Chapter 5

Hyperbolic Equations

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1. Linear Advection Equation

Linearized wave equation: Deceptively “simple-looking” Equation, non-dissipative

\[
\frac{\partial \Theta}{\partial t} + U \frac{\partial \Theta}{\partial x} = 0 \quad \text{(or } \frac{D\Theta}{Dt} = 0 \text{)}
\]

therefore heat is strictly (passively) advected by the flow \(U(x,t)\)

- periodic B.C. \(\Theta(0, t) = \Theta(1, t)\)
- I.C. \(\Theta(x, 0) = \Theta_0(x) = \sin 2\pi x\)

| x = 0 | x = 1 |

- exact solution:
  \[
  \Theta = \Theta_0(x - Ut), \quad \text{wave}
  \]
  \[
  \Theta_t = -U\Theta_0'(x - Ut)
  \]
  \[
  \Theta_x = \Theta_0'(x - Ut)
  \]

Thus, solution is a traveling wave to the right with speed \(U\).

Note: Initial Condition very important in nonlinear case

\[
\left( U\Theta_x \rightarrow \Theta \frac{\partial \Theta}{\partial x} \right)
\]

(shock waves correspond to high wave number.)
Graphically

2. Review of Basic Properties of PDEs

**Dispersion:** Phase speed of structure-solution independent (non-dispersive property) of the wavenumber. Assume solution:

\[
\Theta = \text{Re} \left\{ \left( e^{i2\pi kx} \right) \left( e^{-i2\pi \omega t} \right) \right\} \\
= \text{Re} \left\{ e^{i2\pi k(x-\frac{\omega}{c} t)} \right\}
\]

with amplitude “constant”; \( \omega \) is eigenvalue. (Separation of variables), periodic b.c.

Define phase speed:

\[ C_\phi = \frac{\omega}{k} \]
Plug in:
\[
\begin{align*}
\frac{\partial \Theta}{\partial t} &= -i2\pi \omega \Theta \\
-U \frac{\partial \Theta}{\partial x} &= -2\pi ki\Theta U
\end{align*}
\]
\[
\Rightarrow \omega = kU \Rightarrow \frac{\omega}{k} = C_\phi = U
\]

Thus, waveforms are not dispersed as time progresses, they are also non-dissipative \((\nu \nabla^2 \Theta = 0)\)

In contrast, in heat equation

\[
\frac{\partial \Theta}{\partial t} = \frac{\partial^2 \Theta}{\partial x^2} \rightarrow \Theta = Re\{e^{\sigma t}e^{i2\pi k x}\}
\]

where \(\sigma = \text{eigenvalue.}\)

\[
\frac{\partial \Theta}{\partial t} = \sigma \Theta = -(2\pi k)^2 \Theta = \frac{\partial^2 \Theta}{\partial x^2}
\]

\(\sigma = \text{attenuation factor} \rightarrow \Theta \text{ is damped (dissipative).}\)

In contrast

\[
\Theta_t = \Theta_{xxx} \rightarrow \Theta = Re\{e^{i2\pi k x}e^{-i2\pi t}\}
\]

\[
\frac{\partial \Theta}{\partial t} = -i2\pi \omega \Theta = -i(2\pi k)^3 \Theta = \frac{\partial^3 \Theta}{\partial x^3}
\]

\[
\Rightarrow \frac{\omega}{k} = C_\phi = (2\pi k)^2 \text{ dispersion equation}
\]

Short waves travel \textit{quadratically} faster. Thus, each wave \((k)\) travels at a speed proportional to the square of its wave number.
Order of spatial derivative

(i) $\Theta_t = -\Theta_x$: undamped, non-dispersive

(ii) $\Theta_t = \Theta_{xx}$: damped, dissipative

(iii) $\Theta_t = \Theta_{xxx}$: undamped, dispersive

(iv) $\Theta_t = -\Theta_{xxxx}$: damped, dissipative . . .

