Stochastic modeling of random roughness in shock scattering problems: Theory and simulations

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Abstract

Random roughness is omnipresent in engineering applications and may often affect performance in unexpected way. Here, we employ synergistically stochastic simulations and second-order stochastic perturbation analysis to study supersonic flow past a wedge with random rough surface. The roughness (of length d) starting at the wedge apex is modeled as stochastic process (with zero mean and correlation length A) obtained from a new stochastic differential equation. A multi-element probabilistic collocation method (ME-PCM) on sparse grids is employed to solve the stochastic Euler equations while a WENO scheme is used to discretize the equations in two spatial dimensions. The perturbation analysis is used to verify the stochastic simulations and to provide insight for small values of A, where stochastic simulations become prohibitively expensive. We show that the random roughness enhances the lift and drag forces on the wedge beyond the rough region, and this enhancement is proportional to (d/A) 2. The effects become more pronounced as the Mach number increases. These results can be used in designing smart rough skins for airfoils for maximum lift enhancement at a minimum drag penalty.

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1. Introduction

Virtually any surface can be considered as rough under some fine-scale spatial resolution. Roughness poses naturally a multi-scale modeling problem as the characteristic large scale is often orders of magnitude greater than the roughness height - or correlation length A. Attempting to model roughness in fluid mechanics applications often leads to either over-simplified formulations or prohibitively expensive simulations, so no systematic numerical studies have been published to date. In particular, for supersonic flow past aerodynamic objects with random rough surface even experimental studies are very limited. An intriguing experimental finding published in the Russian literature [1] suggests that roughness enhances lift in airfoils; this was later confirmed by other experimental studies in USA [2] but the highest speeds tested were below the supersonic regime.

Supersonic flow past a smooth wedge is a classical aerodynamics problem, which has been studied extensively [3–7]. The shock path and pressure distribution can be obtained by simple analytical formulas [8]. However, complex shock dynamics is observed when considering a random rough wedge surface. Lighthill [9] and Chu [10] used first-order perturbation analysis to study weak interactions, whereby the shock wave is only slightly perturbed from its base configuration. The first-order theory is adequate only for very small roughness height and does not provide a measure of the mean extra forces induced by roughness since for zero mean height the first-order theory predicts zero mean forces. Here, we employ second-order stochastic perturbation theory coupled with stochastic numerical simulations to study the effect of large and fine random roughness on shock dynamics. The use of the perturbation analysis results is twofold: First, to properly verify the simulations results for small roughness height. Second, to cover the parameter space in the limit of very small values of the correlation length A for which the numerical simulations become prohibitively expensive.

Specifically, to deal with the random roughness, a stochastic mapping technique [11] is employed to transform the original governing equations defined on a random domain into stochastic differential equations defined on a deterministic domain. This allows us to employ well-developed theoretical techniques and recent numerical methods for solving stochastic differential equations in deterministic domains. In particular, a high-order probabilistic collocation method (PCM, [12]) is used to solve the stochastic Euler equations. PCM combines the strengths of Monte Carlo methods and stochastic Galerkin methods. By taking advantage of the existing theory on multivariate polynomial interpolations (see [13,14]), fast convergence is achieved using PCM, when the solutions possess sufficient smoothness in the random space. Additionally, implementation of PCM is straightforward, as it only requires
solutions of the corresponding deterministic problems at pre-selected sampling points. The choice of these sampling or collocation points is based on the sparse grid obtained from the Smolyak algorithm [15]. Sparse grids offer high-order accuracy with convergence rate not as strongly dependent on dimensionality. In the current paper, we extend the stochastic collocation method to a multi-element version (ME-PCM), which is computationally more attractive.

The paper is organized as follows: In the next section, we present the stochastic differential equation that models surface roughness. In Section 3, we give the analytical solutions of the perturbed forces for a full semi-infinite wedge derived from second-order stochastic perturbation analysis. In Section 4, we introduce the high-order stochastic collocation methods on sparse grids for the two-dimensional stochastic Euler equations and also discuss the stochastic mapping for random roughness. In Section 5, we present the analytical results from the second-order stochastic perturbation analysis and numerical simulation results. We conclude in Section 6 with a few remarks. We also include five Appendices (A–E) that provide more details on the analytical results.

2. Modeling random roughness

We denote the roughness length as $d$, and we normalize all length except the correlation length $A$ by the roughness length $d$. We describe the non-dimensional random roughness of correlation length $A$ as a non-dimensional stochastic process $\eta(x)$ through the Karhunen–Loeve (KL) decomposition [16]:

$$h_m(x ; \omega) = \mu(x) + \sum_{i} \sqrt{\lambda_i} \phi_i(x) \xi_i(\omega),$$

where $\mu(x)$ denotes the mean, $\{\xi_i(\omega)\}$ is a set of uncorrelated random variables with zero mean and unit variance, $\omega$ is a random event, and $x$ is the spatial coordinate. Also, $\phi_i(x)$ and $\lambda_i$ are the eigenfunctions and eigenvalues of the covariance kernel $R_m(x_1, x_2)$, respectively, obtained from

$$\int_{\Omega} R_m(x_1, x_2) \phi_i(x_2) dx_2 = \lambda_i \phi_i(x_1).$$

We assume that $\mu(x) = 0$, and the non-dimensional roughness height (distance from the smooth surface) is written as

$$y(x ; \omega) = \epsilon h(x ; \omega) = \epsilon h_m(x ; \omega),$$

where $\epsilon = \max_x (\epsilon h_m)$, $\epsilon$ represents the amplitude of the non-dimensional roughness height, and $h = \frac{h_m}{\epsilon}$ is a second-order stochastic process with zero mean and unit variance. Here $\sigma$ represents the standard deviation.

We can obtain the spatial covariance kernel $R_m(x_1, x_2)$ (following the procedure in [17]) based on the solution of a fourth-order differential equation with stochastic right-hand-side, of the form:

$$\frac{d^4 h_m}{dx^4} + k^4 h_m = f(x),$$

where $x$ is normalized by the roughness length $d$, $k = \frac{x}{d}$, and the random forcing term $f(x)$ is white noise satisfying $E[f(x_1) f(x_2)] = \sigma^2 (x_1 - x_2)$, where $E[\cdot]$ denotes the expectation. Here we consider the case of a finite strip of roughness starting from the apex of the wedge and having length $d$, see Fig. 1. The required boundary conditions for this case are: $h_m(0, \omega) = h_m(0, \omega) = h_m(1, \omega) = h_m(1, \omega) = 0$. The corresponding covariance is given in Appendix A. The eigenfunctions and eigenvalues can be obtained as solutions of the homogeneous equation $\frac{d^4 \psi}{dx^4} + k^4 \psi = 0$ with the boundary conditions $\psi(0) = \psi(1) = \psi(0) = \psi(1) = 0$. Such boundary conditions are chosen due to the assumption for second-order perturbation analysis, which assumes the random roughness and other perturbed quantities are small and smooth in the computational domain. The stochastic process $h_m(x ; \omega)$ can then be represented by the KL expansion

$$h_m(x ; \omega) = \sum_{n=1}^{N} \frac{1}{(\lambda_n^h + k^4)^{1/2}} \psi_n(x) \zeta_n(\omega),$$

where $\psi_n(x) = \cos A_n x - \cosh A_n x - \sin A_n x + \sinh A_n x$, $A_n$ is obtained by solving $\cos A_n \cosh A_n = 1$, and $\{\zeta_n(\omega)\}$ is a set of uncorrelated random variables with zero mean and unit variance. The stochastic perturbation analysis we develop can deal with random variables with different probability density functions. In the numerical results, we use primarily uniform random variables $\zeta_n \in [-\sqrt{3}, \sqrt{3}]$ and random variables with beta distributions: for $x = 1$ and $\beta = 1$, $\zeta_n \in [-\sqrt{2}, \sqrt{2}]$; for $x = 2$ and $\beta = 2$, $\zeta_n \in [-\sqrt{7}, \sqrt{7}]$; for $x = 5$ and $\beta = 5$, $\zeta_n \in [-\sqrt{13}, \sqrt{13}]$.

In order to investigate the effect of roughness granularity, we study three different non-dimensional correlation lengths $A/d = 1$, $A/d = 0.1$ and $A/d = 0.01$. These values determine the number of random dimensions that are required for accurate representation of the random roughness through the KL expansion. Here we are using the following criterion:

$$\sum_{n=1}^{N} \frac{A_n}{(\lambda_n^h + k^4)^{1/2}} \geq 90/\% \sum_{n=1}^{N} \frac{A_n}{(\lambda_n^h + k^4)^{1/2}},$$

based on which we choose the number of dimensions $N$. We arrived at this criterion after considerable testing. If the number of random dimensions is not sufficient, oscillations are observed for both the mean and the variance. For the results we present in this paper we have found that $N = 2$ is required for $A/d = 1$, $N = 12$ for $A/d = 0.1$ and $N = 60$ for $A/d = 0.01$.

