Chapter 8

Navier-Stokes in Primitive Variables

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1. Governing Equations

We refer to \((\vec{v}, p)\) as primitive variables to distinguish from the \((\Psi, \omega)\) formulation. For simplicity we consider the Stokes equations since the nonlinearity can be treated explicitly (in time) and be included as forcing term \((f_x, f_y)\) on the RHS of the equations. In 2D:

\[
\begin{align*}
    x - \text{mom}: & \quad \frac{\partial u}{\partial t} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + f_x, \text{ in } \Omega \\
    y - \text{mom}: & \quad \frac{\partial v}{\partial t} = - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + f_y, \text{ in } \Omega \\
\end{align*}
\]

Continuity:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

Note that the third equation is not an evolution equation but a kinematic constraint, and that it is also valid on boundaries. This is similar to the kinematic equation \(\nabla^2 \Psi = -\omega\) in the \((\Psi, \omega)\) formulation.

Taking the divergence of the momentum equation we obtain:

\[
\frac{\partial}{\partial t}(\nabla \cdot \vec{v}) = -\frac{1}{\rho} \nabla^2 p + \nu \nabla^2 (\nabla \cdot \vec{v}) + \nabla \cdot \vec{f}
\]

and thus,

\[
\nabla^2 p = \rho \nabla \cdot \vec{f}
\]

where \(\vec{f}\) includes nonlinear terms or other forces. In the following we will assume \(f \equiv 0\).

**Pressure Interpretation:** Unlike in compressible flows, in incompressible flows \(p\) is not a thermodynamic quantity. To provide a simple interpretation let us assume that we consider an inviscid flow, so that

\[
\frac{\partial \vec{v}}{\partial t} = -\nabla p \quad \Rightarrow \quad \frac{\vec{v}^{n+1} - \vec{v}^n}{\Delta t} = -\nabla p
\]

\[
\nabla \cdot \vec{v} = 0 \quad \nabla \cdot \vec{v}^{n+1} = 0
\]
\[ \vec{v} \cdot \hat{n} = 0 \text{ on } \partial \Omega \]

Therefore, \( \vec{v} + \nabla \vec{p} = \vec{f} \) where \( \vec{v} \equiv \vec{v}^{n+1} \), \( \vec{p} = \Delta t \vec{p} \) and \( \vec{f} \) contains information from time level \( (n) \). Then forming the inner product

\[
\int_{\Omega} (\vec{v} \cdot \nabla \vec{p}) d\vec{x} = \int_{\Omega} \nabla \cdot (\vec{v} \vec{p}) d\vec{x} - \int_{\Omega} \vec{p} \nabla \cdot \vec{v} d\vec{x} = \int_{\partial \Omega} p\vec{v} \cdot \hat{n} ds = 0
\]

therefore, \( \vec{v} \perp \nabla \vec{p} \) (perpendicular vectors)

**Geometrically:**

![Diagram](image)

In other words, pressure projects \( \vec{f} \) (information) from previous time level to a divergence-free space, i.e., the horizontal axis. Note that \( \vec{f} = \nabla \vec{p} + \vec{v} \), so we have recovered the Weyl’s decomposition of the general vector \( \vec{f} \) into an irrotational (\( \nabla \vec{p} \)) and a solenoidal (\( \vec{v} \)) component.

2. **Stokes Model Problem:**

We consider the decay of a monochromatic (single Fourier mode) wave traveling in a channel.
We can write the \((\vec{v}, p)\) system as:

\[
\begin{bmatrix}
  u^* \\
  v^* \\
  p^*
\end{bmatrix} = \text{Re} \begin{bmatrix}
  u(y, t) \\
  v(y, t) \\
  p(y, t)
\end{bmatrix} e^{i k x} \quad \text{(real parts)}
\]

where \(k\) is specified. Using the diffusive time scale \((\frac{L}{\nu}, \frac{\nu L}{\rho u_0})\) we obtain for the Fourier amplitudes:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -ikp + \left( \frac{\partial^2}{\partial y^2} - k^2 \right) u \\
\frac{\partial v}{\partial t} &= -\frac{\partial p}{\partial y} + \left( \frac{\partial^2}{\partial y^2} - k^2 \right) v \\

iku + \frac{\partial v}{\partial y} &= 0 \\

u(\pm 1, t) = v(\pm 1, t) &= 0
\end{align*}
\]

We can further separate the \(y - t\) variables as:

\[
\begin{bmatrix}
  u \\
  v \\
  p
\end{bmatrix} = e^{\sigma t} \begin{bmatrix}
  \hat{u}(y) \\
  \hat{v}(y) \\
  \hat{p}(y)
\end{bmatrix}
\]

where \(\sigma\) is the decay rate \((\sigma < 0)\). Elimination of \(\hat{v}\) and \(\hat{p}\) gives:

\[
\sigma(D^2 - k^2) \hat{u} = (D^2 - k^2)^2 \hat{u}, \quad -1 < y < 1
\]

where \(D \equiv d/dy\) and \(\hat{u}(\pm 1) = D\hat{u}(\pm 1) = 0\)

The exact solutions are either odd or even in \(y\):

\(\text{Odd:} \quad \hat{u}(y) = \cos a \cosh ky - \cosh k \cos ay\)

\(\text{Even:} \quad \hat{u}(y) = \cos a \cosh ky + \cosh k \cos ay\)
where \( k \tanh k = -a \tan a \)

\((O)\) odd: \( \dot{u}(y) = \sin a \sinh ky - \sinh k \sin ay \)

where \( k \coth k = a \cot a \)

Here we define \( a = \sqrt{-\sigma - k^2} \)

Table: Decay rates of slowest decaying normal mode \((k = 1)\)

<table>
<thead>
<tr>
<th>Type</th>
<th>(a)</th>
<th>(-\sigma)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>2.883356</td>
<td>9.3137</td>
</tr>
<tr>
<td>O</td>
<td>4.423864</td>
<td>20.5706</td>
</tr>
</tbody>
</table>

In the numerical experiments, we solve for

\[
\begin{align*}
\begin{cases}
   u(y, t) \\
   v(y, t) \\
   p(y, t)
\end{cases}
\end{align*}
\]

The corresponding decay rate can then be computed as:

\[ \bar{\sigma} = -\frac{1}{T} \ln \frac{v(y = 0, t + T)}{v(y = 0, t)} \]

where \( T = 0.3 \) (say): then initial energy is reduced by 5 orders after \( T \) for solution \((E)\).
3. Staggered Mesh/Implicit Discretization

We need to solve for

\[
\begin{pmatrix}
u(y, t) \\
v(y, t) \\
p(y, t)
\end{pmatrix}
\]

on the one-dimensional grid defined by \(y \in [-1, 1]\). To avoid specification of the pressure on the boundary we define two meshes: one for velocity (integer points); second for pressure (half points).

\[
y_j = j\Delta y - 1, \quad j = 0, \ldots, N
\]

\[
y_{j+1/2} = (j + 1/2)\Delta y - 1, \quad j = 0, \ldots, (N - 1)
\]

where \(\Delta y = 2/N\). Let \(u_j^{n+1/2} \equiv \frac{1}{2}(u_j^{n+1} + u_j^n)\) (Crank-Nicolson) central-differencing.

Discrete \(x\)-momentum:

\[
M_j^x: \frac{u_j^{n+1} - u_j^n}{\Delta t} = -ik\frac{p_{j+1/2} + p_{j-1/2}}{2} + \left[\frac{u_{j-1}^{n+1/2}}{\Delta y^2} - \left(\frac{2}{\Delta y^2 + k^2}\right) u_j^{n+1/2} + \frac{u_{j+1}^{n+1/2}}{\Delta y^2}\right]
\]

\[j = 1, \ldots, N - 1\]

\(u_0^{n+1} = u_N^{n+1} = 0\)

Discrete \(y\)-momentum:

\[
M_j^y: \frac{v_j^{n+1} - v_j^n}{\Delta t} = -\frac{p_{j+1/2} - p_{j-1/2}}{\Delta y} + \left[\frac{v_{j-1}^{n+1/2}}{\Delta y^2} - \left(\frac{2}{\Delta y^2 + k^2}\right) v_j^{n+1/2} + \frac{v_{j+1}^{n+1/2}}{\Delta y^2}\right]
\]

Note that pressure term is approximated with second-order using neighboring points due to staggering and midpoints. To obtain the same
number of divergence points as the pressure unknowns, we discretize
the continuity equation at the midpoints, i.e.,
\[ ik \frac{u_{j+1}^{n+1} + u_j^{n+1}}{2} + \frac{v_{j+1}^{n+1} - v_j^{n+1}}{\Delta y} = 0, \quad j = 0, \ldots, N - 1 \]
with \( \mathcal{O}(\Delta y^2) \) accuracy.