Therefore: Odd (> 1) derivatives: Dispersion
               Even derivative: Dissipative (damping)

3. Euler-Forward/Center-Difference Scheme

Discretizations ($U = 1$)

\[
\begin{array}{c|c|c|}
  j - 1 & j & j + 1 \\
\hline
\end{array}
\]

\[\Delta_x \Theta_j \equiv \frac{\Theta_{j+1} - \Theta_{j-1}}{2\Delta x}\]

2nd-order: $\Delta_x \Theta_j = \frac{d\Theta_j}{dx_j} + \frac{\Delta x^2}{6} \Theta_{xxx} + \ldots$

\[
\frac{\Theta_{j+1}^n - \Theta_j^n}{\Delta t} + \frac{\Theta_{j+1}^n - \Theta_{j-1}^n}{2\Delta x} = 0, \quad j = 1, \ldots N
\]

$\Theta_0^n = \Theta_N^n$

$\Theta_j^0 = \sin 2\pi x_j$
Taylor-expanding:

\[ T_j = \frac{\Delta t}{2} \frac{\partial^2 \Theta^n}{\partial t^2} + \mathcal{O}(\Delta t^2, \Delta x^2) = \frac{\Delta t}{2} \cdot \frac{\partial}{\partial x} \left( \frac{\partial \Theta}{\partial x} \right) + \mathcal{O}(\Delta t^2, \Delta x^2) \]

(consistency . . .)

The equivalent differential equation (EDE) then is:

\[ \frac{\partial \hat{\Theta}}{\partial t} + \frac{\partial \hat{\Theta}}{\partial x} + T_j = 0 \Rightarrow \frac{\partial \hat{\Theta}}{\partial t} + \frac{\partial \hat{\Theta}}{\partial x} = - \frac{\Delta t}{2} \frac{\partial^2 \hat{\Theta}}{\partial x^2} + \mathcal{O}(\Delta t^2, \Delta x^2) \]

Thus, (EDE) introduces 2\textsuperscript{nd}-order derivative \( \Rightarrow \) dissipation with negative diffusivity \( \Rightarrow \) exponential growth . . .

von Neumann Stability:

\[ \Theta_j^n = \sum_{k=-\infty}^{\infty} a_k^n e^{2\pi i k x_j} \]

Thus,

\[ \Theta_{j+1} - \Theta_{j-1} = \sum a_k e^{2\pi i k \Delta x} (e^{2\pi i k \Delta x} - e^{-2\pi i k \Delta x}) \]

\[ = \sum a_k e^{2\pi i k \Delta x} \cdot 2i \sin 2\pi k \Delta x \]

Plug-in

\[ \sum_k (a_k^{n+1} - a_k^n) e^{2\pi i k x_j} = - \frac{C}{2} \sum_k (2i \sin 2\pi k \Delta x) \cdot a_k^n e^{2\pi i k x_j} \]

where \( C \equiv \frac{U \Delta t}{\Delta x} = \) Courant number or CFL number, Courant-Friedrichs-Lewy (1928).

\[ a_k^{n+1} = a_k^n (1 - iC \sin 2\pi k \Delta x) \]

no A-stable

\[ |a_k^{n+1}| = |a_k^n| \sqrt{1 + C^2 \sin^2 2\pi k \Delta x} \]

But:

\[ |a_k^{n+1}| \leq |a_k^n| (1 + C^2)^{1/2} \leq |a_k^n| e^{C^2/2} \leq |a_k^n| e^{nC^2/2} \]
Thus,

\[ |a_k^n| \leq |a_k^0| e\left(\frac{\Delta t}{\Delta x^2}\right)^n \]

f(n\Delta t), general stability. So, for n \cdot \Delta t fixed can get a stable result for

\[
\left(\begin{array}{c}
\frac{\Delta t}{\Delta x^2} < \text{constant} \\
\Delta t, \Delta x \to 0
\end{array}\right)
\]

As restrictive as for diffusive. Not a practical sheme- instructive to understand instability \rightarrow see plot.

**Stability regions:** (A-stable regions)

re-write

\[
a_k^{n+1} - a_k^n = \left(-i \sin 2\pi k \Delta x \right) \frac{\Delta t}{\Delta x} a_k^n
\]

So, \(\lambda\) is purely imaginary–go to ODE/IVP \rightarrow midpoint rule (Leapfrog)

4. **Euler-Forward/Upwind-Differencing Scheme**

\[
\frac{\Theta_{j+1}^{n+1} - \Theta_{j}^{n+1}}{\Delta t} + \frac{\Theta_{j}^{n+1} - \Theta_{j-1}^{n+1}}{\Delta x} = 0, \ j = 1, \ldots, N
\]

or

\[
\Theta_{j}^{n+1} = \Theta_{j}^{n} - \frac{C}{2} \left(\Theta_{j+1}^{n} - \Theta_{j-1}^{n}\right) + \frac{C}{2} \left(\Theta_{j+1}^{n} - 2\Theta_{j}^{n} + \Theta_{j-1}^{n}\right)
\]

i.e, it looks like we add dissipation at the node \(j\).
EDE:
\[
\frac{\partial \hat{\Theta}}{\partial t} + \frac{\partial \hat{\Theta}}{\partial x} = \frac{1}{2} (\Delta x - \Delta t) \frac{\partial^2 \hat{\Theta}_j}{\partial x^2} + \mathcal{O}(\Delta x^2, \Delta t^2)
\]

If \(\Delta x = \Delta t \Rightarrow C = 1 \Rightarrow\) scheme is exact! (It turns out all high-order terms have the factor \((\Delta x - \Delta t)\).)

