ANALYSIS AND APPROXIMATION OF A FRACTIONAL CAHN-HILLIARD EQUATION

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Abstract. We derive a Fractional Cahn-Hilliard Equation (FCHE) by considering a gradient flow in the negative order Sobolev space $H^{-\alpha}$, $\alpha \in [0, 1]$ where the choice $\alpha = 1$ corresponds to the classical Cahn-Hilliard equation whilst the choice $\alpha = 0$ recovers the Allen-Cahn equation. It is shown that the equation preserves mass for all positive values of fractional order $\alpha$ and that it indeed reduces the free energy. The well-posedness of the problem is established in the sense that the $H^1$-norm of the solution remains uniformly bounded. We then turn to the delicate question of the $L^\infty$ boundedness of the solution and establish an $L^\infty$ bound for the FCHE in the case where the non-linearity is a quartic polynomial. As a consequence of the estimates, we are able to show that the Fourier-Galerkin method delivers a spectral rate of convergence for the FCHE in the case of a semi-discrete approximation scheme. Finally, we present results obtained using computational simulation of the FCHE for a variety of choices of fractional order $\alpha$. It is observed that the nature of the solution of the FCHE with a general $\alpha > 0$ is qualitatively and quantitatively closer to the behaviour of the classical Cahn-Hilliard equation than to the Allen-Cahn equation, regardless of how close to zero be the value of $\alpha$. An examination of the coarsening rates of the FCHE reveals that the asymptotic rate is rather insensitive to the value of $\alpha$ and, as a consequence, is close to the well-established rate observed for the classical Cahn-Hilliard equation.

1. Introduction

A simple model for the phase separation of a binary alloy at a fixed temperature is given by the Cahn-Hilliard equation [5]

$$\frac{\partial u}{\partial t} + (-\Delta)(-\varepsilon^2 \Delta u + F'(u)) = 0 \text{ in } \Omega,$$

where $u$ is an order parameter, $\varepsilon$ is a length scale parameter and $F(s) = \frac{1}{4}(1 - s^2)^2$ is of bistable type, admitting two local minima. The mass in the system is measured by the quantity

$$\int_{\Omega} u(x, t) dx$$

and remains constant in the case of the Cahn-Hilliard equation as one would expect from physical considerations.

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The non-equilibrium state of the system can be expected to evolve in such a way that the free energy functional

\[ E(u) = \int_{\Omega} \left( \frac{\epsilon^2}{2} |\nabla u|^2 + F(u) \right) \]

(1)
decreases in time and approaches a minimum. The Cahn-Hilliard equation can be viewed as a gradient flow \[ \frac{\partial u}{\partial t} = -\nabla V E(u) \]
where \(\nabla V E\) denotes the gradient of \(E\), and \(V\) is an approximately chosen Hilbert space with associated norm \((\bullet, \bullet)_V\). The gradient of \(E\) at a point \(u \in V\) is defined by the equation

\[ (\nabla V E(u), v)_V = \frac{d}{ds} E(u + sv)|_{s=0} \]

Formal computations (ignoring boundary terms for the moment) reveal that the right hand side is given by

\[ \frac{d}{ds} E(u + sv)|_{s=0} = (\epsilon^2 \nabla u, \nabla v)_0 + (F'(u), v)_0 = (-\epsilon^2 \Delta u + F'(u), v)_0 \]

where \((\bullet, \bullet)_0\) denotes the usual \(L^2(\Omega)\)-norm. As a consequence, if one chooses \(V\) to be the space \(L^2(\Omega)\), then the gradient is given by

\[ \nabla_0 E(u) = -\epsilon^2 \Delta u + F'(u) \]

and the associated gradient flow is given by the Allen-Cahn equation [1]:

\[ \frac{\partial u}{\partial t} + (-\epsilon^2 \Delta u + F'(u)) = 0 \text{ in } \Omega. \]

Unfortunately, the Allen-Cahn equation fails to preserve mass. Suppose instead, that we choose \(V\) to be the space \(H^{-1}(\Omega)\) equipped with the norm defined by

\[ (v, w)_{-1} = ((-\Delta)^{-1/2} v, (-\Delta)^{-1/2} w)_0. \]

The gradient with respect to the space \(H^{-1}\) is given by

\[ \nabla_{-1} E(u) = ((-\Delta)(-\epsilon^2 \Delta u + F'(u)) \]

and the associated gradient flow is the Cahn-Hilliard equation given above which, as mentioned earlier, does preserve the mass.

The derivation of the Cahn-Hilliard equation as a gradient flow in the dual space \(H^{-1}\) may appear somewhat ad hoc. Why not, for instance, look at the gradient flow in the space \(H^{-\alpha}\) where \(\alpha > 0\) instead? Taking the norm on the space \(H^{-\alpha}\) to be defined by

\[ (v, w)_{-\alpha} = ((-\Delta)^{-\alpha/2} v, (-\Delta)^{-\alpha/2} w)_0 \]

results in a gradient given by

\[ \nabla_{-\alpha} E(u) = ((-\Delta)\alpha(-\epsilon^2 \Delta u + F'(u)) \]

The fractional order operator \((-\Delta)^\alpha\) is defined below, while the justifications for the above steps are provided in the Appendix. The associated gradient flow in the general case \(\alpha > 0\) will be referred to as the Fractional Cahn-Hilliard Equation (FCHE).
We consider the following FCHE for $0 \leq \alpha \leq 1$ with periodic boundary condition:

$$
\frac{\partial u}{\partial t}(x, t) + (-\Delta)^\alpha (-\varepsilon^2 \Delta u(x, t) + f(u(x, t))) = 0, \quad (x, t) \in \Omega \times (0, T],
$$

(2)

$u(\cdot, t)$ is $2\pi$-periodic for all $t \in (0, T]$,

$u(x, 0) = u_0(x), \quad x \in \Omega,$

where $\Omega = (0, 2\pi)^2$, $\varepsilon$ is a positive constant, $x = (x_1, x_2)$, and $f(u) = F'(u) = u^3 - u$.

The choice $\alpha = 1$ corresponds to the classical Cahn-Hilliard equation whilst the choice $\alpha = 0$ gives the Allen-Cahn equation.