3. Stochastic perturbation analysis

We consider the perturbation from the mean location of an oblique shock in supersonic flow past a rough half-wedge with a finite roughness strip of length $d$ while the rest of the wedge is smooth; a schematic of this problem and notation are shown in Fig. 1. We assume that: (1) The random wedge roughness is small, and correspondingly the perturbation of the shock slope is small. (2) The oblique shock is attached to the wedge. (3) The flow between the shock and the wedge is adiabatic. The domain of solution is between the perturbed shock and the location of the unperturbed shock corresponding to a smooth wedge surface.

![Fig. 1. Sketch of supersonic flow past a wedge with rough surface: Definition of coordinate system and notation; shown is also a perturbed shock path and the location of the unperturbed shock corresponding to a smooth wedge surface.](image-url)
to obtain the flow state after the shock [8]. In first-order perturbation
theory, the shock and wedge boundary conditions are imposed at the unperturbed shock location and at the smooth wedge sur-
face. However, in the second-order theory, these conditions are im-
posed at the perturbed surfaces in an iterative manner in order to
account for the correct perturbed shock location. We denote the
half-wedge angle by $\theta_0$, the unperturbed shock angle for smooth
wedge by $\chi_0$, the shock angle for rough wedge at location $x$
by $\chi(x; \omega)$, and the angle between $v_x$ and $u_t$ by $\theta(x; \omega)$ satisfying $\tan \theta = \frac{v_x}{u_t}$ where $v_x$ and $u_t$ are the velocity right after the shock perpen-
dicular and parallel to the wedge, respectively. For a smooth
wedge, $\chi$ and $\theta$ degenerate to $\chi_0$ and $\theta_0$. We denote the incoming
flow velocity by $w_1$ with its normal component $u_1 = w_1 \sin \chi$, and we also denote the velocity there by $w_2$ and its normal com-
ponent to the shock by $u_2 = w_2 \sin(\chi - \theta) = w_1 \cos \chi \tan(\chi - \theta)$. Here, $u_1, u_2$ are the normal components of the velocities going into
and coming out of the shock. Fig. 1 shows a sketch of a typical per-
turbed shock path induced by the randomly rough boundary.

Isentropic flow between the wedge and the shock is governed
by the steady Euler equation along with the isentropic condition:

$$
\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} + \rho c^2 \left( \frac{e}{c^2} \right) = 0,
$$

(6a)

$$
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \rho^2 \frac{1}{\gamma} = 0,
$$

(6b)

$$
\frac{\partial c}{\partial x} + \frac{\partial c}{\partial y} = 0,
$$

(6c)

$$
\frac{\partial q}{\partial x} + \frac{\partial q}{\partial y} = 0.
$$

(6d)

On the wedge surface ($y = h(x; \omega)$), the slip boundary condition
($\frac{\partial y}{\partial t} = \eta \frac{\partial}{\partial x}$) is employed, where $v_w$ and $u_w$ are the velocity perpendic-
ular and parallel to the wedge. Using the Rankine–Hugoniot relations
on the shock ($y = \chi(x)$) we obtain:

$$
\frac{p_w}{\rho_w} = \frac{1}{1 + \gamma} (M_1^2 \sin^2 \chi - 1),
$$

(7a)

$$
\frac{\rho_w}{\rho_1} = \frac{u_2}{u_1} \left( \frac{(\gamma - 1)M_1^2 \sin^2 \chi + 2}{(\gamma + 1)M_1^2 \sin^2 \chi} \right),
$$

(7b)

$$
\tan(\chi - \theta) = \tan \chi \left( \frac{(\gamma - 1)M_1^2 \sin^2 \chi + 2}{(\gamma + 1)M_1^2 \sin^2 \chi} \right) = T(M_1, \chi),
$$

(7c)

where the subscripts 1 and 2 correspond to the state going into
and coming out of the shock, respectively. Also, $M_1 = \frac{\sqrt{\gamma}}{c_1}$ and $c_2^2 = \frac{\gamma}{\gamma - 1}$.

3.1 First-order theory

The domain of solution is between the perturbed shock and the
wedge surface. We use the subscript ‘2’ for the flow state after
the shock and take the x-axis along the surface of the ‘unperturbed’
wedge. Let

$$
\frac{v}{w_2} = 1 + \epsilon' w, \quad \frac{v}{w_2} = \epsilon' v, \quad \frac{p}{w_2} = 1 + \epsilon' p, \quad \frac{\rho}{w_2} = 1 + \epsilon' \rho, \quad s = \frac{s}{w_2} = 1 + \epsilon' s.
$$

(8)

On the wedge surface, we have $v_w = \frac{\sqrt{\gamma}}{c_1}$ and using the Rankine–
Hugoniot relations (Eqs. 7a, 7b and 7c), we obtain the interface condi-
tions after the shock:

$$
\nu_i = \nu_f = F(M_1, \chi_0),
$$

(9)

where $F(M_1, \chi_0)$ and $G(M_1, \chi_0)$ are defined as

$$
F(M_1, \chi_0) = \frac{d\nu}{d\chi} = 1 - \frac{1}{\gamma M_1^2} \frac{c}{\gamma},
$$

(10a)

$$
G(M_1, \chi_0) = \frac{\sqrt{M_1^2 - 1} - M_1^2}{\gamma M_1^2} \frac{1}{\gamma M_1^2} \frac{1}{\gamma M_1^2} \frac{2\gamma M_1^2 \sin 2\chi_0 - \gamma + 2\gamma M_1^2 \sin^2 \chi_0}{},
$$

(10b)

where $T = \tan(\chi - \theta)$, $T_0 = \tan(\chi_0 - \theta_0)$, $M_1 = \frac{\sqrt{\gamma}}{c_1}$, $c_1^2 = \gamma \frac{\rho_1}{\rho_w}$, the ratio of specific heats is $\gamma = 1.4$ and $M_2 = \frac{M_1^2}{\gamma M_1^2} \frac{p_w}{\rho_w} \frac{\partial u}{\partial y}$.

All the flow variables after the shock are linearized in Eq. (8). To
quantify the region of validity for the stochastic perturbation anal-
ysis, we use the standard deviation of the stochastic roughness as a
measure. To simplify the problem with small perturbation, we have
employed the following assumption:

$$
\epsilon = \epsilon (x, y, \omega, \theta),
$$

(11)

where $\epsilon$ denotes the standard deviation. The region of validity with
respect to the roughness amplitude $\epsilon$ for the stochastic perturbation
analysis is discussed in Appendix C.

Substituting Eq. (8) into the steady Euler equations (Eqs. (6a)–
(6d)), we obtain the linearized small perturbation equations:

$$
\frac{\partial v'}{\partial y} + \frac{M_1^2 - 1}{\gamma M_2^2} \frac{\partial p'}{\partial x} = 0,
$$

(12a)

$$
\frac{\partial w'}{\partial y} + \frac{1}{\gamma M_2^2} \frac{\partial p'}{\partial x} = 0,
$$

(12b)

$$
\frac{\partial v'}{\partial y} + \frac{1}{\gamma M_2} \frac{\partial p'}{\partial y} = 0,
$$

(12c)

$$
\frac{\partial (p' - \gamma \rho')}{\partial x} = 0.
$$

(12d)

The normal component of $v'$ can be obtained by solving Eqs. (12a)
and (12b)

$$
\frac{\partial v'}{\partial y} = \frac{1}{\gamma M_2^2} \frac{\partial}{\partial y} \left( \frac{p'}{\gamma \rho'} \right) = 0
$$

(13a)

$$
\frac{\partial w'}{\partial y} + \frac{1}{\gamma M_2^2} \frac{\partial p'}{\partial x} = \text{constant, } s' = p' - \gamma \rho' = \text{constant.}
$$

(13b)

Then, $v'$, $p'$, $w'$, $\rho'$ and $s'$ can be obtained by solving Eqs. (13a) and (13b).