\(C < 1 : \Delta t \leq \Delta x \Rightarrow\) forward heat equation \(\rightarrow\) stable scheme but of \(\mathcal{O}(\Delta t, \Delta x)\)

\(C > 1:\) instability, “negative” heat equation \(\rightarrow\) exponential growth.

**von Neumann stability:**
\[
\begin{align*}
a^n_k &= a^n_k \left[ 1 - C(1 - e^{-2\pi i k \Delta x}) \right] \\
|a^{n+1}_k| &= |a^n_k| |1 - C(1 - e^{-2\pi i k \Delta x})| \\
&= |a^n_k| \left[ 1 + 2(C^2 - C)(1 - \cos 2\pi k \Delta x) \right]^{1/2}
\end{align*}
\]

Need: \(2C^2 - 2C \leq 0 \Rightarrow C \leq 1\)

**Stability diagram:**
\[
\frac{a^{n+1}_k - a^n_k}{\Delta t} = \frac{1 - e^{-2\pi i k \Delta x}}{\Delta x} a^n_k
\]
\[
\lambda \Delta t = -C(1 - e^{-2\pi i k \Delta x}) \\
= -C(1 - \cos 2\pi k \Delta x) - iC \sin 2\pi k \Delta x
\]
The damping due to upwind brings the \( \lambda \Delta t \) into the stability region of the Euler scheme.

This scheme is the opposite of the previous one—too stable:

\[
\frac{\partial \hat{\Theta}}{\partial t} + U \frac{\partial \hat{\Theta}}{\partial t} = \nu_{eff} \frac{\partial^2 \hat{\Theta}}{\partial x^2} + \ldots
\]

where

\[
\nu_{eff} = \frac{U \Delta x}{2} (1 - C) \sim U \Delta x
\]

Solution will be so damped in time that wave will disappear \( \rightarrow \) unphysical

(see plot) \( \rightarrow C = 1/2 \)

Although \( C = 1 \rightarrow \) exact not practical \( \left( \frac{U}{\Delta x} \right) \) (non-uniform mesh), may vary!

**Upwind**

“Transportive Property” (Roache): The effect of a perturbation in a transport property is advected only in the direction of the velocity.

- Upwind scheme \( \rightarrow \) transportive
- Centered Difference \( \rightarrow \) non-transportive
* First-order upwind:

\[ \frac{\partial \Theta}{\partial t} + U \frac{\partial \Theta}{\partial x} \Rightarrow \frac{\Delta \Theta_j}{\Delta t} = - \frac{U_j \Theta_j - U_{j-1} \Theta_{j-1}}{\Delta x} \]

* Transportive, but not conservative when \( U \) changes sign.

* Second-order upwind:

\[ \frac{\Delta \Theta_j}{\Delta t} = - \frac{U_R \Theta_R - U_L \Theta_L}{\Delta x} \]

\[ U_R = \frac{1}{2} (U_{j+1} + U_j) ; U_L = \frac{1}{2} (U_{j-1} + U_j) \]

\[ \Theta_R = \Theta_j, U_R > 0 \]

\[ \Theta_L = \Theta_{j-1}, U_L > 0 \]

* Transportive and Conservative, also (almost) second-order.

5. Lax-Friedrichs Scheme: (Lax, 1954)

\[ \Theta_j^{n+1} = \frac{1}{2} (\Theta_{j+1}^n + \Theta_{j-1}^n) - \frac{C}{2} (\Theta_{j+1}^n - \Theta_{j-1}^n) \]

Note: Odd-even oscillations.
This can be considered as a correction to the unstable central scheme:

\[ \Theta_{j}^{n+1} - \Theta_{j}^{n} = \frac{1}{2}(\Theta_{j+1}^{n} - 2\Theta_{j}^{n} + \Theta_{j-1}^{n}) - \frac{C}{2}(\Theta_{j+1}^{n} - \Theta_{j}^{n}) \]

**von Neumann analysis**

\[ a_{k}^{n+1} = a_{k}^{n}[\cos 2\pi k\Delta x - iC \sin 2\pi k\Delta x] \]

stability for \( C \leq 1 \). For \( C = 1 \) we obtain \( |a_{k}^{n+1}| = |a_{k}^{n}| \Rightarrow \) no dissipation errors.

