherwise.} 
\end{cases}
\]

Unlike the earlier DNS study where the flow is two-dimensional, in this case the true flow is three-dimensional but only a two-dimensional slice is visualized via PIV. Correspondingly, imposing the divergence-free condition on the two-dimensional POD modes is not appropriate. To this end, we will employ the divergent POD modes and let the penalty terms “counteract” the divergent contributions (first term of Eq. (4)); how accurate is this procedure will be tested by the results presented in the next section. Intuitively, it can be justified as the penalty term controls effectively the boundary mass fluxes (on \(\Gamma_1\) and \(\Gamma_2\)), and by adjusting the value of \(\tau_1\) and \(\tau_2\), respectively, we can counteract any mass sources or sinks due to the pressure contributions in the domain interior. The pressure contributions from the boundaries vanish due to periodicity and Dirichlet boundary conditions, similarly to DNS case. Regarding the representation of the time-dependent velocity boundary condition at \(\Gamma_1\) and \(\Gamma_2\), we have found that it is accurate to use a Fourier series with 16 Fourier modes to represent the time-periodic forcing at those boundaries. A systematic investigation of this has been presented in [5].

The Galerkin projection of the two-dimensional governing equations leads to the dynamical system:

\[
\frac{d a_j}{dt} = f_j(a) - G_j(a)
\]
with $a = [a_1, a_2, \ldots, a_M]$, where $M$ is the number of POD modes. The term $f_j(a)$ has the same form as in Eq. (6). The boundary penalty term $G_j^*(a)$ is modified as follows:

$$G_j^*(a) = \tau_1 \left( \int_{\Gamma_1} \Phi_i(y|\Gamma_1) \Phi_j(y|\Gamma_1) \, dy - \int_{\Gamma_1} (U_{\infty} - U_0(y|\Gamma_1)) \Phi_j(y|\Gamma_1) \, dy \right) + \tau_2 \left( \int_{\Gamma_2} \Phi_i(y|\Gamma_2) \Phi_j(y|\Gamma_2) \, dy - \int_{\Gamma_2} (U_{\infty} - U_0(y|\Gamma_2)) \Phi_j(y|\Gamma_2) \, dy \right)$$

### 3.3. Transformation to an autonomous system

In the next section we will show how to track periodic branches of the dynamical systems described by Eqs. (5) and (8). However, in order to effectively use the AUTO dynamical system package [13] to track the periodic branch, these non-autonomous systems need to be transformed to autonomous systems. To this end, we introduce the nonlinear oscillators

$$\frac{dp}{dt} = p + \hat{\beta}q - p(p^2 + q^2), \quad \frac{dq}{dt} = q - \hat{\beta}p - q(p^2 + q^2).$$

This particular system has an asymptotically stable solution given by

$$p(t) = \sin(\hat{\beta}t) \quad \text{and} \quad q(t) = \cos(\hat{\beta}t).$$

We then incorporate the nonlinear oscillator to the POD-penalty system in order to obtain an equivalent autonomous system.
3.3.1. DNS: POD-penalty autonomous system

We recall that in Eq. (6) the only time-dependent term is $U_{\omega_0}$, which is given by Eq. (1). The equivalent autonomous system is

$$ \frac{d a_j}{d t} = f_j(a) - G_j(a, p), \quad \frac{d p}{d t} = p + \omega q - \omega^2 p^2 + q^2, \quad \frac{d q}{d t} = q - \omega p - \omega q(p^2 + q^2), $$

where $a = [a_1, a_2, \ldots, a_M], f_j(a)$ is given by Eq. (6) and $G_j(a, p)$ is now defined as

$$ G_j(a, p) = \tau_1 \left\{ a_i \int_{y_1}^{y_2} \phi_j(y_{1|y}) \cdot \phi_j(y_{2|y}) dy - \int_{y_1}^{y_2} \left( (1 + A p, 0) - U_0(y_{1|y}) \right) \cdot \phi_j(y_{2|y}) dy \right\}. $$

Therefore, we have replaced the term $\sin(\omega t)$ in Eq. (1) with $p(t)$.