3.1 Discrete Poisson Equation:
We need to satisfy the discrete divergence-free condition, i.e., \( d_{j+1/2}^{n+1} = 0 \), where
\[ d_{j+1/2}^{n+1} \equiv ik \frac{u_{j+1}^{n+1} + u_j^{n+1}}{2} + \frac{v_{j+1}^{n+1} - v_j^{n+1}}{\Delta y} \]
and thus we form the discrete pressure equation by taking the divergence of the discrete momentum equation, i.e.,
\[ ik \frac{M_{j+1}^x + M_j^x}{2} + \frac{M_{j+1}^y - M_j^y}{\Delta y}, \quad j = 1, \ldots, (N - 2) \]
(Note that \( M_N^x \) is not defined)
\[
\frac{d_{j+1/2}^{n+1} - d_{j+1/2}^{n}}{\Delta t} = -\frac{1}{2} \left[ \frac{d_{j-1/2}^{n+1}}{\Delta y^2} - \left( \frac{2}{\Delta y^2} + k^2 \right) d_{j+1/2}^{n+1} + \frac{d_{j+3/2}^{n+1}}{\Delta y^2} \right]
\]
\[
= -\frac{1}{\Delta y^2} \left( \overline{\mathcal{P}}_{j-1/2} - 2\overline{\mathcal{P}}_{j+1/2} + \overline{\mathcal{P}}_{j+3/2} \right)
+ \frac{k^2}{4} \left( \overline{\mathcal{P}}_{j+3/2} + 2\overline{\mathcal{P}}_{j+1/2} + \overline{\mathcal{P}}_{j-1/2} \right)
\]
\( j = i, \ldots, N - 2 \)

Let us define
\[ \mathcal{L}^0 \overline{\mathcal{P}}_{j+1/2} = \frac{\overline{\mathcal{P}}_{j-1/2}}{\Delta y^2} - \frac{2}{\Delta y^2} \overline{\mathcal{P}}_{j+1/2} + \frac{\overline{\mathcal{P}}_{j+3/2}}{\Delta y^2} \]
\[-\frac{k^2}{4} \left( \overline{\mathcal{P}}_{j-1/2} + 2\overline{\mathcal{P}}_{j+1/2} + \overline{\mathcal{P}}_{j+3/2} \right) \]
Then by solving:
\[ \mathcal{L}^0 \overline{\mathcal{P}}_{j+1/2} = 0, \quad j = 1, \ldots, N - 2 \]
the discrete divergence equation is the difference equation for the parabolic equation
\[
\frac{\partial d}{\partial t} - \frac{1}{2} \nabla^2 d = 0
\]
with homogeneous b.c. and thus it has a unique solution:
\[
d_{j+1/2}^{n+1} \equiv 0 \quad j = 0, \ldots, N - 1
\]
Note that \(\mathcal{L} \nabla^2 p = 0\) differs from the discrete analog of \(\nabla^2 p = 0\), i.e.,
\[
\mathcal{L} \nabla^2 p_{j+1/2} = \frac{\Delta y}{\Delta t} \frac{\partial^2 p_{j+1/2}}{\partial y^2} = -\frac{2}{\Delta y^2} p_{j+1/2} + \frac{p_{j+3/2}}{2} - k^2 p_{j+1/2}
\]
In summary, we solve the following difference equations:

1. \(\mathcal{L} \nabla^2 p_{j+1/2} = 0, (j = 1, \ldots, N - 2),
\[
d_{j+1/2}^{n+1} = d_{N-1/2}^{n+1} = 0
\]

2. \[
\begin{align*}
\frac{\partial u_j}{\partial t} & = -D_x p_{j+1/2} \equiv -ik \frac{p_{j+1/2} + p_{j-1/2}}{2} \\
\frac{\partial v_j}{\partial t} & = -D_y p_{j+1/2} \equiv \frac{p_{j+1/2} - p_{j-1/2}}{\Delta y} \\
& j = 1, \ldots, N - 1
\end{align*}
\]

3. \[
\begin{align*}
\frac{u_{j+1/2}}{\Delta y^2} - \left( \frac{2}{\Delta y^2} + k^2 + \frac{2}{\Delta t} \right) u_{j+1/2} & = -\frac{u_j + u_{j+1}}{\Delta t} \\
& = \mathcal{R}_{u_{j+1/2}}^{n+1} \\
\frac{v_{j+1/2}}{\Delta y^2} - \left( \frac{2}{\Delta y^2} + k^2 + \frac{2}{\Delta t} \right) v_{j+1/2} & = -\frac{v_j + v_{j+1}}{\Delta t} \\
& = \mathcal{R}_{v_{j+1/2}}^{n+1}
\end{align*}
\]
\[
\begin{align*}
\frac{u_j}{\Delta y^2} - \left( \frac{2}{\Delta y^2} + k^2 + \frac{2}{\Delta t} \right) u_j & = -\frac{u_j + u_{j+1}}{\Delta t} \\
\frac{v_j}{\Delta y^2} - \left( \frac{2}{\Delta y^2} + k^2 + \frac{2}{\Delta t} \right) v_j & = -\frac{v_j + v_{j+1}}{\Delta t}
\end{align*}
\]
where \(u_0^{n+1/2} = v_0^{n+1/2} = u_N^{n+1/2} = v_N^{n+1/2} = 0\)
3.2 Green’s Functions Implementation

(G): Pre-processing step

\[
\mathcal{L}^c p_{j+1/2}^k = 0; \quad j = 1, \ldots, N - 2
\]

\[
P_{j/2}^k = \begin{cases} 
1, k = 1 \\
0, k = 2 
\end{cases} 
\]

\[
P_{N-1/2}^k = \begin{cases} 
0, k = 1 \\
1, k = 2 
\end{cases} 
\]

(2)

\[
\frac{\hat{U}_j^k}{\Delta t} = -D_x p_{j+1/2}^k; \quad \frac{\hat{V}_j^k}{\Delta t} = -D_y p_{j+1/2}^k
\]

(3)

\[
H U_j^k = -\frac{\hat{U}_j^k}{\Delta t}; \quad H V_j^k = -\frac{\hat{V}_j^k}{\Delta t}
\]

\[
U_0^k = U_N^k = V_0^k = V_N^k = 0
\]

Solve for Greens’s functions \((U_j^k, V_j^k, P_j^k)\) and store them (if necessary). The cost is \(6 \times \mathcal{O}(N)\) for the 6 tridiagonal solvers.

(I): Time-Stepping, \(\Delta t\)

(1)

\[
\mathcal{L}^c p_{j+1/2}^I = 0, \quad P_{1/2}^I = P_{N-1/2}^I = 0 \ (\text{arbitrary})
\]

(here \(p^I \equiv 0\) but usually the Poisson equation is forced on the RHS).

(2)

\[
\frac{\hat{U}_j^I - u_j^n}{\Delta t} = -D_x p_{j+1/2}^I; \quad \frac{\hat{V}_j^I - v_j^n}{\Delta t} = -D_y p_{j+1/2}^I
\]

(3)

\[
H U_j^I = -\frac{u_j^n + \hat{U}_j^I}{\Delta t}; \quad H V_j^I = -\frac{v_j^n + \hat{V}_j^I}{\Delta t}
\]

We now need to satisfy the zero boundary-divergence constraint, i.e.,

\[
d_{1/2}^{n+1} = d_{N-1/2}^{n+1} = 0, \text{ where}
\]

\[
d_{j+1/2}^{n+1} = D_{j+1/2}^I + \sum_{k=1}^{2} \alpha \phi_k D_{j+1/2}^k, \quad j = 0, \ldots, N - 1
\]
and

\[ D_{1/2}^I = i k \frac{U_1^I + U_0^I}{2} + \frac{V_1^I - V_0^I}{\Delta y} \]

Compute \([\alpha]\) from \([\alpha] = [G]^{-1} \begin{bmatrix} -D_{1/2}^I \\ -D_{N-1/2}^I \end{bmatrix}\) where

\[
[G] = \begin{bmatrix}
D_{1/2}^1 & D_{1/2}^2 \\
D_{N-1/2}^1 & D_{N-1/2}^2
\end{bmatrix}
\]

The matrix \([G]\) can be formed and inverted at the preprocessing step \((G)\) so that only matrix-vector multiplies are performed in time-stepping. Note that the time step \(\Delta t\) has to remain constant during time-step. In addition, all the eigenfunctions \((U^k, V^k, P^k)\) are required in order to compute, for example:

\[ u_j^{n+1} = U_j^I + \sum_{k=1}^{2} \alpha_k U_j^k \]

If storage is at premium \((U^k, V^k, P^k)\) can be recomputed every time step. This slows down the execution time by about a factor of 3.

**Example:** \(k = 1, \Delta t = 0.01, N = 50 (\Delta y = 0.04)\) then \(\sigma_G = -9.3145\) versus \(\sigma_{exact} = -9.3137\)

### 3.3 MAC Grid: Two-dimensional Staggered Mesh

The MAC grid (for Marker And Cell) was first proposed by Harlow and Welch (1965). It is an extension of the 1D staggered grid to 2D and 3D. Here we form cells \((i,j)\) with the pressure assigned at the center of the cell and the velocities \((u,v)\) at the midpoints. (Notice that we change the notation from the 1D example).
For example, evaluate:

\[
\frac{\partial p}{\partial x}|_{i+1/2,j} = \frac{p_{i+1,j} - p_{i,j}}{\Delta x} + O(\Delta x^2)
\]

\[
\frac{\partial^2 u}{\partial x^2}|_{i+1/2,j} = \frac{u_{i+3/2,j} - 2u_{i+1/2,j} + u_{i-1/2,j}}{\Delta x^2} + O(\Delta x^2)
\]

The RHS of the Poisson pressure equation contains terms like \(\frac{\partial^2 u^2}{\partial x^2}\) due to non-linear contribution in Navier-Stokes. These terms are:

\[
\frac{\partial^2 u^2}{\partial x^2}|_{i+1/2,j} = \frac{u_{i+1,j}^2 - 2u_{i,j}^2 + u_{i-1,j}^2}{\Delta x^2}
\]

where

\[
u_{i+1,j} = \frac{1}{2}(u_{i+3/2,j} + u_{i+1/2,j})
\]

The product terms \((uv)_{i+1/2,j+1/2}\) are evaluated as the product of averages, not as the average of products, i.e.

\[
(uv)_{i+1/2,j+1/2} = \frac{1}{2}(u_{i+1/2,j} + u_{i+1/2,j+1})\frac{1}{2}(v_{i+1,j+1/2} + v_{i,j+1/2})
\]

The full discrete x-momentum equation in flux-form is then

\[
\frac{u_{i+1/2,j}^{n+1} - u_{i+1/2,j}^n}{\Delta t} + \frac{u_{i+1,j}^2 - u_{i,j}^2}{\Delta x} + \frac{(uv)_{i+1/2,j+1/2} - (uv)_{i+1/2,j-1/2}}{\Delta y}
\]

\[
= -\frac{p_{i+1,j} - p_{i,j}}{\Delta x} + Re^{-1}\left[\frac{u_{i+3/2,j} - 2u_{i+1/2,j} + u_{i-1/2,j}}{\Delta x^2}\right]
\]
\[
\frac{\partial^2(u^2)}{\partial x^2} - 2 \frac{\partial^2(uv)}{\partial x \partial y} - \frac{\partial^2(v)}{\partial y^2} - \frac{\partial d}{\partial t} + Re^{-1} \nabla^2 d
\]

where
\[
d \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}
\]

(also referred to as dilation term in compressible flows).