**EDE:**

\[ \frac{\partial \Theta}{\partial t} + U \frac{\partial \Theta}{\partial x} = \frac{\Delta x^2}{2\Delta t} (1 - C^2) \Theta_{xx} + \frac{U \Delta x^2}{3} (1 - C^2) \Theta_{xxx} + \ldots \]

For \( C = 1 \), exact scheme \( \Rightarrow \) no dispersion errors either. The dispersion errors can be also measured by the ratio

\[ \epsilon_{\phi} = \frac{\tan^{-1}(C \tan \phi)}{C \phi}, (\phi = 2\pi k\Delta x) \]

where for \( C = 1 \) \( \Rightarrow \epsilon_{\phi} = 1 \Rightarrow \) no dispersion errors.

**Numerical viscosity:**

\[ \nu_{eff} = \frac{U}{2C} \Delta x (1 - C^2) \]

\[ = \frac{U \Delta x}{2} (1 - C) \left( 1 + C \right) \]

Therefore, the diffusion of this scheme is higher than upwind by the factor \( \left( \frac{1+C}{C} \right) \).

**Phase errors:**

\[ \epsilon_{\phi} = \frac{\omega}{kU} \cong 1 + \frac{\phi^2}{3} (1 - C^2) + O(\Delta x^4) \]

leading errors.

\[ \Theta^{n+1}_j = \frac{C}{2} (1 - C) \Theta^{n+1}_{j-2} + C (2 - C) \Theta^{n+1}_{j-1} + \left( 1 - \frac{3C}{2} + \frac{C^2}{2} \right) \Theta^n_j \]

which can be re-written as:

\[ \Theta^{n+1}_j = \Theta^n_j - \frac{C}{2} (1 - C) \Theta^{n+1}_{j-1} + \frac{1}{2} C (1 - C) \left( \Theta^n_j - 2 \Theta^n_{j-1} + \Theta^n_{j-2} \right) \]

1st-order upwind

dissipation at \((j-1)\)

EDE:

\[ \Theta_t + U \Theta_x = \frac{U}{6} \Delta x^2 (1 - C)(2 - C) \Theta_{xxx} \]

\[ - \frac{C}{8} \Delta x^3 (1 - C)^2 (2 - C) \left( \frac{\partial^4 \Theta}{\partial x^4} \right) + \ldots \]

There is no second-order derivative and therefore dissipation is reduced compared to first-order upwind.

von Neumann:

\[ a_k^{n+1}_k = a_k^n \left\{ 1 - 2C[1 - (1 - C) \cos \phi] \sin^2 \phi / 2 \right. \]

\[ - \left. iC \sin \phi [1 + 2(1 - C) \sin^2 \phi / 2] \right\} \]
For stability $C \leq 2$ thus,

$$\left| \frac{a_k^{n+1}}{a_k^n} \right|^2 = 1 - 4C(1 - C^2)(2 - C)\sin^4 \phi/2$$

**Phase error:**

$$\epsilon_\phi = \frac{\omega/k}{U} \approx 1 + \frac{1}{6}(1 - C)(2 - C)\phi^2 + \mathcal{O}(\phi^4)$$

leading errors for $C < 1$, lagging error for $C > 1$.

Note: Combine LW and WB to reduce phase error for $0 < C < 1$.

7. **Lax-Wendroff Scheme (1960):** $\mathcal{O}(\Delta t^2, \Delta x^2)$

Expand in time

$$\theta_j^{n+1} = \theta_j^n + \Delta t (\theta_t)_j^n + \frac{\Delta t^2}{2} (\theta_{tt})_j^n + \mathcal{O}(\Delta t^3)$$

replace

$$\theta_t = U^2 \theta_{xx} \text{ (Second-Order Wave Equation)}$$

thus,

$$\theta_j^{n+1} = \theta_j^n - U\Delta t (\theta_x)_j^n + \frac{U^2 \Delta t^2}{2} (\theta_{xx})_j^n + \mathcal{O}(\Delta t^3)$$

or

$$\theta_j^{n+1} = \theta_j^n - \frac{C}{2} (\theta_{j+1}^n - \theta_{j-1}^n) + \frac{C^2}{2} \frac{(\theta_{j+1}^n - 2\theta_j^n + \theta_{j-1}^n)}{\text{dissipation}}$$

LW and WB are the only unique 2nd-order schemes on 3pt stencils.

