TEMPERED FRACTIONAL STURM–LIOUVILLE EIGENPROBLEMS

MOHSEN ZAYERNOURI†, MARK AINSWORTH‡, AND GEORGE EM KARNIADAKIS‡

Abstract. Continuum-time random walk is a general model for particle kinetics, which allows for incorporating waiting times and/or non-Gaussian jump distributions with divergent second moments to account for Lévy flights. Exponentially tempering the probability distribution of the waiting times and the anomalously large displacements results in tempered-stable Lévy processes with finite moments, where the fluid (continuous) limit leads to the tempered fractional diffusion equation. The development of fast and accurate numerical schemes for such nonlocal problems requires a new spectral theory and suitable choice of basis functions. In this study, we introduce two classes of regular and singular tempered fractional Sturm–Liouville problems of two kinds (TFSLP-I and TFSLP-II) of order \( \nu \in (0, 2) \). In the regular case, the corresponding tempered differential operators are associated with tempering functions \( p_I(x) = \exp(2\tau) \) and \( p_{II}(x) = \exp(-2\tau) \), \( \tau \geq 0 \), respectively, in the regular TFSLP-I and TFSLP-II, which do not vanish in \([-1, 1]\). In contrast, the corresponding differential operators in the singular setting are associated with different forms of \( p_I(x) = \exp(2\tau)(1-x)^{1+\beta} \) and \( p_{II}(x) = \exp(-2\tau)(1-x)^{1+\alpha}(1+x)^{1+\beta} \), vanishing at \( x = \pm 1 \) in the singular TFSLP-I and TFSLP-II, respectively. The aforementioned tempered fractional differential operators are both of tempered Riemann–Liouville and tempered Caputo type of fractional order \( \mu = \nu/2 \in (0, 1) \). We prove the well-posedness of the boundary-value problems and that the eigenvalues of the regular tempered problems are real-valued and the corresponding eigenfunctions are orthogonal. Next, we obtain the explicit eigensolutions to TFSLP-I and -II as nonpolynomial functions, which we define as tempered Jacobi poly-fractonomials. These eigenfunctions are orthogonal with respect to the weight function associated with TFSLP-I and -II. Finally, we introduce these eigenfunctions as new basis (and test) functions for spectrally accurate approximation of functions and tempered-fractional differential operators. To this end, we further develop a Petrov–Galerkin spectral method for solving tempered fractional ODEs, followed by the corresponding stability and convergence analysis, which validates the achieved spectral convergence in our simulations.

Key words. regular/singular tempered fractional Sturm–Liouville operators, well-posedness, tempered Jacobi poly-fractonomial, spectral approximation, Petrov–Galerkin method

AMS subject classifications. 34L10, 58C40, 34K28, 65M70, 65M60

DOI. 10.1137/140985536

1. Introduction. Continuum-time random walk (CTRW) is a rigorous and general mathematical model for particle kinetics, which incorporates waiting times and/or non-Gaussian jump distributions with divergent second moments to account for the anomalous jumps called Lévy flights [20, 8, 19]. The continuous limit for such models leads to a fractional in time and/or space diffusion equation [21, 23, 11]. However, in practice, many physical processes take place in bounded domains in finite times and have finite moments. Therefore, the divergent second moments may not be applicable to such processes. In order to overcome this modeling barrier, there are different techniques such as discarding the very large jumps and employing truncated Lévy flights [17], or, adding a high-order power-law factor [27]. However, the most popular, and

---

*Submitted to the journal’s Methods and Algorithms for Scientific Computing section September 8, 2014; accepted for publication April 24, 2015; published electronically July 21, 2015. This work was supported by the Collaboratory on Mathematics for Mesoscopic Modeling of Materials (CM4) at PNNL funded by the Department of Energy, by OSD/MURI, and by NSF/DMS.

†Division of Applied Mathematics, Brown University, Providence, RI 02912 (mohsen.zayernouri@brown.edu, mark.ainsworth@brown.edu).

‡Corresponding author. Division of Applied Mathematics, Brown University, Providence, RI 02912 (george_karniadakis@brown.edu).
perhaps most rigorous, approach to get finite moments is exponentially tempering the probability of large jumps of Lévy flights, which results in tempered-stable Lévy processes with finite moments [8, 5, 18, 26]. The corresponding fluid (continuous) limit for such models yields the tempered fractional diffusion equation, which complements the previously known models in fractional calculus.

In order to develop efficient numerical schemes for such nonlocal operators, it is important to first formulate a proper spectral theory for tempered fractional eigen-problems. In the standard calculus, the Sturm–Liouville problem (SLP) has been a fruitful resource for the development of spectral methods, spectral/hp element methods, and the theory of self-adjoint operators [2, 35]. However, mostly integer order differential operators in SLPs have been used, and such operators do not include any fractional differential operators. Over the last decade, it has been demonstrated that many systems in science and engineering can be modeled more accurately by employing fractional-order rather than integer-order derivatives [7, 28, 9, 16]. In most fractional Sturm–Liouville formulations, the ordinary derivatives in a traditional SLP are replaced with fractional derivatives, and the resulting problems are approximated using a variety of numerical schemes [1, 10, 22]. However, approximating such an infinite-dimensional nonlocal operator in a finite-dimensional space can be challenging. It turns out that the linear systems resulting from these numerical methods quickly become ill-conditioned, which may suffer from round-off errors and the pseudospectra. That would prohibit computing the eigenvalues and eigenfunctions with the desired precision.

Establishing the basic properties of fractional Sturm–Liouville problems (FSLPs) such as orthogonality of the eigenfunctions, the nature of eigenvalues, etc., is the first step. Qi and Chen in [24], and Atanackovic and Stankovic in [4], considered a classical Sturm–Liouville operator including a sum of the left- and right-sided fractional derivatives. Bas and Metin [6], Klimek and Agrawal [13], Zayernouri and Karniadakis [30], and Rivero, Trujillo, and Velasco [25] defined different classes of fractional Sturm–Liouville operators and investigated the properties of the corresponding eigenfunctions and the eigenvalues. In [12], the exact eigensolutions are obtained in terms of the standard Legendre polynomials. In addition, in [14], variational methods for the FSLP are investigated. Recently, Zayernouri and Karniadakis in [30] formulated regular and singular FSLPs of kinds I and II and obtained explicit eigensolutions, in the form of Jacobi poly-fractonomials given by

\begin{equation}
(1.1) \quad P_{n}^{\alpha,\beta,\mu}(x) = (1 + x)^{-\beta + \mu - 1} P_{n-1}^{\alpha - \mu + 1, -\beta + \mu - 1}(x), \quad x \in [-1, 1],
\end{equation}

with \( \mu \in (0, 1) \), \(-1 \leq \alpha < 2 - \mu \), and \(-1 \leq \beta < \mu - 1 \), which represent the eigenfunctions of the singular problem of the first kind, and

\begin{equation}
(1.2) \quad P_{n}^{\alpha,\beta,\mu}(x) = (1 - x)^{-\alpha + \mu - 1} P_{n-1}^{-\alpha - \mu + 1, \beta - \mu + 1}(x), \quad x \in [-1, 1],
\end{equation}

in which \(-1 < \alpha < \mu - 1 \) and \(-1 < \beta < 2 - \mu \), and \( \mu \in (0, 1) \), are eigenfunctions of the singular problem of the second kind. In (1.1) and (1.2), \( P_{n-1}^{\alpha - \mu + 1, -\beta + \mu - 1}(x) \) are the standard Jacobi polynomial of parameters \( \alpha - \mu + 1 \) and \( -\beta + \mu - 1 \), and moreover, \( P_{n-1}^{-\alpha + \mu - 1, \beta - \mu + 1}(x) \) denote Jacobi polynomials of the corresponding parameters \( -\alpha + \mu - 1 \) and \( \beta - \mu + 1 \). They showed that these eigenfunctions have similar properties to those with a nonfractional setting such as orthogonality, recurrence relations, fractional derivatives, and integration formula. Jacobi poly-fractonomials have been successfully employed as basis and test functions in developing spectrally accurate Petrov–Galerkin spectral and discontinuous spectral element methods in.
and fractional spectral collocation methods in [33, 34] for a variety of fixed and variable-order fractional ODEs/PDEs, including multiterm problems and the nonlinear space-fractional Burgers equation. While most of the existing schemes have been developed for one-dimensional or two-dimensional problems, and rarely for three-dimensional cases, Zayernouri, Ainsworth, and Karniadakis have recently developed a unified Petrov–Galerkin spectral method with a unified fast solver in [29] for solving elliptic, parabolic, and hyperbolic fractional PDEs in high-dimensional (up to 10-dimensional) problems.

In this study, we consider FSLPs corresponding to tempered fractional boundary-value problems and formulate both regular and singular tempered fractional Sturm–Liouville problems (TFSLPs) of two kinds. We prove the well-posedness of the boundary-value problems, that the eigenvalues of the tempered problems are real-valued, and that the corresponding eigenfunctions are orthogonal, and we obtain the corresponding explicit eigenfunctions as tempered Jacobi poly-fractonomials. We employ these eigenfunctions as basis (and test) functions in a Petrov–Galerkin spectral method for approximating tempered fractional ODEs (TFODEs).

The organization of the paper is as follows. In section 2, we provide the preliminary definitions in fractional calculus and the underlying functions spaces. In section 3, we examine the well-posedness of the boundary-value problems, and in section 4, we introduce the regular tempered eigenproblems and prove that the corresponding eigenfunctions are real-valued and orthogonal with respect to the associate weight functions. Subsequently in section 5, we consider the corresponding singular tempered boundary-value problem and obtain the explicit solutions along with their properties. Next, in section 6, we examine the approximation properties of the eigenfunctions. Finally, we employ the eigenfunctions to the first and second problems as basis and test functions for numerical approximation of TFODEs and carry out stability and convergence studies. We conclude with a summary and discussion in section 7.

2. Definitions. We start with some preliminary definitions of fractional and tempered fractional calculus following [23, 8, 18]. The left-sided and the right-sided tempered Riemann–Liouville fractional integrals of order $\mu \in (0,1)$ are defined as

\[
(\text{RLI}_{xL}^\mu,\tau)f(x) = \left( e^{-\tau x} \text{RLI}_{xL}^\mu e^{\tau x} \right)f(x) = \frac{1}{\Gamma(\mu)} \int_{xL}^{x} e^{-\tau(x-s)}f(s)ds, \quad x > x_L,
\]

and

\[
(\text{RLI}_{xR}^\mu,\tau)f(x) = \left( e^{\tau x} \text{RLI}_{xR}^\mu e^{-\tau x} \right)f(x) = \frac{1}{\Gamma(\mu)} \int_{x}^{x_R} e^{-\tau(s-x)}f(s)ds, \quad x < x_R,
\]

respectively, where $\Gamma$ represents the Euler gamma function. When $\tau = 0$, the tempered fractional integrations (2.1) and (2.2) reduce to the standard Riemann–Liouville fractional integrations $(\text{RLI}_{xL}^\mu)f(x)$ and $(\text{RLI}_{xR}^\mu)f(x)$, respectively.