In order to define the fractional Laplacian $(-\Delta)^\alpha$, we use the Fourier decomposition. For any $u \in L^2_{\text{per}}(\Omega)$,

$$
u(x) = \sum_{k,l \in \mathbb{Z}} \hat{u}_{kl} e^{ikx_1 + ilx_2},
$$

(3)

where $i^2 = -1$, and the Fourier coefficients are given by

$$
\hat{u}_{kl} = \langle u, e^{ikx_1 + ilx_2} \rangle = \frac{1}{(2\pi)^2} \int_{\Omega} u e^{ikx_1 + ilx_2} dx.
$$

The fractional Laplacian is then defined by

$$
(-\Delta)^\alpha u = \sum_{k,l \in \mathbb{Z}} (k^2 + l^2)^\alpha \hat{u}_{kl} e^{ikx_1 + ilx_2}.
$$

(4)

Fractional order Allen-Cahn and Cahn-Hilliard models have been considered by other authors. A fractional extension of a mass-conserving Allen-Cahn phase field model has been studied in [23]. Bosch and Stoll [2] proposed a fractional inpainting model based on a fractional order vector-valued Cahn-Hilliard equation. The Fourier spectral method has been widely used to numerically approximate the solution of the classical Cahn-Hilliard equation [9, 10, 15, 16, 24] and to study the coarsening dynamics [8, 17, 26]. Alternative approaches include the Legendre spectral Galerkin [23, 25] or the Petrov Galerkin method with general Jacobi functions [20] for Riesz fractional partial differential equations. The Fourier spectral method is a natural choice for the approximation of problems where the fractional derivative is defined by spectral decomposition and the fact that (at least for the linear terms) the operator is diagonal offers possibilities for efficient solvers [3].

In the present work, we analyse the FCHE (2) arising as a gradient flow of the free energy functional $E$ in the negative order fractional space $H^{-\alpha}$. In particular, we show that the equation preserves mass for all positive values of fractional order $\alpha$ and that it indeed reduces energy. The well-posedness of the problem is established in the sense that the $H^1$-norm of the solution remains uniformly bounded. We then turn to the more interesting, and delicate, question of the $L^\infty$ boundedness of the solution.

The uniform $L^\infty$-boundedness of the solution is of particular interest both from physical and computational considerations. For instance, $L^\infty$-boundedness is used in order to prove the convergence of the Fourier-Galerkin scheme for the approximation of the classical Cahn-Hilliard equation where a key step requires a uniform bound on the quantity $\max |F''(s)|$ where $s$ lies between $u(t)$ and the Fourier-Galerkin approximation $u_N(t)$. Previous work dealing with the convergence of numerical schemes for approximating the classical Cahn-Hilliard equation has assumed that such an estimate holds in the analysis [11], or have simply assumed that $F$ has
quadratic growth [22] (giving the uniform estimate). Caffarelli et. al. [4] showed that the solution of the classical Cahn-Hilliard equation remains bounded in $L_\infty$ in the case where the non-linear term $F$ is of quadratic type. Subsequently, He et. al. [15] weakened the hypothesis to allow the case when $F$ is a quartic polynomial and used it to establish the convergence of the Fourier-Galerkin approximation in the case of the classical Cahn-Hilliard equation.

The analysis of the uniform $L_\infty$-boundness in the case of the FCHE is more delicate because the underlying regularity of the solution is only $H^{1+\alpha}$ rather than the $H^2$-regularity for the classical equation. This creates technical issues when attempting to use the Gagliardo-Nirenberg type estimate as in [15] to obtain the $L_\infty$-boundedness. Nevertheless, we succeed in establishing an $L_\infty$ bound for the FCHE in the case where $F$ is a quartic polynomial. The same estimate holds for the Fourier spectral approximation using the same arguments for the finite dimensional subspace. Combining the $L_\infty$ estimate for the infinite dimensional case leads to a uniform estimate for max $|F''(s)|$ where $s$ lies between $u(t)$ and the Fourier-Galerkin approximation $u_N(t)$ without any additional assumptions on the problem or the data.

As a consequence of the estimates, we are able to show that the Fourier-Galerkin delivers a spectral rate of convergence for the FCHE in the case of a semi-discrete approximation scheme. We show that the same estimate holds for a simple fully discrete scheme. Finally, we present results obtained using computational simulation of the FCHE for a variety of choices of fractional order $\alpha$. It is observed that the nature of the solution of the FCHE with a general $\alpha > 0$ is qualitatively (and quantitatively) closer to the behavior of the classical Cahn-Hilliard equation than to the Allen-Cahn equation, regardless of how close to zero be the value of $\alpha$. An examination of the coarsening rates of the FCHE reveals that the asymptotic rate is rather insensitive to the value of $\alpha$ and, as a consequence, is close to the well-established rate observed for the classical Cahn-Hilliard equation.

\section{Properties of Fractional Cahn-Hilliard Equation}

Let $C^\infty_{\text{per}}(\Omega)$ be the set of all restrictions onto $\Omega$ of all real-valued, $2\pi$-periodic, $C^\infty$-functions on $\mathbb{R}^2$. For each $s \geq 0$, let $H^s_{\text{per}}(\Omega)$ be the closure of $C^\infty_{\text{per}}(\Omega)$ in the usual Sobolev norm $\| \cdot \|_s$ with $\|u\|_s^2 = \sum_{k,l \in \mathbb{Z}} \hat{u}_{kl}^2(1 + k^2 + l^2)^s$. Note that $H^0_{\text{per}}(\Omega) = L^2_{\text{per}}(\Omega)$. In particular, we use $\| \cdot \|$ to denote the usual $L^2$-norm $\| \cdot \|_0$.

The weak form of (2) is obtained in the usual way by multiplying a test function $v \in H^{1+\alpha}_{\text{per}}(\Omega)$ on both sides of (2) and integrating over $\Omega$. Using fractional integration by parts (see Lemma 8), we arrive at the weak formulation of (2): find $u \in H^{1+\alpha}_{\text{per}}(\Omega)$, such that

$$
\left( \frac{\partial u}{\partial t}, v \right) + \varepsilon^2 (-\Delta)^{1+\alpha/2} u, (-\Delta)^{1+\alpha/2} v \right) + (f(u), (-\Delta)^{\alpha} v) = 0, \forall v \in H^{1+\alpha}_{\text{per}}(\Omega).
$$

\subsection{Stability.}
The Allen-Cahn equation ($\alpha = 0$) fails to preserve mass. The next results show that several properties of the solution of the standard Cahn-Hilliard equation are also enjoyed by the Fractional Cahn-Hilliard equation.

\textbf{Lemma 1.} Let $0 < \alpha \leq 1$, then the FCHE is mass conserving, i.e., $\int_{\Omega} u(x,t)dx$ is a constant for all $t \geq 0$. 