On the staggered grid:
\[
d_{ij} = \frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x} + \frac{v_{i,j+1/2} - v_{i,j-1/2}}{\Delta y}
\]

and the Poisson pressure equation is discretized at the node \((i,j)\) where pressure is defined.

To implement boundary conditions, we need to extrapolate one component in the \(x\) or \(y\) direction. For example, let \((j)\) be a wall surface, then \(u\) is defined there, so \(u_{i+1/2} = 0 \forall i\) but \(v\) is defined on the \(j+1/2\). By setting a fictitious node (outside the domain) \(j-1/2\) we then set:
\[
v_{i,j-1/2} = -v_{i,j+1/2}
\]

which makes the average velocity at the boundary
\[
v_{i,j} = \frac{1}{2}(v_{i,j+1/2} + v_{i,j-1/2}) = 0
\]

satisfying the no-slip condition. Note that this treatment of boundary condition produces errors \(O(\Delta y)\) in the diffusion of \(v\) (e.g., \(\frac{\partial^2 v}{\partial y^2}\)). This problem is illustrated in the following example for the 1D problem:
\[
\frac{\partial^2 v}{\partial y^2} = 1
\]
with \( v(0) = 0 \) and \( v(1) = 1 \). Then, \( \nu_0 = 0 = \frac{1}{2}(v_1 + v_{-1}) \), set \( v_{-1} = -v_1 \). This leads to first-order accuracy even if a second-order differencing is used everywhere else.

The marker particles used in MAC method are streaklines similar to smoke visualizations. Their positions are monitored through their Lagrangian coordinates, e.g.

\[
\begin{align*}
\frac{dx_p}{dt} &= u_p \\
\frac{dy_p}{dt} &= v_p \\
\Rightarrow \quad x_p^{n+1} &= x_p^n + u_p^n \Delta t \\
y_p^{n+1} &= y_p^n + v_p^n \Delta t
\end{align*}
\]

(Also, higher-order time integrators can be used here.) In general, we need to interpolate for \((u_p, v_p)\) giving their coordinates. A second-order interpolation is as follows:

\[
\Delta y_p \\
\Delta x_p
\]

Define: \( X \equiv \frac{\Delta x_p}{\Delta y_p}, Y \equiv \frac{\Delta y_p}{\Delta y_p} \), then:

\[
u_p = \frac{X}{2}(u_6 - u_4) + (u_2 - u_8) + \frac{1}{2}[X^2(u_6 + u_4 - 2u_5) + Y^2(u_2 + u_8 - 2u_5) + \frac{1}{2}XY(u_3 + u_7 - u_1 - u_9)]
\]

(relation due to Choin and Street, 1969/70, (JCP)).

In summary, the advantages of using a non-staggered grid are:
- Pressure is not defined on the boundaries and thus no b.c. is required.

- Solution of \( \nabla^2 p = f \) is obtained on a standard 5-point molecule which guarantees uniqueness.

- Second-order accuracy is obtained with neighboring points.

The disadvantages are:

- Only one component of the velocity is defined on the boundary.

- Complexity especially in 3D.

- No easy extensions to higher-order differencing.

4. Non-Staggered Mesh/Implicit Discretization

4.1 Consistent 2D Poisson Equation

Let us consider a single grid and assign velocity and pressure at the same node \((i, j)\). The momentum equation then reads:

\[
\frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t} = - \frac{p_{i+1,j} - p_{i-1,j}}{2\Delta x} + \ldots
\]

and apply the continuity equation, i.e.,

\[
d_{ij}^{n+1} \equiv \frac{u_{i+1,j}^{n+1} - u_{i-1,j}^{n+1}}{2\Delta x} + \frac{v_{i,j+1}^{n+1} - v_{i,j-1}^{n+1}}{2\Delta y} = 0
\]
Now, to derive the consistent (discrete) Poisson equation for pressure we take the discrete divergence of momentum equations. This will lead to:
\[
\frac{1}{\Delta x^2}(\vec{p}_{i+1,j} + \vec{p}_{i,j+2} - 4\vec{p}_{i,j} + \vec{p}_{i-1,j} + \vec{p}_{i,j-2}) = f^n
\]
to be satisfied in order to have \( \delta_{ij}^{n+1} = 0 \). This molecule, however, produces 4 independent pressure networks, e.g.,

Since the pressure is determined up to a constant on each submesh (hydrostatic mode) and the constants are arbitrary we get multiple solutions ("checkerboard" instability). Note that a "diagonal" molecule (like in FEM) produces a "checkerboard" instability with two independent pressure submeshes, i.e.,

4.2 Discretization of Stokes Model Problem

Consider again the decaying Fourier mode in a channel. The 1D discretization provides a good model to study the effects of a single grid:

On this grid, we then have the following algorithm:

1. Poisson pressure equation

\[
\mathcal{L} \vec{p}_i = 0 \Rightarrow \left[ \frac{\vec{p}_{j-2}}{(2\Delta y)^2} - \left( \frac{2}{(2\Delta y)^2} + k^2 \right) \vec{p}_j + \frac{\vec{p}_{j+2}}{(2\Delta y)^2} \right] = 0
\]
\[ j = 1, \ldots, N - 1 \]
\[ y = +1 \]
\[ y = -1 \]
with b.c.
\[ d_{N+1}^{n+1} = 0 \Rightarrow \frac{1}{\Delta y} \left[ -\frac{3}{2} y_0^{n+1} + 2 v_1^{n+1} - \frac{1}{2} v_2^{n+1} \right] + ik u_0 = 0 \]
(second-order, one-sided derivative)
\[ d_0^{n+1} = 0 \Rightarrow \frac{1}{\Delta y} \left[ \frac{3}{2} v_N^{n+1} - 2 v_{N-1}^{n+1} + \frac{1}{2} v_{N-2}^{n+1} \right] + ik u_N = 0 \]
(2) Velocity update due to pressure correction
\[ \frac{\hat{u}_j - u_j^n}{\Delta t} = -\frac{ik p_j}{D_{\nu p_j}} \frac{\hat{v}_j - v_j^n}{\Delta t} = -\frac{p_{j+1} - p_{j-1}}{2 \Delta y} \frac{v_j^n}{D_{\nu p_j}} \]
(3) Velocity update due to viscous terms
\[ \frac{u_j^{n+1} - \hat{u}_j}{\Delta t} = \frac{1}{2} \left( \frac{u_{j+1}^{n+1/2} - 2 u_j^{n+1/2} + u_{j-1}^{n+1/2}}{\Delta y^2} \right) \]
where
\[ u_j^{n+1/2} = \left( u_j^{n+1} + u_j^n \right) / 2 \]
or
\[ \frac{u_j^{n+1/2} - \left( \frac{2}{\Delta y^2} + k^2 + \frac{2}{\Delta t} \right) u_j^{n+1/2} + \frac{u_{j+1}^{n+1/2}}{\Delta y^2}}{H u_j^{n+1/2}} = -\frac{u_j^n + \hat{u}_j}{\Delta t} \]
\[ j = 1, \ldots, N - 1 \]

and
\[ v_0^{n+1} = u_N^{n+1} = 0 \]

Similarly,
\[ H v_j^{n+1/2} = -\frac{v^n_j + \dot{v}_j}{\Delta t}, \quad j = 1, \ldots, N - 1 \]
\[ v_0^{n+1} = v_N^{n+1} = 0 \]

The discrete divergence \((d_j^{n+1})\) satisfies:
\[ \frac{d_j^{n+1} - d_j^n}{\Delta t} = -\mathcal{L}^c \bar{p}_j + \frac{1}{2} \mathcal{L} d_j^{n+1} + \frac{1}{2} \mathcal{L} d_j^n \]
\[ j = 1, \ldots, N - 1 \]
\[ d_0^{n+1} = d_N^{n+1} = 0 \]

where \(\mathcal{L}\) is the discrete standard Laplacian \((= \Delta_{yy}^0)\). If \(\mathcal{L}^c \bar{p}_j = 0\) then due to homogeneity

\[ \Rightarrow d_j^{n+1} = 0, \quad \forall \ j = 0, \ldots, N \]

If we instead solve \(\mathcal{L} \bar{p}_j = 0\), then \((\mathcal{L}^c - \mathcal{L}) \bar{p}_j = O(\Delta y^2)\), thus
\[ \frac{d_j^{n+1} - d_j^n}{\Delta t} = O(\Delta y^2) \Rightarrow d_j^{n+1} \sim O(\Delta y^2) \]

**Example:** The numerical solution with the \(\mathcal{L} \bar{p}_j = 0\) gives a unique pressure solution \((\Delta t = 0.01, \Delta y = 0.04, N = 50)\) gives \(\sigma_{NS} = -9.3107\) versus \(\sigma_C = -9.3145\) versus \(\sigma_{exact} = -9.3137\).