**von Neumann** $\Rightarrow a_k^{n+1} = a_k^n [1 - iC \sin 2\pi k \Delta x - C^2 (1 - \cos 2\pi k \Delta x)]$.

For stability $\Rightarrow C \leq 1$

**Phase errors:**

$$\omega/k = 1 - \frac{1}{6}(1 - C^2)\phi^2 + \mathcal{O}(\phi^4)$$
lagging errors.

- Phase error is largest at high frequencies
- EDE:

\[
\hat{\Theta}_t + U \hat{\Theta}_x = -\frac{U}{6} \Delta x^2 (1 - C^2) \hat{\Theta}_xxx - \frac{U \Delta x^3}{8} \cdot C(1 - C^2) \hat{\Theta}_xxxx + \mathcal{O}(\Delta x^4)
\]

8. Crank-Nicolson/Center-Differencing Scheme

Provide (implicit → A-stable) stability

\[
\frac{\Theta_j^{n+1} - \Theta_j^n}{\Delta t} + \frac{1}{2} \Delta x \Theta_j^{n+1} + \frac{1}{2} \Delta x \Theta_j^n = 0
\]

Or

\[
\frac{\Delta t}{2} \frac{\partial^2 \Delta}{\partial x^2} \frac{1}{2} \Delta t \frac{\partial \Delta}{\partial t} = -\frac{1}{2} \Delta t \frac{\partial^2 \Theta}{\partial x^2} \mathcal{O}(\Theta t^2, \Delta x^2)
\]

Implicit part determines computational complexity/work

\[
\frac{1}{\Delta t} \Theta_j^{n+1} + \frac{1}{4 \Delta x} (\Theta_j^{n+1} - \Theta_j^{n+1})
\]

⇒ diagonal dominance if \( \frac{1}{4 \Delta x} + \frac{1}{4 \Delta x} \geq \frac{1}{\Delta t} \Rightarrow [C < 2] \). So stable (inexpensive → no pivots) system inversion limits time stepping here.

**von Neumann stability:**

\[
a_k^{n+1} = \left( \frac{1 - iC}{2} \sin 2\pi k \Delta x \right) a_k^n \Rightarrow |a_k^{n+1}| = |a_k^n| \forall C, k
\]
neutral stability: absolute but weak stability
\[ \lambda \Delta t = \frac{-c_1^2 \sin^2 2\pi k \Delta x - i \frac{c_2}{2} \sin 2\pi k \Delta x}{1 - c_1^2 \sin^2 2\pi k \Delta x} \]

\[ \frac{\partial^2 \Theta}{\partial t^2} + \frac{\partial^2 \Theta}{\partial x^2} = \left( -\frac{1}{12} \Delta t^2 + \frac{1}{6} \Delta x^2 \right) \frac{\partial^4 \Theta}{\partial x^4} \]

(all \( \Theta_{xx}, \Theta_{xxx}, \) etc vanish). Assume: \( \hat{\Theta} = Re \left\{ e^{2\pi ikx} e^{-2\pi i\omega t} \right\} \)

\[ -2\pi i \omega = -2\pi ik - \left( \frac{1}{12} \Delta t^2 + \frac{1}{6} \Delta x^2 \right) (-i)(2\pi k)^3 \]

\[ \Rightarrow \frac{\omega}{k} = 1 - \left( \frac{1}{12} \Delta t^2 + \frac{1}{6} \Delta x^2 \right) (2\pi k)^2 \]

Phase error

\( C_\phi = 1 \) is correct phase-speed. Thus, \( \Delta \phi = - \left( \frac{1}{12} \Delta t^2 + \frac{1}{6} \Delta x^2 \right) (2\pi k)^2 \) = error in period/peak.

Numerical experiment: \( C = 1 \rightarrow \) shift every period 0.0165 to the left (plot) \( \rightarrow \) numerical dispersion \( k = 1 \) mode (I.C.)

- As \( k \uparrow \) phase errors in small waves
- Worst \( k = N/2 \rightarrow \Theta \sim Re \left\{ e^{2\pi ik \Delta x} \right\} Re \left\{ e^{2\pi i \frac{N}{2}} \right\} = Re \left\{ e^{\pi i j} \right\}. \)
(2\Delta x waves)

and \( \frac{\Theta_{j+1} - \Theta_{j-1}}{2\Delta x} = 0 \Rightarrow \) mode never moves \( \rightarrow \) should move with speed \( C_\phi = 1 \).