Proof. Take \( v \equiv 1 \) in equation (5), then since \( \alpha > 0 \), we have \((-\triangle)^\alpha u = 0\), so
\[
\frac{d}{dt}(u, 1) = \frac{d}{dt} \int_{\Omega} u dx = 0.
\]
It follows that the mass is conserved, i.e.,
\[
\int_{\Omega} u(x, t) dx = \int_{\Omega} u(x, 0) dx = \int_{\Omega} u_0(x) dx.
\]
\( \square \)

The next result shows that FCHE, like the Allen-Cahn and Cahn-Hilliard equations, does not result in an increase in the free energy functional (1).

Lemma 2. The energy is non-increasing for all \( 0 \leq \alpha \leq 1 \):
\[
E(u(t)) \leq E(u_0), \quad \forall \, t \geq 0.
\] (6)
Proof. Direct computation gives
\[
\frac{d}{dt} E(u) = \frac{d}{dt} \int_{\Omega} \left( \frac{\varepsilon^2}{2} |\nabla u|^2 + F(u) \right) dx = \left( \varepsilon^2 \nabla u, \nabla \frac{\partial u}{\partial t} \right) + \left( f(u), \frac{\partial u}{\partial t} \right)
\]
\[
= \varepsilon^2 (-\triangle) u, \frac{\partial u}{\partial t} \right) + \left( f(u), \frac{\partial u}{\partial t} \right) = \varepsilon^2 (-\triangle) u + f(u), \frac{\partial u}{\partial t} \right) = - \varepsilon^2 (-\triangle) u + f(u), (\varepsilon^2 (-\triangle) u + f(u)) \right) = - \|(-\triangle)^{\alpha/2} (\varepsilon^2 (-\triangle) u + f(u)) \|^2 \leq 0.
\]
and the result follows. \( \square \)

We can now establish the stability and continuous dependence of the weak solution as follows:

Theorem 1. The solution of FCHE is stable and satisfies for all \( 0 < \alpha \leq 1 \)
\[
\|u(t)\|_1 \leq C(1 + \|u_0\|_1), \quad \forall \, t > 0,
\] (7)
where \( C \) is a positive constant independent of \( t \).
Proof. Observe that \( F \) satisfies
\[
\frac{s^4}{8} - \frac{1}{4} \leq F(s) \leq \frac{3s^4}{8} + \frac{3}{4}, \quad s \in \mathbb{R},
\]
and hence,
\[
\frac{1}{2} \varepsilon^2 \|\nabla u(t)\|^2 + \frac{1}{8} \int_{\Omega} |u(t)|^4 dx \leq E(u(t)) + \frac{1}{4}, \quad \forall \, t > 0.
\]
Using the fact that the energy is non-increasing gives
\[
E(u(t)) + \frac{1}{4} \leq E(u_0) + \frac{1}{4} \leq \frac{1}{2} \varepsilon^2 \|\nabla u_0\|^2 + \frac{3}{8} \int_{\Omega} |u_0|^4 dx + 1.
\]
Thanks to the following Sobolev inequality, valid for all \( s > \frac{1}{2} \),
\[
\|v\|_{L^4} \leq C\|v\|_s, \quad \forall \, v \in H^s_{per}(\Omega),
\]
we arrive at
\[
\frac{1}{2} \varepsilon^2 \|\nabla u(t)\|^2 \leq \frac{1}{2} \varepsilon^2 \|\nabla u_0\|^2 + \frac{3}{8} \|u_0\|^4_{L^4} + 1 \leq \frac{1}{2} \varepsilon^2 \|\nabla u_0\|^2 + C\|u_0\|^4_s + 1.
\]
Finally, thanks to the Poincaré inequality, using conservation of mass gives
\[
\|u(t)\| \leq \frac{1}{|\Omega|} \int_{\Omega} u_0 dx + \|u(t) - \frac{1}{|\Omega|} \int_{\Omega} u dx\|
\]
\[
\leq \frac{1}{|\Omega|} \int_{\Omega} u_0 dx + C\|\nabla u(t)\|.
\]
The result now follows since the result would then follow for general $v$ using
\[
\|u(t)\| \leq \frac{1}{|\Omega|} \int_{\Omega} u_0 dx + C\|\nabla u(t)\|.
\]

2.2. $L_\infty$ boundness. In the case of one space dimension, the Sobolev inequality $\|u\|_\infty \leq C\|u\|_s$ is valid for all $s > \frac{1}{2}$ which, thanks to Theorem 1, means that the solution of FCHE is bounded for all $t \geq 0$. Unfortunately, the same argument cannot be used in two space dimensions owing to the failure of the Sobolev inequality (which requires $s > 1$ in 2D). Nevertheless, we will show that the boundness of $\|u(t)\|_\infty$ also holds true for the two dimensional problem by using a different argument. The key estimate needed is the following Sobolev inequality. An analogous result in the case $\Omega = \mathbb{R}^d$ is given in [14, Exercise 6.1.2].

**Lemma 3.** Let $v : [0 : 2\pi]^d \to \mathbb{R}$ be periodic. Then for $-\infty < 2\mu < d < 2\nu < \infty$, there holds for $C = C(\mu, \nu, d) > 0$,
\[
\|v\|_\infty \leq C\{\|v\| + \|(-\Delta)^{\nu/2} v\|^{\frac{d-2\mu}{d-\nu}} \|(-\Delta)^{\nu/2} v\|^{\frac{2\mu-d}{d-\nu}}\}.
\]