The corresponding tempered Riemann–Liouville fractional derivative of order $\mu$ with tempering parameter $\tau \geq 0$ is given by

\[
(\text{RLD}_{xL}^\mu,\tau)f(x) = \left( e^{-\tau x} \text{RLD}_{xL}^\mu e^{\tau x} \right)f(x) = \frac{e^{-\tau x}}{\Gamma(1-\mu)} \frac{d}{dx} \int_{xL}^{x} e^{\tau s}f(s)ds, \quad x > x_L,
\]
and
\begin{equation}
(\mathcal{RL}_{x\in[0,x_R]}^\mu)^{\tau}\!f(x) = \left( e^{\tau x} \mathcal{RL}_{x\in[0,x_R]}^\mu e^{-\tau x} \right) f(x) = \frac{e^{\tau x}}{\Gamma(1-\mu)} \frac{d}{dx} \int_x^{x_R} e^{-\tau s} f(s) ds \frac{d}{ds} \bigg|_{s=x} \frac{d}{ds} \bigg|_{s=x} \frac{(s-x)\mu}{x}, \quad x < x_R.
\end{equation}

An alternative approach in defining the tempered fractional derivatives is based on the left- and right-sided tempered Caputo derivatives of order \( \mu \in (0,1) \), defined, respectively, as
\begin{equation}
(\mathcal{C}_{x\in[0,x_R]}^\mu)^{\tau}\!f(x) = \left( e^{\tau x} \mathcal{C}_{x\in[0,x_R]}^\mu e^{-\tau x} \right) f(x) = \frac{e^{\tau x}}{\Gamma(1-\mu)} \int_x^{x_R} [e^{-\tau s} f(s)]' ds \frac{d}{ds} \bigg|_{s=x} \frac{d}{ds} \bigg|_{s=x} \frac{(s-x)\mu}{x}, \quad x > x_L,
\end{equation}
and
\begin{equation}
(\mathcal{T}_{x\in[0,x_R]}^\mu)^{\tau}\!f(x) = \left( e^{-\tau x} \mathcal{T}_{x\in[0,x_R]}^\mu e^{-\tau x} \right) f(x) = \frac{e^{-\tau x}}{\Gamma(1-\mu)} \int_x^{x_R} [e^{-\tau s} f(s)]' ds \frac{d}{ds} \bigg|_{s=x} \frac{d}{ds} \bigg|_{s=x} \frac{(s-x)\mu}{x}, \quad x < x_R.
\end{equation}

Similarly, if \( \tau = 0 \), the tempered fractional derivatives (2.3) and (2.4) reduce to the standard Riemann–Liouville fractional integrations \((\mathcal{RL}_{x\in[0,x_R]}^\mu)^{\tau}\!f(x)\) and \((\mathcal{RL}_{x\in[0,x_R]}^\mu)^{\tau}\!f(x)\), and in addition, (2.5) and (2.6) reduce to the Caputo fractional integrations \((\mathcal{C}_{x\in[0,x_R]}^\mu)^{\tau}\!f(x)\) and \((\mathcal{T}_{x\in[0,x_R]}^\mu)^{\tau}\!f(x)\), respectively. The corresponding relationships between the tempered Riemann–Liouville and tempered Caputo fractional derivatives when \( \mu \in (0,1) \) are given by
\begin{equation}
(\mathcal{TRL}_{x\in[0,x_R]}^\mu)^{\tau}\!f(x) = \frac{f(x_L)}{\Gamma(1-\mu)(x-x_L)} + (\mathcal{TC}_{x\in[0,x_R]}^\mu)^{\tau}\!f(x)
\end{equation}
and
\begin{equation}
(\mathcal{TRL}_{x\in[0,x_R]}^\mu)^{\tau}\!f(x) = \frac{f(x_R)}{\Gamma(1-\mu)(x-x_R)} + (\mathcal{TC}_{x\in[0,x_R]}^\mu)^{\tau}\!f(x).
\end{equation}

These definitions coincide with each other when boundary-values vanish. Moreover, letting \( \Omega = [x_L, x_R] \), the corresponding fractional integrations by parts for the aforementioned fractional derivatives are obtained as
\begin{equation}
\left( f(x), \mathcal{RL}_{x\in[0,x_R]}^\mu g(x) \right)_{\Omega} = \left( g(x), \mathcal{C}_{x\in[0,x_R]}^\mu f(x) \right)_{\Omega} - f(x) \left( \mathcal{TRL}_{x\in[0,x_R]}^\mu g(x) \right)_{x=x_L}
\end{equation}
and
\begin{equation}
\left( f(x), \mathcal{RL}_{x\in[0,x_R]}^\mu g(x) \right)_{\Omega} = \left( g(x), \mathcal{T}_{x\in[0,x_R]}^\mu f(x) \right)_{\Omega} + f(x) \left( \mathcal{TRL}_{x\in[0,x_R]}^\mu g(x) \right)_{x=x_L},
\end{equation}
where \((\cdot,\cdot)_{\Omega}\) represents the standard \( L^2 \) inner product. We note that (2.9) and (2.10) are also valid when the fractional derivatives and integrals are replaced with their tempered counterparts.

By \( H^s(\mathbb{R}) \), \( s \geq 0 \), we denote the fractional Sobolev space on \( \mathbb{R} \), defined as
\begin{equation}
H^s(\mathbb{R}) = \{ v \in L^2(\mathbb{R}) \mid (1 + |\omega|^2)^{s/2} \mathcal{F}(v)(\omega) \in L^2(\mathbb{R}) \},
\end{equation}
which is endowed with the norm
\begin{equation}
\| \cdot \|_{s,\mathbb{R}} = \|(1 + |\omega|^2)^{s/2} \mathcal{F}(-)\|_{L^2(\mathbb{R})},\]

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.
where $\mathcal{F}(v)$ represents the Fourier transform of $v$. Subsequently, we denote by $H^s([-1, 1])$, $s \geq 0$, the fractional Sobolev space on the finite closed interval $[-1, 1]$, defined as
\begin{equation}
H^s([-1, 1]) = \{ v \in L^2(\mathbb{R}) \mid \exists \tilde{v} \in H^s(\mathbb{R}) \text{ s.t. } \tilde{v}|_{[-1, 1]} = v \},
\end{equation}
with the norm
\begin{equation}
\| v \|_{s, [-1, 1]} = \inf_{\tilde{v} \in H^s(\mathbb{R}), \tilde{v}|_{[-1, 1]} = v} \| \cdot \|_{s, \mathbb{R}}.
\end{equation}
We note that the definition of $H^s([-1, 1])$ and the corresponding norm rely on the Fourier transformation of the function. Other useful norms associated with $H^s([-1, 1])$ have been introduced in [15],
\begin{equation}
\| \cdot \|_{l, s, [-1, 1]} = \left( \| \cdot \|_{L^2([-1, 1])}^2 + \| RL_x D_\mu^\ast(\cdot) \|_{L^2([-1, 1])}^2 \right)^{1/2},
\end{equation}
and
\begin{equation}
\| \cdot \|_{r, s, [-1, 1]} = \left( \| \cdot \|_{L^2([-1, 1])}^2 + \| RL_x D_\mu(\cdot) \|_{L^2([-1, 1])}^2 \right)^{1/2},
\end{equation}
such that $\| \cdot \|_{l, s, [-1, 1]}$, $\| \cdot \|_{r, s, [-1, 1]}$, and $\| \cdot \|_{s, [-1, 1]}$ are shown to be equivalent. The following lemmas provide alternative ways of carrying out fractional integration-by-parts, equivalent to (2.9) and (2.10) in the following.

**Lemma 2.1** (see [15]). For all $0 < \xi \leq 1$, if $u \in H^1([a, b])$ such that $u(a) = 0$, and $w \in H^{\xi/2}([a, b])$, then
\begin{equation}
a_s D_s^\xi u \cdot w_{\Omega} = \left( a_s D_s^\xi u, w \right)_{\Omega},
\end{equation}
where $(\cdot, \cdot)_{\Omega}$ represents the standard inner product in $\Omega = [a, b]$.

**Lemma 2.2** (see [15]). For all $0 < \xi \leq 1$, if $u \in H^1([a, b])$ such that $u(b) = 0$ and $w \in H^{\xi/2}([a, b])$, then
\begin{equation}
\left( s_b D_b^\xi u, w \right)_{\Omega} = \left( s_b D_b^\xi u, a_s D_s^\xi w \right)_{\Omega}.
\end{equation}

**Remark 2.3.** It is easy to check that Lemmas 2.1 and 2.2 also hold when the standard Riemann–Liouville fractional derivatives are replaced with their corresponding tempered derivatives.