**4.3 Pressure Boundary condition**

Consider the domain \(\Omega\) and the inhomogeneous Stokes equation, i.e.,
\[ \frac{\partial \vec{v}}{\partial t} = -\nabla p + \text{Re}^{-1} \nabla^2 \vec{v} + \vec{f} \]

The normal component of momentum gives Neumann b.c. for pressure:

\[ \frac{\partial \vec{v} \cdot \hat{n}}{\partial t} = -\frac{\partial p}{\partial n} + \text{Re}^{-1} \nabla^2 \vec{v} \cdot \hat{n} + \vec{f} \cdot \hat{n} \]

Assume impermeable walls for simplicity and zero forcing

\[ \frac{\partial p}{\partial n} = \text{Re}^{-1} \nabla^2 \vec{v} \cdot \hat{n} \]

In this form, the pressure and velocity fields are still coupled, except in the limit of \textit{very high} Reynolds number ($\nabla^2 \vec{v}$ bounded) where we obtain:

\[ \frac{\partial p}{\partial n} = 0, \]

inviscid-type b.c. If we treat explicitly the coupled pressure b.c., i.e.

\[ \frac{\partial p_n^{n+1}}{\partial n} = \frac{1}{\text{Re}} (\nabla^2 \vec{v})^n \cdot \hat{n} \]

(Euler-forward) then the RHS is known at time level ($n + 1$). Unfortunately this formulation leads to instability. However, we can write:

\[ \frac{\partial \vec{p}_n^{n+1}}{\partial n} = \frac{1}{\text{Re}} [\nabla (\nabla \cdot \vec{v}_n^{n+1}) - \nabla \times \omega^n] \cdot \hat{n} \]

where $\vec{\omega} = \nabla \times \vec{v}$, or

\[ \frac{\partial \vec{p}_n^{n+1}}{\partial n} = -\frac{1}{\text{Re}} \nabla \times \omega^n \cdot \hat{n} \]

(rotational form) and this formulation is stable because the soleinoidal term was imposed at the time level ($n + 1$). A generalization can
be obtained for the Navier-Stokes equation by including explicitly the nonlinear term \( N(\vec{v}) \) and use high-order integration, i.e.,

\[
\frac{\partial p}{\partial n} = -\sum_{q=0}^{J_c} \beta q \left[ \frac{\nabla \times \vec{\omega}}{Re} + N(v) \right]^{n-q} \cdot \hat{n},
\]

where \( J_c = 2 \) or \( 3 \) for Adams-Bashforth. (Solvability condition is needed for Neumann). A Dirichlet boundary condition for the pressure can be obtained by considering the tangential component of the pressure equation: (Stokes);

\[
\frac{\partial p}{\partial s} = \frac{1}{Re} \nabla^2 \vec{v} \cdot \hat{s}
\]

ignoring body forces and assuming zero velocity on \( \partial \Omega \). If we integrate along the boundary \((s)\), then we obtain Dirichlet b.c. for pressure. A question is then raised if the pressure solution with Neumann b.c. satisfies the tangential b.c. For the Navier-Stokes we have the system:

\[
\nabla^2 p = -\nabla \cdot [(\vec{u} \cdot \nabla)\vec{v}] \text{ on } \Omega, t \geq 0
\]

\[
\frac{\partial p}{\partial n} = Re^{-1} \nabla^2 u_n - \left[ \frac{\partial u_n}{\partial t} + \vec{v} \cdot \nabla u_n \right] \text{ on } \partial \Omega, t \geq 0
\]

We assume that we can apply the Poisson equations on \( \partial \Omega \), i.e.,

\[
\nabla^2 p = \frac{\partial^2 p}{\partial n^2} + \frac{\partial^2 p}{\partial s^2} = -\nabla \cdot [(\vec{v} \cdot \nabla)\vec{v}]
\]

\[
= - \left[ \frac{\partial}{\partial n} (\vec{v} \cdot \nabla v_n) + \frac{\partial}{\partial s} (\vec{v} \cdot \nabla v_n) \right]
\]

The Neumann b.c. (differentiate in \( \hat{n} \)):

\[
\frac{\partial^2 p}{\partial n^2} = \frac{\partial}{\partial n} \left[ \frac{\partial}{\partial n} Re^{-1} \nabla^2 v_n - \left( \frac{\partial v_n}{\partial t} + \vec{v} \cdot \nabla v_n \right) \right]
\]

and thus:

\[
\frac{\partial^2 p}{\partial s^2} = -\frac{\partial}{\partial s} (\vec{v} \cdot \nabla v_n) - Re^{-1} \nabla^2 \frac{\partial v_n}{\partial n} + \frac{\partial}{\partial t} \left( \frac{\partial v_n}{\partial n} \right)
\]
From continuity in \((n, s)\):

\[
\frac{\partial v_n}{\partial n} + \frac{\partial v_s}{\partial s} = 0
\]

thus:

\[
\frac{\partial^2 p}{\partial s^2} = -\frac{\partial}{\partial s}(\bar{v} \cdot \nabla v_s) + \frac{\partial}{\partial s}\left[Re^{-1} \nabla^2 v_s - \frac{\partial v_s}{\partial t}\right]
\]

or

\[
\frac{\partial}{\partial s}\left\{\frac{\partial v_s}{\partial t} + \bar{v} \cdot \nabla v_s + \frac{\partial p}{\partial s} - Re^{-1} \nabla^2 v_s\right\} = 0
\]

\[
\Rightarrow \frac{\partial p}{\partial s} = Re^{-1} \nabla^2 v_s - \bar{v} \cdot \nabla v_s - \frac{\partial v_s}{\partial t}, \text{ QED}
\]

Therefore, if the Neumann b.c. is applied to the pressure equation, a sufficiently smooth solution may also satisfy the Dirichlet b.c.. This is, in general true for \(t > 0\). At \(t = 0\), only the Neumann pressure b.c. is the appropriate one as there is no sufficient smoothness in the solution to guarantee the proof.

**An inviscid flow example:**


Consider the inviscid stagnation flow:

\( u = x; \ v = -y \)

**Neumann boundary conditions:**

At: \( x = 0 \)
\[
\frac{\partial p}{\partial x} = 0, \text{ symmetry}
\]

\( x = 1 \)
\[
\frac{\partial p}{\partial x} = -uu_x - vu_y
\]

At: \( y = 0, 1 \):
\[
\frac{\partial p}{\partial y} = -uv_x - vv_y
\]
The corresponding Poisson equation for the pressure is:

\[ p_{xx} + p_{yy} = -\left( (u u_x + v u_y)_x + (u v_x + v v_y)_y \right) \int_{\Omega} \]

where the solvability (compatibility) condition, i.e.,

\[ \int_{\Omega} \int f dA = \int_{\partial \Omega} \frac{\partial p}{\partial n} ds \]

should be satisfied with the Neumann condition.

**Dirichlet boundary conditions:**

\[ p = -\frac{1}{2}(u^2 + v^2), \text{ at } y = 0, y = 1, x = 1 \]

\[ \text{At } x = 0 \Rightarrow \frac{\partial p}{\partial x} = 0 \]

Numerical solution gives identical solution to machine accuracy. It is interesting to note that in the numerical experiment on a 40 \times 40 grid using central finite differencing if we solve the Poisson equation using SOR, the Dirichlet problem is faster if the analytical velocity field is used on the RHS, whereas the Neumann problem is faster if we use the numerically obtained velocity field. This, perhaps, is another indication of the robustness of the Neumann b.c.