3.3.2. Experiment: POD-penalty autonomous system

The transformation of the non-autonomous system to an equivalent autonomous one for the case of POD-penalty system derived from experimental data is somewhat more complicated. For this POD-penalty system, we have the representation of the velocity vectors at the boundaries $\Gamma_1$ and $\Gamma_2$ in the form of Fourier series as

$$ U^{\omega_0}_{\omega_0}(y, t) = A^{\omega_0,0} + \sum_{n=1}^{N} \left[ A^{\omega_0,n} \cos \left( \frac{n \pi y}{\Gamma_1} \right) + B^{\omega_0,n} \sin \left( \frac{n \pi y}{\Gamma_1} \right) \right], $$

where $T$ is the period, $N$ the number of Fourier modes (for this case $N = 16$), $y$ a grid boundary point, and $i = 1$ or 2 for velocity vectors at the boundaries $\Gamma_1$ and $\Gamma_2$, respectively.

We then can transform Eq. (8) into an equivalent autonomous system as follows:

$$ \frac{d a_j}{d t} = f_j(a) - G_j(a, p, q), \quad \frac{d p}{d t} = p_a + \frac{n \pi}{T} p_a - p_a(p_a^2 + q_a^2), \quad \frac{d q}{d t} = q_a - \frac{n \pi}{T} p_a - q_a(p_a^2 + q_a^2), $$

where with $a = [a_1, a_2, \ldots, a_M], f_j(a)$ is the same as Eq. (6), $p = [p_1, p_2, \ldots, p_N], q = [q_1, q_2, \ldots, q_N]$ and $n = 1 \ldots N$. Note that here $\omega = n \pi T$.

Correspondingly, $G_j(a, p, q)$ is defined as

$$ G_j(a, p, q) = \tau_1 \left\{ a_i \int_{y_1}^{y_2} \phi_j(y_{1|y}) \cdot \phi_j(y_{2|y}) dy - \tau_1 \left( \sum_{k=1}^{n_1} A^{\omega_1,k}_{\omega_1} + \sum_{n=1}^{N} A^{\omega_0,n}_{\omega_1} q_n + B^{\omega_0,n}_{\omega_1} p_n - U_0(y_{1|y}) \right) \cdot \phi_j(y_{2|y}) dy \right\} $$

$$ + \tau_2 \left\{ a_i \int_{y_2}^{y_2} \phi_j(y_{2|y}) \cdot \phi_j(y_{2|y}) dy \right\} $$

$$ - \tau_2 \left( \sum_{k=1}^{n_2} A^{\omega_1,k}_{\omega_1} + \sum_{n=1}^{N} A^{\omega_0,n}_{\omega_1} q_n + B^{\omega_0,n}_{\omega_1} p_n - U_0(y_{1|y}) \right) \cdot \phi_j(y_{2|y}) dy \right\}. $$

Here $n_j$ is the number of grid points on $\Gamma_1$ and $\Gamma_2$, and $w_0$ is the weight for the trapezoid integration.
4. Results

4.1. DNS: Galerkin POD-penalty system

The Galerkin POD-penalty systems for Reynolds number $Re = 100$ and 500 are derived by employing 100 snapshots per period for both cases. We first present representative results of the stability of these solutions and subsequently we investigate their accuracy.

4.1.1. Stability of periodic solutions

Here, we study stability of the solutions of the Galerkin POD-penalty model through bifurcation analysis. We choose the bifurcation parameter to be the penalty constant $\tau_1$. In order to use the AUTO bifurcation tracking package, for this case, the asymptotically stable periodic solution must be provided—this of course is not known a priori. To this end, we assume a constant (typically large) value of $\tau_1$ and obtain the corresponding solution of the non-autonomous system. However, it is not certain that this solution will have the same period as the forcing period or even being periodic [21–23]. To overcome this, we will study the stability of the particular solution for that specific value of $\tau_1$ using Poincaré maps, following the work of [24–26]. Specifically, we have obtained the stability of the periodic solution using Poincaré maps, which we used to find the return times of the periodic solution, following procedures outlined in [24,25]. We then employed the algorithm in [26] in order to find the Floquet multipliers of the periodic solution.