3. **Well-posedness.** Let $\Omega = [-1, 1]$, $\mu \in (0, 1)$, and
\begin{equation}
\mathcal{L}_I(\cdot) := RL_x D_\mu \left[ p_I(x) - C_x D_\mu(\cdot) \right],
\end{equation}
in which $p_I(x)$ is nonvanishing and continuous in $\Omega$. We define the bilinear form
\begin{equation}
a_I(u, v) = (\mathcal{L}_I u, v)_{\Omega}
\end{equation}
for some $v(x)$. Assuming $u(-1) = 0$, we have $\mathcal{L}_I = RL_x D_\mu \left[ p_I(x) - \frac{RL_x D_\mu}{-1} \right]$ by virtue of the property (2.7). By plugging it into (3.2) and carrying out the fractional integration-by-parts (2.9), we obtain
\begin{equation}
a_I(u, v) = (\mathcal{L}_I u, v)_{\Omega}
\end{equation}
\begin{align}
&= \left( RL_x D_\mu \left[ p_I(x) - \frac{RL_x D_\mu}{-1} u(x) \right], v \right)_{\Omega} \\
&= \left( p_I(x) \frac{RL_x D_\mu}{-1} u, \frac{RL_x D_\mu}{-1} v \right)_{\Omega} - \frac{RL_x D_\mu}{-1} \frac{RL_x D_\mu}{-1} u(x)_{x = -1} \\
&= \left( p_I(x) \frac{RL_x D_\mu}{-1} u, \frac{RL_x D_\mu}{-1} v \right)_{\Omega}
\end{align}
if we further assume that \( v(1) = \frac{RL}{x}D^\mu u \bigg|_{x=1} = 0 \). Now, let
\[
U_I = \{ u \in C(\Omega) \mid \| -\frac{1}{x} D^\mu u \|_{L^2(\Omega), p_I(x)} < \infty, \text{ and } u(1) = \frac{RL}{x}D^\mu u \bigg|_{x=1} = 0 \}
\]
and
\[
V_I = \{ v \in C(\Omega) \mid \| -\frac{1}{x} D^\mu v \|_{L^2(\Omega), p_I(x)} < \infty, \text{ and } v(1) = 0 \}.
\]
We observe that \( V_I \supseteq U_I \). Therefore, for convenience and in order to adopt a Galerkin (rather than a Petrov–Galerkin) method, we choose \( U_I = U_I \cap V_I \) to be the space of trial and test functions \( u \) and \( v \), respectively. Hence, taking \( L_I u = \lambda w_I(x) u \) with respect to the positive weight function \( w_I(x) \), the regular eigenvalue problem of the first kind reads as follows: find \( u \in U_I \) such that
\[
a_I(u, v) := (L_I u, v)_\Omega = (\lambda u, v)_{\Omega, w_I(x)} \quad \forall v \in U_I,
\]
or equivalently,
\[
\left( \frac{RL}{x}D^\mu u, \frac{RL}{x}D^\mu v \right)_{\Omega, p_I(x)} - \lambda (u, v)_{\Omega, w_I(x)} = 0 \quad \forall v \in U_I.
\]
Following similar steps, we define the fractional differential operator
\[
L_{II}(\cdot) := \frac{RL}{x}D^\mu \left[ p_{II}(x) \frac{C}{x} D^\mu \left( \cdot \right) \right],
\]
in which the nonvanishing \( p_{II}(x) \) is continuous in \( \Omega \). The corresponding bilinear form \( a_{II}(u, v) \) is then defined as
\[
a_{II}(u, v) = (L_{II} u, v)_\Omega.
\]
Now, assuming \( u(1) = 0 \), it is easy to verify that \( L_{II} = \frac{RL}{x}D^\mu \left[ p_{II}(x) \frac{RL}{x}D^\mu (\cdot) \right] \) by virtue of the property (2.8). By plugging it into (3.9) and carrying out the fractional integration-by-parts (2.10), we obtain
\[
a_{II}(u, v) = \left( p_{II}(x) \frac{RL}{x}D^\mu u, \frac{RL}{x}D^\mu v \right)_\Omega - v(x) \frac{RL}{x}D^\mu \left( \frac{RL}{x}D^\mu u \right) \bigg|_{x=1}
\]
\[
= \left( p_{II}(x) \frac{RL}{x}D^\mu u, \frac{RL}{x}D^\mu v \right)_\Omega,
\]
when \( v(1) = \frac{RL}{x}D^\mu \left( \frac{RL}{x}D^\mu u \right) \bigg|_{x=1} = 0 \). Now, let
\[
U_{II} = \{ u \in C(\Omega) \mid \| \frac{RL}{x}D^\mu u \|_{L^2(\Omega), p_{II}(x)} < \infty, \text{ and } u(1) = \frac{RL}{x}D^\mu u \bigg|_{x=1} = 0 \},
\]
and
\[
V_{II} = \{ v \in C(\Omega) \mid \| \frac{RL}{x}D^\mu v \|_{L^2(\Omega), p_{II}(x)} < \infty, \text{ and } v(1) = 0 \}.
\]
Here, we also observe that \( V_{II} \supseteq U_{II} \). Therefore, we choose \( U_{II} = U_{II} \cap V_{II} \) to be the trial and test function spaces. Subsequently, by \( L_{II} u = \lambda w_{II}(x) u \) with respect to the positive weight function \( w_{II}(x) \), the second regular fractional eigenvalue problem reads as follows: find \( u \in U_{II} \) such that
\[
\left( \frac{RL}{x}D^\mu u, \frac{RL}{x}D^\mu v \right)_{\Omega, p(x)} - \lambda (u, v)_{\Omega, w_{II}(x)} = 0 \quad \forall v \in U_{II}.\]
Remark 3.1. The construction of the aforementioned eigenvalue problems of the first and second kinds can be similarly done by replacing all left- and right-sided fractional derivatives/integrals to their corresponding tempered counterparts.

Theorem 3.2. Let \( p_I(x) = p_{II}(x) = 1 \). Then, the regular eigenvalue problem of the first kind, \( L_I u(x) = \lambda w_I(x) u(x) \), subject to the boundary conditions \( u(-1) = RL \mu (RL D^{\mu}_{x} u)|_{x=1} = 0 \), and the second regular eigenvalue problem, \( L_{II} u(x) = \lambda w_{II}(x) u(x) \), subject to \( u(1) = RL \mu (RL D^{\mu}_{x} u)|_{x=-1} = 0 \), are well-posed.

Proof. We first observe that \( U_I \) is a Hilbert space; moreover, \( a_I(u,v) \) and \( a_{II}(u,v) \) are linear and continuous. To this end, we need the following conclusion by Lemma 2.4 in [15], which states that there are positive constants \( C_1 \) and \( C_2 \) such that for any \( w \in H^\mu([a,b]) \),

\[
C_1 \int_a^b a_I RL D^{\mu}_{x} w(x) RL D^{\mu}_{x} w(x) dx \leq \| RL D^{\mu}_{x} w(x) \|^2 \leq C_2 \int_a^b a_I RL D^{\mu}_{x} w(x) RL D^{\mu}_{x} w(x) dx,
\]

in which \( RL D^{\mu}_{x} \) can be either \( RL D^{\mu}_{x} \) or \( RL D^{\mu}_{R} \). Hence, \( a_I(u,v) \geq C_1 \| u \|_{U_I} \) and \( a_{II}(u,v) \geq C_{II} \| u \|_{U_{II}} \), so the bilinear forms are coercive and by the Lax–Milgram lemma these problems are well-posed.

4. Regular TFSLPs of kinds I and II. After this preparation and setting the underlying spaces, we now introduce the following regular TFSLP of order \( \nu = 2 \mu \in (0,2) \):

\[
(4.1) \quad TRL D^{\mu,\tau}_{x} p_i(x) TC D^{\mu,\tau}_{x} F^{(i)}(x) + \lambda w_i(x) F^{(i)}(x) = 0, \quad x \in [x_L, x_R],
\]

where \( i = 1, 2 \), with \( i = 1 \) denoting the regular TFSLP of the first kind, in which \( TRL D^{\mu,\tau}_{x} \equiv TRL D^{\mu,\tau}_{x_L} \) and \( TC D^{\mu,\tau}_{x} \equiv TC D^{\mu,\tau}_{x} \). Moreover, \( i = 2 \) corresponds to the regular TFSLP of the second kind where \( TRL D^{\mu,\tau}_{x} \equiv TRL D^{\mu,\tau}_{x_L} \) and \( TC D^{\mu,\tau}_{x} \equiv TC D^{\mu,\tau}_{x_R} \). In such setting, \( \mu \in (0,1) \), \( \tau \geq 0 \), \( p_i(x) \neq 0 \), and \( w_i(x) \) is a nonnegative weight function. In addition, we assume that \( p_i \) and \( w_i \) are real-valued continuous functions in the interval \([x_L, x_R]\). The tempered fractional boundary-value problem (4.1) is subject to the following boundary conditions:

\[
(4.2) \quad a_1 F^{(i)}(x_L) + a_2 TRL D^{\mu,\tau}_{x} p_i(x) TC D^{\mu,\tau}_{x} F^{(i)}(x) |_{x=x_L} = 0,
\]

\[
(4.3) \quad b_1 F^{(i)}(x_R) + b_2 TRL D^{\mu,\tau}_{x} p_i(x) TC D^{\mu,\tau}_{x} F^{(i)}(x) |_{x=x_R} = 0,
\]

where \( a_1^2 + a_2^2 \neq 0 \), \( b_1^2 + b_2^2 \neq 0 \). Here, \( TRL D^{\mu,\tau}_{x} \equiv TRL D_{x_L}^{\mu,\tau} \) when \( i = 1 \) for the regular TFSLP of the first kind, while \( TRL D^{\mu,\tau}_{x} \equiv TRL D_{x_L}^{\mu,\tau} \) when \( i = 2 \) for the regular TFSLP of the second kind.

Theorem 4.1. The eigenvalues of the regular TFSLP of kinds I and II (4.1) subject to the nonlocal boundary conditions (4.2) and (4.3) are real, and the eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight functions \( w_i(x) \).

Proof. We first consider the first regular problem when \( i = 1 \):

\[
TRL D^{\mu,\tau}_{x_L} p_i(x) TC D^{\mu,\tau}_{x} F^{(i)}(x) + \lambda w_i(x) F^{(i)}(x) = 0.
\]
Now, by the definition of the tempered derivatives $\mathcal{T}^{RL}_{x}D_{x}^\mu_{x,R}$ and $\mathcal{T}^{TC}_{x}D_{x}^\mu_{x,R}$, we can rewrite the first problem as

\[
e^{\tau x} \mathcal{T}^{RL}_{x}D_{x}^\mu_{x,R} e^{-\tau x} \left[ p_1(x) e^{-\tau x} \mathcal{C}_{xL} D_{x}^\mu e^{\tau x} F^{(1)}_{\lambda}(x) \right] + \lambda w_1(x) F^{(1)}_{\lambda}(x) = 0,
\]

where we multiply both sides by $e^{-\tau x}$ to obtain

\[
e^{\tau x} \mathcal{T}^{RL}_{x}D_{x}^\mu_{x,R} \left[ e^{-2\tau x} p_1(x) \mathcal{C}_{xL} D_{x}^\mu e^{\tau x} F^{(1)}_{\lambda}(x) \right] + \lambda e^{-2\tau x} w_1(x) \left( e^{\tau x} F^{(1)}_{\lambda}(x) \right) = 0.
\]

By taking $P_1(x) \equiv e^{-2\tau x} p_1(x)$ and $W_1(x) \equiv e^{-2\tau x} w_1(x)$, and $\Phi^{(1)}_{\lambda}(x) \equiv e^{\tau x} F^{(1)}_{\lambda}(x)$, we transform the tempered boundary-value problem of the first kind to

\[
(4.4) \quad \mathcal{T}^{RL}_{x}D_{x}^\mu_{x,R} \left[ P_1(x) \mathcal{C}_{xL} D_{x}^\mu \Phi^{(1)}_{\lambda}(x) \right] + \lambda W_1(x) \Phi^{(1)}_{\lambda}(x) = 0.
\]