**Proof.** It suffices to show that for $v$ with vanishing average value, i.e., $\bar{v} = 0$, there holds
\[
\|v\|_\infty \leq C\|(-\Delta)^{\nu/2} v\|^{\frac{d-2\mu}{d-\nu}} \|(-\Delta)^{\nu/2} v\|^{\frac{2\mu-d}{d-\nu}},
\]
since the result would then follow for general $v$ using
\[
\|v\|_\infty \leq \|v - \bar{v}\|_\infty + |\bar{v}| \leq \|v - \bar{v}\|_\infty + C\|v\|.
\]
If $v$ has vanishing average, then the Fourier expansion of $v$ is given by
\[
v(x) = \sum_{|k| \neq 0} \hat{v}_k e^{ikx},
\]
where $k = (k_1, \cdots, k_d)$ denotes the multi-index and $x = (x_1, \cdots, x_d)$. Thus, for any $N \in \mathbb{N}$
\[
\frac{1}{2} \|v\|_\infty^2 \leq \frac{1}{2} \left( \sum_{|k| \neq 0} |\hat{v}_k|^2 \right) \leq \left( \sum_{0 < |k| \leq N} |\hat{v}_k|^2 \right) + \left( \sum_{|k| > N} |\hat{v}_k|^2 \right),
\]
and then the first term is bounded by
\[
\left( \sum_{0 < |k| \leq N} |\hat{v}_k|^2 \right) \leq \left( \sum_{0 < |k| \leq N} |k|^{-2\mu} \right) \left( \sum_{0 < |k| \leq N} |k|^{2\nu} |\hat{v}_k|^2 \right)
\]
\[
\leq \left( \sum_{0 < |k| \leq N} |k|^{-2\mu} \right) \|(-\Delta)^{\nu/2} v\|^2,
\]
and the second term is bounded similarly by
\[
\left( \sum_{|k| > N} |\hat{v}_k|^2 \right) \leq \left( \sum_{|k| > N} |k|^{-2\nu} \right) \|(-\Delta)^{\nu/2} v\|^2.
\]
Now for $d - 2\mu > 0$,
\[
\sum_{0 < |k| \leq N} |k|^{-2\mu} \leq \int_{|x| \leq N} \frac{dx}{|x|^{2\mu}} = \int_{0}^{N} r^{-2\mu} r^{d-1} dx = \frac{N^{d-2\mu}}{d - 2\mu},
\]
whilst for $2\nu - d > 0$,
\[
\sum_{|k| > N} |k|^{-2\nu} \leq \int_{|x| > N} \frac{dx}{|x|^{2\nu}} = \int_{N}^{\infty} r^{-2\nu} r^{d-1} dr = \frac{N^{d-2\nu}}{2\nu - d}.
\]
Hence, for all $N \in \mathbb{N}$
\[
\frac{1}{2} \|v\|_{\infty}^2 \leq \frac{N^{d-2\mu}}{d - 2\mu} \|(-\Delta)^{\mu/2} v\|^2 + \frac{N^{d-2\nu}}{2\nu - d} \|(-\Delta)^{\nu/2} v\|^2.
\]
If $\|(-\Delta)^{\mu/2} v\| = 0$, then the result is trivial. Otherwise, since $\nu > \mu$, we have $\|(-\Delta)^{\nu/2} v\| \geq \|(-\Delta)^{\mu/2} v\|$ thanks to Lemma 10 and we can choose $N \in \mathbb{N}$ to be given by
\[
N = \left[ \frac{\|(-\Delta)^{\mu/2} v\|}{\|(-\Delta)^{\nu/2} v\|} \right]^{1/(\nu - \mu)}
\]
where $[f]$ denotes the integer part of $f$. With this choice we obtain
\[
\|v\|_{\infty}^2 \leq C(\mu, \nu, n) \left( \frac{\|(-\Delta)^{\nu/2} v\|}{\|(-\Delta)^{\mu/2} v\|} \right)^{\frac{n-2\mu}{2-n}} \|(-\Delta)^{\nu/2} v\|^2,
\]
and the result follows. \qed

A consequence of Lemma 3 is the following special case in two space dimensions $d = 2$, with $\mu = 1 - \alpha, \nu = 1 + \alpha$ for $\alpha > 0$:

**Lemma 4.** For $\alpha > 0$, if $v \in H^{1+\alpha}_{pt} (\Omega)$, then
\[
\|v\|_{\infty}^2 \leq C(\alpha) \left( \|v\|^2 + \|(-\Delta)^{(1-\alpha)/2} v\| \|(-\Delta)^{(1+\alpha)/2} v\| \right).
\]

The main result of this section is a uniform $L_{\infty}$-bound on the solution of FCHE given in the following theorem:

**Theorem 2.** Let $u$ be the solution of FCHE, then there holds for all $0 < \alpha \leq 1$
\[
\|u(t)\|_{\infty} + \|(-\Delta)^{(1+\alpha)/2} u(t)\| \leq C(u_0),
\]
where $C(u_0)$ is a constant that depends on $u_0$ but not on $t$.

We begin by establishing the following weaker variant of the estimate (11):

**Lemma 5.** Let $u$ be the solution of (5), then for $0 < \alpha \leq 1$, the following estimate
\[
\int_{t}^{t+1} \|(-\Delta)^{(1+\alpha)/2} u(\tau)\|^2 d\tau \leq C(u_0), \quad \forall t \geq 0,
\]
holds, where $C(u_0)$ depends on $u_0$ but not on $t$.

**Proof.** Taking $v = u$ in (5) gives
\[
(\frac{\partial u}{\partial t}, u) + \varepsilon^2 \|(-\Delta)^{(1+\alpha)/2} u\|^2 + \langle f(u), (-\Delta)^{\alpha} u \rangle = 0,
\]
or, equally well,
\[ \varepsilon^2 \|(-\Delta)^{(1+\alpha)/2} u(t)\|^2 = -\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 - (f(u(t)), (-\Delta)^\alpha u(t)). \]

Integrating from \(t\) to \(t + 1\), we have
\[
\varepsilon^2 \int_t^{t+1} \|(-\Delta)^{(1+\alpha)/2} u(\tau)\|^2 d\tau = \frac{1}{2} \|u(t)\|^2 - \frac{1}{2} \|u(t + 1)\|^2
- \int_t^{t+1} (f(u(\tau)), (-\Delta)^\alpha u(\tau)) d\tau. \tag{13}
\]

To obtain the estimate (12), let us first consider the last term of (13). We claim that the following estimate holds.

\[
(u^3, (-\Delta)^\alpha u)
\leq \hat{C}(\alpha) \kappa \|\nabla u\|^2 \|\nabla u\|^2 + \frac{1}{2} \varepsilon^2 \|(-\Delta)^{1-\alpha/2} u\|^2 \|(-\Delta)^{(1+\alpha)/2} u\|^2 + \frac{1}{2\kappa} \|\nabla u\|^2, \tag{14}
\]

where \(\hat{C}(\alpha)\) is a constant depends on \(\alpha\) and \(\kappa\) is a positive constant to be determined. If \(0 < \alpha \leq \frac{1}{2}\), then we can prove (14) by arguing as follows:

\[
(u^3, (-\Delta)^\alpha u) \leq \frac{\kappa}{2} \|u^3\|^2 + \frac{1}{2\kappa} \|(-\Delta)^{1-\alpha/2} u\|^2 \|(-\Delta)^{(1+\alpha)/2} u\|^2 + \frac{1}{2\kappa} \|(-\Delta)^\alpha u\|^2.
\]

Then using the estimate (10), we obtain

\[
(u^3, (-\Delta)^\alpha u)
\leq \hat{C}(\alpha) \kappa \|\nabla u\|^2 \|\nabla u\|^2 + \frac{1}{2} \varepsilon^2 \|(-\Delta)^{1-\alpha/2} u\|^2 \|(-\Delta)^{(1+\alpha)/2} u\|^2 + \frac{1}{2\kappa} \|(-\Delta)^\alpha u\|^2.
\]