Let us examine a specific case to illustrate this approach. We consider the low Reynolds number $Re = 100$ case with the number of POD modes $M = 6$ and integrate Eq. (5) for a few values of the penalty parameter, say in the range of $\tau_1 \in [2000, 3000]$. We found, through the Poincaré map, that in this range an asymptotically stable periodic solution does indeed exist. Hence, we can choose any value of $\tau_1 \in [2000, 3000]$ to apply AUTO in order to study stability of periodic solutions more systematically. We also found that the period of the asymptotic state is $T = 5.9988$ while the corresponding period from the full DNS is identical to this value. As we will show in the next section, agreement in the period does not imply agreement in the flow field dynamics between the DNS and the low-dimensional system.

Repeating this procedure with higher truncations at $M = 10$ and 20 at the same Reynolds number, we found similarly asymptotically stable periodic states, which can be used as starting points for the AUTO bifurcation analysis. A similar study was performed for the POD-penalty system at Reynolds number $Re = 500$ for $M = 6, 10$ and 20 POD modes. With the penalty parameter $\tau_1 = 3000$, the system posses an asymptotically stable solution with period $T = 4.5454$, which is in agreement with the results from the full DNS. Other large values of the penalty parameter yield similar results. After obtaining the asymptotically stable periodic solution, we use it as a starting point for AUTO, and track the stability of the periodic solution by decreasing $\tau_1$ until loss of stability is detected. This produces the lowest value of the penalty parameter that guarantees stability. Specifically, for all the systems examined here loss of stability shows bifurcation into a torus. The results of this analysis for both Reynolds numbers are listed in Table 1, where in the third column the minimum values of $\tau_1$ for stability are presented.

<table>
<thead>
<tr>
<th>$Re$</th>
<th>Modes</th>
<th>Lowest penalty parameter for stability</th>
<th>Best penalty parameter for accuracy</th>
<th>$\geq 2.5 \times 10^3$</th>
<th>$\geq 3.0 \times 10^3$</th>
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<tr>
<td>100</td>
<td>6</td>
<td>5.21427</td>
<td>5200</td>
<td>2000</td>
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</tr>
<tr>
<td></td>
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<td>5.23467</td>
<td>6000</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>3.20842</td>
<td>6500</td>
<td></td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>6</td>
<td>1.39258</td>
<td>9900</td>
<td>9000</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
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<td></td>
<td>$\geq 3.0 \times 10^3$</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.27168</td>
<td>$\geq 3.0 \times 10^3$</td>
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</tbody>
</table>
Fig. 4. Simulation using $M=10$ POD modes for $Re = 100$, $\tau_1 = 5 \times 10^5$. DNS: $\triangle$; POD-penalty simulation: solid line.

4.1.2. Accuracy of periodic solutions

We now turn our attention to the accuracy of the flow dynamics predicted by the low-dimensional system at different values of the $\tau_1$ parameter. In Fig. 4 we plot the phase portraits predicted by the POD-penalty system for $\tau_1 = 500,000$ against the DNS corresponding results for the system with truncation at $M = 10$. At this value of $\tau_1$ an asymptotically stable state is obtained with the correct time period but as can be observed in this plot the accuracy in the flow dynamics is poor. In order to improve this accuracy we first define a relative error for each penalty parameter by

$$E_{\tau_1} = \frac{\sqrt{\sum_{i=1}^{M} (a^*_i - Q^*_i)^2}}{\sqrt{\sum_{i=1}^{M} (Q^*_i)^2}}.$$

(10)

where $a^*_i$ and $Q^*_i$ are the maximum of the predicted POD modal coefficients corresponding to the low-dimensional system and DNS, respectively.

In Fig. 5 we plot the results of the bifurcation analysis for $Re = 100$ and $M = 10$ for the first four POD modes. We see that for the higher values of the penalty parameter $\tau_1$ the accuracy of $a^*_i$ compared to $Q^*_i$ is worse than for
the lower values of $\tau_1$. Similar results are also observed for the case of $Re = 100$ with $M = 6$ and 20, and also at $Re = 500$ with $M = 6$. However, this trend is not universal. For example, for the truncation $M = 20$ at $Re = 500$ (see results in Fig. 6), the agreement in the flow dynamics is good for large values of $\tau_1$ but there is a lower bound of the penalty parameter $\tau_1$ below which this agreement is lost. Similar results were obtained at $Re = 500$ with $M = 10$. Therefore, it is the combination of the penalty parameter and truncation parameter for certain complexity in the flow dynamics (here governed by the Reynolds number) that determines the quality of the prediction in the POD-penalty system.