The variable $v'$ measures the deviation of a streamline from its
unperturbed direction, which is the direction of the wedge surface.
We denote by $\Delta \theta = \theta' - \theta$ and $\Delta x = x' - x$. From Eqs. (10a) and (10b), we obtain the boundary conditions after the shock for $v'$ and $p'$ as shown in Eq. (9). Thus, at the shock, Eq. (13a) becomes:

$$

v' = \frac{M_1^2 - 1}{\gamma M_2^2} \frac{p'}{\gamma \rho'} = \frac{F(M_1, \chi_0) + G(M_1, \chi_0)}{\gamma (x; \omega)},
$$

(14)

along $\frac{\theta}{\Delta x} = \frac{1}{\sqrt{\gamma M_2^2}}$. Eq. (14) will allow us to calculate the deviation
of the shock angle $\Delta x$ for any perturbation on the wedge surface
through the Riemann invariants $j$ and $s$. Applying the inviscid condition
that the normal velocity of the fluid is the same as the normal velocity
of the wedge surface, we have $v_w = s_w$. We denote the slope of the
characteristics $j = s - \frac{M_1^2 - 1}{\gamma M_2^2}$, where $m > T_0 = \tan(\chi_0 - \theta_0)$. Now applying Eq. (13a) between the two points $x$ and $x_1$, and between $x_1$ and $x_2$ (see Fig. 2), we obtain:

$$

v_w(x_1; \omega) + \frac{M_1^2 - 1}{\gamma M_2^2} p_w(x_1; \omega) = \frac{F(M_1, \chi_0) + G(M_1, \chi_0)}{\gamma (x; \omega)},
$$

(15a)

$$

v_w(x_1; \omega) - \frac{M_1^2 - 1}{\gamma M_2^2} p_w(x_1; \omega) = \frac{F(M_1, \chi_0) - G(M_1, \chi_0)}{\gamma (x; \omega)}.
$$

(15b)
First, we obtain a correction to the first-order solution to account for the correct shock path location instead of the mean location. Using an iterative procedure we start with \( z'(x; \omega) \) (Eq. (18)) and \( p'_w(x; \omega) \) (Eq. (17)), and upon convergence we obtain the corrected perturbed shock path \( z'(x; \omega) \) and corresponding corrected pressure \( \tilde{p}_w(x; \omega) \). To this end, we have to re-define \( \beta \) in Eq. (17), i.e. \( \beta = x_m/x \), where \( x_m, x \) are the distances from the apex of a reflection pair of points on the wedge surface \( (x_m < x) \). Let
\[
\frac{u}{W_2} = 1 + \Delta u, \quad \frac{v}{W_2} = \Delta v, \quad \frac{p}{p_z} = 1 + \Delta p, \quad \frac{p}{p_z} = 1 + \Delta p,
\]
\[
\frac{s}{S_2} = 1 + \Delta s,
\]
where \( \Delta w = \epsilon \tilde{W} + \epsilon^2 \tilde{W}^2 \), etc., are the total first- and second-order corrections. Substituting Eq. (20) into the steady Euler equations (Eqs. (6a)–(6d)), we obtain the second-order small perturbation equations:
\[
\frac{\partial}{\partial y} w + \tilde{M}_w^2 \frac{\partial \tilde{p}}{\partial x} = R_1(x, y), \quad \frac{\partial}{\partial x} w + \tilde{M}_w^2 \frac{\partial \tilde{p}}{\partial y} = R_2(x, y), \quad \frac{\partial}{\partial x} \left( \tilde{p} - v^2 \right) = R_3(x, y), \quad \frac{\partial}{\partial x} \left( \tilde{p} - v^2 \right) = R_4(x, y),
\]
where
\[
R_1(x, y) = -\frac{1}{\gamma \tilde{M}_w^2} (\tilde{M}_w^2 \tilde{p} + \tilde{p} + (\tilde{M}_w^2 + 1) \tilde{W}) \frac{\partial \tilde{p}}{\partial x} + \tilde{W} \frac{\partial \tilde{p}}{\partial x} + \left( \tilde{W} \frac{\partial \tilde{p}}{\partial x} - \frac{1}{\gamma \tilde{M}_w^2} \frac{\partial \tilde{p}}{\partial y} \right),
\]
and
\[
R_3(x, y) = \frac{1}{\gamma \tilde{M}_w^2} \left( \tilde{W} \frac{\partial \tilde{p}}{\partial y} + (\tilde{M}_w^2 + 1) \tilde{W} \frac{\partial \tilde{p}}{\partial y} \right).
\]
We note that the \( \tilde{p} \) denotes a converged first-order state that is corrected due to the shock path update. On the wedge surface, we have \( v'_w = \frac{\partial}{\partial y} v'_w = \tilde{W} \frac{\partial}{\partial y} \). Using the Rankine–Hugoniot relations (Eqs. (7a) and (7c)), we obtain \( p'_w \) and \( v'_w \) on the shock path based on the first-order corrected shock path,
\[
v'_w = F(M_1, \tilde{x}_o) \tilde{x}' + F_2(M_1, \tilde{x}_o) \tilde{x}^2, \quad \tilde{p}'_w = G(M_1, \tilde{x}_o) \tilde{x}' + G_2(M_1, \tilde{x}_o) \tilde{x}^2,
\]
where \( \tilde{z}' = \tilde{z}'(x; \omega) \), \( G_1(M_1, \tilde{x}_o) = \frac{20}{(\gamma + 1)^2 \tilde{M}_w^2 + 1} + \frac{20}{(\gamma + 1)^2 \tilde{M}_w^2 + 1} \), and \( F_2(M_1, \tilde{x}_o) = \frac{2}{(\gamma + 1)^2 \tilde{M}_w^2 + 1} + \frac{20}{(\gamma + 1)^2 \tilde{M}_w^2 + 1} \). Two linearized characteristic equations and the linear Bernoulli integral can be derived from the steady state Euler equations with isentropic condition, i.e.,
\[
\frac{\partial}{\partial x} w + \frac{\partial}{\partial y} \frac{w'}{\gamma \tilde{M}_w^2 + 1} = R_3 \frac{\partial}{\partial x} R_1, \quad \frac{\partial}{\partial x} w + \frac{\partial}{\partial y} \frac{w'}{\gamma \tilde{M}_w^2 + 1} = R_4 \frac{\partial}{\partial x} R_1
\]
and
\[
F_2(M_1, \tilde{x}_o) = \frac{20}{(\gamma + 1)^2 \tilde{M}_w^2 + 1} + \frac{20}{(\gamma + 1)^2 \tilde{M}_w^2 + 1}.
\]
From Eq. (22), we get
\[
v'_w = \frac{\gamma \tilde{M}_w^2 - 1}{\gamma \tilde{M}_w^2} \tilde{p}'_w = (F + G) \tilde{x}' + (F_2 + G_2) \tilde{x}^2,
\]
(24)
(25)
Hence, the perturbed shock path is obtained defined in Eq. (16a). Similarly, from Eqs. (25) and (27), we obtain the unperturbed shock path and forcing terms in the second-order perturbation equations and are non-dimensionalized by the shock mean position defined in Eq. (18). Adaptively update the shock location until the numerical error of the shock location is small enough.

Thus, we have

\[ z(x; \omega) = \bar{x}(x; \omega) + \epsilon \bar{z}(x; \omega) + \epsilon^2 \bar{z}^2(x; \omega), \]

where \( \bar{x} \) and \( \bar{z} \) are defined implicitly by Eq. (31), \( T_0 = \tan(\theta_0) \), \( \bar{x} \) is the corrected variable from \( x \) defined in Eq. (27), and \( \bar{z} \) is the unperturbed shock path and the perturbed shock path. Hence, the perturbed shock path is obtained

\[ \Delta L(x; \omega) = \epsilon \left( h \sin \theta_0 - \cos \theta_0 \int_0^x \tilde{p} w \, dx_1 \right) + \epsilon \left( \int_0^x \left( \tilde{p} w \sin \theta_0 - \tilde{p}^w \cos \theta_0 \right) \, dx_1 \right), \]  

\[ \Delta D(x; \omega) = \epsilon \left( h \cos \theta_0 + \sin \theta_0 \int_0^x \tilde{p} w \, dx_1 \right) + \epsilon \left( \int_0^x \left( \tilde{p} w \sin \theta_0 + \tilde{p}^w \sin \theta_0 \right) \, dx_1 \right). \]

Clearly, the perturbed lift has a mean value \( \propto \epsilon^2 \) whereas the corresponding standard deviation scales \( \propto \epsilon \). The first-order theory predicts zero mean perturbed lift since the assumed roughness has zero mean. We summarize here the results for the perturbed wall pressure distribution for the two extreme cases of correlation length \( A \):

\[ A / d \ll 1 : \quad E[\Delta p_w] \propto \epsilon^2 \left( \frac{\partial h}{\partial x} \right)^2 \propto \epsilon^2 \left( A / d \right)^{-2}, \]

\[ A / d \gg 1 : \quad E[\Delta p_w] \propto \epsilon \left( \frac{\partial h}{\partial x} \right) \propto \epsilon \left( A / d \right)^{-1}, \]

These results show a strong dependence of the perturbed wall pressure on the granularity of the roughness for small correlation length but no dependence at all for large correlation length. The perturbed lift force on the wedge follows similar scaling laws for the mean but the standard deviation is different. For example, in the small correlation limit the variance of the perturbed lift is independent of \( A / d \). In the next section we will examine the validity of these results in comparisons with stochastic numerical simulations; see also Appendix C.