**4.4 Classical splitting scheme:**


We consider again the Stokes model problem on a non-staggered grid. Here we test the accuracy of the inviscid-type pressure b.c., i.e., \( \frac{\partial p}{\partial y} = 0 \) at \( y = \pm 1 \)

**Step 1:** Pressure equation

\[ (\Delta_{yy} - k^2) p_j = \frac{1}{\Delta l} (\Delta_y v_j^n + i k u^n), \quad j = 1, \ldots, N - 1 \]

At the boundaries \( \frac{\partial p}{\partial y} = 0 \), which can be used in the pressure equation:

\[ j = 0 : -\left( \frac{2}{\Delta y^2} + k^2 \right) p_0 + \frac{2 p_1}{\Delta y^2} = 0 \]
\[ j = N : \frac{2 \overline{p}_{N-1}}{\Delta y^2} - \left( \frac{2}{\Delta y^2} + k^2 \right) \overline{p}_N = 0 \]

**Step 2:** Velocity update due to pressure correction

\[
\frac{\dot{u}_j - u^n_j}{\Delta t} = -ik\overline{p}_j; \quad \frac{\dot{v}_j - v^n_j}{\Delta t} = -\frac{\overline{p}_{j+1} - \overline{p}_{j-1}}{2\Delta y}, \quad j = 1, \ldots, N-1
\]

**Step 3:** Velocity update due to viscous correction

\[
\frac{u^{n+1}_j - \dot{u}_j}{\Delta t} = -\frac{1}{2} \nabla^2 u^{n+1/2}_j
\]

\[ u^{n+1}_0 = u^{n+1}_N = 0 \]

Similarly for \( v^{n+1/2} \). The equation for divergence in semi-discrete form is then:

\[
\frac{d^{n+1} - d^n}{\Delta t} = -\nabla^2 \overline{p} + \nabla^2 \frac{d^{n+1}}{2} + \nabla^2 \frac{d^n}{2}
\]

Assuming that at time level \( n \):

\[ d^n = 0 \Rightarrow \left( \nabla^2 - \frac{2}{\Delta t} \right) d^{n+1} = 0 \text{ in } \Omega \]

However \( d^{n+1} \) at \( y = \pm 1 \) is not specified and thus it is of order \( O(1) \). The solution is then driven by the non-zero b.c. and thus we develop normal boundary layers of order

\[ \delta \sim \sqrt{\Delta t} \]

or for N-S

\[ \delta \sim \sqrt{\frac{\Delta t}{Re}} \]
But
\[
d_w \sim \delta \left( \frac{\partial d}{\partial n} \right)_w = \mathcal{O}\left( \frac{\partial v}{\partial n} \right) \Rightarrow v \sim d_w \delta \sim \left( \frac{\partial d}{\partial n} \right)_w \cdot \Delta t
\]

Thus, velocity is controlled by \( \Delta t \) so although the divergence (pressure) errors may be \( \mathcal{O}(\Delta t^{1/2}) \) at the boundary the velocity errors are \( \mathcal{O}(\Delta t) \). Notice that the use of high-order integration (e.g., CN here) still produces and \( \mathcal{O}(\Delta t) \) accuracy due to the inviscid (incorrect) type b.c.

Graphically: (Stokes problem)

4.5 High-order splitting scheme

To increase the time accuracy of the splitting algorithm we next consider the b.c.

\[
\frac{\partial \overline{p}}{\partial y} = \left[ \frac{\partial^2 v}{\partial y^2} - \kappa^2 v \right]_n \hat{y}, \text{ Laplacian Form}
\]

\[
\frac{\partial \overline{p}}{\partial y} = - \left( \nabla \times \omega^n \right) \cdot \hat{y}, \text{ Rotational Form}
\]

where in the continuous case the RHS is identical. However, here we want to examine what happens if the RHS is non-zero and in the form shown here. We can obtain an even higher accuracy by including multiple levels \( (n), (n - 1), \) etc. on RHS. We can analyze these two b.c exactly by performing the following eigenvalue analysis:
**Continuous Problem:** The normal modes \((\hat{u}, \hat{v}, \hat{p})(y)\) satisfy:
\[
\nabla^2 \hat{p}_i = 0
\]
\[(\sigma - \nabla^2)\nabla^2 \hat{v}_i = 0, \ i = 1, \ldots
\]

In semi-discrete form (using Euler-Backwards) we obtain:
\[
\frac{\hat{v}^{n+1} - \hat{v}^n}{\Delta t} = -\nabla \hat{p} + \nabla^2 \hat{v}^{n+1}
\]

and assuming an amplification factor \(\mu\) for the normal modes we can write:
\[
(\hat{u}^n, \hat{v}^n, \hat{p}^n) = \mu^n (\bar{u}_i, \bar{v}_i, \bar{p}_i)
\]

and thus
\[
\frac{\mu - 1}{\Delta t} \bar{v}_i = -\mu \nabla \bar{p}_i + \mu \nabla^2 \bar{v}_i
\]

and the decoupled equations are:
\[
\nabla^2 \hat{p}_i = 0 \text{ and } (\sigma - \nabla^2)\nabla^2 \hat{v}_i = 0
\]

where
\[
\sigma = \frac{\mu - 1}{\Delta t}
\]

**Splitting Formulation:**

\[
(1) \quad \frac{\nu^* - \bar{v}}{\Delta t} = -\mu \nabla \hat{p} \\
\nabla \cdot \nu^* = 0
\]

\[
(2) \quad \frac{\bar{\nu} - \nu^*}{\Delta t} = \nabla^2 \bar{v}
\]

where \(\nu^*\) is the intermediate field and \(\hat{\mu}\) is the amplification factor of the splitting scheme. Eliminating \(\bar{v}\) and applying the \(\nabla \times \nabla\) we obtain:
\[
\left[ \frac{\mu - 1}{\Delta t} - \nabla^2 \right] \nabla^2 \nu^* = (\sigma - \nabla^2) \nabla^2 \nu^* = 0
\]
However, the final velocity $\tilde{v}$ satisfies:

$$(\tilde{\sigma} - \nabla^2) \nabla^2 \left( \frac{\tilde{\mu}}{\Delta t} - \nabla^2 \right) \tilde{v} = 0$$

Therefore, $v^*$ satisfies an equation similar to the continuous problem and therefore it has two non-divergent modes. However, $\tilde{\sigma}$ has an extra mode due to $\left( \frac{\tilde{\mu}}{\Delta t} - \nabla^2 \right)$ which produces a boundary layer of thickness $\delta \sim \sqrt{\Delta t}$ (splitting error). For the 1D eigenproblem we consider we have

$$\nabla^2 = D^2 - k^2 \text{ where } D \equiv d/dy$$

$$\nabla^2 - \tilde{\sigma} = D^2 + \tilde{a}^2, \text{ where } \tilde{a}^2 = -k^2 - \tilde{\sigma}$$

$$\nabla^2 - \frac{\tilde{\mu}}{\Delta t} = D^2 - \lambda^2, \text{ where } \lambda^2 = k^2 + \frac{\tilde{\mu}}{\Delta t}$$

The general solution is then:

$$v^* = A^* \cosh ky + B^* \cos \tilde{a} y$$
$$\tilde{v} = \tilde{A} \cosh ky + \tilde{B} \cos \tilde{a} y + \tilde{C} \cosh \lambda y$$

where the modes $(k, \tilde{a})$ are the non-divergent ones and $(\lambda)$ is the spurious divergent mode. Consider the first non-divergent mode:

$$v^*_k = A^* \cosh ky; \quad \tilde{v}_k = \tilde{A} \cosh ky$$

since:

$$\frac{\tilde{\mu} \tilde{v} - v^*}{\Delta t} = \nabla^2 \tilde{v} = 0 \Rightarrow \tilde{A} = \frac{A^*}{\tilde{\mu}}$$

and from continuity:

$$\tilde{u}_k = -\frac{D \tilde{v}_k}{ik} = i \frac{A^*}{\tilde{\mu}} \sinh ky.$$  

Similarly, the second mode gives: $\tilde{B} = B^*$ and

$$\tilde{u}_{\alpha} = -i \frac{\tilde{a}}{k} \tilde{B} \sin \tilde{a} y$$
The third mode is: \( v_\lambda^* = 0 \) and \( \tilde{v}_\lambda = \tilde{C} \cosh \lambda y \), thus from \( \tilde{v}_\lambda^* - \tilde{v}_\lambda = -\nabla p \tilde{\mu} \Delta t \Rightarrow \tilde{v}_\lambda = \tilde{\mu} \Delta t \nabla p \Rightarrow \nabla \times \tilde{v}_\lambda = 0 \Rightarrow D \tilde{u}_\lambda = ik \tilde{v}_\lambda \). The final mode then is

\[
\tilde{v} = \frac{A^*}{\tilde{\mu}} \cosh ky + B^* \cos \tilde{a}y + C^* \cosh \lambda y
\]

\[
\tilde{u} = \frac{iA^*}{\tilde{\mu}} \sinh ky - \frac{i\tilde{a}B^*}{k} \sin \tilde{a}y + iC^* \frac{k}{\lambda} \sinh \lambda y
\]

At \( y = 1 \Rightarrow \tilde{u} = \tilde{v} = 0 \), no-slip \( \Rightarrow \):

\[
C^* = -\frac{A^* \cosh k(\tilde{a} \tan \tilde{a} + k \tanh k)}{\tilde{\mu} \cosh \lambda [\tilde{a} \tan \tilde{a} + (k^2/\lambda) \tanh \lambda]}
\]

But \( \tilde{a} \tan \tilde{a} - a \tan a = \mathcal{O}(\Delta \sigma) \), where \( \Delta \sigma = \sigma - \tilde{\sigma} \) Thus,

\[
C^* \sim \Delta \sigma
\]

So the error in growth rate that characterizes the time-accuracy of the scheme is directly proportional to the divergent mode. The non-zero pressure b.c. is then:

\[
\tilde{\mu} \frac{\partial \tilde{p}}{\partial \hat{n}} = \hat{n} \cdot \nabla^2 \tilde{v} \quad \text{Laplacian}
\]

\[
\mu \frac{\partial \tilde{p}}{\partial \hat{n}} = -\hat{n} \cdot \nabla \times \nabla \times \tilde{v} \quad \text{(Rotational)}
\]

which if combined with \( v^* - \tilde{v} = -\tilde{\mu} \Delta t \frac{\partial \tilde{p}}{\partial \hat{n}} \) at \( y = 1 \):

\[
v^* + \Delta t D^2 \tilde{v} = 0 \quad \text{(Laplacian)}
\]

\[
v^* - ik \Delta t D \tilde{u} = 0 \quad \text{(Rotational)}
\]

substitute \( v^*, \tilde{v}, \tilde{u} \) to obtain:

\[
A^* \left( 1 + \frac{\Delta t k^2}{\tilde{\mu}} \right) \cosh k + B^*(1 - \Delta t \tilde{a}^2) \cos \tilde{a} + C^* \lambda^2 \Delta t \cosh \lambda = 0 \quad \text{(Laplacian)}
\]

\[
A^* \left( 1 + \frac{\Delta t k^2}{\tilde{\mu}} \right) \cosh k + B^*(1 - \Delta t \tilde{a}^2) \cos \tilde{a} + C^* k^2 \Delta t \cosh \lambda = 0 \quad \text{(Rotational)}
\]
To satisfy th b.c. \( \tilde{u} = \tilde{v} = 0 \) in addition to the above equation for \( A^*, B^*, C^* \) non-zero we require that

\[
\text{(Laplacian)} \left| \begin{array}{ccc}
\mu + \Delta t k^2 & 1 - \tilde{a}^2 \Delta t & \lambda^2 \Delta t \\
1 & 1 & 1 \\
k \tanh k & -\tilde{a} \tan \tilde{a} & k^2 \tanh \lambda \end{array} \right| = 0
\]

Rotational: substitute \((\lambda^2 \Delta t)\) with \((k^2 \Delta t)\). Further expansion shows that the amplification factor \(\mu\) is approximated with higher-order in the rotational case and only with first-order in the Laplacian case.