Since \(\alpha \leq \frac{1}{2}\), (14) holds for \(0 < \alpha \leq \frac{1}{2}\) thanks to (39). In the case \(\frac{1}{2} < \alpha \leq 1\) we argue as follows: By virtue of (37) and (38), we get

\[
(u^3, (-\Delta)^\alpha u) = ((-\Delta)^{1/2} u^3, (-\Delta)^{\alpha-1/2} u).
\]

Using the Cauchy-Schwarz and Young inequalities, we obtain

\[
((-\Delta)^{1/2} u^3, (-\Delta)^{\alpha-1/2} u) \leq \frac{\kappa}{2} \|(-\Delta)^{1/2} u^3\|^2 + \frac{1}{2\kappa} \|(-\Delta)^{\alpha-1/2} u\|^2.
\]

Observing that,
\[
\|(-\Delta)^{1/2} v\|^2 = \sum_{k,l \in \mathbb{Z}} (k^2 + l^2) \varepsilon^2 \|\nabla v\|^2, \quad \forall v \in H^1_{\text{per}}(\Omega),
\]

we have

\[
\frac{\kappa}{2} \|(-\Delta)^{1/2} u^3\|^2 = \frac{\kappa}{2} \|\nabla u^3\|^2 = \frac{\kappa}{2} \|3u^2 \nabla u\|^2 \leq \frac{9\kappa}{2} \|u\|_{\infty}^4 \|\nabla u\|^2,
\]

then using the estimate (10) again and noting that \(\alpha - \frac{1}{2} \leq \frac{1}{2}\), the estimate (14) follows for \(\frac{1}{2} < \alpha \leq 1\). Furthermore, to handle the linear part, we use

\[
(u, (-\Delta)^\alpha u) = \|(-\Delta)^{\alpha/2} u\|^2 \leq \|\nabla u\|^2. \tag{15}
\]

If \(u\) is a constant, then estimate (12) holds trivially. Otherwise, we can choose \(\kappa\) such that \(\hat{C}(\alpha) \kappa \|\nabla u\|^2 \|(-\Delta)^{(1-\alpha)/2} u\|^2 = \varepsilon^2, \) i.e., \(\kappa = \varepsilon^2 / (2\hat{C}(\alpha) \|\nabla u\|^2 \|(-\Delta)^{(1-\alpha)/2} u\|^2).\) Thanks to \(\|u(t)\|_1 \leq C(1 + \|u_0\|_1)\) (see Theorem 1), and using (39) again, we have
that $\kappa$ is a bounded constant which depends on $u_0$ but not on $t$. Moreover, by combining (13), (14) and (15), we obtain

$$\frac{\varepsilon^2}{2} \int_0^{t+1} \|(\Delta)^{(1+\alpha)/2}u(\tau)\|^2 d\tau \leq C(\alpha, \|u_0\|_1).$$

Hence, the estimate (12) holds. \hfill \square

Now we can finally give the proof of Theorem 2.

**Proof.** (Proof of Theorem 2) Taking $v = (\Delta)^{1+\alpha}u$ in (5) and using (37) yields

$$0 = \frac{d}{dt} \|(\Delta)^{(1+\alpha)/2}u\|^2 + \varepsilon^2 \|(\Delta)^{1+\alpha}u\|^2 + \left( (\Delta)^\alpha f(u), (\Delta)^{1+\alpha}u \right)$$

$$- \frac{d}{dt} \|(\Delta)^{(1+\alpha)/2}u\|^2 + \frac{\varepsilon^2}{2} \|(\Delta)^{1+\alpha}u\|^2$$

$$+ \frac{\varepsilon^2}{2} \|(\Delta)^{1+\alpha}u + (\Delta)^\alpha f(u), (\Delta)^{1+\alpha}u \rangle. \quad (16)$$

Notice that

$$\|\varepsilon^2(\Delta)^{1+\alpha}u + (\Delta)^\alpha f(u)\|^2$$

$$\leq \left( \varepsilon^2(\Delta)^{1+\alpha}u, \varepsilon^2(\Delta)^{1+\alpha}u + 2(\Delta)^\alpha f(u) \right)$$

$$= 2\varepsilon^2 \|(\Delta)^{1+\alpha}u + (\Delta)^\alpha f(u)\|^2. \quad (17)$$

By equation (16) and (17), we obtain

$$\frac{d}{dt} \|(\Delta)^{(1+\alpha)/2}u\|^2 + \frac{\varepsilon^2}{2} \|(\Delta)^{1+\alpha}u\|^2 + \frac{1}{2\varepsilon^2} \|(\Delta)^\alpha f(u)\|^2$$

$$= \frac{1}{2\varepsilon^2} \|(\Delta)^{1+\alpha}f(u)\|^2 \leq \frac{1}{2\varepsilon^2} \|(\Delta)^{(1+\alpha)/2}f(u)\|^2$$

thanks to Lemma 10. Now $f(u) = u^3 - u$, and so with the help of (41), we have

$$\|(\Delta)^{(1+\alpha)/2}f(u)\|^2 \leq C_0 \|u\|_\infty^4 \|(\Delta)^{(1+\alpha)/2}u\|^2 + \|(\Delta)^{(1+\alpha)/2}u\|^4,$$  \hfill (19)

where $C_0$ is a constant that depends on $\alpha$. By (10) we have

$$\|u\|_\infty^2 \leq C(\alpha) \|u\|^2 + \|(\Delta)^{(1-\alpha)/2}u\|^2 \|(\Delta)^{(1+\alpha)/2}u\|.$$  \hfill (20)

Moreover, since $(1-\alpha)/2 \leq 1/2$, then by Theorem 1, we know that

$$\|(\Delta)^{(1-\alpha)/2}u\| \leq \|u\|_1 \leq C(1 + \|u_0\|_1). \quad (21)$$

Hence, by (19)-(21), we obtain

$$\|(\Delta)^{(1+\alpha)/2}f(u)\|^2 \leq C_1 \|(\Delta)^{(1+\alpha)/2}u\|^4 + C_2 \|(\Delta)^{(1+\alpha)/2}u\|^2,$$

where $C_1$ depends on $u_0$ but not on $t$. Therefore, by (18), we get

$$\frac{1}{2\varepsilon^2} \|(\Delta)^{(1+\alpha)/2}u\|^2$$

$$\leq \frac{1}{2\varepsilon^2} \|(\Delta)^{(1+\alpha)/2}u\|^2 + \frac{\varepsilon^2}{2} \|(\Delta)^{1+\alpha}u\|^2 + \frac{1}{2\varepsilon^2} \varepsilon^2 \|(\Delta)^{1+\alpha}u + (\Delta)^\alpha f(u)\|^2$$

$$\leq \frac{1}{2\varepsilon^2} \left( C_1 \|(\Delta)^{(1+\alpha)/2}u\|^4 + C_2 \|(\Delta)^{(1+\alpha)/2}u\|^2 \right).$$
That is
\[
\frac{d}{dt} \|(-\Delta)^{(1+\alpha)/2}u\|^2 \leq \frac{1}{\varepsilon} \left( C_1 \|(-\Delta)^{(1+\alpha)/2}u\|^4 + C_2 \|(-\Delta)^{(1+\alpha)/2}u\|^2 \right),
\]
and the bound on \(\|(-\Delta)^{(1+\alpha)/2}u(t)\|\) follows thanks to estimate (12) and the uniform Gronwall Lemma 12 proved later. The bound on \(\|u(t)\|_\infty\) follows thanks to the estimate (10) proved in Lemma 4. □