The overall procedure to obtain the second-order perturbed variables is as follows:

1. Calculate the first-order perturbed shock path based on the shock mean position defined in Eq. (18). Adaptively update the shock location until the numerical error of the shock location is small enough.
2. Obtain the first-order perturbed flow variables based on the first-order adaptively updated shock location (the quantities).
3. Calculate the second-order perturbed flow variables.
4. Obtain the perturbed lift force by employing the first-order and the second-order perturbed flow variables.

We can readily extend our results to a full semi-infinite wedge, see Appendix B. In Appendices C, D and E we provide more results from the perturbation analysis related to the validity of such analysis, to Mach wave reflections at the shock surface and scaling laws, respectively.

4. Stochastic numerical methods

The classical polynomial Chaos represents a second-order stochastic process by a spectral expansion based on Hermite orthogonal polynomials in terms of Gaussian random variables. Its use in solving stochastic differential equations was pioneered by Ghanem and Spanos [18] who employed a Galerkin projection to derive an equivalent system of deterministic equations; this can, typically, be solved with standard numerical techniques. Xiu and Karniadakis [19] developed the generalized Polynomial Chaos (gPC), which employs a broader family of trial bases based on the orthogonal polynomials from the Askey scheme. A general second-order random process \( T(\omega) \) can be expressed by gPC as
\[ T(\omega) = \sum_{k=0}^{N} \vec{T}_k \phi_k(\vec{\xi}(\omega)), \]  

(34)

where \( \omega \) is the random event and the family \( \{ \phi_k \} \) is an orthogonal basis in terms of the random vector \( \vec{\xi}(\omega) \) with the following orthogonality relation

\[ \langle \phi_i, \phi_j \rangle = \langle \phi_i^T \rangle \delta_{ij}, \]

where \( \delta_{ij} \) is the Kronecker delta, and \( \langle \cdot \rangle \) denotes the ensemble average with respect to the probability density function (PDF) of \( \xi \). For a certain random vector \( \xi \), the gPC basis \( \{ \phi_i \} \) can be chosen in such a way that its weight function has the same form as the PDF of \( \xi \). For example, Gaussian random variables are associated with the Hermite polynomials, uniform random variables are associated with the Legendre polynomials, etc. An important aspect of the above chaos expansion is that the random processes are decomposed into a set of deterministic functions in the spatiotemporal variables multiplied by the random basis polynomials, which are independent of these variables.

4.1. Collocation projection

The probabilistic collocation method (PCM) was first introduced by Tatang in [20] and has been extended in [21,12] recently. We consider the stochastic equation \( L(\vec{x}, t; \vec{\xi}(\omega); u) = f(\vec{x}, t; \vec{\xi}(\omega)) \) with a general (nonlinear) differential operator \( L \), where \( \vec{x} \in \mathbb{R}^d \), \( d = 1, 2, 3 \), indicates the physical space and \( t \) the time. In contrast to Galerkin projection, in the collocation formulation we employ Delta functions \( \delta(\vec{\xi} - \vec{\zeta}_k) \) as test functions, \( k = 0, \ldots, M - 1 \), where \( \{ \vec{\zeta}_k \} \) is a proper set of grid (quadrature) points on the support of \( \vec{\xi}(\omega) \) and \( M \) is the number of grid points. By applying the collocation projection on both sides of the equation, we obtain:

\[ L(\vec{x}, t; \vec{\zeta}_k; u) = f(\vec{x}, t; \vec{\zeta}_k). \]  

(35)

Obviously, the resulting set of deterministic equations is uncoupled and Eq. (35) has the same form as the original equation. In particular, if the operator \( L \) is nonlinear, the Galerkin projection formulation leads to a system of ordinary or partial differential equations with the unknowns \( u \) being coupled. For complex fluid dynamical systems, e.g., compressible flows, it is more efficient to use the collocation projection to obtain the governing equations.

4.2. High dimensionality and sparse grids

Due to its highest degree of exactness, Gauss quadrature can be efficient for one-dimensional integration. However, the number of random dimensions increases very fast as the correlation length decreases and if the number of random dimensions is not adequate, erroneous oscillations appear for both the mean and the variance. In multi-dimensions, using grid sets based on tensor products of one-dimensional constructs leads to prohibitively large number of collocation points. In this work, we use the Smolyak algorithm [15], which is a linear combination of tensor product formulas and the resulting grid set has a significantly smaller number of grids compared to the full tensor product rule. Recently, Xiu and Hesthaven [12] constructed a PCM extension based on sparse grids using the Smolyak algorithm [15]. Sparse grids do not depend strongly on the dimensionality of the random space, and hence, it is more suitable for applications with large dimensional random inputs.

Let us consider the numerical integration of function \( f(\vec{x}) \) over the \( d \)-dimensional domain \( \Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_d \), where the \( \Gamma_i \) can possibly be unbounded. Smolyak’s algorithm is given by the linear combination of the tensor product polynomials in the following way

\[ A(q, d_r) = \sum_{q-d_1=0}^{d_r-1} \sum_{q-d_1=0}^{d_r} (-1)^{q-j} \binom{d_r-1}{q-j} \langle \psi^j \rangle \langle \psi^{d_r-j} \rangle, \]

(36)

where \( d_r \) is the number of random dimensions and the sparseness parameter \( q \geq d_r \) determines the order of the algorithm and \( |r| = l_1 + l_2 + \cdots + l_d \). Here \( \psi^j \) denotes a one-dimensional quadrature rule for each \( \Gamma_r \).

4.2.1. Nested sparse grids

The extreme points of the Chebyshev polynomials are employed by Clenshaw–Curtis formulas. The set of sparse grid points for \( \Gamma_r, d_r \) is nested, i.e., \( d_r < d_{r+1} \). The number of points is given by \( m_r = 2^{d_r+1} - 1, i \geq 2 \). These nodes are given as

\[ s_j = -\cos \left( \frac{\pi j}{m_r-1} \right), \quad s_k = 0, j = 1, \ldots, m_r \]  

(38)

and the corresponding weights are

\[ o_j = \frac{2}{m_r-1} \left( 1 + 2 \sum_{j=1}^{m_r/2} \frac{1}{1 - 4 \cos^2 \left( \frac{\pi j}{m_r} \right)} \right), \quad 2 \leq j \leq m_r - 1, \]

\[ o_j = o_{m_r} = \frac{1}{m_r(m_r-2)}. \]

(39)

where \( \sum^r \) represents that the last term of the sum is halved.

To simulate the supersonic flow past a rough wedge problem, we employ PCM on sparse grids based on the Smolyak algorithm [15], which results in significantly less number of collocation points. Specifically, for correlation length \( A/d = 0.1 \), which requires 12 dimensions, the number of grid points is 313 instead of 531,441 grid points for full tensor products. Convergence with the sparse grids is achieved by increasing the level parameter of each sparse grid. An adaptive procedure was employed to guarantee that the numerical stochastic error is smaller than a specified threshold by comparing the difference between two successive sparse grid levels.

4.2.2. Non-nested sparse grids

An alternative way to generate sparse nodal sets is to use Gaussian quadrature points by Gaussian formulas. By choosing \( 2^{-1} + 1 \) number of Gaussian quadrature points, a \( 2^{d} \) degree of exactness can be achieved. The nested property may be lost, which makes the nodal set larger, compared with the nested nodal set. However, for random inputs with arbitrary PDFs, the sparse quadrature rule can be constructed in such a way that the weight function coincides with the PDF. Gaussian quadrature nodal sets for arbitrary weight functions can be also obtained in an efficient way, see [22].