### 4.6 Other Splitting Schemes:

Consider a domain \(\Omega\) (\(\tilde{v}_N = 0\) or periodic). A time step is then split into:

(i) Nonlinear step:

\[
\frac{\tilde{v} - \tilde{v}^n}{\Delta t} = \sum_{j=0}^{J} \beta_j [\langle \tilde{v} \cdot \nabla \rangle \tilde{v}^{n-q}]
\]

no b.c.

(ii)

\[
\frac{\hat{v} - \tilde{v}}{\Delta t} = \frac{Re^{-1}}{2} \left[ \nabla^2 \hat{v} + \nabla^2 \tilde{v} \right]
\]

with b.c. \(\hat{v} = \Delta t \nabla p^n\) on \(\partial \Omega\) (weakly unstable).

(iii)

\[
\frac{\tilde{v}^{n+1} - \hat{v}}{\Delta t} = -\nabla p^{n+1}, \nabla \tilde{v}^{n+1} = 0
\]

\[\Rightarrow \nabla^2 p^{n+1} = \nabla \cdot \left( \frac{\hat{v}}{\Delta t} \right)\]

and

\[
\frac{\partial p}{\partial n} = \hat{v} \cdot \hat{n}
\]

for solvability. The b.c. in step (ii) is not correct, instead use:

\(\hat{v} \cdot \hat{s} = (\Delta t \nabla p^n) \cdot \hat{s}\), (tangential) and \(\hat{v} \cdot \hat{n} = 0\), normal.
To appreciate the choice of this boundary condition we consider \( \hat{u} \) as an approximation to \( u^*(x, t_{n+1}) \) where the continuous function \( u^*(x, t) \) satisfies

\[
\frac{\partial u^*}{\partial t} = \NL^* + \Re^{-1} \nabla^2 u^*
\]

\( u^*(x, t_n) = u(x, t_n) \)

Therefore,

\[
\hat{u} \approx u^*(x, t_n + \Delta t) = u^*(x, t_n) + \Delta t \frac{\partial u^*}{\partial t} + \frac{1}{2} \Delta t^2 \frac{\partial^2 u^*}{\partial t^2} + \ldots
\]

\[
= u(x, t_n) + \Delta t(N \NL^* + \Re^{-1} \nabla^2 u^*) + \frac{1}{2} \Delta t^2 \frac{\partial}{\partial t} \left( N \NL^* + \Re^{-1} \nabla^2 u^* \right) + \ldots
\]

\[
= u(x, t_n) + \Delta t(N \NL + \Re^{-1} \nabla^2 u) + \frac{1}{2} \Delta t^2 (N \NL + \Re^{-1} \nabla^2 u) + \ldots
\]

\[
= u(x, t_n) + \Delta t \left( \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} \right) + \frac{1}{2} \Delta t^2 \left( \frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial t} \frac{\partial p}{\partial x} \right) + \ldots
\]

\[
= \underbrace{u(x, t_{n+1})}_{\hat{u}^{n+1}} + \Delta t \frac{\partial p}{\partial x} + \mathcal{O}(\Delta t^2)
\]

Thus, if \( u^{n+1} = 0 \Rightarrow \hat{u} = \Delta t \frac{\partial p}{\partial x} \), similarly for \( \hat{v} = \Delta t \frac{\partial p}{\partial y} \)

5. **Staggered Mesh/Explicit Discretization**

Here we treat the viscous terms explicitly although the pressure is still treated implicitly. The semi-discrete system has the form:

\[
\frac{\hat{v}^n - \hat{v}^{n-1}}{\Delta t} = -\nabla \tilde{p} + \nabla^2 \hat{v}^n
\]

\[\nabla \cdot \hat{v}^{n+1} = 0\]

\[\hat{v} = 0\]

In two substeps, the fully discrete system is:

\[
(1) \quad \frac{\hat{u}_j - u^n_j}{\Delta t} = \frac{u^n_{j+1} - u^n_{j-1}}{2 \Delta y^2} - \left( \frac{2}{\Delta y^2} + k^2 \right) u^n_j + \frac{u^n_{j+1} - u^n_{j-1}}{\Delta y^2}, \quad j = 1, \ldots, N - 1
\]
(2) \[ M_j^x: \quad \frac{u_j^{n+1} - \hat{u}_j}{\Delta t} = -ik\frac{\overline{p}_{j-1/2} + \overline{p}_{j+1/2}}{2} \]
\[ M_j^y: \quad \frac{v_j^{n+1} - \hat{v}_j}{\Delta t} = -\frac{\overline{p}_{j+1/2} - \overline{p}_{j-1/2}}{\Delta y}, \quad j = 1, \ldots, N - 1 \]

Also:
\[ d_j^{n+1} = 0, \quad j = 0, 1, \ldots, (N - 1) \]

To decouple the equations, we take the discrete divergence:
\[ ik\frac{M_j^x + M_j^y}{2} + \frac{M_j^y + M_j^y}{\Delta y}, \quad j = 1, \ldots, (N - 2) \]
\[ \frac{1}{\Delta y^2} \left[ \overline{p}_{j+3/2} - 2\overline{p}_{j+1/2} + \overline{p}_{j-1/2} \right] - \frac{k^2}{4} \left[ \overline{p}_{j+3/2} + 2\overline{p}_{j+1/2} + \overline{p}_{j-1/2} \right] \]
\[ = \frac{\hat{d}_{j+1/2}}{\Delta t} - \frac{d_j^{n+1}}{\Delta t}, \quad j = 1, \ldots, (N - 2) \]

At the boundaries:
\[ 0: \quad d_1^{n+1} = \frac{ik}{2} \left[ \frac{u_1^{n+1} + \hat{u}_0^{n+1}}{2} + \frac{v_1^{n+1} - v_0^{n+1}}{\Delta y} \right] \]
\[ = \frac{ik}{2} \left[ \frac{\hat{u}_1}{2} - ik\Delta t \frac{\overline{p}_{3/2} + \overline{p}_{1/2}}{2} \right] + \frac{1}{\Delta y} \left[ \hat{v}_1 - \Delta t \frac{\overline{p}_{3/2} - \overline{p}_{1/2}}{\Delta y} \right] = 0 \]

or:
\[ 0: \quad \frac{1}{\Delta y^2} (\overline{p}_{3/2} - \overline{p}_{1/2}) - \frac{k^2}{4} (\overline{p}_{3/2} + \overline{p}_{1/2}) = \frac{1}{\Delta t} \left( \hat{v}_1 + ik\frac{\hat{u}_1}{2} \right) \]

So even at the boundaries the velocity-pressure equations are completely decoupled. This is due to the explicit treatment of the viscous terms.

In order to investigate what is the actual b.c. that the pressure satisfies at the boundaries (by extrapolation) we perform the following analysis:

Let
\[ \tilde{p}_1 \equiv \frac{\partial p}{\partial y} \big|_1 = \frac{p_{3/2} - p_{1/2}}{\Delta y}; \quad \tilde{p}_1 \equiv p(y_1) = \frac{p_{3/2} + p_{1/2}}{2} \]
\[ \tilde{p}_0 \equiv \frac{\partial p}{\partial y} \big|_0; \quad p_0 \equiv p(y_0) \]
then 0: becomes;

\[
\frac{1}{\Delta y} p_1' - \frac{k^2}{4} p_1 = \frac{1}{\Delta t} \left( \frac{1}{\Delta y} \dot{v}_1 + ik\frac{\dot{u}_1}{2} \right) + \frac{p_0'}{\Delta y} - \frac{k^2 p_0}{2} - \frac{1}{\Delta y} \left( \frac{1}{\Delta t} \dot{v}_1 - p_0' \right) + \frac{i k}{2} \left( \frac{\dot{u}_1}{\Delta t} + ik p_0 \right)
\]

\[
(\nabla^2 p)_{1/2} = \frac{1}{\Delta y} \left( \frac{\dot{v}_1}{\Delta t} - p_0' \right) + \frac{i k}{2} \left( \dot{u}_1 + ik p_0 \right)
\]

As \( \Delta y \to 0 \), then \( \frac{1}{\Delta y} \left( \frac{\dot{v}_1}{\Delta t} - p_0' \right) \) dominates. But

\[
\dot{v}_1 = \underbrace{v_1^n}_{O(\Delta y^2)} + \Delta t \underbrace{\frac{\partial^2 v_1^n}{\partial y^2}}_{O(\Delta y^2)} - \Delta t \underbrace{k^2 v_1^n}_{O(\Delta y^2)}
\]

Therefore:

\[
(\nabla^2 p)_{1/2} \sim \frac{1}{\Delta y} \frac{\partial^2 v_0}{\partial y^2} \Rightarrow \frac{\partial p}{\partial y} \sim \frac{\partial^2 v}{\partial y^2}
\]

In other words, the staggered grid generates the correct pressure b.c. for the pressure, i.e, the Neumann b.c. derived from the normal component of the N-S applied at a Dirichlet boundary.