In Fig. 7, the relative error, defined in Eq. (10), is plotted against the penalty parameter in order to determine the best penalty parameter for the case $Re = 100$ with $M = 10$. The best penalty parameter is found to be approximately 6000. A qualitatively different result is provided in Fig. 8, where the relative error for the case $Re = 500$ with...
Fig. 7. Left: relative error for the POD-penalty model with respect to DNS with $M = 10$ POD modes for $Re = 100$, and a close-up on the right.

$M = 20$ is plotted against the penalty parameter. Here we find that the error decreases monotonically with the penalty parameter but above some value, much greater than the stability bound, the accuracy saturates.

A summary of our studies to determine the best values of the penalty parameter $\tau_1$ for the best possible accuracy is presented in the last column of Table 1. Following the results of the bifurcation analysis, we then performed integration of the POD-penalty system using the parameters in Table 1. The corresponding results for two typical cases are shown in Fig. 9 for $Re = 100$ and Fig. 10 for $Re = 500$. For the latter case we observed the following: for the simulation of the POD-penalty system with $M = 10$, the system can predict correctly up to the fifth mode while the prediction of higher modes is erroneous. When we increase the number of modes in the system to $M = 20$, the accuracy of prediction of the dynamics is much better, i.e. good accuracy is now obtained up to 15th mode, see Fig. 11. We recall that these two cases correspond to a lower bound in the penalty parameter for best accuracy as determined by the bifurcation analysis. This finding could possibly suggest that for cases with a lower bound for the most effective penalty parameter a higher truncation is required to achieve better accuracy; it is not clear if this result will be true in other flow problems, however. We also note that integrating the POD-penalty Galerkin system

Fig. 8. Relative error for the POD-penalty model with respect to DNS with $M = 20$ POD modes for $Re = 500$, and a close up on the right.
requires approximately 10 periods to reach an asymptotically stable state even if the “exact” DNS conditions are used in the initialization process.

In summary, the lesson learned from the DNS study is that above a certain threshold in the value of penalty parameter, stability of the periodic solution is obtained. However, the best accuracy may be obtained for specific values of the penalty parameter that seem to depend strongly on the flow dynamics and the truncation in the POD-penalty low-dimensional system.

4.2. Experiment: Galerkin POD-penalty system

In this section, we will present the results for the POD-penalty for the experimental data. We have used two truncations in the number of modes here, $M = 6$ and $12$. The number of Fourier modes that represents the periodic time-dependent boundary condition is set to $N = 16$. For this POD-penalty model, there are two penalty parameters $\tau_1$ and $\tau_2$ that need to be specified. However, we will adopt here a procedure where we track these penalty parameters by fixing one of them at a point that the asymptotically stable periodic solution for the POD-penalty system is obtained. As in the previous study with DNS data, this asymptotically stable periodic solution might not be accurate
but it will be used as a starting point for AUTO to track the minimum value of either $\tau_1$ and $\tau_2$ for asymptotic stability of the periodic solutions.

4.2.1. Stability of periodic solutions

From preliminary numerical experiments for both $M = 6$ and $12$ we have determined that at $\tau_1 = 1000$ and $\tau_2 = 1000$ the corresponding POD-penalty systems possess an asymptotically stable solution; see Fig. 12 for these specific parameters. As in the previous case with the DNS data, the periodic solution is then studied through the Poincaré map to examine its stability. We have found that the periodic solution for these specific penalty parameters is indeed asymptotically stable with period of $T = 0.89$, which is in agreement with the data from PIV.

Having determined the starting point for AUTO, the tracking of stability of the periodic solution is then performed by fixing $\tau_1 = 1000$ and decreasing $\tau_2$ until loss of stability is detected. We then switch the role of $\tau_1$ and $\tau_2$ and perform an analogous analysis. This study produces the lowest values of the penalty parameter for stability, and the corresponding results are presented in Table 2. We have found that for both values of $M = 6$ and $12$ the periodic solution loses its stability when one of the Floquet multipliers crosses the unit circle at $-1$. This was also observed.
Fig. 11. Higher modes from the simulation using $M = 20$ POD modes for $Re = 500$, $\tau_1 = 4 \times 10^6$. DNS: $\triangle$, POD-penalty simulation: solid line.

for $M = 6$ with $\tau_2 = 1000$ fixed while varying $\tau_1$. However, in the case of the truncation with $M = 12$ the periodic solution loses its stability when one of the Floquet multipliers crosses the unit circle at 1.