9. Adams-Bashforth/Center Differencing Scheme

\[
\frac{a_k^{n+1} - a_k^n}{\Delta t} = \sum_{q=0}^{M} \alpha_q \lambda_k a_k^{n-q}, \quad M = 1, 2
\]

2\textsuperscript{nd}-order:

\[
\frac{\Theta_j^{n+1} - \Theta_j^n}{\Delta t} + \frac{3}{2} \Delta x \Theta_j^n - \frac{1}{2} \Delta x \Theta_j^{n-1} = 0
\]

with corresponding equivalent differential equation (EDE)

\[
\frac{\partial^{2} \hat{\Theta}}{\partial t^2} + \frac{\partial^{2} \hat{\Theta}}{\partial x^2} = \left( \frac{5\Delta t^2}{12} - \frac{\Delta x^2}{6} \right) \frac{\partial^3 \hat{\Theta}}{\partial x^3} + \ldots
\]

dispersive.

(Note, here: \( \Theta_{xxx}, \Theta_{vi} \) etc are not all zero \( \rightarrow \) amplitude errors too!)

von Neumann stability:

\[
\Theta_j^n = \sum a_k^n e^{2\pi ikx_j}
\]

\[
a_k^{n+1} + a_k^n \left( -1 + \frac{3}{2} iC \sin 2\pi k \Delta x \right) + a_k^{n-1} \left( -\frac{iC}{2} \sin 2\pi k \Delta x \right) = 0
\]
Let
\[ \mu \equiv \frac{a_k^{n+1}}{a_k^n} \Rightarrow \mu^2 + \mu \left( -1 + \frac{3}{2}i \sin 2\pi k \Delta x \right) + \left( \frac{-iC}{2} \sin 2\pi k \Delta x \right) = 0 \]

\( \mu \) is complex. Two roots \((C \ll 1)\):

\[ \mu_+ = 1 - iC \sin 2\pi k \Delta x - \frac{C^2}{2} \sin^2 2\pi k \Delta x - \frac{1}{4}iC^3 \sin^3 2\pi k \Delta x - \frac{1}{8}C^4 \sin^4 2\pi k \Delta x + \ldots \]

(physical mode)

\[ \mu_- = -\frac{1}{2}iC \sin 2\pi k \Delta x + \frac{1}{2}C^2 \sin^2 2\pi k \Delta x + \ldots \]

(numerical mode)

Note that both \( \mu_+ , \mu_- \) depend on I.C. \( \Rightarrow \) the numerical mode is damped relatively to the physical mode. For \( C \) small \( \rightarrow |\mu_-| \ll 1 \). (Not A-stable.)

to leading order:

\[ |\mu_+| = \left( 1 + \frac{1}{2}C^4 \right)^{1/2} + \text{h.o.t. in } C \]

\[ |a_k^n| \approx |a_k^0| (1 + \frac{1}{2}C^4)^{1/2} \]

\[ \leq |a_k^0| e^{\frac{C^4 \Delta x}{4}} = |a_k^0| e^{\frac{\Delta x (\Delta + \Delta)}{4}} \]

which is generally stable.

Scheme is similar to EF/CD but here growth is proportional to \( C^4 \) not \( C^2 \) and requires only \( \Delta t^3 < \Delta x^4 \) not \( \Delta t < \Delta x^2 \) as in EF.

**Example-plot:** \( C = 0.1 \), Growth of 1,000 after 250,000 time steps. Most unstable mode. An unstable scheme which is usable for \( C \) small.

**3rd AB-CD:**

\[ \frac{\Theta^{n+1} - \Theta^n}{\Delta t} + \frac{23}{12} \Delta x \Theta^n_j - \frac{16}{12} \Delta x \Theta^{n-1}_j + \frac{5}{12} \Delta x \Theta^{n-2}_j = 0 \]
EDE:
\[
\frac{\partial \Theta^2}{\partial t} + \frac{\partial \Theta}{\partial x} = -\Delta x^2 \frac{\partial^3 \Theta}{\partial x^3} \text{ dispersive} + O(\Delta t^3) \frac{\partial^4 \Theta}{\partial x^4} \text{ diffusive} + \ldots O(\Delta t \Delta x^2)
\]

if high-order spatial discretization is used → small dispersion! (e.g., spectral). As stable as upwind.

\[
\lambda \Delta t = -iC \sin 2\pi k \Delta x (< 0.723) \Rightarrow C < 0.723
\]

10. **Effects of Boundary Conditions**

\[
\begin{align*}
\Theta_t + \Theta_x &= 0; \quad \Theta(x, t = 0) = \sin 2\pi x \quad \text{I.C.} \\
0 < x < 1 &\quad \Theta(x = 0, t) = -\sin 2\pi t \quad \text{b.c (one)}
\end{align*}
\]
The left boundary \( x = 0 \) generated the information that is then propagated through the domain. No b.c. is needed at \( x = 1 \) outflow.