Although sparse grid nodal set has significantly small number of points, the number of sparse grid points between different levels also increases very fast, especially for high dimensions. For random roughness described by 12 random dimensions, the number of sparse grid points for level 2 is 313 but the points for level 3 is 2649. Another option to improve resolution is to use multi-element probabilistic collocation method (ME-PCM), that is to decompose the random space into finite elements to achieve h-refinement in random dimensions. ME-PCM is especially useful for such problems with low regularity in the parametric space or with random frequencies.
4.3. Multi-element probabilistic collocation method (ME-PCM)

Based on gPC, the multi-element generalized polynomial chaos (ME-gPC) with Galerkin projection was first presented in [23]. ME-PCM is an extension of ME-gPC with collocation projection, which decomposes the random space into finite elements as in the deterministic finite element method. To illustrate this idea here we consider the one-dimensional case. We assume that the support \([a, b]\) of the one-dimensional random variable \(\zeta\) is decomposed into elements \(\varepsilon_k := [a_k, b_k]\). We then define a new random variable \(\varsigma_k, k = 1, 2, \ldots, N_e\), in each random element, \(\varepsilon_k\), as
\[
\varsigma_k = \frac{b_k - a_k}{2} \zeta_k + \frac{b_k + a_k}{2},
\]
with a re-scaled PDF
\[
f_k(\varsigma_k) = \frac{f(\zeta_k)}{\text{Pr}(\zeta_k \in \varepsilon_k)} \frac{b_k - a_k}{2}, \quad k = 1, \ldots, N_e,
\]
where \(N_e\) is the number of random elements and \(f(\zeta)\) is the PDF of \(\zeta\), \(\text{Pr}(\zeta_k \in \varepsilon_k) = \int_{\varepsilon_k} f(\zeta)\,d\zeta\) is the probability that \(\zeta\) is located in random element \(\varepsilon_k\). Such decomposition of the random space can be easily extended to multi-random-dimensions. Specifically, \(\varsigma_k\) is mapped to the space \([-1, 1]^n\). The desired random field \(u(\zeta)\) is approximated locally within each random element by ME-PCM, where the degree of perturbation is effectively decreased by the above linear transform from \(O(1)\) to \(O(\frac{1}{\varsigma})\). Subsequently, we gather the information from all random elements with appropriate weight to obtain the statistics of \(u(\varsigma)\). Compared with PCM, ME-PCM is especially favorable for such problems with discontinuities in the random space or long-time integration. The general procedure is as follows:

1. Obtain the sparse nodal set and corresponding weights for each random element.
2. Evaluate the solution \(u(x; \tau)\) at each point \(\tau\) using the deterministic solver.
3. Calculate the moments of the local solution \(u_k\),
\[
E[u_k(x)] = \sum_{i \in \Omega} u_i(x) c^i(\tau),
\]
where \(c^i\) are the weights corresponding to \(v_k\).
4. Calculate the moments of the global solution,
\[
E[u(x)] = \sum_{k=1}^{N_e} \text{Pr}(\xi \in \varepsilon_k) E[u_k(x)].
\]

In this paper, ME-PCM is employed to solve supersonic flow past a wedge with large random roughness described as a random process with uniform random variables. The \(d\)-dimensional random vector has constant PDF, \((\xi)^d\). Thus, there is no need to construct a new orthogonal basis on-the-fly for each element. In the current work we will employ both nested sparse nodal sets and non-nested Gaussian sparse nodal sets.

4.4. Mapping of the random domain

We solve the Euler equations using a boundary-fitted domain that involves a transformation based on a random Jacobian. Specifically, as shown in Fig. 3, the stochastic mapping of the random physical domain \(D(x, y; \omega)\) onto a computational domain \(E(\varsigma_1, \varsigma_2)\) can be defined from the solutions of
\[
\frac{\partial^2 y}{\partial \varsigma_1^2} + \frac{\partial^2 y}{\partial \varsigma_2^2} = 0,
\]
where we set \(\varsigma_1 = x\), see details in [18,11,19]. We represent the surface roughness by a KL decomposition, which here acts as a boundary condition. The transformed stochastic boundary value problem in the fixed stochastic rough domain \(E\) can be solved by a variety of techniques. As the correlation length of the random roughness decreases, the spatial resolution has to be increase proportionally to capture the smallest scale of the random roughness. Due to the decrease of the computational grid size \(Ax\), the time step \(\Delta t\) also has to be sufficiently small to satisfy the CFL condition. Therefore, the computational cost will dramatically increase by decreasing the correlation length of the random roughness.

5. Results and discussion

In the simulations, we employ a fifth-order weighted essentially non-oscillatory (WENO) scheme [24] for spatial discretization with 1000 \(\times\) 1000 grid points in the domain \([0.6] \times [0.4]\). The time step \(dt\) is \(10^{-3}\) and steady state is achieved by time-marching. The stochastic simulations are based on a multi-element probabilistic collocation method (ME-PCM) and multi-dimensional integration using sparse grids. We now present results based on perturbation analysis and on stochastic numerical simulations for the following conditions: The semi-infinite wedge, see Fig. 1, is truncated after 6d while the rough region is \([0.1d]\); the angle of the wedge is \(\theta_0 = 14.7436\), the angle of the unperturbed shock for Mach number.
2 is $\chi_0 = 45^\circ$ and the angle of the unperturbed shock for Mach number $M_1 = 8$ is $\chi_0 = 20.5755^\circ$. (in the following we will take $d = 1$). We consider two values of the inflow Mach number, i.e., $M_1 = 2$ and 8. The non-dimensional sound speed is $C_1 = 1$ and the pressure is $p_1 = 1$ at inflow. For the smooth wedge shock problem, the region after the oblique shock is uniform. All the physical quantities can be obtained from the Rankine–Hugoniot relations, hence the outflow boundary conditions can be set up accurately.

The standard deviations of the non-dimensional perturbed lift and drag forces along the wedge surface for inflow $M_1 = 2$ and 8 are presented in Figs. 4 and 5, respectively. For roughness height $\epsilon = 0.003$ and correlation length $A/d = 0.1$ the numerical results based on the full solution of the stochastic nonlinear Euler equations agree well with the second-order stochastic perturbation solutions (denoted as “Analytical Soln” in the plot). For small $\epsilon$, the standard deviation of the perturbed lift and drag forces scales $\propto \epsilon$, which has been verified by both the second-order stochastic perturbation analysis and numerical simulations. As the correlation length $A/d$ decreases, a sharp discontinuity is observed at $x = 0$ and 1. The random roughness at each point between 0 and 1 can be approximately treated as an independent random variable with the same variance for very small $A/d$, which explains why the standard deviation of perturbed forces for small $A/d$ has almost constant value for $x \in (0, 1)$. Since $\sigma(h(0; \epsilon)) = \sigma(h(1; \epsilon)) = 0$ and $r \ll 1$, the standard deviations of the perturbed lift and drag are close to zero at $x = 0$ and $x = 1$. Another discontinuity can also be observed at $x = 4.23$ for $M_1 = 2$, as shown in Fig. 4 and at $x = 2.84$ for $M_1 = 8$, as shown in Fig. 5; this corresponds to the boundary of the first reflection region as predicted by the stochastic perturbation analysis.

Comparing Figs. 4 and 5, we observe that the standard deviation of the perturbed lift is greater than the perturbed drag for both $M_1 = 2$ and 8. Beyond the rough region, the standard deviation of the perturbed lift and drag contributed by the reflection is much smaller than the one within the rough region. In Fig. 4 different $A/d$ has a maximum value of the standard deviation of the perturbed lift and drag forces, while in Fig. 5 smaller $A/d$ has larger maximum value. The stochastic perturbation analysis can provide more insight into the lift and drag distribution. Note that the specific values of reflection index $r$ for $M_1 = 2$ and 8 are $r_{M_1=2} = 0.0657$ and $r_{M_1=8} = -0.01098$, respectively. Since $r$ is about 1$\%$ for $M_1 = 2$, the non-dimensional perturbed pressure $p_u$ can be approximated as $p_u \approx \frac{M_1^2}{\sqrt{M_1^2 - 1}}$. Thus, the standard deviation of the perturbed lift and drag forces can be expressed approximately as $\sigma(\Delta L)/\epsilon \approx \sigma(h)(\sin \theta_0 - \frac{M_1^2}{\sqrt{M_1^2 - 1}} \cos \theta_0)$ and $\sigma(\Delta D)/\epsilon \approx \sigma(h)(\cos \theta_0 + \frac{M_1^2}{\sqrt{M_1^2 - 1}} \sin \theta_0)$.

Fig. 4. Standard deviation of the non-dimensional perturbed (a) lift and (b) drag forces along the wedge surface for inflow $M_1 = 2$.

Fig. 5. Standard deviation of the non-dimensional perturbed (a) lift and (b) drag forces along the wedge surface for inflow $M_1 = 8$. 
Since the maximum of $\sigma(h)$ is 1 and does not depend on the correlation length $A/d$, the standard deviations of the perturbed lift and drag forces have the same maximum value for $M_1 = 2$. For $M_1 = 8$, the reflection index $r$ is much larger than the one for $M_1 = 2$ and smaller $A/d$ has larger contribution to the standard deviation of the perturbed lift and drag forces, which explains why smaller $A/d$ corresponds to larger maximum value of the standard deviation of the perturbed lift and drag forces.

