Denote all polynomials of order less than or equal to \( N \) by \( \mathbb{P}_N \) over the interval \([-1, 1]\).

**Proposition.** For all \( v \in \mathbb{P}_N \) with \( v(1) = 0 \), there exists a constant \( C \) independent of \( N \) such that

\[
\int_{-1}^{1} (1 + x)v_x^2 \, dx \leq CN^2 \int_{-1}^{1} v^2 \, dx.
\]

(0.1)

**Proof.** Note that \( v = \sum_{n=0}^{N-1} a_n (L_n - L_{n+1}) \) and \( v(1) = 0 \) since \( L_n(1) = L_{n+1} = 1 \).

\[
\int_{-1}^{1} v^2 \, dx = \int_{-1}^{1} \left[ \sum_{n=0}^{N-1} a_n (L_n - L_{n+1}) \right]^2 \, dx \\
= \int_{-1}^{1} \left[ a_0 L_0 + \sum_{n=1}^{N-1} (a_n - a_{n-1}) L_n - a_{N-1} L_N \right]^2 \, dx \\
= 2a_0^2 + \sum_{n=0}^{N-1} (a_n - a_{n-1})^2 \frac{2}{2n+1} + a_{N-1}^2 \frac{2}{2N+1}.
\]

Noting that

\[
-(1 + x)(L_n - L_{n+1})_x = (n + 1)(L_n + L_{n+1}),
\]

we have

\[
\int_{-1}^{1} (1 + x)v_x^2 \, dx = \int_{-1}^{1} (1 + x) \left[ \sum_{n=0}^{N-1} a_n (L_n - L_{n+1})_x \right]\left[ \sum_{n=0}^{N-1} a_n (L_n - L_{n+1})_x \right] \, dx \\
= \int_{-1}^{1} \left[ \sum_{n=0}^{N-1} -\frac{(n + 1)a_n}{n+1}(L_n + L_{n+1}) \right]\left[ \sum_{n=0}^{N-1} a_n (L_n - L_{n+1})_x \right] \, dx \\
= \int_{-1}^{1} \left[ \sum_{n=0}^{N-1} -\frac{(n + 1)a_m a_n}{n+1}(L_n + L_{n+1})(L_m - L_{m+1})_x \right] \, dx \\
= -\sum_{m,n=0}^{N-1} \frac{(n + 1)a_m a_n}{n+1} \int_{-1}^{1} (L_n + L_{n+1})(L_m - L_{m+1})_x \, dx \\
= 2 \sum_{n=0}^{N-1} a_n^2(n+1),
\]

where we used the fact that

\[
\int_{-1}^{1} (L_n + L_{n+1})(L_m - L_{m+1})_x \, dx = -2\delta_{m,n}.
\]

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Hence
\[ \frac{\int_{-1}^{1} v^2 \, dx}{\int_{-1}^{1} (1 + x)v^2 \, dx} \geq \frac{a_{2N-1}^2 \sum_{n=0}^{2N+1} a_n^2 (n + 1)}{\sum_{n=0}^{N-1} a_n^2} = O(N^{-2}). \]

It remains to verify that (0.2). Recall that
\[ (2n + 1)xL_n = nL_{n-1} + (n + 1)L_{n+1} \quad (0.3) \]
and
\[ (2n + 1)L_n = (L_{n+1} - L_{n-1})_x. \quad (0.4) \]
Taking derivative with respect to \( x \) in (0.3) leads to
\[ (2n + 1)x(L_n)_x + (2n + 1)L_n = n(L_{n-1})_x + (n + 1)(L_{n+1})_x. \quad (0.5) \]
Rewrite (0.5) as, by (0.4),
\[ (2n + 1)x(L_n)_x + (2n + 1)L_n = (2n + 1)(L_{n-1})_x + (n + 1)((L_{n+1} - L_{n-1})_x) \]
\[ = (2n + 1)(L_{n-1})_x + (n + 1)(n + 1) L_n, \]
and thus simplifying gives
\[ x(L_n)_x = (L_{n-1})_x + nL_n. \quad (0.6) \]
Similarly, we have
\[ x(L_n)_x = (L_{n+1})_x - (n + 1)L_n \quad (0.7) \]
By (0.6) and (0.7), we have (0.2). □