6. Steady N-S: The Artificial Compressibility Method

6.1 Basic Idea

\[
NL(v) + \nabla p = Re^{-1} \nabla^2 v; \nabla \cdot v = 0
\]


Perturbed Continuity Equation:

(1) \[
\frac{\partial p}{\partial t} + c^2 \nabla \cdot v = 0
\]

\(c^2 = \) arbitrary constant.
• This equation imposes $\nabla \cdot v = 0$ when steady state is reached.
• $c^2$ is chosen to ensure the existence of a steady numerical solution.
• (1) can be derived from compressible N-S with state law: $p = c^2 \rho$, $c^2 = \text{const.}$

**Convergence:**

$$\max \left( \frac{1}{\Delta t} |u^{n+1} - u^n|, \frac{1}{\Delta t} |v^{n+1} - v^n|, \frac{1}{c^2 \Delta t} |p^{n+1} - p^n| \right) < \epsilon$$

### 6.2 MAC discretization

**Discretization with MAC**

**Explicit in time**

\[
\begin{align*}
  x - \text{mom} : & \quad \frac{1}{\Delta t} \left( \frac{u^{n+1}_{i+1/2,j} - u^n_{i+1/2,j}}{\Delta t} \right) + \frac{\text{NL in } u}{\Delta t} \frac{\Delta x}{\Delta t} p^n_{i+1/2,j} = \frac{1}{Re} \nabla^2_n u^n_{i+1/2,j} \\
  y - \text{mom} : & \quad \frac{1}{\Delta t} \left( \frac{v^{n+1}_{i,j+1/2} - v^n_{i,j+1/2}}{\Delta t} \right) + \frac{\text{NL in } v}{\Delta t} \frac{\Delta y}{\Delta t} p^n_{i,j+1/2} = \frac{1}{Re} \nabla^2_n v^n_{i,j+1/2}
\end{align*}
\]

**Continuity:**

\[
\frac{1}{\Delta t} \left( \frac{p^{n+1}_{i,j} - p^n_{i,j}}{\Delta t} \right) + c^2 \left( \Delta^1_x u^{n+1}_{ij} + \Delta^1_y v^{n+1}_{ij} \right) = 0
\]

**Definitions:**

\[
\begin{align*}
  \Delta^1_x f_{\ell,m} & = \frac{1}{\Delta x} \left( f_{\ell+1/2,m} - f_{\ell-1/2,m} \right) \\
  \Delta^1_y f_{\ell,m} & = \frac{1}{\Delta y} \left( f_{\ell,m+1/2} - f_{\ell,m-1/2} \right)
\end{align*}
\]
\( \ell, m \) may not be integers.

\[
\nabla^2 f_{\ell, m} = \Delta_{xx} f_{\ell, m} + \Delta_{yy} f_{\ell, m} \\
\Delta_{xx} f_{\ell, m} = \frac{f_{\ell+1, m} - 2f_{\ell, m} + f_{\ell-1, m}}{\Delta x^2} \\
\Delta_{yy} f_{\ell, m} = \frac{f_{\ell, m+1} - 2f_{\ell, m} + f_{\ell, m-1}}{\Delta y^2}
\]

Treatment of non-linear terms:

\[
a(u, v) = \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) \quad \text{or} \quad a(u, v) = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \\
b(u, v) = \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(v^2) \quad \text{or} \quad b(u, v) = u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}
\]

Conservative \hspace{1cm} Non-conservative

Non-conservative form:

\[
a^n_{i+1/2, j} = u^n_{i+1/2, j} \Delta^0_x u^n_{i+1/2, j} + \hat{v}^n_{i+1/2, j} \Delta^0_y u^n_{i+1/2, j} \\
b^n_{i, j+1/2} = \hat{u}^n_{i, j+1/2} \Delta^0_x v^n_{i, j+1/2} + v^n_{i, j+1/2} \Delta^0_y v^n_{i, j+1/2}
\]

where

\[
\hat{u}_{i, j+1/2} = \frac{1}{4}(u_{i+1/2, j} + u_{i+1/2, j+1} + u_{i, j+1/2, j} + u_{i-1/2, j}) \\
\hat{v}_{i+1/2, j} = \frac{1}{4}(v_{i+1, j+1/2} + v_{i+1, j+1/2} + v_{i, j+1/2} + v_{i, j+1/2, j})
\]

\[
\Delta^0_x f_{\ell, m} = \frac{1}{2\Delta x}(f_{\ell+1, m} - f_{\ell-1, m}) \\
\Delta^0_y f_{\ell, m} = \frac{1}{2\Delta y}(f_{\ell, m+1} - f_{\ell, m-1})
\]
Conservative form: (momentum and energy are conserved)

\[ a_{i+1/2,j}^n = \Delta_x^1(u^2)_{i+1/2,j}^n + \Delta_y^1(uv)_{i+1/2,j}^n \]

\[ b_{i,j+1/2}^n = \Delta_x^1(uv)_{i,j+1/2}^n + \Delta_y^1(v^2)_{i,j+1/2}^n \]

where

\[ (u^2)_{ij} = \frac{1}{4}(u_{i+1/2,j} + u_{i-1/2,j})^2 \]

\[ (uv)_{ij} = \frac{1}{4}(u_{i+1/2,j+1} + u_{i+1/2,j})(v_{i+1,j+1/2} + v_{i,j+1/2}) \]

\[ (v^2)_{ij} = \frac{1}{4}(v_{i,j+1/2} + v_{i,j-1/2})^2 \]

Note, that the conservative formulation is preferable since the continuity equation has been altered. Thus, during the transient period the two formulations are not equivalent.

6.3 The choice of parameter \( c \):


Consider the 1D inviscid subsystem:

\[ \frac{1}{c^2} \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} = 0 \quad \text{(continuity)} \]

\[ \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial p}{\partial x} = 0 \quad \text{(conservative form)} \]
In matrix form:

\[
\begin{align*}
\frac{\partial}{\partial t} \begin{bmatrix} p \\ u \\ w \end{bmatrix} + \begin{bmatrix} \frac{c^2}{A} \\ 1 \\ 2u \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} p \\ u \\ w \end{bmatrix} &= 0
\end{align*}
\]

or

\[
\frac{\partial W}{\partial t} + A \frac{\partial W}{\partial x} = 0
\]

The eigenvalues of \( A \) are: \( \lambda_{\pm} = u \pm \sqrt{u^2 + c^2} \) both real \( \Rightarrow \) hyperbolic system. The pseudo-sonic speed is \( a = \sqrt{u^2 + c^2} \) and the system remains subsonic \( (u < a) \). Disturbances are propagated with the pseudo-pressure wave speed.

For distance \( d \) the pressure waves require a time of the order:

\[
\Delta t_p \approx \frac{d}{(a - U)}
\]

We compare with diffusion effects which would propagate over a distance \( \delta \sim \sqrt{\nu t} \approx 2\sqrt{\nu t} \) thus

\[
\Delta t_{\nu} = \frac{\delta^2}{4\nu} = \frac{U \cdot d}{\nu} \cdot \frac{\delta^2}{4Ud}
\]

or

\[
\Delta t_{\nu} = \frac{\delta^2}{4Ud} Re
\]

and \( U = \) characteristic velocity. For good convergence we require

\[
\Delta t_p \ll \Delta t_{\nu}
\]

or

\[
\frac{d}{(a - U)} \ll Re \frac{\delta^2}{4Ud}
\]

\[
\Rightarrow \frac{4}{Re} \left( \frac{d}{\delta} \right)^2 \ll \frac{a - U}{U} = \frac{a}{U} - 1
\]

\[
\Rightarrow \left[ 1 + \frac{4}{Re} \frac{d}{\delta} \right]^2 \ll \frac{a^2}{U^2} = \frac{U^2 + c^2}{U^2}
\]
\[
\Rightarrow \frac{c^2}{U^2} \gg \left(1 + \frac{4}{Re} \frac{d}{\delta}\right)^2 - 1
\]

**Example:** duct flow,

\[
\frac{d}{\delta} = 25, \ Re = 100
\]

\[
\frac{c}{U} \gg \sqrt{3}
\]

\[
Re = 1000 \Rightarrow \frac{c}{U} \gg 0.46
\]

Below that value the system diverges. At very large values the system diverges too. An upper bound may be

\[
\frac{1}{\Delta t} > c
\]

(External flows are more forgiving!)