In Figs. 6 and 7, we investigate convergence on sparse grids for the perturbed mean lift and drag along the wedge surface for inflow $M_1 = 2$ and 8, respectively. For $A/d = 1$, fast convergence is observed for both the means of the perturbed lift and drag forces with inflow $M_1 = 2$ and 8. For fine roughness (smaller correlation length, e.g., $A/d = 0.1$), a larger number of sparse grid points is necessary to achieve convergence, as expected.

In Figs. 8 and 9 we present convergence results with respect to the number of elements in the random space ($N_e$). Convergence can be achieved by either increasing the sparse grid level or by increasing the number of random elements $N_e$. For correlation length $A/d = 0.1$, $N = 12$ random dimensions are used. Using the sparse grids, there are 25 sparse grid points for level 1, 313 points for level 2, and 2649 points for level 3; the latter is very expensive for large-scale simulations. ME-PCM can be employed in such case, with relative lower computational cost. For example, ME-PCM with two random elements ($N_e = 2$) is performed and only 626 sparse grid points are used to achieve accurate results.

We now examine what effect the type of distribution has on our results. From perturbation analysis, we know that for small random roughness the mean solution does not depend on the distribution of the random variables. However, for large roughness, the mean perturbed lift force may depend on the distribution. In Fig. 10, we compare the mean perturbed lift force with uniform and beta distributions ($\beta = 2$ and 5). Larger mean perturbed lift force is observed for random variables with uniform distribution. We can readily extend our results to a full semi-infinite wedge, see Appendix B and Fig. 13, where the upper surface is smooth whereas the lower one has a strip of random roughness of the type that we studied in this paper. A net mean lift is obtained whereas due to symmetry and inviscid assumptions there is no net lift for a full semi-infinite wedge with both surfaces smooth. The mean and standard deviation of lift-to-drag ratio of a full semi-infinite wedge along the wedge surface for inflow $M_1 = 2$ and correlation length $A/d = 1$ and 0.1.

![Fig. 6. Sparse grids convergence of the mean of the non-dimensional perturbed (a) lift and (b) drag along the wedge surface for inflow $M_1 = 2$ and correlation length $A/d = 1$ and 0.1.](image)

![Fig. 7. Sparse grids convergence of the mean of the non-dimensional perturbed (a) lift and (b) drag along the wedge surface for inflow $M_1 = 8$ and correlation length $A/d = 1$ and 0.1.](image)
6. Summary and discussion

In supersonic flows, the shock dynamics is very sensitive to the boundary conditions, transport coefficients, and geometric smoothness of the wetted surface. Moreover, shock dynamics in-
volves strong nonlinear waves which make it quite difficult to predict accurately and efficiently. Here we have combined stochastic numerical simulations with second-order stochastic perturbation analysis to study the effect of the random roughness on the scattering of shock waves and the associated modification to the lift and drag forces on the wedge surface and corresponding lift-to-drag ratio for a half- and a full semi-infinite wedge. For small perturbations, we have derived analytical solutions using second-order stochastic perturbation analysis. For large perturbations, we resorted to multi-element probabilistic collocation (ME-PCM) simulations on sparse grids.

A summary of our findings is as follows: First, we have used the analytical solutions derived from the second-order stochastic perturbation analysis to verify the correctness of the numerical simulations of the stochastic nonlinear Euler equations. Under small perturbation, the numerical results match with the analytical solutions. However, for large roughness height, the numerical solution deviates significantly from the theory. Specifically, the dependence of the standard deviation of the perturbed lift and drag forces is predicted by the second-order stochastic perturbation analysis and is also verified by the stochastic numerical simulations. Another particularly revealing finding is the increase of the maximum of the standard deviation of the perturbed lift and drag with the granularity of roughness for \( M_1 = 8 \). In contrast, the maximum of the standard deviation of the perturbed lift and drag is independent of the granularity of roughness for \( M_1 = 2 \). Further-

Fig. 11. (a) Mean and (b) standard deviation of the lift-to-drag ratio of a full semi-infinite wedge for inflow \( M_1 = 2 \).

Fig. 12. (a) Mean and (b) standard deviation of the lift-to-drag ratio of a full semi-infinite wedge for inflow \( M_1 = 8 \).

Fig. 13. Sketch of supersonic flow past a full semi-infinite wedge: definition of coordinate system; the lower half wedge is rough with a perturbed shock path and the upper half wedge is smooth with an unperturbed shock.
more, we extended our results to a full semi-infinite wedge, see Fig. 13, where the upper surface is smooth whereas the lower one has a strip of random roughness of the type that we studied in this paper. Positive values of the mean of lift-to-drag ratio are obtained beyond the rough region for both inflows, i.e., $M_1 = 2$ and 8, and for correlation lengths $A/d = 1$ and 0.1.

Finally, we remind the reader that in this study we refer to the wave lift and drag and did not take into account the subtle structural changes near the wall due to viscous stresses nor did we consider the real gas effect [6].

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**Appendix A. Stochastic representation of random roughness**

The spatial covariance kernel $R_{ob}$ used in the present work is based on the solution of Eq. (4). The corresponding covariance computed from the solution of Eq. (4) is

$$E[h_{ob}(x_1,\omega)h_{ob}(x_2,\omega)] = \frac{\sqrt{2}(x_1, x_2)}{64k'(2 - \cos2k'x - \cos\sqrt{2k'}x)}, \quad (42)$$

where $J(x_1, x_2) = F(x_1 - x_2) + F(2 - |x_1 - x_2|) - F(x_1 + x_2) - F(2 - x_1 - x_2)$ and $F(x)$ is defined as

$$F(x) = -12f_1(t_1) - 60f_1(t_1) - f_2 + 6f_3 - 24f_4 - 8af_5 - 4f_6 - 8f_7, \quad (43)$$

where $a = \sqrt{2k'}, t_1 = \sqrt{2}t, t_2 = \sqrt{2}t(x - 2), t_3 = \sqrt{2}(x - 4), f_1(x) = \sinh x - \sin x, f_2 = (80 - 20t_1)\cosh t_1 - \cos t_1, f_3 = \cos t_1 \sinh t_1 - \cos t_1 \cosh t_1 - \sin t_1 \cosh t_1 + \sin t_1 \cosh t_1, f_4 = \sin t_1 \cosh t_2 + \cos t_1 \sinh t_2 + \sin t_1 \cosh t_2 + \cos t_1 \cosh t_2, f_5 = \sin t_1 \sinh t_1 + \sin t_1 \cosh t_1 - \sin t_1 \cosh t_1 - \sin t_1 \sinh t_1, f_6 = t_1(\cosh t_1 + \cosh t_1 - \cos t_1 \cosh t_1 - \cos t_1 \cosh t_1) - f_7 = (\cos t_1 \cosh t_2 + \cos t_1 \cosh t_2 + \cos t_1 \cosh t_2 + \cos t_1 \cosh t_2) - f_7 = (\cos t_1 \cosh t_2 + \cos t_1 \cosh t_2 + \cos t_1 \cosh t_2 + \cos t_1 \cosh t_2).

The eigenfunctions and eigenvalues of the covariance kernel can be obtained by substituting Eq. (42) into Eq. (2).

Let $A/d \ll 1$ and consider only the region $0 < x < \frac{1}{2}$, then $F(x)$ in Eq. (43) can be simplified as

$$F(x) = \frac{\sqrt{2}k'(3k'x) - 2}{2} \left(12 + 6 \left[\frac{\sin \sqrt{2k'x}}{\sqrt{2k'}} - \frac{\cos \sqrt{2k'x}}{\sqrt{2k'}}\right]\right). \quad (44)$$

Substituting Eq. (44) into Eq. (42), we observe that $d$ is eliminated from $E[h_{ob}(x,\omega)h_{ob}(y,\omega)]$ and $A$ is the characteristic length parallel to the wedge surface. On the other hand, if $A/d \gg 1$, since $k = \frac{4\sqrt{2}}{M_2} = 0$, a Taylor expansion can be employed in Eq. (42) for $x_1 = x_2$ and $k$ around 0 to obtain the following simple formula

$$E[h_{ob}^2(x,\omega)] = \frac{x^4(1 - x)^4(3 + x - x^3)}{1260}. \quad (45)$$

Obviously, $A$ does not appear in Eq. (45), hence, $d$ is the characteristic length. For $A/d \sim 1$, the characteristic length is a combination of both $A$ and $d$. Note that in Eqs. (44) and (45), $x$ is normalized with $d$.