Clearly, $P_1(x) \neq 0$ and $W_1(x)$ are continuous functions in $[x_L, x_R]$. Moreover, we obtain the boundary conditions corresponding to this change of variables (functions) from (4.2) and (4.3) by employing the definitions of the tempered fractional integration as

\[
al_1 \Phi^{(1)}_{\lambda}(x_L) + a_2 \mathcal{T}^{RL}_{x}D_{x}^\mu_{x,R} \left[ P_1(x) \mathcal{C}_{xL} D_{x}^\mu \Phi^{(1)}_{\lambda}(x) \right] \big|_{x=x_L} = 0,
\]

\[
b_1 \Phi^{(1)}_{\lambda}(x) + b_2 \mathcal{T}^{RL}_{x}D_{x}^\mu_{x,R} \left[ P_1(x) \mathcal{C}_{xL} D_{x}^\mu \Phi^{(1)}_{\lambda}(x) \right] \big|_{x=x_R} = 0.
\]

Following [13], the transformed eigenproblem has real-valued eigenvalues and eigenfunctions. Moreover, the corresponding eigenfunctions to distinct eigenvalues are orthogonal with respect to $W_1(x)$. Clearly, the transformed problem (4.4) subject to (4.5) and (4.6) shares the same eigenvalues with the original problem (4.1) subject to (4.2) and (4.3), when $i = 1$. Therefore, the eigenvalues of the regular TFSLP of the first kind are real-valued. Moreover, let $\xi^{(1)}_1$ and $\xi^{(1)}_2$ be eigenfunctions corresponding to two distinct eigenvalues $\lambda_1$ and $\lambda_2$. Then, by the orthogonality of the transformed problem, we have

\[
\int_{x_L}^{x_R} \xi^{(1)}_1(x) \xi^{(1)}_2(x) W_1(x) dx = 0,
\]

which can be rewritten by the inverse transformation of the eigenfunctions as

\[
\int_{x_L}^{x_R} \left( e^{\tau x} \xi^{(1)}_1(x) \right) \left( e^{\tau x} \xi^{(1)}_2(x) \right) e^{-2\tau x} w_1(x) dx = 0
\]

or

\[
\int_{x_L}^{x_R} \xi^{(1)}_1(x) \xi^{(1)}_2(x) w_1(x) dx = 0,
\]

where $\xi^{(1)}_1(x)$ and $\xi^{(1)}_2(x)$ are the corresponding real-valued eigenfunctions associated with the distinct eigenvalues $\lambda_1$ and $\lambda_2$.

When $i = 2$, we follow similar steps by taking $P_2(x) \equiv e^{2\tau x} p_2(x)$ and $W_2(x) \equiv e^{2\tau x} w_2(x)$, and $\Phi^{(2)}_{\lambda}(x) \equiv e^{-\tau x} F^{(2)}_{\lambda}(x)$, and through the transformation of the tempered boundary-value problem of the second kind, we complete the proof. \qed
4.1. Regular tempered eigenproblems. We specifically solve two regular TFSLPs, denoted by regular TFSLP-I and -II of order \( \nu = 2\mu \in (0, 2) \), by choosing particular forms \( p_i(x) \) and \( w_i(x) \). To this end, the following tempered nonlocal differential operator is defined:

\[
L_i^{\mu, \tau} := RL \{ e^{(-1)^{i+1}2\tau x} C_D^{\mu, \tau}(\cdot) \}, \quad i = 1, 2,
\]

where \( L_1^{\mu} := RL_x \{ e^{2\tau x} C_D^{\mu}(\cdot) \} \) in the regular TFSLP-I; for the case of the regular TFSLP-II, we reverse the order of the right-sided and left-sided tempered derivative for the inner and outer fractional derivatives in the operator, i.e., \( L_2^{\mu} := RL_x \{ e^{-2\tau x} C_D^{\mu}(\cdot) \} \), where \( \mu \in (0, 1) \). We note that the term \( e^{(-1)^{i+1}2\tau x} \neq 0 \) \( \forall x \in [x_L, x_R] \) yields the regularity character to the tempered boundary-value problem. That being defined, we consider the regular TFSLP (-I and -II) as

\[
\text{(4.8)} \quad L_i^{\mu, \tau} F^{(i)}(x) + \lambda e^{(-1)^{i+1}2\tau x} (1 - x)^{-\mu} (1 + x)^{-\mu} F^{(i)}(x) = 0, \quad i = 1, 2, \quad x \in [-1, 1].
\]

We shall solve (4.8) subject to a homogeneous Dirichlet and a homogeneous fractional integro-differential boundary condition

\[
\text{(4.9)} \quad TRL_{x=1}^{1-\mu, \tau} \left[ e^{2\tau x} TC_{-1}D_x^{\mu}(F^{(1)}(x)) \right] |_{x=1} = 0,
\]

and

\[
\text{(4.10)} \quad TRL_{x=-1}^{1-\mu, \tau} \left[ e^{-2\tau x} TC_{x}D^{\mu}(F^{(2)}(x)) \right] |_{x=-1} = 0,
\]

which are enforced on the regular TFSLP-I and TFSLP-II, respectively.

4.2. Explicit eigensolutions to the regular TFSLP-I and -II. Next, we obtain the analytical solution \( F^{(i)}(x) \) to the regular TFSLP-I and II, (4.8), subject to the homogeneous Dirichlet and integro-differential boundary conditions (4.9) and (4.10).

**Theorem 4.2.** The exact eigenfunctions to (4.8), when \( i = 1 \), i.e., the regular TFSLP-I, subject to (4.9), are given by

\[
F^{(1)}_n(x) = e^{-\tau x} (1 + x)^{\mu} P^{\mu}_{n-1}(x) \quad \forall n \geq 1,
\]

and the corresponding distinct eigenvalues are

\[
\lambda^{(1)}_n = -\frac{\Gamma(n + \mu)}{\Gamma(n - \mu)} \quad \forall n \geq 1.
\]

Moreover, the exact eigenfunctions to (4.8), when \( i = 2 \), i.e., the regular TFSLP-II, subject to (4.10), are given as

\[
F^{(2)}_n(x) = e^{\tau x} (1 - x)^{\mu} P^{\mu}_{n-1}(x) \quad \forall n \geq 1
\]

with the corresponding distinct eigenvalues given by

\[
\lambda^{(2)}_n = \lambda^{(1)}_n = -\frac{\Gamma(n + \mu)}{\Gamma(n - \mu)} \quad \forall n \geq 1.
\]
Proof. First, we prove (4.11) and (4.12). Clearly, \( F_n^{(1)}(-1) = 0 \). Therefore, by (2.7), we substitute \( \mathcal{T}C \mathcal{D}_x^{\mu,\tau} \) by \( \mathcal{T}RL \mathcal{D}_x^{\mu,\tau} \), hence,

\[
\{ e^{2\tau x} \mathcal{T}C \mathcal{D}_x^{\mu,\tau} F_n^{(1)}(x) \}_{x=+1} = \{ e^{2\tau x} \mathcal{T}RL \mathcal{D}_x^{\mu,\tau} F_n^{(1)}(x) \}_{x=+1}
\]

\[
= \{ e^{2\tau x} \mathcal{T}RL \mathcal{D}_x^{\mu,\tau} \left( e^{-\tau x} (1+x)^\mu P_{n-1}^{-\mu}(x) \right) \}_{x=+1}
\]

Following [3, 30] and for \( \mu > 0, \alpha > -1, \beta > -1 \) \( \forall x \in [-1,1] \) we have

\[
\mathcal{R}L \mathcal{T}_x \left\{ (1+x)^\alpha P_n^{\alpha,\beta}(x) \right\} = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta+\mu+1)} (1+x)^{\beta+\mu} P_{n-\mu,\beta+\mu}(x)
\]

and

\[
\mathcal{R}L \mathcal{T}_x \left\{ (1-x)^\alpha P_n^{\alpha,\beta}(x) \right\} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\mu+1)} (1-x)^{\alpha+\mu} P_{n+\mu,\beta-\mu}(x).
\]

The relation (4.16) can be reduced to the case when \( \alpha = +\mu \) and \( \beta = -\mu \) as

\[
\mathcal{R}L \mathcal{T}_x \left\{ (1+x)^{-\mu} P_n^{\mu,-\mu}(x) \right\} = \frac{\Gamma(n-\mu+1)}{\Gamma(n+1)} P_n(x),
\]

where \( P_n(x) = P_n^0(x) \) represents the Legendre polynomial of degree \( n \). On the other hand, we can set \( \alpha = \beta = 0 \) in (4.16) and take the fractional derivative \( \mathcal{R}L \mathcal{D}_x^{\mu} \) on both sides of (4.16) to obtain

\[
\mathcal{R}L \mathcal{D}_x^{\mu} \left\{ (1+x)^\mu P_n^{-\mu}(x) \right\} = \frac{\Gamma(n+\mu+1)}{\Gamma(n+1)} P_n(x).
\]

Now, by carrying out the fractional Riemann–Liouville derivative in the bracket of (4.15) using (4.19), we obtain

\[
\{ e^{x_\tau} \mathcal{R}L \mathcal{T}_x^{1-\mu} \left[ \frac{\Gamma(n+\mu)}{\Gamma(n)} P_{n-1}(x) \right] \}_{x=+1} = e^{x_\tau} \frac{\Gamma(n+\mu)}{\Gamma(n)} \left\{ e^{x_\tau} \mathcal{R}L \mathcal{T}_x^{1-\mu} \left[ P_{n-1}(x) \right] \right\}_{x=+1}.
\]

By working out the fractional integration using (4.17) (when \( \alpha = \beta = 0 \)), we obtain

\[
\{ e^{x_\tau} (1-x)^\mu P_{n-1}^{\mu,-\mu}(x) \}_{x=+1} = 0,
\]

hence \( F_n^{(1)}(x) \) satisfies the boundary conditions.