### 6.4 The Turkel Preconditioner


Consider the 2D inviscid part of Navier-Stokes. We will try to generalize the artificial compressibility methods:

\[
\frac{1}{c^2} \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2 + p) + \frac{\partial}{\partial y}(uv) = 0
\]

\[
\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(v^2 + p) = 0
\]

or in vector form:

\[
P^{-1} \frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = 0
\]
where

\[ U = \begin{pmatrix} p \\ u \\ v \end{pmatrix}, \quad F = \begin{pmatrix} u \\ u^2 + p \\ uv \end{pmatrix}, \quad G = \begin{pmatrix} v \\ uv \\ v^2 + p \end{pmatrix} \]

and

\[ P^{-1} = \begin{bmatrix} \frac{1}{c^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

which is the inverse of the preconditioning matrix \( P \). We can rewrite the system as:

\[ \frac{\partial U}{\partial t} + P \left\{ \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right\} = 0 \]

In quasi-linear form we get:

\[ \frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} + B \frac{\partial U}{\partial y} = 0 \]

where

\[ A = P \frac{\partial F}{\partial U} = \begin{bmatrix} 0 & c^2 & 0 \\ 1 & 2u & 0 \\ 0 & u & v \end{bmatrix} \]

\[ B = P \frac{\partial G}{\partial U} = \begin{bmatrix} 0 & 0 & c^2 \\ 0 & v & u \\ 1 & 0 & 2v \end{bmatrix} \]

This forms a purely hyperbolic system since the matrix \( D = \xi A + \eta B, \quad (\xi, \eta \in (-\infty, \infty)) \) has real eigenvalues and it can be diagonalized by a nonsingular matrix \( T \) whose columns are the eigenvector of \( D \).
Turkel modified $P$ as follows:

$$
P^{-1} = \begin{bmatrix}
\frac{1}{c^2} & 0 & 0 \\
\frac{(\alpha + 1)u}{c^2} & 1 & 0 \\
\frac{(\alpha + 1)v}{c^2} & 0 & 1
\end{bmatrix}
$$

by modifying the unsteady terms in momentum equation. Its inverse is:

$$
P = \begin{bmatrix}
c^2 & 0 & 0 \\
-(\alpha + 1)u & 1 & 0 \\
-(\alpha + 1)v & 0 & 1
\end{bmatrix}
$$

where $\alpha$ is a new parameter. For $\alpha = -1$ we recover the standard artificial compressibility method. The new $A, B$ are:

$$
A = P \frac{\partial F}{\partial U} = \begin{bmatrix}
0 & c^2 & 0 \\
1 & (1 - \alpha)u & 0 \\
0 & -\alpha v & u
\end{bmatrix};
B = \begin{bmatrix}
0 & 0 & c^2 \\
0 & v & -\alpha u \\
1 & 0 & (1 - \alpha)v
\end{bmatrix}
$$

The corresponding matrix to be examined for hyperbolicity is:

$$
D = \xi A + \eta B
$$

which has eigenvalues that are also all real.

$$
\lambda_1 = \frac{W}{1 - \alpha}; \lambda_2 = \frac{1}{2}(W + a); \lambda_3 = \frac{1}{2}(W - a)
$$

where

$$
W = (1 - \alpha)(\xi u + \eta v) \\
a^2 = W^2 + 4c^2(\xi^2 + \eta^2)
$$

Note that for the compressible Euler the corresponding eigenvalues are:

$$
A : \lambda_1 = u \\
\lambda_2 = u + a, \quad a = \text{(speed of sound)} \\
\lambda_3 = u - a
$$
The preconditioner then changes the eigenstructure of the Jacobian matrices which in turn govern the stability and convergence of hyperbolic systems using explicit time integration.

For fast convergence the objective is to minimize the maximum ratio of wave speeds of the system. The wave speeds are found by taking the Fourier transform of the quasilinear equation and substitute the \( x, y \) component of the Fourier transform variable for \( \xi \) and \( \eta \), respectively in the eigenvalue relations. In Fourier space, the minimax ratio of wave speeds is obtained from an algebraic equation. Turkel’s analysis finally gives:

\[
c^2 = \begin{cases} 
(2 - \alpha)(u^2 + v^2), & \alpha < 1 \\
\alpha(u^2 + v^2), & \alpha \geq 1 
\end{cases}
\]

Note that for the artificial compressibility:

\[
c^2 = 3(u^2 + v^2)
\]

Numerical experiment suggests that:

\[
c^2 = \beta(u^2 + v^2), \quad 1 \leq \beta \leq 5
\]

Note that \( c \) scales with the velocity here unlike the classical artificial compressibility methods where \( c = \text{constant globally} \). Finally Turkel found that the optimum \( c = 1 \) is not true in experiment.

6.5 Stability of Artificial Compressibility

I. * linearize:

\[
\begin{align*}
\bar{u}_{i+1/2,j} &= \bar{u}_{i,j+1/2} = u_0 = \text{const} \\
\bar{u}_{i,j+1/2} &= \bar{u}_{i+1/2,j} = v_0 = \text{const}
\end{align*}
\]

Disregard the pressure, then each momentum equation is of the advection-diffusion type

\[
f_{ij}^{n+1} = \left[ I - \Delta t (u_0 \Delta x_x + u_0 \Delta y_y - \frac{1}{Re} \nabla^2 h) \right] f_i^n
\]

\((\Delta x = \Delta y)\)
Conditions of stability

\[
\frac{4 \Delta t}{Re \Delta x^2} \leq 1
\]

\[
\frac{1}{4} (|u_0| + |v_0|)^2 \Delta t Re \leq 1
\]

II. Neglect convective contributions \((u_0 = v_0 = 0)\):

\[
u_{i+1/2,j}^{n+1} = \left( I - \frac{\Delta t}{Re} \nabla \right) \nabla u_{i+1/2,j}^n - \Delta t \Delta x p_{i+1/2,j}^n
\]

\[
v_{i,j+1/2}^{n+1} = \left( I - \frac{\Delta t}{Re} \nabla \right) \nabla v_{i,j+1/2}^n - \Delta t \Delta y p_{i,j+1/2}^n
\]

\[
p_{i,j}^{n+1} = p_{i,j}^n - \Delta t \Delta c^2 \left( \nabla u_{i,j}^{n+1} + \nabla v_{i,j}^{n+1} \right)
\]

von Neumann Analysis, construct amplification matrix, its spectral radius < 1

\[
4 \frac{\Delta t}{\Delta x^2} \left( \frac{1}{Re} + \frac{\Delta t c^2}{2} \right) \leq 1, \quad c^2 > 0
\]

Experiments: good convergence when \(c^2\) is large. Enhance convergence by using the most recently obtained values, i.e.,

\[
\frac{1}{\Delta t} \left( u_{i+1/2,j}^{n+1} - u_{i+1/2,j}^n \right) + \hat{u}_{i+1/2,j}^n \left( \frac{u_{i+3/2,j}^n - u_{i-1/2,j}^n}{2 \Delta x} \right)
\]

\[
+ \hat{v}_{i+1/2,j}^n \left( \frac{v_{i+1/2,j+1}^n - v_{i+1/2,j-1}^n}{2 \Delta y} \right) + \frac{1}{\Delta x} \left( p_{i+1,j}^n - p_{i,j}^n \right)
\]

\[
- \frac{1}{Re} \left( \frac{u_{i+3/2,j}^n - 2u_{i+1/2,j}^n + u_{i-1/2,j}^n}{\Delta x^2} + \frac{u_{i+1/2,j+1}^n - 2u_{i+1/2,j}^n + u_{i+1/2,j-1}^n}{\Delta y^2} \right) = 0
\]

The equivalent differential equation is

\[
K \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} - \frac{1}{Re} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0
\]

with

\[
K = 1 - \Delta t \left( \frac{u + v}{2 \Delta x} + \frac{2}{Re \Delta x^2} \right)
\]
parabolic equation for \( K > 0 \)

Convergence: neglect convective/pressure \( \Rightarrow \) successive relaxation of a Laplace equation with

\[
0 < \omega = \frac{4 \Delta t}{Re \Delta x^2} < 2
\]

\[
\frac{2 \Delta t}{Re \Delta x^2} < 1
\]

condition is two times less restrictive than previous

\( K < 1 \) \( \Rightarrow \) \((u + v)Re \Delta x + 4 > 0\)

6.6 The original Chorin (1967) scheme:

Leapfrog-Dufort-Frankel

\[
\frac{1}{2 \Delta t} (u_{i+1/2,j}^{n+1} - u_{i+1/2,j}^{n-1}) + a_{i+1/2,j}^n + \Delta x \Delta p_{i+1/2,j}^n - \frac{1}{Re} \nabla_h^2 u_{i+1/2,j}^n
\]

\[
\frac{\Delta t^2}{Re} \left( \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \Delta t u_{i+1/2,j}^n = 0
\]

\[
\frac{1}{2 \Delta t} (v_{i,j+1/2}^{n+1} - v_{i,j+1/2}^{n-1}) + b_{i,j+1/2}^n + \Delta y \Delta p_{i,j+1/2}^n - \frac{1}{Re} \nabla_h^2 v_{i,j+1/2}^n
\]

\[
\frac{\Delta t^2}{Re} \left( \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \Delta t v_{i,j+1/2}^n = 0
\]

\[
\frac{1}{2 \Delta t} (p_{i,j}^{n+1} - p_{i,j}^{n-1}) + c^2 (\Delta x u_{i,j}^n + \Delta y u_{i,j}^n) = 0
\]

where

\[
\Delta t f^n = \frac{1}{\Delta t^2} (f^{n+1} - 2f^n + f^{n-1})
\]

main error

\[
\frac{\Delta t^2}{Re} \left( \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \frac{\partial^2 y}{\partial x^2} \to 0
\]

in steady state
Chorin suggested: \[ \Delta t < \gamma \cdot c \cdot \Delta x, \]

where: \[ \gamma = 0.6 \text{ in 3D} \]