**Appendix B. Analytical results for a full-wedge**

Based on the results of the half-wedge, we can readily obtain analytical results for a full semi-infinite wedge, where the upper surface is smooth whereas the lower one has a strip of random roughness of the type that we studied here. Different from the coordinate system as we build on the half semi-infinite wedge, the $x'$ axis is on the horizontal surface as shown in Fig. 13. Here we define $\Delta l$ and $\Delta d$ as the non-dimensional perturbed lift and drag due to the rough surface on the lower wedge. The total lift and drag of the full semi-infinite wedge as $L(x',\omega)$ and $D(x',\omega)$. The total lift and drag forces are:

$$L(x',\omega) = -x' + \int_0^{\sqrt{\pi}} p_w(x';\omega) \left(\cos \theta_0 - \frac{\partial \chi(x';\omega)}{\partial x_1} \sin \theta_0\right) dx_1,$$

$$D(x',\omega) = x' \tan \theta_0 + \int_0^{\sqrt{\pi}} p_w(x';\omega) \left(\sin \theta_0 + \frac{\partial \chi(x';\omega)}{\partial x_1} \cos \theta_0\right) dx_1,$$

where $p_w(x';\omega)$ is the non-dimensional total pressure on the lower rough wedge surface and the perturbed lift $\Delta l(x',\omega)$ and drag $\Delta D(x',\omega)$ are given as

$$\Delta l(x',\omega) = \left(-h \sin \theta_0 + \cos \theta_0 \int_0^{\sqrt{\pi}} \tilde{p}_w dx_1\right), \quad \Delta D(x',\omega) = \left(h \cos \theta_0 + \sin \theta_0 \int_0^{\sqrt{\pi}} \tilde{p}_w dx_1\right), \quad (46)$$

where $\tilde{p}_w$ is the corrected variable from the non-dimensional first-order perturbed pressure on the wedge surface, $p'_w$ and $p''_w$ is the corresponding second-order perturbed pressure. The lift-to-drag ratio of the full semi-infinite wedge is

$$\frac{L(x',\omega)}{D(x',\omega)} = \frac{-h \sin \theta_0 + \cos \theta_0 \int_0^{\sqrt{\pi}} \tilde{p}_w dx_1 - h \sin \theta_0 + f_1}{2x' \tan \theta_0} \left(\cos \theta_0 \left(\int_0^{\sqrt{\pi}} \tilde{p}_w dx_1 - h \sin \theta_0 + f_1\right)\right) \quad (48)$$

where

$$f_1 = \int_0^{\sqrt{\pi}} \left(p'_w \cos \theta_0 - \frac{\partial \chi}{\partial x_1} \sin \theta_0\right) dx_1 + \frac{\cos \theta_0}{2x'} \left(h^2 \cos \theta_0 - h \frac{\cos 2\theta_0}{\sin \theta_0} \int_0^{\sqrt{\pi}} \tilde{p}_w dx_1 - \cos \theta_0 \left(\int_0^{\sqrt{\pi}} \tilde{p}_w dx_1\right)^2\right).$$

**Appendix C. Validity of perturbation analysis**

The assumptions for the perturbation analysis are stated in Eq. (11). To quantify the region of validity for the stochastic perturbation analysis, we use the standard deviation of the stochastic roughness as a measure. The region of validity with respect to the roughness amplitude $\epsilon$ for the stochastic perturbation analysis can be obtained by considering Eq. (11). For a fixed wedge angle, the region of validity depends on two parameters: (1) the kernel of the random roughness, which depends on the correlation length $A$ and roughness length $d$; and (2) the inflow Mach number $M_1$. Due to inflow Mach number effect, $\frac{\Delta \epsilon}{\epsilon}$ has the lowest value compared to the other flow variables in the entire region of $M_1, \theta_0$ and $A/d$. This can also be observed from the characteristics, i.e., along the characteristics, $\epsilon' = \frac{\Delta \epsilon}{\epsilon} \nu' > \nu$, especially for $M_2 \gg 1$.\]
case. Thus, the region of validity for the stochastic perturbation analysis is

$$\epsilon \leq \epsilon_{\text{max}} = \frac{0.1}{\sqrt{\sigma(p')}}$$  \hspace{1cm} (49)$$

where $p'$ is defined in Eq. (19a). In Fig. 15a, it is observed that in the fully correlated case ($\epsilon_{\text{max}}$ is proportional to a constant ($M_1$ and $\theta_b$ are fixed) and $\epsilon_{\text{max}}$ is not a function of correlation length $A$. However, in the weakly correlated case ($\epsilon_{\text{max}}$ is a linear function of $A/d$ if $M_1$ and $\theta_b$ are fixed). In between, there is a transition region which connects the region.

Mathematically, the results in Fig. 15a can be interpreted as follows: for fixed wedge angle $\theta_b = 14.7436^\circ$, from Fig. 14 we know the reflection index $r \ll 1$ for all inflow Mach numbers. For $r \ll 1$ case, Eq. (49) is simplified as

$$\epsilon \leq \frac{\sqrt{M_2^2 - 1}}{\sqrt{\sigma(p')}}$$  \hspace{1cm} (50)$$

For $A/d \ll 1$ and $r \ll 1$, Eq. (49) can be rewritten as,

$$\epsilon \leq \frac{\sqrt{M_2^2 - 1}}{M_1} \times \frac{1}{\sqrt{\sigma(p')}}$$  \hspace{1cm} (50)$$

For $A/d \gg 1$ and $r \ll 1$, Eq. (49) can be expressed as, $\epsilon \leq \frac{\sqrt{M_2^2 - 1}}{\sqrt{\sigma(p')}}$ where $c \approx 0.032137$.

Physically, the results in Fig. 15a can be explained as follows: On the random rough wedge surface, we have $v_b = \frac{M_1}{20}$ and to satisfy the small perturbation assumption, we require $v_b < 1$ or $\frac{M_1}{20} < 1$. The physical meaning of $\epsilon_{\text{max}}$ $< 1$ is that the ratio of the characteristic length perpendicular to the wedge and the one parallel to the wedge has to be a small number (if $M_1$ and $\theta_b$ are fixed). Recall that $d$ gives the characteristic length perpendicular to the wedge surface. For fully correlated stochastic roughness case, $d$ is the characteristic length parallel to the wedge surface. Thus, we observe $\frac{d_{\text{max}}}{d_{\text{max}}} = \epsilon_{\text{max}} = c$ in the fully correlated case. However, for the weakly correlated stochastic roughness case, $A$ is the characteristic length parallel to the wedge surface. Therefore, for the weakly correlated case, our $\epsilon_{\text{max}}$ is a small constant.

In Fig. 15b, we can see that for wedge angle $\theta_b = 14.7436^\circ$, $\epsilon_{\text{max}}$ has the peak value at $M_1 = 2$. At $M_1 = 1.634$, the shock is detached from the wedge and the flow after the shock is transonic and therefore non-uniform; a sharp decay of $\epsilon_{\text{max}}$ can be observed. For large $M_1$, $\epsilon_{\text{max}}$ decreases monotonically with increasing $M_1$. Therefore, using Eq. (50), we can appreciate that the region of validity is smaller as the flow after the shock approaches transonic or hypersonic states.

In Fig. 14, we verify that the infinite series in Eq. (16a) is convergent in a realistic aerodynamic range. The Mach number before the shock varies from 1.4 to 20. From the plot we can see that the reflection index $r$ is approximately within the $[0.7, 0.7]$ range. Actually, in most of the region, except where the after shock region is almost transonic, the reflection index $r$ is bounded within $[0.2, 0.2]$. Thus, the infinite series in Eq. (16a) is always convergent. The physical meaning of $r = \frac{\epsilon}{\epsilon_{\text{max}}}$ is that it represents the strength of the characteristics reflection at the shock surface; the reflection index $r$ in the $M_1 = 2$ case is approximately 0.01 corresponding to very weak reflections. A nearly linear mapping between the rough wedge perturbation and the corresponding perturbed shock path can be observed. However, the reflection index in the $M_1 = 8$ case is approximately 0.066 corresponding to relatively larger reflections, compared to the $M_1 = 2$ case.

**Appendix D. Weak Mach wave reflection at shock surface**

Consider a double wedge as shown in Fig. 16. Define $p = P_0/\sigma(p')$ and $v = W_0/\sigma(v')$. Since there is no perturbation on shock path $a$, for any point $B$ on the wedge $b$, we obtain the perturbed pressure $p_b$ using the characteristics method,

$$\frac{\sqrt{M_2^2 - 1}}{\sqrt{M_1^2}} \frac{p_b}{\rho_b} = v_b = \tan \theta_b$$  \hspace{1cm} (51)$$

Fig. 14. Reflection index $r$ as a function of wedge angle $\theta_b$ and inflow Mach number $M_1$.

Fig. 15. (a) Region of validity for stochastic perturbation analysis as a function of $A/d$. (b) Region of validity for stochastic perturbation analysis as a function of $M_1$. 