Next, we show that (4.11) indeed satisfies (4.8), when \( i = 1 \), with the eigenvalues (4.12). First, we multiply both sides of (4.8) by \( e^{-x_\tau} \) and then take the fractional integration of order \( \mu \) on both sides when \( i = 1 \) to obtain

\[
e^{-x_\tau} e^{2x_\tau} \mathcal{T}C \mathcal{D}_x^{\mu,\tau} F_n^{(1)}(x) = -\lambda \mathcal{R}L \mathcal{T}_x^{\mu} \left\{ e^{-x_\tau} e^{2x_\tau} (1-x)^{-\mu} (1+x)^{-\mu} F_n^{(1)}(x) \right\}.
\]
Substituting (4.11) and replacing the Caputo derivative by the Riemann–Liouville one, we obtain
\[ RL_1 \tau^\mu \left[ (1 + x)^\mu \ P_{n-1}^{-\mu,\mu}(x) \right] = -\lambda RL_1 \tau^\mu \left\{ (1 - x)^{-\mu} \ P_{n-1}^{-\mu,\mu}(x) \right\}. \]

Finally, the fractional derivative on the left-hand side and the fractional integration on the right-hand side are worked out using (4.18) and (4.19), respectively, as
\[ \frac{\Gamma(n + \mu)}{\Gamma(n)} P_{n-1}(x) = -\lambda \frac{\Gamma(n - \mu)}{\Gamma(n)} P_{n-1}(x), \]
and therefore
\[ \lambda \equiv \lambda_n^{(1)} = -\frac{\Gamma(n + \mu)}{\Gamma(n - \mu)} \forall n \geq 1. \]

The orthogonality of the eigenfunctions (4.11) with respect to \( w_1(x) = e^{2\tau x} (1 - x)^{-\mu} (1 + x)^{-\mu} \) is checked as
\[
\int_{-1}^{1} w_1(x) F_k^{(1)}(x) F_j^{(1)}(x) dx = \int_{-1}^{1} w_1(x) e^{-2\tau x} \left[ (1 + x)^\mu \right]^2 \ P_{k-1}^{-\mu,\mu}(x) \ P_{j-1}^{-\mu,\mu}(x) dx
\]
\[ = \int_{-1}^{1} (1 - x)^{\mu} (1 + x)^{\mu} \ P_{k-1}^{-\mu,\mu}(x) \ P_{j-1}^{-\mu,\mu}(x) dx
\]
\[ = C_k^{-\mu,\mu} \delta_{kj}, \]
where \( C_k^{-\mu,\mu} \) denotes the orthogonality constant corresponding to Jacobi polynomials with parameters \( ^{-\mu,\mu} \). Similar steps can be taken for the second tempered eigenproblem when \( i = 2 \).

The eigenfunctions of the regular TFSLP-I and -II are nonpolynomial functions due to their particular structure, i.e., the multiplier \( e^{\pm \tau x} (1 \pm x)^\mu \). These eigenfunctions in fact generalize the existing Jacobi poly-fractonomial functions \( (1 \pm x)^\mu \ P_{n-1}^{\pm,\pm}(x) \), introduced in [30] as the exact eigenfunctions of the regular fractional (nontempered) Sturm–Liouville problems (RFSLP-I and -II). We note that when the tempering parameter \( \tau = 0 \), they reduce to the Jacobi poly-fractonomials. Here, to distinguish the tempered eigenfunctions from them, we refer to \( F_n^{(1)}(x) \) as the tempered Jacobi poly-fractonomial. Clearly, when \( \tau = 0 \) and \( \mu \to 1 \) or \( 0 \), the tempered Jacobi poly-fractonomials (4.11) and (4.13) coincide with the well-known Jacobi polynomials, which are the eigenfunctions of the standard (integer-order) SLP.

Remark 4.3. The regular TFSLP-I and -II share the same eigenvalues with RFSLP-I and -II in [30]. We note that the growth in the magnitude of eigenvalues with respect to \( n \) is dependent on the fractional derivative order \( \mu \). The asymptotic values are obtained as
\[
|\lambda_n^{(i)}| = \begin{cases} 
n^2, & \mu \to 1, \\
n, & \mu \to 1/2, \\
1, & \mu \to 0. \end{cases}
\]
Hence, there are two modes of growth in the magnitude of \( \lambda_n^{(i)} \), the sublinear mode corresponding to \( 0 < \mu < 1/2 \) and the superlinear subquadratic mode corresponding to \( 1/2 < \mu < 1 \). In particular, when \( \mu = 1/2 \), the eigenvalues grow linearly with \( n \).
4.3. Properties of the eigenfunctions of the regular TFSLP-I and -II.

Next, we list a number of important properties of the solutions to the regular TFSLP-I and II in (4.8):

- Recurrence relations:
  \[ F_1^{(i)}(x) = e^{-\tau x}(1 + x)^\mu, \]
  \[ F_2^{(i)}(x) = e^{-\tau x}(1 + x)^\mu(x + \mu), \]
  \[ a_nF_{n+1}^{(i)}(x) = b_nxF_{n}^{(i)}(x) - c_nF_{n-1}^{(i)}(x), \]
  \[ a_n = 4n^2(n - 1), \]
  \[ b_n = 2n(2n - 1)(2n - 2), \]
  \[ c_n = 4n(n - 1 + \mu)(n - 1 + \mu), \]

where the upper signs correspond to \( i = 1 \), solution to the regular TFSLP-I, and the lower signs correspond to the regular TFSLP-II when \( i = 2 \).

- Orthogonality:
  \[ \int_{-1}^{1} e^{-(1)^{+1}2\tau x}(1 - x)^{-\mu}(1 + x)^{-\mu} F_k^{(i)}(x)F_m^{(i)}(x)dx = J_k^{\mu_1,\beta_1}, \]

\[ J_k^{\alpha_1,\beta_1} = \frac{2}{2k-1} \frac{\Gamma(k + \alpha_1)\Gamma(k + \beta_1)}{(k-1)!\Gamma(k)}, \]

where \((\alpha_1, \beta_1) = (-\mu, \mu)\) and \((\alpha_2, \beta_2) = (\mu, -\mu)\).

- Fractional derivatives:
  \[ T_{-1}^{RLD_x^{\nu,\tau}} F_n^{(1)}(x) = TC_{-1}^{D_x^{\nu,\tau}} F_n^{(1)}(x) = e^{-\tau x} \frac{\Gamma(n + \mu)}{\Gamma(n)} P_{n-1}(x), \]
  \[ T_{x}^{RLD_x^{\nu,\tau}} F_n^{(2)}(x) = TC_{x}^{D_x^{\nu,\tau}} F_n^{(2)}(x) = e^{\tau x} \frac{\Gamma(n + \mu)}{\Gamma(n)} P_{n-1}(x), \]

where \( P_{n-1}(x) \) denotes the standard Legendre polynomial of order \( n - 1 \).

- Special values:
  \[ F_n^{(1)}(-1) = 0, \quad F_n^{(1)}(+1) = e^{-\tau} 2^\mu \left( \frac{n - 1 - \mu}{n - 1} \right), \]
  \[ F_n^{(2)}(+1) = 0, \quad F_n^{(2)}(-1) = e^{\tau} (-1)^{n-1} F_n^{(1)}(+1). \]

5. Singular tempered fractional problems.

In the regular setting \( p_i(x) \) did not vanish in \([-1, 1]\), and here we aim to generalize it to singular eigenproblems, in which \( p_i(x) = e^{(-1)^{+1}2\tau x}(1 - x)^{\alpha+1}(1 + x)^{\beta+1} \) that vanishes at the boundary points. We present the definition of the singular TFSLP-I and singular TFSLP-II of order \( \nu = 2\mu \in (0, 2) \), associated this time with the additional parameters \(-1 < \alpha < 2 - \mu \) and \(-1 < \beta < \mu - 1 \) in the singular TFSLP-I \((i = 1)\) and \(-1 < \alpha < \mu - 1 \) and \(-1 < \beta < 2 - \mu \) in the singular TFSLP-II \((i = 2)\), for \( x \in [-1, 1] \) as

\[ R_{L} D^{\nu,\tau} \left\{ e^{(-1)^{+1}2\tau x}(1 - x)^{\alpha+1}(1 + x)^{\beta+1} C D^{\nu,\tau} P^{(i)}(x) \right\} + \lambda^{(i)} e^{(-1)^{+1}2\tau x}(1 - x)^{\alpha+1-\mu}(1 + x)^{\beta+1-\mu} P^{(i)}(x) = 0, \]
where $\tau \geq 0$, $\mu \in (0,1)$ and $i = 1, 2$. Similar to the regular setting, $i = 1$ denotes the singular TFSLP-I in which $RL\mathcal{D}_{x}^{\mu, \tau} \equiv TRL\mathcal{D}_{x}^{\mu, \tau}$ and $TC\mathcal{D}_{x}^{\mu, \tau} \equiv TC\mathcal{D}_{x}^{\mu, \tau}$; also, $i = 2$ corresponds to the singular TFSLP-II, where $TRL\mathcal{D}_{x}^{\mu, \tau} \equiv TRL\mathcal{D}_{x}^{\mu, \tau}$ and $TC\mathcal{D}_{x}^{\mu, \tau} \equiv TC\mathcal{D}_{x}^{\mu, \tau}$. The singular problem (5.1) is subject to

$$\begin{align*}
\{ TRL\mathcal{D}_{x}^{1-\mu, \tau} p_i(x) & \}_{x=\tau} = 0, \\
\{ TC\mathcal{D}_{x}^{\mu, \tau} p_i(x) & \}_{x=\tau} = 0,
\end{align*}
$$

(5.2)

in which we follow the same definitions as in the regular problems, i.e., $TRL\mathcal{D}_{x}^{1-\mu, \tau} \equiv TRL\mathcal{D}_{x}^{1-\mu, \tau}$ when $i = 1$ in the singular TFSLP-I, and $TRL\mathcal{D}_{x}^{1-\mu, \tau} \equiv TRL\mathcal{D}_{x}^{1-\mu, \tau}$ in case of $i = 2$ in the singular TFSLP-II; $p_i(x) = e^{-}\left((-1)^{i+1}2x(1-x)^{\alpha+1}(1+x)^{\beta+1}\right)$ and $w_i(x) = e^{-}\left((-1)^{i+1}2x(1-x)^{\alpha+1}(1+x)^{\beta+1-\mu}\right)$ in (5.1), which is a nonnegative function.

**Theorem 5.1.** The eigenvalues of the singular TFSLP-I and -II (5.1)–(5.3) are real-valued, and the eigenfunctions corresponding to distinct eigenvalues of (5.1)–(5.3) are orthogonal with respect to the weight function

$$w_i(x) = e^{-}\left((-1)^{i+1}2x(1-x)^{\alpha+1}(1+x)^{\beta+1-\mu}\right).$$

Moreover, the exact eigenfunctions of the singular TFSLP-I (5.1)–(5.3), when $i = 1$, are given as

$$\begin{align*}
P_n^{(1)}(x) = (1)P_n^{\alpha, \beta, \mu}(x) = e^{-}\left[(1+x)^{-\beta+\mu-1}\right] P_{n-1}^{\alpha-1, -\beta+\mu-1}(x),
\end{align*}
$$

(5.4)

and the corresponding distinct eigenvalues are

$$\Lambda_n^{(1)} = (1)\Lambda_n^{\alpha, \beta, \mu} = \frac{\Gamma(n-\beta+\mu-1)\Gamma(n+\alpha+1)}{\Gamma(n-\beta-1)\Gamma(n+\alpha-\mu+1)}.$$

(5.5)

In addition, the exact eigenfunctions to the singular TFSLP-II (5.1)–(5.3), in case of $i = 2$, are given as

$$\begin{align*}
P_n^{(2)}(x) = (2)P_n^{\alpha, \beta, \mu}(x) = e^{\tau x}\left(1-x\right)^{-\alpha+\mu-1} P_{n-1}^{\alpha-1, -\beta-\mu+1}(x),
\end{align*}
$$

(5.6)

and the corresponding distinct eigenvalues are

$$\Lambda_n^{(2)} = (2)\Lambda_n^{\alpha, \beta, \mu} = \frac{\Gamma(n-\alpha+2\mu-1)\Gamma(n+\beta+1)}{\Gamma(n-\alpha+\mu-1)\Gamma(n+\beta-\mu+1)}.$$

(5.7)

**Proof.** By the definition of the tempered fractional derivatives and integrations, it is easy to check that (5.1) can be transformed into the singular fractional Sturm–Liouville in [30] through $W_i^{(2)} = e^{-}\left((-1)^{i+1}2x\right)w_i(x)$, $P_i^{(2)} = e^{-}\left((-1)^{i+1}2x\right)p_i(x)$, and $P^{(i)}(x) = e^{-}\left((-1)^{i+1}2x\right)P^{(i)}(x)$. Then, readily, the proof is complete following Theorems 4.1 and 4.2 in [30].