3. Fourier Galerkin Semi-discretization in Space

Let \(X_N = \text{span}\{e^{ikx_1+ilx_2}, -N \leq k, l \leq N\}\), then the Fourier spectral-Galerkin approximation for (5) consists of finding \(u_N \in X_N\), such that,
\[
\left( \frac{\partial u_N}{\partial t}, v \right) + \varepsilon^2 (-\Delta)^{(1+\alpha)/2}u_N, (-\Delta)^{(1+\alpha)/2}v \right) + \left( f(u_N), (-\Delta)^\alpha v \right) = 0, \quad \forall v \in X_N.
\]
with initial condition \(u_N(0) = \Pi_N u_0\). Here \(\Pi_N\) is the usual \(L^2\)-projection operator in \(X_N\), namely
\[
(u - \Pi_N u, v) = 0, \quad \forall v \in X_N.
\]

All arguments used for the continuous problem (5) discussed in Section 2 remain valid for the Fourier Galerkin problem (22). In particular, we note that the Fourier spectral-Galerkin method preserves mass, i.e.
\[
\int_{\Omega} u_N(x, t) dx = \int_{\Omega} u_N(x, 0) dx = \int_{\Omega} \Pi_N u_0(x) dx = \int_{\Omega} u_0(x) dx,
\]
and the monotonicity of the energy in the sense
\[
\frac{d}{dt} E(u_N) = -\|(-\Delta)^\frac{\alpha}{2} (\varepsilon^2 (-\Delta)^\alpha u_N + f(u_N))\|_0^2 \leq 0, \quad \forall t > 0.
\]
Moreover, we also have the same stability estimate for \(u_N(x, t)\):

**Corollary 1.** The Fourier spectral Galerkin method (22) is stable and it holds for all \(0 < \alpha \leq 1\)
\[
\|u_N(t)\|_1 \leq C(1 + \|u_0\|_1), \quad \forall t > 0,
\]
where \(C\) is independent of \(t\) and \(N\).

Similarly, applying the same argument as the one used in the continuous case, we have that

**Corollary 2.** Let \(u_N\) be the solution of weak problem (22), then there holds for all \(0 < \alpha \leq 1\) and \(t > 0\)
\[
\|u_N(t)\|_\infty + \|(-\Delta)^{(1+\alpha)/2}u_N(t)\| \leq C(u_0),
\]
where \(C(u_0)\) is a constant depends on \(u_0\) but not on \(N\) and \(t\).

3.1. \(L_2\)-projection estimate. The operator \(\Pi_N\) commutes with the fractional Laplacian operator \((-\Delta)^\alpha:\)

**Lemma 6.** For any \(u \in H^s_{per}(\Omega)\) and \(s \geq 0\), we have
\[
\Pi_N (-\Delta)^s u(x) = (-\Delta)^s \Pi_N u(x).
\]
In particular, the \(L_2\) projection is uniformly stable in \(H^s_{per}(\Omega),\)
\[
\|\Pi_N u\|_{2s} \leq C\|u\|_{2s}.
\]
Proof. Expanding $u$ in the form (3), then by the definition of $\Pi_N$, we know that
\[
\Pi_N u(x) = \sum_{k,l=-N}^{N} \hat{u}_{kl} e^{ikx_1 + ilx_2}.
\]
We have thus,
\[
(-\triangle)^s \Pi_N u(x) = \sum_{k,l=-N}^{N} \hat{u}_{kl} (k^2 + l^2)^s e^{ikx_1 + ilx_2}.
\]
On the other hand, we have
\[
\Pi_N (-\triangle)^s u(x) = \Pi_N \sum_{k,l\in\mathbb{Z}} \hat{u}_{kl} (k^2 + l^2)^s e^{ikx_1 + ilx_2} = \sum_{k,l=-N}^{N} \hat{u}_{kl} (k^2 + l^2)^s e^{ikx_1 + ilx_2}
\]
as required. Finally, if $u$ is a constant, then $\|\Pi_N u\|_2 = \|u\|_2$, otherwise,
\[
\|\Pi_N u\|_2 \leq C \|(-\triangle)^s u\| \leq C \|u\|_2,
\]
and hence, (27) holds. □

The following approximation result is standard in the case of integer order norms:

Lemma 7. Suppose that $u \in H^\nu_{per}(\Omega)$, then the following estimate holds for all $0 \leq \mu \leq \nu$,
\[
\|u - \Pi_N u\|_{\mu} \leq cN^{\nu-\mu}\|u\|_{\nu}.
\]
Proof. We begin with the case where $\Omega = (0, 2\pi)$ so that $u$ is a univariate function.

Let $\Pi_N^x$ be the $L^2$ projection in the one dimensional case. In this case we have $u - \Pi_N^x u = \sum_{|k|>N} \hat{u}_k e^{ikx}$, and so,
\[
\|u - \Pi_N^x u\|_{\mu}^2 \leq \sum_{|k|>N} (1 + |k|^2)^{\mu} \hat{u}_k^2 \leq cN^{-2(\nu-\mu)}\|u\|_{\nu}^2.
\]
Now, in the case $\Omega = (0, 2\pi)^2$, $\Pi_N = \Pi_N^x \Pi_N^y$, where $\Pi_N^x$ and $\Pi_N^y$ are the $L^2$ projections in the $x_1$ and $x_2$ variables respectively, and the error estimate (28) can be obtained using the following triangle inequality
\[
\|u - \Pi_N u\|_{\mu} \leq \|u - \Pi_N^x u\|_{\mu} + \|\Pi_N^y (u - \Pi_N^x u)\|_{\mu}.
\]
along with stability of the $L^2$ projection in $H^\mu$ (Lemma 6). □