\[
\begin{array}{cccccc}
  j = 0 & 1 & \cdots & N - 1 & N & N + 1 \\
  x = 0 & & & & & "phantom" \text{ node}
\end{array}
\]

C-N/CD:

\[
\frac{\Theta_j^{n+1} - \Theta_j^n}{\Delta t} + \frac{1}{2} \left( \frac{\Theta_{j+1}^{n+1} - \Theta_{j-1}^{n+1}}{2\Delta x} + \frac{\Theta_{j+1}^n - \Theta_{j-1}^n}{2\Delta x} \right) = 0, \quad j = 1, \ldots, N - 1
\]

(stability-not a problem)

\[
\Theta_0^{n+1} = -\sin 2\pi t^{n+1}; \quad \Theta_j^0 = \sin 2\pi x_j \quad \text{(B.C./I.C.)}
\]

**What to do at** \( j = N \)?

extrapolate linearly:

\[
\Theta_{N+1} = \Theta_N + \Delta x \frac{\Theta_N - \Theta_{N-1}}{\Delta x} = 2\Theta_N - \Theta_{N-1}
\]

\[
\frac{\Theta_{N+1} - \Theta_N}{\Delta x} = \frac{\Theta_N - \Theta_{N-1}}{\Delta x},
\]

first-order, one-sided derivative still \( \mathcal{O}(\Delta x^2) \) **overall**. Therefore

\[
\frac{\Theta_{N+1}^n - \Theta_N^n}{\Delta t} + \frac{1}{2} \left( \frac{\Theta_N^{n+1} - \Theta_{N-1}^{n+1}}{\Delta x} + \frac{\Theta_N^n - \Theta_{N-1}^n}{\Delta x} \right) = 0
\]

**Explicit schemes**: Consider also stability of the spurious (artificial) outflow condition.

AB2 is conditionally stable for heat equation.

\[
\frac{\Theta_j^{n+1} - \Theta_j^n}{\Delta t} + \frac{3}{2} \left( \frac{\Theta_{j+1}^n - \Theta_{j-1}^n}{2\Delta x} \right) - \frac{1}{2} \left( \frac{\Theta_{j+1}^{n-1} - \Theta_{j-1}^{n-1}}{2\Delta x} \right) = 0
\]

\[
\begin{align*}
\Theta_0^{n+1} &= -\sin 2\pi t^{n+1} \\
\Theta_j^0 &= \sin 2\pi x_j \\
&\quad \{ \ j = 1, \ldots, N - 1 \}
\end{align*}
\]
\[ j = N; \quad \frac{\Theta_{N}^{n+1} - \Theta_{N}^{n}}{\Delta t} + 3 \frac{1}{2} \left( \frac{\Theta_{N}^{n} - \Theta_{N-1}^{n}}{\Delta x} \right) - \frac{1}{2} \left( \frac{\Theta_{N-1}^{n-1} - \Theta_{N-1}^{n-1}}{\Delta x} \right) = 0 \]

dissipative

some error at outflow, but stable result \( \rightarrow \) plot \( (C = 0.1) \)

Leapfrog:

\[ \frac{\Theta_{j}^{n+1} - \Theta_{j}^{n-1}}{2\Delta t} + \frac{\Theta_{j+1}^{n} - \Theta_{j-1}^{n}}{2\Delta x} = 0, \quad j = 1, \ldots, N - 1 \]

\[ \frac{\Theta_{N}^{n+1} - \Theta_{N}^{n-1}}{2\Delta t} + \frac{\Theta_{N}^{n} - \Theta_{N-1}^{n}}{\Delta x} = 0 \]

dissipative. \( C = 0.1 \) \( \rightarrow \) unstable plot \( \rightarrow 2\Delta x \) wave instability.

Thus, upwinding with \( AB \) \( \rightarrow \) stable scheme. Upwinding with leapfrog \( \rightarrow \) unstable scheme.

11. Finite Element Discretizations

[Diagram of 1D discretization with nodes 1, 2, and 3]
\( \mathcal{R}^k = \text{residual in element } k := \int_{x^k} \, w_i \left( \frac{\partial \Theta_j^k}{\partial t} + \sum_{j=1}^{2} \Theta_j^k \frac{\partial h_j}{\partial r} \right) \, dx \) (weak form of equations.)