**Remark 5.2.** In the earlier work [30], the standard singular SLP was generalized to the fractional (nontempered) Sturm–Liouville problems (SFSLP-I and -II) and the singular Jacobi poly-fractonomials of the first and second kinds given in (1.1) and (1.2), which were explicitly obtained as the eigenfunctions. Here, the tempered Jacobi poly-fractonomials involve another parameter, i.e., $\tau \geq 0$, which further generalizes the existing singular fractional eigenfunctions, and they in fact complete the whole family of tempered Jacobi poly-fractonomials. Moreover, we note that the singular TFSLP-I and -II also share the same eigenvalues, hence the same asymptotic values, with SFSLP-I and -II as in [30].
5.1. Properties of the eigensolutions to the singular TFSLP-I and -II.

We list a number of properties of the eigensolutions to the singular TFSLP-I and -II as follows:

• Recurrence relations:

A recurrence relations is obtained for the tempered Jacobi poly-fractonomials $(1)_{P_n}^{\alpha,\beta,\mu}(x)$ of the first kind, when $i = 1$,

\[
(1)_{P_1}^{\alpha,\beta,\mu}(x) = e^{-\tau x}(1 + x)^{-\beta - \mu - 1},
\]

\[
(1)_{P_2}^{\alpha,\beta,\mu}(x) = \frac{1}{2} e^{-\tau x}(1 + x)^{-\beta - \mu - 1} [\alpha + \beta - 2\mu + 2 + (\alpha - \beta + 2)x],
\]

\[
\vdots
\]

\[
a_n(1)_{P_{n+1}}^{\alpha,\beta,\mu}(x) = (b_n + c_n x)(1)_{P_n}^{\alpha,\beta,\mu}(x) - d_n(1)_{P_{n-1}}^{\alpha,\beta,\mu}(x),
\]

\[
a_n = 2n(n + \alpha - \beta)(2n + \alpha - \beta - 2),
\]

\[
b_n = (2n - \alpha + \beta - 1)(\alpha - \beta)(\alpha + \beta - 2\mu + 2),
\]

\[
c_n = (2n + \alpha + \beta)(2n + \alpha - \beta - 1)(2n + \alpha - \beta - 2),
\]

\[
d_n = 2(n - \alpha + \mu - 2)(n + \beta - \mu)(2n + \alpha + \beta),
\]

and the second kind when $i = 2$,

\[
(2)_{P_1}^{\alpha,\beta,\mu}(x) = e^{-\tau x}(1 - x)^{-\alpha + \mu - 1},
\]

\[
(2)_{P_2}^{\alpha,\beta,\mu}(x) = \frac{1}{2} e^{-\tau x}(1 - x)^{-\alpha + \mu - 1} [-\alpha - 2\mu - 2 + (-\alpha + \beta + 2)x],
\]

\[
\vdots
\]

\[
a_n^*(2)_{P_{n+1}}^{\alpha,\beta,\mu}(x) = (b_n^* + c_n^* x)(2)_{P_n}^{\alpha,\beta,\mu}(x) - d_n^*(2)_{P_{n-1}}^{\alpha,\beta,\mu}(x),
\]

\[
a_n^* = 2n(n + \alpha + \beta)(2n - \alpha + \beta - 2),
\]

\[
b_n^* = (2n - \alpha + \beta - 1)(\alpha - \beta)(\alpha + \beta - 2\mu + 2),
\]

\[
c_n^* = (2n + \alpha + \beta)(2n + \alpha - \beta - 1)(2n + \alpha - \beta - 2),
\]

\[
d_n^* = 2(n + \alpha - \mu)(n - \beta + \mu - 2)(2n + \alpha - \beta).
\]

• Orthogonality:

\[
\int_{-1}^{1} e^{\pm 2\tau x}(1 - x)^{\alpha + 1 - \mu}(1 + x)^{\beta + 1 - \mu} (1)_{P_k}^{\alpha,\beta,\mu}(x) (1)_{P_j}^{\alpha,\beta,\mu}(x) dx = (i)_{C_k^{\alpha,\beta}} \delta_{kj},
\]

\[
(1)_{C_k^{\alpha,\beta}} = \frac{2^{\alpha - \beta + 1}}{2k + \alpha - \beta - 1} \frac{\Gamma(k + \alpha - \mu + 1)\Gamma(k - \beta + \mu - 1)}{(k - 1)!\Gamma(k + \alpha - \beta)},
\]

\[
(2)_{C_k^{\alpha,\beta}} = \frac{2^{\alpha + \beta + 1}}{2k - \alpha + \beta - 1} \frac{\Gamma(k - \alpha + \mu - 1)\Gamma(k + \beta - \mu + 1)}{(k - 1)!\Gamma(k - \alpha + \beta)}.
\]

• Fractional derivatives:

\[
RL_{-1} x^{-\beta + \mu + 1, \tau} (1)_{P_n}^{\alpha,\beta,\mu}(x) = C_{D_n^{-\beta + \mu + 1}}^{\alpha,\beta,\mu}(x) = e^{\tau x} \frac{\Gamma(n + \mu)}{\Gamma(n)} P_{n-1}^{\beta - \alpha,0}(x),
\]

\[
RL_{x^\tau}^{\alpha + \mu - 1} (2)_{P_n}^{\alpha,\beta,\mu}(x) = C_{D_n^{-\alpha + \mu - 1}}^{\alpha,\beta,\mu}(x) = e^{-\tau x} \frac{\Gamma(\alpha + \mu)}{\Gamma(n)} P_{n-1}^{\beta - \alpha,0}(x),
\]

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.
where $P_{n-1}^{\alpha-\beta}(x)$ and $P_{n}^{0,\beta-\alpha}(x)$ denote the standard Jacobi polynomials.

- Special values:
  \[
  (1) P_n^{\alpha,\beta}(x) = 0, \quad (1) P_n^{\alpha,\beta}(+1) = 2^{-\beta+\mu-1} \left(\frac{n + \alpha - \mu}{n - 1}\right) e^{-\tau},
  \]
  \[
  (2) P_n^{\alpha,\beta}(+1) = 0, \quad (2) P_n^{\alpha,\beta}(x) = 2^{-\alpha+\mu-1} \left(\frac{n + \beta - \mu}{n - 1}\right) e^{-\tau}.
  \]

6. Approximability of the tempered eigenfunctions. We introduce the tempered eigenfunctions (regular and singular) as new basis functions in $L^2_{\alpha}(\mathbb{R})$. Then, we discuss how fast the expansion coefficients of the approximation would decay.

**Theorem 6.1.** The set of regular eigenfunctions $\{F_n^{(i)}(x) : n = 1, 2, \ldots\}$ and the set of singular eigenfunctions $\{P_n^{\alpha,\beta}(x) : n = 1, 2, \ldots\}$ each forms a basis for the infinite-dimensional Hilbert space $L^2_{\alpha}(\mathbb{R})$, and the corresponding eigenvalues $\lambda_n^{(i)}$ and $\Lambda_n^{(i)}$ are simple.

**Proof.** It suffices to prove the regular part with the singular part following similar steps. Let $f(x) \in L^2_{\alpha}(\mathbb{R})$ and then clearly $g(x) = e^{\pm \tau x}(1 \pm x)^{-\mu} f(x) \in L^2_{\alpha}(\mathbb{R})$ as well when $\mu \in (0, 1)$. Considering the upper signs to correspond to the regular TFSLP-I, $i = 1$, and the lower signs to correspond to the case $i = 2$, i.e., the regular TFSLP-II, we have

\[
\left\| \sum_{n=1}^{N} a_n F_n^{(i)}(x) - f(x) \right\|_{L^2_{\alpha}(\mathbb{R})} = \left\| e^{\pm \tau x}(1 \pm x)^\mu \left( \sum_{n=1}^{N} a_n P_n^{\pm \mu}(x) - e^{\pm \tau x}(1 \pm x)^{-\mu} f(x) \right) \right\|_{L^2_{\alpha}(\mathbb{R})}.
\]

Therefore by the Weierstrass theorem, the set of tempered eigenfunctions $\{F_n^{(i)}(x) : n = 1, 2, \ldots\}$ is dense in the Hilbert space and it forms a basis for $L^2_{\alpha}(\mathbb{R})$.