3.2. Error estimate. In order to analyze the error in the Fourier Galerkin scheme, we follow the usual approach and consider the function $e_N(t) \in \mathbb{X}_N$ given by
\[
e_N(t) = \Pi_N u(t) - u_N(t),
\]
so that
\[
e_N(0) = \Pi_N u_0 - \Pi_N u_0 = 0.
\]
Thus, for \( v_N \in \mathcal{X}_N \), with the help of (23), we have
\[
\left( \frac{\partial e_N}{\partial t}, v_N \right) + (\varepsilon^2 (-\Delta)^{(1+\alpha)/2} e_N, (-\Delta)^{(1+\alpha)/2} v_N) = (\Pi_N \frac{\partial u}{\partial t}, v_N) + \left( \varepsilon^2 \Pi_N (-\Delta)^{(1+\alpha)/2} u, (-\Delta)^{(1+\alpha)/2} v_N \right)
- \left( \frac{\partial u_N}{\partial t}, v_N \right) - \left( \varepsilon^2 (-\Delta)^{(1+\alpha)/2} u_N, (-\Delta)^{(1+\alpha)/2} v_N \right)
= \left( \frac{\partial u}{\partial t}, v_N \right) + \varepsilon^2\left( (-\Delta)^{(1+\alpha)/2} u, (-\Delta)^{(1+\alpha)/2} v_N \right) + \left( f(u_N), (-\Delta)^\alpha v_N \right)
= \left( f(u) - f(u_N), (-\Delta)^\alpha v_N \right).
\]
Taking \( v_N = e_N \) in above equation, and using the Cauchy-Schwarz and Young inequalities, we get
\[
\frac{1}{2} \frac{d}{dt} \|e_N\|^2 + \varepsilon^2 \|(-\Delta)^{(1+\alpha)/2} e_N\|^2 \leq \frac{1}{2\varepsilon^2} \|f(u) - f(u_N)\|^2 + \frac{\varepsilon^2}{2} \|(-\Delta)^\alpha e_N\|^2
\leq \frac{1}{2\varepsilon^2} \|f(u) - f(u_N)\|^2 + \frac{\varepsilon^2}{2} \|(-\Delta)^{(1+\alpha)/2} e_N\|^2,
\]
which implies that
\[
\frac{d}{dt} \|e_N\|^2 + \varepsilon^2 \|(-\Delta)^{(1+\alpha)/2} e_N\|^2 \leq \frac{1}{\varepsilon^2} \|f(u) - f(u_N)\|^2. \tag{29}
\]
Let \( t \) be fixed and note that
\[
|f(u) - f(u_N)| \leq |u - u_N| \cdot M_N(t),
\]
where \( M_N(t) = \max\{|f'(s)|, s \text{ is between } u(t) \text{ and } u_N(t)\} \). Then, thanks to Theorem 2 and Corollary 2, we have
\[
M_N(t) \leq L, \tag{30}
\]
where \( L \) is a constant independent of \( N \) and \( t \) (but depending on \( \|u_0\|_1 \)). Thus,
\[
|f(u(t)) - f(u_N(t))| \leq L\|u(t) - u_N(t)\|, \quad \forall \, t > 0,
\]
which gives for all \( t > 0 \)
\[
\|f(u(t)) - f(u_N(t))\| \leq L\|u(t) - u_N(t)\| \leq L(\|u(t) - \Pi_N u(t)\| + \|e_N(t)\|).
\]
Inserting this estimate into the earlier estimate (29), we can get
\[
\frac{d}{dt} \|e_N\|^2 + \varepsilon^2 \|(-\Delta)^{(1+\alpha)/2} e_N\|^2 \leq \frac{L}{\varepsilon^2} \left\{ \|u(t) - \Pi_N u(t)\|^2 + \|e_N\|^2 \right\}.
\]
It follows that
\[
\frac{d}{dt} \left\{ e^{-Lt/\varepsilon^2} \|e_N\|^2 \right\} \leq \frac{L}{\varepsilon^2} e^{-Lt/\varepsilon^2} \|u(t) - \Pi_N u(t)\|^2,
\]
which in turn gives
\[
\|e_N\|^2 \leq \frac{L}{\varepsilon^2} \int_0^t e^{L(t-\tau)/\varepsilon^2} \|u(\tau) - \Pi_N u(\tau)\|^2 d\tau,
\]
since \( e_N(0) = 0 \). Finally, with the help of (28), we have
\[
\|e_N\|^2 \leq \frac{L}{\varepsilon^2} N^{-2r} e^{Lt/\varepsilon^2} \int_0^t \|u(\tau)\|^2 d\tau, \quad t \geq 0.
\]
By the triangle inequality and (28), we obtain the following error estimate for the Fourier Galerkin scheme:
Theorem 3. Let \( u(t) \) and \( u_N(t) \) be the solutions of problem (5) and (22), if \( u \in H^1_{\text{per}}(\Omega) \cap H^r(\Omega) \), then the error estimate holds for \( 0 < \alpha \leq 1 \):

\[
\| u(t) - u_N(t) \| \leq C e^{T/\varepsilon^2} N^{-r} \left\{ \| u(t) \|_r + \left( \int_0^t \| u(\tau) \|_r^2 \right)^{1/2} \right\}.
\]

Observe that the rate of convergence is independent of \( \alpha \). Moreover, if \( u \in H^r \) for arbitrary \( r \), then the usual spectral rate of convergence holds.

### 4. Numerical examples

4.1. Implementation. For the time integration, we use a first-order semi-implicit scheme in time [22]. For a given partition \( 0 = t_0 < t_1 < \cdots < t_K = T, \ t_n = n\delta t, n = 0, 1, \cdots, K \), where \( \delta t \) is the time step, and \( K = T/\delta t \), we consider the Fourier Galerkin scheme (22) in space, and the first-order time semi-implicit scheme given by

\[
\frac{1}{\delta t} (u_n^{n+1} - u_n^n, v) + \left( (-\Delta)^{\alpha/2} (\varepsilon^2 (-\Delta) u_n^{n+1}) + f(u_n^n), (-\Delta)^{\alpha/2} v \right) = 0, \ \forall \ v \in \mathbb{X}_N.
\]

where \( u_n^n \) denotes the numerical approximation to \( u_N(t) \) at time \( t_n \). The stability of this scheme follows standard arguments given in Appendix C.