Linear basis:

\[
R_i^k = \int_{-1}^{1} h_i \left( \sum_{j=1}^{2} \frac{\partial \Theta_j^k}{\partial t} h_i + \sum_{j=1}^{2} \Theta_j^k \frac{\partial h_j}{\partial r} \frac{L^k}{2} \right) \, dr
\]

or

\[
R_i^k = \sum_{j=1}^{2} B_{ij} \Theta_j^k + \sum_{j=1}^{2} D_{ij} \Theta_j^k
\]

where

\[
[B_{ij}] = \frac{L^k}{2} \int_{-1}^{1} h_i h_j \, dr = L^k \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}, \text{ mass matrix}
\]

\[
[D_{ij}] = \int_{-1}^{1} h_i \frac{\partial h_j}{\partial r} \, dr = \begin{bmatrix} -1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}, \text{ derivative non-symmetric matrix}
\]
Direct stiffness, \( w_k = \) top hat

\[
\begin{align*}
0 & \quad 1 & 2 \\
\bullet & \quad \bullet & \quad \bullet \\
W_2 & \quad \quad \quad W_2 \\
\end{align*}
\]

Matrix-Form:

\[
\Delta x \begin{bmatrix}
1/3 & 1/6 \\
1/6 & 1/3 & 1/3 & 1/6 \\
1/6 & 1/3 & 1/3 \\
-1/2 & 1/2 \\
-1/2 & -1/2 + 1/2 & 1/2 \\
-1/2 & -1/2 + 1/2 \\
\end{bmatrix}
\frac{\partial \Theta_j}{\partial t} + \Theta_j = 0
\]

or

\[
\frac{1}{6} \frac{\partial \Theta_{j-1}}{\partial t} + \frac{2}{3} \frac{\partial \Theta_j}{\partial t} + \frac{1}{6} \frac{\partial \Theta_{j+1}}{\partial t} + \frac{\Theta_{j+1} - \Theta_{j-1}}{2 \Delta x} = 0
\]

\( j = \) global node.

Crank-Nicolson in time:

\[
\frac{1}{6} \left( \Theta_{j-1}^{n+1} + 4 \Theta_j^{n+1} + \Theta_{j+1}^{n+1} \right) - \frac{1}{6} \left( \Theta_{j-1}^n + 4 \Theta_j^n + \Theta_{j+1}^n \right) + \frac{1}{2} \left( \Theta_{j+1}^{n+1} - \Theta_{j-1}^{n+1} \right) + \frac{\Theta_{j+1}^n - \Theta_{j-1}^n}{2 \Delta x} = 0
\]
Comments:

1. (Lumping $B$ or using collocation ⇒ retrieve finite difference)

2. Equivalent Differential Equation:

\[
\frac{\partial \Theta_j^n}{\partial t} + \frac{\partial \hat{\Theta}_j^n}{\partial x} = -\frac{\Delta t^2}{12} \frac{\partial^3 \hat{\Theta}_j^n}{\partial x^3} + O(\Delta t^3, \Delta x^2 \Delta t, \Delta x^4, \Delta t^4, \ldots)
\]

no $\Delta x^2$. Compact in space 4th-order with a 2nd-order molecule!

Accuracy in space $O(\Delta x^4)$!

3. Neutrally stable $|a_k^{n+1}| = |a_k^n|

4. Dispersion errors $O(\Delta t^2) \frac{\partial^2 \Theta}{\partial x^2}$

Contrast with C-N/FD dispersion $(O(\Delta t^2) + O(\Delta x^2)) \frac{\partial^2 \Theta}{\partial x^2}$

**Finite elements-hyperbolic equation - B.C.**

\[
\begin{pmatrix}
1/3 & 1/3 \\
1/6 & 1/3
\end{pmatrix}
\begin{pmatrix}
\partial \Theta_j \\
\partial x \\
\partial \Theta_j \\
\partial t
\end{pmatrix}
= \begin{pmatrix}
-1/2 & 1/2 \\
1/2 & 1/2
\end{pmatrix}
\begin{pmatrix}
\partial \Theta_j \\
\partial x
\end{pmatrix} = 0
\]

\[
\frac{1}{6} \frac{\partial \Theta_{N-1}}{\partial t} + \frac{1}{3} \frac{\partial \Theta_N}{\partial t} + \frac{1}{2} \frac{\theta_N - \Theta_{N-1}}{\Delta x} = 0
\]

in “condensed” form:

\[
\frac{\partial \Theta_N}{\partial t} + \left( \frac{\Theta_N - \Theta_{N-1}}{\Delta x} \right) = 0
\]

same as finite difference.

- For complicated geometries use FEM ⇒ maps
- B.C. of lower order, in general
- Stability depends on time-stepping, e.g. leapfrog is unstable here.