To show the simplicity of the eigenvalues, assume that corresponding to the eigenvalue $\lambda_n^{(i)}$, there exists another eigenfunction $F_j^{(i)}(x) \in L^2_{\alpha}(\mathbb{R})$ in addition to $F_j^{(i)}(x)$, which by Theorem 4.1 is orthogonal to the rest of the eigenfunctions $F_n^{(i)}(x), n \neq j$. By the density of the eigenfunctions set, i.e., (6.1), we can represent $F_j^{(i)}(x)$ as

\[
F_j^{(i)}(x) = \sum_{n=1}^{\infty} a_n F_n^{(i)}(x).
\]
Now, by multiplying both sides by $F_k^{(i)}(x)$, $k = 1, 2, \ldots$ and $k \neq j$, and integrating with respect to the weight function $w_i(x)$ we obtain

$$
(6.2) \int_{-1}^{1} w_i(x) F_k^{(i)}(x) F_k^{(i)}(x) dx = \sum_{n=1}^{\infty} a_n \int_{-1}^{1} w_i(x) F_k^{(i)}(x) F_n^{(i)}(x) dx = a_k C_k \neq 0,
$$

which contradicts Theorem 4.1, and this completes the proof. $\square$

### 6.1. Spectral approximation using singular tempered basis $(i)P_n^{\alpha,\beta,\mu}(x)$, $\mu \in (0, 1)$. Next, we study the approximation properties of the family of Jacobi poly-fractonomials $(i)P_n^{\alpha,\beta,\mu}(x)$ by representing $f(x) \in L^2_{w}[-1, 1]$ as

$$
(6.3) \quad f(x) \approx \sum_{n=1}^{N} \hat{f}_n (i)P_n^{\alpha,\beta,\mu}(x), \quad x \in [-1, 1].
$$

Here, we need to understand how fast the expansion coefficients $\hat{f}_n$ decay by $N$. To this end, we multiply (6.3) by $L_i^{\alpha,\beta,\mu}(i)P_k^{\alpha,\beta,\mu}(x)$, $k = 1, 2, \ldots, N$, and carry out the integration over $[-1, 1]$ to obtain

$$
\int_{-1}^{1} f(x) L_i^{\alpha,\beta,\mu}(i)P_k^{\alpha,\beta,\mu}(x) dx = \int_{-1}^{1} \left( \sum_{n=1}^{N} \hat{f}_n (i)P_n^{\alpha,\beta,\mu}(x) \right) L_i^{\alpha,\beta,\mu}(i)P_k^{\alpha,\beta,\mu}(x) dx,
$$

where $L_i^{\alpha,\beta,\mu}(i)P_k^{\alpha,\beta,\mu}(x)$ on the right-hand side can be substituted by the right-hand side of (5.1), i.e., $-\Lambda_n^{(i)} w(x) (i)P_k^{\alpha,\beta,\mu}(x)$ as

$$
\int_{-1}^{1} f(x) L_i^{\alpha,\beta,\mu}(i)P_k^{\alpha,\beta,\mu}(x) dx = \sum_{n=1}^{N} -\hat{f}_n \Lambda_n^{(i)} \int_{-1}^{1} (1 - x)^{\alpha+1-\mu}(1 + x)^{\beta+1-\mu} (i)P_n^{\alpha,\beta,\mu}(x) (i)P_k^{\alpha,\beta,\mu}(x),
$$

and by the orthogonality property (5.8) we get

$$
(6.4) \quad \hat{f}_k = \frac{-1}{\Lambda_k^{(i)}} \int_{-1}^{1} f(x) RL \mu \left\{ (1 - x)^{\alpha+1}(1 + x)^{\beta+1} C \mu (i)P_k^{\alpha,\beta,\mu}(x) \right\} dx.
$$

Now, by carrying out the fractional integration-by-parts using Lemmas 2.1 and 2.2, or equivalently using (2.9) and (2.10), we obtain

$$
(6.5) \quad \hat{f}_k = \frac{-1}{\Lambda_k^{(i)}} \int_{-1}^{1} (1 - x)^{\alpha+1}(1 + x)^{\beta+1} \left( C \mu (i)P_k^{\alpha,\beta,\mu}(x) \right) \left( C \mu f(x) \right) dx.
$$

Again, by Lemmas 2.1 and 2.2, we obtain

$$
\hat{f}_k = \frac{-1}{\Lambda_k^{(i)}} \int_{-1}^{1} (i)P_k^{\alpha,\beta,\mu}(x) RL \mu \left\{ (1 - x)^{\alpha+1}(1 + x)^{\beta+1} C \mu f(x) \right\} dx,
$$
or equivalently
\[
\hat{f}_k = \frac{-1}{(\alpha^\beta L_k^\alpha)^{1/2}} \int_{-1}^{1} (i)^{w,\beta} f(x) L_i^{\alpha,\beta}(x) dx,
\]
if denoted by \( f(x) = L_i^{\alpha,\beta}[f(x)] \in L_w^2[-1,1] \). By carrying out the fractional integration-by-parts another \((m-1)\) times, and setting \( f_{(m)}(x) = L_i^{\alpha,\beta}[f_{(m-1)}(x)] \in L_w^2[-1,1] \), we obtain
\[
\hat{f}_k \approx C_{\text{sing}} \frac{\| f_{(m)}(x) \|_{L_w^2}}{|A_k^{(1)}|^{m}}, \quad k = 1, 2, \ldots, N.
\]
Consequently, when \( m \to \infty \) and \( f_{(m)}(x) \in L_w^2[-1,1] \), we recover the spectral decay of the expansion coefficients \( \hat{f}_k \).

### 6.2. Numerical approximation
Here, we test the performance of the tempered Jacobi poly-fractonomials as basis functions in approximating some tempered functions, involved with some fractional character, also in developing a Petrov–Galerkin method for TFODEs, followed with the well-posedness and convergence analyses.

#### 6.2.1. Function approximation
Following Theorem 6.1, we employ tempered eigenfunctions as a complete basis, and next we examine their efficiency in approximating \( f(x) \in L_w^2[-1,1] \). We note that both regular and singular tempered bases share the same structure. Hence, taking \( \alpha, \beta \to -1 \), we represent \( f(x) \) in \( x \in [-1,1] \) as
\[
f(x) \approx \sum_{n=1}^{N} \hat{f}_n (i)^{w-1,\alpha}(x) = \sum_{n=1}^{N} \hat{f}_n e^{-\tau x}(1+x)^\mu P_n^{-\mu,\mu}(x).
\]
In order to obtain the unknown coefficients \( \hat{f}_n \), we multiply both sides of (6.7) by \( u_l(x) F_{m}^{(1)}(x) \) and integrate over \([-1,1]\), where by the orthogonality property of the eigenfunctions (see Theorem 4.1), we obtain
\[
\hat{f}_m = \frac{1}{J_m} \left( f(x), F_{m}^{(1)}(x) \right)_{L_w^2([-1,1])},
\]
where \( J_m \) is given by (4.23).

In Figure 1, we plot the log-log \( L^2 \)-error of approximating \( f(x) = e^{-x}(1+x)^{1/2} \), the simplest tempered fractonomial, versus \( N \), the number of terms in the expansion (6.7) when, instead, the Legendre polynomials are used as the basis functions. While only one term is needed to accurately capture \( f(x) \) employing the tempered poly-fractonomials, this plot exhibits the deficiency of using standard polynomials to approximate functions whose (higher) derivatives happen to be singular on the boundaries. Moreover, in Figure 2, we compare the performance of tempered and Legendre bases in approximating \( f(x) = \sin(\pi e^{-x}(1+x)^{1/2}) \) (left), and \( f(x) = (1+x)^{2/3}\exp(-x)\sin(\pi x) \) (right), where the tempered poly-fractonomials outperform the Legendre polynomials by orders of magnitude.

#### 6.2.2. Petrov–Galerkin method for tempered fractional differential equations
We now test the efficiency of the tempered poly-fractonomials in solving
some tempered fractional differential equations. To this end, we develop a Petrov–Galerkin spectral method for the following problem:

\[
\frac{T_{RL}^\mu,\tau}{-1}D_x^{2\mu,\tau}u(x) = f(x), \quad x \in (-1, 1],
\]

\[
u(-1) = 0,
\]

where \( \tau \geq \) and \( 2\mu \in (0, 1) \). We choose \( U_N \subset U_I \) to be the space of basis (trial) functions, defined in terms of the following eigenfunction of the first kind as

\[
U_N = \text{span}\{F_n^{(1)}(x) \quad \forall n \geq 1\},
\]

whose elements, i.e., any linear combinations of \( F_n^{(1)}(x) = e^{-\tau x}(1 + x)\mu \ P_{n-\nu}^{-\mu}(x) \), satisfy the left-boundary condition. Here, we note that when \( \alpha, \beta \to -1 \), the singular bases approach the regular ones. For simplicity and to avoid dealing with extra parameters \( \alpha \) and \( \beta \), we set them to \(-1\). Moreover, we consider \( V_N \subset U_{II} \) as the space of test functions to be constructed as

\[
V_N = \text{span}\{F_n^{(2)}(x) \quad \forall n \geq 1\},
\]

where we recall \( F_n^{(2)}(x) = e^{\tau x}(1 + x)^\mu \ P_{n-\nu}^{\mu}(x) \). Noting Remark 2.3, it is straightforward to check that \( \forall u \in U_N \) and \( w \in V_N \), we have the following bilinear form:

\[
a(u, w) = \left( \frac{T_{RL}^\mu,\tau}{-1}D_x^{2\mu,\tau}u, w \right)_{\Omega} = \left( \frac{T_{RL}^\mu,\tau}{-1}D_x^{\mu,\tau}u, \frac{T_{RL}^\mu,\tau}{-1}D_x^{\mu,\tau}w \right)_{\Omega}.
\]
TEMPERED FRACTIONAL STURM–LIOUVILLE EIGENPROBLEMS

Fig. 3. Petrov–Galerkin scheme for TFODE (6.9): $L^2$-error versus $N$, the number of expansion terms in (6.14), corresponding to the limit fractional orders $2\mu = 1/10$ and $2\mu = 9/10$. Here, the exact solutions are $u(x) = e^{-x}(1 + x)^5$ (left) and $u(x) = e^{1+x+x^2}(1 + x)^2$ (right).

Hence, we consider the corresponding weak form of (6.9), which reads as follows: find $u_N \in U_N$ such that

\begin{equation}
(6.13) \quad a(u_N, w) := \left( TRL_{-1}^{\mu} u_N, TRL_{x1}^{\mu} w \right)_\Omega = \left( f, w \right)_\Omega \quad \forall w \in V_N,
\end{equation}

which becomes equivalent to the strong form (6.9) when $u_N \approx u$ possesses sufficient smoothness. Next, we seek the approximate solution $u_N$ of the form

\begin{equation}
(6.14) \quad u_N(x) = \sum_{n=1}^{N} b_n F_n^{(1)}(x),
\end{equation}

where $b_n$ are the unknown expansion coefficients. By plugging (6.14) into (6.13), which enforces the residual $R_N(x) = TRL_{-1}^{\mu} u_N(x) - f(t)$ to be $L^2$-orthogonal to all elements in $V_N$, we obtain

\begin{equation}
\sum_{n=1}^{N} b_n \left( TRL_{-1}^{\mu} F_n^{(1)}(x), TRL_{x1}^{\mu} F_k^{(2)}(x) \right)_\Omega = \left( f(x), F_k^{(2)}(x) \right)_\Omega, \quad k = 1, 2, \ldots, N,
\end{equation}

which yields a diagonal stiffness matrix on the left-hand side, due to (4.24) and (4.25), whose diagonal entries are given by $\gamma_k = \left( \frac{\Gamma(k+\mu+1)}{\Gamma(k+1)} \right)^2 \frac{2}{2k-1}$. Consequently, we obtain the expansion coefficients as

\begin{equation}
(6.15) \quad b_k = \frac{1}{\gamma_k} \left( f, F_k^{(2)}(x) \right)_\Omega.
\end{equation}

In Figure 3, we study the convergence of the proposed Petrov–Galerkin scheme. We plot the log-log $L^2$-error versus $N$, the number of expansion terms in (6.14), corresponding to the limit fractional orders $2\mu = 1/10$ and $2\mu = 9/10$, considering two different exact solutions: (i) $u(x) = e^{-x}(1 + x)^5$ (left) and (ii) $u(x) = e^{1+x+x^2}(1 + x)^2$ (right). These plots show the spectral mode of convergence in the Petrov–Galerkin spectral method. In what follows, we further provide the stability and convergence analysis of the method.
6.3. Stability and convergence analysis. We carry out the discrete stability analysis given the pair of $U_N \subset U_I$ and $V_N \subset V_I$.