4.2.2. Accuracy of periodic solutions

In order to find the most effective values of the penalty parameters that produces the most accurate asymptotically stable periodic solution compared to the data from PIV, we also track the periodic branch by increasing $\tau_1$ or

<table>
<thead>
<tr>
<th>Modes</th>
<th>Lowest penalty parameter for stability</th>
<th>Best penalty parameters for accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M = 6$</td>
<td>$\tau_1 = 1000$</td>
<td>$\tau_2 = 1000$</td>
</tr>
<tr>
<td>$M = 12$</td>
<td>$201.237$</td>
<td>$27.935$</td>
</tr>
<tr>
<td></td>
<td>$\geq 2 \times 10^7$</td>
<td>$\geq 5 \times 10^7$</td>
</tr>
</tbody>
</table>

* Corresponding asymptotically stable periodic solution has a relative error greater than 5%.
$(\tau_2)$ while fixing the other one at 1000. Then, we compute the relative error in order to find the best value of the penalty parameter for accuracy. In Fig. 13, the relative error is presented for the case of $M = 12$ with fixed $\tau_2 = 1000$. A summary of best values of the penalty parameter for all cases is presented in the last column of Table 2. Using the values from the table we then integrate in time the POD-penalty system to reach the asymptotic stable periodic states. The results of such simulations are presented in Figs. 14 and 15 as phase portraits. There is very good agreement with the corresponding experimental data with the higher truncation giving higher accuracy, as expected.

5. Summary

We have developed a Galerkin POD-penalty method to construct low-dimensional dynamical systems for unsteady fluid flows with time-dependent boundary conditions. Penalized boundaries are incorporated directly in the Galerkin statement of the Navier–Stokes equations, and thus information about the pressure field on such boundaries is not required. The resulting dynamical system is non-autonomous, so we couple it to an equivalent nonlinear oscillator in order to study its stability using standard bifurcation analysis.
Fig. 13. Relative error for the POD-penalty model from PIV data. Here, $M = 12$ with fixed $\tau_2$, and a close up on the right.

Fig. 14. Simulation using $M = 6$ POD modes for PIV data, $\tau_1 = 7.5 \times 10^4$, $\tau_2 = 1000$. PIV: $\triangle$; POD-penalty simulation: solid line.
We study the stability and accuracy of periodic solutions using the penalty parameter(s) as bifurcation parameter(s). We consider two prototype flows based on results from direct numerical simulations (flow past a circular cylinder), and from experiments (wave–structure interaction). The results from both studies are qualitatively similar.

We find that there is a threshold value of the penalty parameter above which asymptotic stability of the periodic solution is guaranteed. This is an expected result, similar to what is known for numerical discretizations of Navier–Stokes equations. The surprising, however, finding is that the accuracy of the solution predicted by the Galerkin POD-penalty system does not improve as the penalty parameter increases, as it is the case for full numerical discretizations. Instead, there is a specific range within which the solution is accurate. In particular, depending on the number of modes (i.e. truncation) and the flow complexity (i.e. Reynolds number) the best solution may correspond to a specific value of the penalty parameter or a range well above the threshold value for stability.

In numerical discretizations that employ the penalty approach to impose Dirichlet or other type of boundary conditions, as the penalty parameter approaches a very large number (e.g. inverse of machine precision) the boundary conditions are imposed exactly, i.e. in a strong form. Correspondingly, the error in the solution scales inversely proportional to the penalty parameter. Our findings here suggest that for low-dimensional systems, imposing the boundary conditions in a strong form may lead to an erroneous solution. Similar trends have been observed in spectral penalty methods for simulations of high Reynolds number turbulence at relatively low resolution [27]. This
can have great consequences in constructing effective low-dimensional dynamical systems as well as in formulating proper boundary conditions in large-eddy simulations. However, generalization of this conclusion to other flow problems has to be tested very carefully.

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