Appendix E. Scaling laws

We now derive simplified asymptotic expressions for the statistical measures of perturbed lift and drag. We note that if $A/d \ll 1$ and consider only $0 < x < \frac{1}{3}$ region, $F(x)$ in Eq. (43) can be simplified as in Eq. (44). Substituting Eq. (44) into Eq. (42), we obtain the following variances:

$$
\mathbb{E}[h_{\omega}^2] = \frac{\sqrt{2}}{16\sqrt{r}} \left(3 - \left(6 + 3(\sin \sqrt{2kx} - \cos \sqrt{2kx})ight) e^{-\sqrt{2kx}}\right).
$$

$$
\mathbb{E} \left(\frac{\partial h_{\omega}}{\partial x}\right) = \frac{\sqrt{2}}{16\sqrt{r}} \left(1 - \left(2 - (\sin \sqrt{2kx} + \cos \sqrt{2kx}) + 2\sqrt{2kx} \left(1 - \sin \sqrt{2kx}\right) e^{-\sqrt{2kx}}\right)\right).
$$

It is interesting to notice the behavior of the variance $\mathbb{E}[h_{\omega}^2]$ for $A/d \ll 1$ in the two limiting cases: $\mathbb{E}[h_{\omega}^2] \to \frac{2e^{-1/2}}{\sqrt{16\sqrt{r}}}$ if $x \to 0$ ($x \ll \frac{1}{2}$); while $\mathbb{E}[h_{\omega}^2] \to \frac{2e^{-1/2}}{\sqrt{16\sqrt{r}}}$ if $x \to \frac{1}{2}$ ($x \gg \frac{1}{2}$) Thus, considering the symmetric properties around $x = \frac{1}{2}$, we observe that $\mathbb{E}[h_{\omega}^2]$ grows as $x^2$ around $x = 0$ and $x = 1$. After a transition region, $\mathbb{E}[h_{\omega}^2]$ grows as a constant at $x = \frac{1}{2}$. Therefore, for $A/d \ll 1$, the non-dimensional roughness height in Eq. (3), $\mathbb{E}[\left(\left(\frac{\partial h_{\omega}}{\partial x}\right)^2\right)] = \frac{1}{2} \mathbb{E}[\left(\frac{\partial h_{\omega}}{\partial x}\right)^2]$, where $\mu = \frac{\sqrt{2}}{16\sqrt{r}}$. These results for $A/d \gg 1$ have been given in Eq. (45). Similarly, $\mathbb{E}[\left(\frac{\partial h_{\omega}}{\partial x}\right)^2]$ can also be derived, i.e.,

$$
\mathbb{E} \left(\frac{\partial h_{\omega}}{\partial x}\right) = \frac{\sqrt{2}}{16\sqrt{r}} \left(1 - \left(2 - (2 - \sqrt{2kx} + \cos \sqrt{2kx}) + 2\sqrt{2kx} \left(1 - \sin \sqrt{2kx}\right) e^{-\sqrt{2kx}}\right)\right).
$$

Therefore, for $A/d \gg 1$, the non-dimensional roughness height

$$
\mathbb{E} \left(\frac{\partial h_{\omega}}{\partial x}\right) = \frac{\sqrt{2}}{16\sqrt{r}} \left(1 - \left(2 - (2 - \sqrt{2kx} + \cos \sqrt{2kx}) + 2\sqrt{2kx} \left(1 - \sin \sqrt{2kx}\right) e^{-\sqrt{2kx}}\right)\right).
$$

It is obvious that as $A/d \to 1$, $\mathbb{E}[\left(\frac{\partial h_{\omega}}{\partial x}\right)^2]$ does not depend on $\frac{1}{d}$. From the second-order perturbation analysis, all the flow variables can be derived as a function of either $h(x; \omega)$ or $\frac{\partial h}{\partial x}$. Considering Eqs. (45) and (57) are all independent of $\frac{1}{d}$ for $A/d \to 1$, we can then conclude that all the flow variables are independent of $\frac{1}{d}$.

The reflection index $r$ defined in Eq. (16b) is $r < 0.1$ in most of the region in Fig. 14 except the transonic region $(M_2 \approx 1)$. To simplify the formula, we consider the $r = 0$ case. Eq. (17) can be simplified as $p_{\omega} = \frac{\gamma M_{\infty}^2}{\sqrt{\rho_{\infty}}}$. Therefore, the standard deviation of $\Delta p_{\omega}$ for $A/d \ll 1$ is

$$
\sigma(\Delta p_{\omega}) = \frac{\gamma M_{\infty}^2}{\sqrt{\rho_{\infty}}} \left[1 - \sqrt{\frac{\gamma M_{\infty}^2}{\sqrt{\rho_{\infty}}} - 1}\right] \frac{d}{A}. \quad (58)
$$

where $\mu = \frac{\sqrt{2}}{16\sqrt{r}}$ and $\mathbb{E}[\left(\frac{\partial h_{\omega}}{\partial x}\right)^2]$ is defined in Eq. (56). Eq. (58) is valid for $r < 1$, $A/d \gg 1$ and most of the region of $x$, except the region around $x = 0$ and $x = 1$. It is interesting to note that for $r < 1$ and $A/d \ll 1$, $\frac{\partial h_{\omega}}{\partial x} \approx \frac{\partial h}{\partial x}$. A similar behavior can also be observed for $\sigma(\Delta h_{\omega})$, $\sigma(\Delta u_{\omega})$, $\sigma(\Delta u_{\omega})$, and $\sigma(\Delta d_{\omega})$.

Similarly, for $r = 0$ case, Eq. (33a) can be simplified as

$$
\sigma(\Delta l_{\omega}) = \left|\sin \theta_0 - \frac{\gamma M_{\infty}^2}{\sqrt{\rho_{\infty}}} \cos \theta_0\right| \frac{\sigma(h)}{\epsilon}.
$$

Therefore, the standard deviation of $\Delta l_{\omega}$ for $A/d \ll 1$ is

$$
\sigma(\Delta l_{\omega}) = \left|\sin \theta_0 - \frac{\gamma M_{\infty}^2}{\sqrt{\rho_{\infty}}} \cos \theta_0\right| \frac{\sigma(h)}{\epsilon}. \quad (59)
$$
Thus, for \( r \ll 1 \) and \( A/d \ll 1 \), \( \sigma(\Delta L) \) is independent of \( \frac{A}{d} \) except the region around \( x = 0 \) and \( x = 1 \). \( \sigma(\Delta D) \) and \( \sigma(z) \) also have similar properties.

For \( r = 0 \) case, Eq. (28) can be rewritten as

\[
\rho_w(x; \omega) = \frac{\gamma M_2^2}{\sqrt{M_2^2 - 1}} \left( G_2 - F_2 \right) \left( \frac{\partial h}{\partial x} \right)^2 \left( \frac{\partial h}{\partial x} \right) + \nu_w(x; \omega)
\]

\[
- \int_0^x R_2(x_1, -mx_1) \, dx_1.
\]

(60)

Considering \( E[R_2] \propto E\left[ \frac{\partial h}{\partial x} \right] \propto e^{-2kx} \to 0 \) and \( \nu_w = -\frac{1}{\sqrt{M_2^2 - 1}} \left( \frac{\partial h}{\partial x} \right)^2 \), we have,

\[
E[\Delta \rho_w] \approx E[\nu_w]
\]

\[
= q(G_2 - F_2) E\left[ \left( \frac{\partial h}{\partial x} \right)^2 \right] - \frac{1}{\sqrt{M_2^2 - 1}} E\left[ \frac{\partial h}{\partial x} \right]^2
\]

\[
\approx q(G_2 - F_2) - \frac{1}{\sqrt{M_2^2 - 1}} \frac{d^2}{3A} \approx \frac{d^2}{A^2}.
\]

(61)

Hence, for \( r \ll 1 \) and \( A/d \ll 1 \), \( \frac{\Delta \rho_w}{\rho_w} \propto \frac{d^2}{A} \). Similar behavior can also be observed for all other mean flow variables: \( E[\Delta \rho_w], E[\Delta u_w], E[\Delta v_w] \) and \( E[\Delta z], E[\Delta A] \) in summary, the mean flow variables scale with \( r^2 \) and all the standard deviations scale with \( \epsilon \). For \( A/d \gg 1 \), the means and standard deviations are independent of \( \frac{A}{d} \). However, for \( A/d \ll 1 \), the means scale with \( \frac{A}{d} \). The standard deviations of \( \Delta \rho_w, \Delta u_w, \Delta v_w \) and \( \Delta \rho_w \) scale with \( \frac{A}{d} \) while the standard deviations of \( z, \Delta L \) and \( \Delta D \) are independent of \( \frac{A}{d} \).

References