Case I. $0 < \mu < 1/2$. We represent $u_N$ as

\begin{equation}
    u_N(x) = \sum_{n=1}^{N} \hat{u}_n F_n^{(1)}(x)
\end{equation}

and choose $v_N$ to be the following linear combination of elements in $V_N$:

\begin{equation}
    v_N(x) = \sum_{k=1}^{N} \hat{v}_k F_k^{(2)}(x).
\end{equation}

Hence, we obtain

\begin{equation}
    a(u_N, v_N) = \left( -1 \mathbb{D}_x^{\mu,\tau} u_N , x \mathbb{D}_1^{\mu,\tau} v_N \right)_{\Omega}
    = \sum_{n=1}^{N} \hat{u}_n \sum_{k=1}^{N} \hat{v}_k \int_{-1}^{1} -1 \mathbb{D}_x^{\mu,\tau} F_n^{(1)}(x) x \mathbb{D}_1^{\mu} F_k^{(2)}(x) \, dx
    = \sum_{n=1}^{N} \hat{u}_n \frac{\Gamma(n + \mu)}{\Gamma(n)} \sum_{k=1}^{N} \hat{v}_k \frac{\Gamma(k + \mu)}{\Gamma(k)} \int_{-1}^{1} P_{n-1}(x) P_{k-1}(x) \, dx
    = \sum_{n=1}^{N} \hat{u}_n^2 \left( \frac{\Gamma(n + \mu)}{\Gamma(n)} \right)^2 \frac{2}{2n - 1}.
\end{equation}

Moreover, we have

\begin{equation}
    \|v_N\|_V^2 = \| x \mathbb{D}_1^{\mu,\tau} v_N \|_{L^2([-1, 1])}^2 = \int_{-1}^{1} \left( \sum_{k=1}^{N} \hat{v}_k e^{\tau x} \frac{\Gamma(k + \mu)}{\Gamma(k)} P_{k-1}(x) \right)^2 \, dx,
\end{equation}

by which we observe that

\begin{equation}
    \|v_N\|_V^2 \geq C_1^2 \int_{-1}^{1} \left( \sum_{k=1}^{N} \hat{v}_k \frac{\Gamma(k + \mu)}{\Gamma(k)} P_{k-1}(x) \right)^2 \, dx = C_1^2 a(u_N, v_N),
\end{equation}

\begin{equation}
    \|v_N\|_V^2 \leq C_2^2 \int_{-1}^{1} \left( \sum_{k=1}^{N} \hat{v}_k \frac{\Gamma(k + \mu)}{\Gamma(k)} P_{k-1}(x) \right)^2 \, dx = C_2^2 a(u_N, v_N),
\end{equation}

where $C_1 = \min_{-1 \leq x \leq 1} (|e^{\tau x}|)$ and $C_2 = \max_{-1 \leq x \leq 1} (|e^{\tau x}|)$. We can obtain similar results for $\|u_N\|_U^2$ as

\begin{equation}
    c_1^2 a(u_N, v_N) \leq \|u_N\|_U^2 \leq c_2^2 a(u_N, v_N),
\end{equation}

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.
where \( c_1 = \min_{-1 \leq x \leq 1} (|e^{-tx}|) \) and \( c_2 = \max_{-1 \leq x \leq 1} (|e^{-tx}|) \). Hence, there exists a positive constant \( C \) such that \( \|v_N\|_V \leq C\|u_N\|_U \), by which and using the right inequality of (6.21) we obtain

\[
(6.22) \quad \sup_{v_N \in V_N} \frac{a(u_N, v_N)}{\|v_N\|_V} \geq \frac{1}{c_2} \frac{\|u_N\|_U^2}{C\|u_N\|_U} = \frac{1}{Cc_2^2}\|u_N\|_U \quad \forall u_N \in U_N,
\]

that is, stability is guaranteed for \( \beta = \frac{1}{Cc_2^2} \). Therefore, Cea's lemma holds,

\[
(6.23) \quad \|u - u_N\|_U \leq \left( 1 + \frac{M}{\beta} \right) \|u - \tilde{u}_N\|_U \quad \forall u_N \in U_N,
\]

in which the continuity constant \( M = 1 \). Next, in order to obtain the corresponding (projection) error estimates, we expand the exact solution \( u \), when \( 2\mu \in (0,1) \), in terms of the following infinite series of tempered Jacobi poly-fractional moments:

\[
(6.24) \quad u(x) = \sum_{n=1}^{\infty} \hat{u}_n e^{-tx}(1 + x)^{\mu} P_{n-1}^{\mu}(x).
\]

Here, we would like to bound \( \|u - u_N\|_U \) in terms of higher-order derivatives. We first note that

\[
-1D_x^{r+\mu} u(x) = e^{-tx}\left[-1D_x^{r+\mu} e^{-tx} u(x)\right] = e^{-tx}\left[\frac{d^r}{dx^r} -1D_x^\mu D_x^{-r} e^{-tx} u(x)\right]
\]

\[
= \sum_{n=1}^{\infty} \hat{u}_n \frac{\Gamma(n + \mu)}{\Gamma(n)} e^{-tx} \frac{d^r}{dx^r} [P_{n-1}(x)],
\]

where

\[
\frac{d^r}{dx^r} [P_{n-1}(x)] = \begin{cases} \frac{(n-1+r)!}{2r(n-1)!} P_{n-1-r}^{r,r}(x), & r \leq n, \\ 0, & r > n. \end{cases}
\]

Therefore,

\[
-1D_x^{r+\mu} u(x) = \sum_{n=r}^{\infty} \hat{u}_n \frac{\Gamma(n + \mu)}{\Gamma(n)} \frac{(n-1+r)!}{2r(n-1)!} e^{-tx} P_{n-1-r}^{r,r}(x).
\]

Hence,

\[
\|(1-x)^{r/2}(1+x)^{r/2} -1D_x^{r+\mu} u(x)\|_{L^2((-1,1))}
\]

\[
= \int_{-1}^{1} (1-x)^r (1+x)^r \left( \sum_{n=r}^{\infty} \hat{u}_n \frac{\Gamma(n + \mu)}{\Gamma(n)} \frac{(n-1+r)!}{2r(n-1)!} e^{-tx} P_{n-1-r}^{r,r}(x) \right)^2 dx
\]

\[
\geq \sum_{n=r}^{\infty} \left[ \frac{\hat{u}_n \Gamma(n + \mu)}{\Gamma(n)} \frac{(n-1+r)!}{2r(n-1)!} \right]^2 \int_{-1}^{1} e^{-2tx} (1-x)^r (1+x)^r (P_{n-1-r}^{r,r}(x))^2 dx
\]

\[
\geq C \sum_{n=r}^{\infty} \left[ \frac{\hat{u}_n \Gamma(n + \mu)}{\Gamma(n)} \frac{(n-1+r)!}{2r(n-1)!} \right]^2 \int_{-1}^{1} (1-x)^r (1+x)^r (P_{n-1-r}^{r,r}(x))^2 dx
\]

\[
= C \sum_{n=r}^{\infty} \left[ \frac{\hat{u}_n \Gamma(n + \mu)}{\Gamma(n)} \right]^2 \frac{2}{2n+1} \frac{(n-1+r)!}{(n-1-r)!}.
\]
where \(C = \min_{x \in [-1,1]} e^{-2rx} = e^{-2\tau}\). Noting that \(\frac{(n-1+r)!}{(n-1-r)!}\) is minimized when \(n = N + 1\), we have

\[
\|u - u_N\|_2^2 \leq \sum_{n=N+1}^{\infty} \left[ \hat{a}_n \frac{\Gamma(n + \mu)}{\Gamma(n)} \right]^2 \leq \sum_{n=N+1}^{\infty} \left[ \hat{a}_n \frac{\Gamma(n + \mu)}{\Gamma(n)} \right]^2 \frac{(n-1+r)!}{(n-1-r)!} \frac{(N-r)!}{(N+r)!} \\
\leq \frac{1}{C} \frac{(N-r)!}{(N+r)!} \| -1D_x^{r+\mu}u(x)\|_{L^2([-1,1])}^2 \\
\leq cN^{-2r} \| -1D_x^{r+\mu}u(x)\|_{L^2([-1,1])}^2,
\]

(6.25)

where \(r \geq 1\) and \(2\mu \in (0,1)\).

**7. Summary.** The development of fast and accurate global schemes for tempered nonlocal problems requires a proper spectral theory providing efficient choice of basis functions. Such operators appear in the continuum-time random walk model by incorporating waiting times and/or non-Gaussian jump distributions with divergent second moments to account for Lévy flights. Exponentially tempering such distributions results in tempered-stable Lévy processes with finite moments and continuous limit arising as tempered fractional diffusion equation.

In this study, we first presented a regular tempered fractional Sturm–Liouville (TFSLP) problem of two kinds, regular TFSLP-I and regular TFSLP-II, of order \(\nu \in (0,2)\) employing both tempered Riemann–Liouville and tempered Caputo fractional derivatives of order \(\mu = \nu/2 \in (0,1)\). We formulated the boundary-value problem by establishing the well-posedness of that, and then proving that the eigenvalues of the regular tempered problems are real-valued and the corresponding eigenfunctions are orthogonal. Next, we explicitly obtained the eigensolutions to the regular TFSLP-I and -II defined as tempered Jacobi poly-fractonomials. These eigenfunctions were shown to be orthogonal with respect to the weight function associated with the regular TFSLP-I and -II. We also showed that when the tempering parameter \(\tau = 0\), such eigenfunctions reduce to the regular Jacobi poly-fractonomials introduced in [30]. We demonstrated that such tempered eigensolutions enjoy many other attractive properties, such as recurrence structure and having exact fractional derivatives and integrals.

In addition, we extended the fractional operators to a new family of singular TFSLPs of two kinds, singular TFSLP-I and singular TFSLP-II. Subsequently, we obtained the eigensolutions to the singular TFSLP-I and -II analytically, also as nonpolynomial functions, hence completing the whole family of the tempered Jacobi poly-fractonomials. Finally, we introduced the approximation properties of such eigenfunctions by introducing them as new basis (and test) functions. Moreover, we developed a Petrov–Galerkin spectral method for solving tempered fractional ODEs, for which the corresponding stability and convergence analyses were carried out.

**REFERENCES**