Writing

\[
u_n^N = \sum_{k,l=-N/2}^{N/2} \hat{u}_{kl}^n e^{ikx_1+iqx_2}, \ n = 0, 1, \cdots, K,
\]

and taking \( v = e^{ipx_1+iqx_2} \), \( p, q = -N/2, \cdots, N/2 \) we get, thanks to the definition of fractional Laplacian,

\[
\frac{\hat{u}_{kl}^{n+1} - \hat{u}_{kl}^n}{\delta t} + (k^2 + l^2)^{1+\alpha} \hat{u}_{kl}^{n+1} + \varepsilon^2 (k^2 + l^2)^\alpha \hat{f}_{kl}^n = 0, \ k, l = -N/2, \cdots, N/2,
\]

where \( \hat{f}_{kl}^n \) are the Fourier coefficients of \( f(u_n^n) \). In order to use the scheme (33), we need only compute the non-linear term involving \( f \). To do this, we use the De-aliasing technique described in [18, Section 4.3.2] along with the FFT. More precisely, the De-aliasing technique removes the aliasing errors arising from dealing with the product of two functions.

We briefly describe the main idea in the univariate setting. Let \( v, w \in L^2_0(0, 2\pi) \), the basic idea of De-aliasing is to represent the product \( vw \) by an interpolant \( q = I_M(vw) \) of sufficiently high order, \( M \), such that there are no aliasing errors. We compute the mesh point values of the interpolant and its coefficients efficiently with the FFT.

Let \( x_j = 2\pi j/N, j = 0, 1, \cdots, N-1 \) and

\[
v(x) = \sum_{k=-N/2}^{N/2} \hat{v}_k e^{ikx}, \ w(x) = \sum_{k=-N/2}^{N/2} \hat{w}_k e^{ikx}.
\]

Define new padded coefficients

\[
\hat{v}_k = \begin{cases} \hat{v}_k, & |k| \leq N/2, \\ 0, & |k| > N/2, \end{cases} \quad \hat{w}_k = \begin{cases} \hat{w}_k, & |k| \leq N/2, \\ 0, & |k| > N/2. \end{cases}
\]
It is easy to see that
\[(\hat{vw})_m = \sum_{k=-N/2}^{N/2} \hat{v}_k \hat{w}_{m-k} = \sum_{k=-N/2}^{N/2} \hat{v}_k \hat{w}_{m-k}.\]

We then compute a set of mesh point values at \(M \geq N\) mesh points \(y_j = 2\pi j/M,\)
\[v_j = \sum_{k=-M/2}^{M/2-1} \hat{v}_k e^{iky_j}, \quad w_j = \sum_{k=-M/2}^{M/2-1} \hat{w}_k e^{iky_j}, \quad j = 0, 1, \ldots, M - 1,
\]
and obtain the Fourier coefficients of the product \(vw\) given by \(q_j = v_j w_j, j = 0, 1, \ldots, M - 1,\) using the discrete Fourier transform
\[\hat{q}_k = \frac{1}{M} \sum_{j=0}^{M-1} q_j e^{-iky_j}, \quad k = -M/2, \ldots, M/2 - 1.\]

The coefficients of the interpolant and the exact coefficients of the product are related by the aliasing formula
\[\hat{q}_k = (\hat{vw})_k + \sum_{l \in \mathbb{Z}, l \neq 0} (\hat{vw})_{k+LM} = (\hat{vw})_k + \sum_{l \in \mathbb{Z}, l \neq 0} \sum_{p=-N/2}^{N/2} \hat{v}_p \hat{w}_{k+LM-p}. \quad (34)\]

It is well-known that, with a proper choice of \(M\), we can eliminate the last (aliasing) term in (34) in which \(\hat{q}_k = (\hat{vw})_k\). Note that \(\hat{w}_m = 0\) for \(m > N/2\). Thus, if \(M\) is chosen such that \(|k + M - p| > N/2\) for \(|k|, |p| \leq N/2\), then the aliasing term vanishes for all \(l \neq 0\). The worst case occurs when \(k = -N/2\) and \(p = N/2\), which gives
\[-\frac{N}{2} - \frac{N}{2} + M > \frac{N}{2} \Rightarrow M > \frac{3N}{2}.\]

So if the product in physical space is computed with at least \(3N/2 + 2\) \((M\) has to be even), the interpolant \(q(x)\) using the values at those points matches the product \(vw\) exactly, and the Fourier interpolation coefficients \(\hat{q}_k\) match \((\hat{vw})_k\). Therefore, the computation of \((\hat{vw})_k\) using the De-aliasing technique requires two FFTs each of which is of order \(M \log M\). For a product of three functions (which we need for the FCHE), one can simply apply the De-aliasing technique twice.

4.2. Numerical examples. We present several numerical examples to demonstrate the accuracy of the Fourier-Galerkin scheme and to illustrate the behavior of the solutions to problem (2).

Example 1. Verification of spectral convergence rate. Let us first test the convergence with respect to \(N\). We choose \(\varepsilon^2 = 1/10\) with initial condition
\[u_0(x, y) = \sin x \cos y. \quad (35)\]

The \(L_2\) errors at time \(T = 1\) for different values of fractional order \(\alpha\) are shown in Figure 1 on a semi-log scale. The true solution is unknown and we therefore use the Fourier Galerkin approximation in the case \(N = 128\) as a reference solution, with the time step is set to be small enough to ensure the time discretization error can be ignored.
As we can see in Figure 1, the spectral convergence with respect to $N$ is observed, and the rate of convergence is independent of the fractional order $\alpha$ as predicted in Theorem 3.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{error_logscale}
\caption{Spatial $L_2$ errors at time $T = 1$ for FCHE with $\varepsilon^2 = 1/10$ using initial condition (35).}
\end{figure}

Example 2. Coalescence of two kissing bubbles [19]. We simulate the coalescence of two kissing bubbles, i.e., the initial condition is

$$u_0(x, y) = \begin{cases} 1, & (x + 1)^2 + y^2 \leq 1 \text{ or } (x - 1)^2 + y^2 \leq 1, \\ -1, & \text{otherwise}. \end{cases}$$

Set $\varepsilon^2 = 1/100$, $N = 128$, $\delta t = 0.01$.

In order to illustrate the effect of changing the value of fractional derivative $\alpha$ on the character of the solution of FCHE, we present plots of the solution for different values of fractional order $\alpha$ at various times in Figure 2. The solutions were obtained using the Fourier Galerkin method with $N = 128$ and a time step size $\delta t = 0.01$. Theorem 3 shows that the scheme is convergent and we assume that the numerical solutions provides an accurate approximation of the true solution. We observe that in all cases, as time evolves, the two bubbles coalesce into a single bubble. The classical Allen-Cahn equation does not conserve mass, which means that, (see Figure 2 (A)) in the case $\alpha = 0$, the bubble shrinks and finally disappears. On the other hand, when $\alpha = 1$, i.e., the classical Cahn-Hilliard equation (which does preserve mass), we see that the shape of the bubble reaches a steady circular shape. More interestingly, we observe that, as expected from the mass conservation for $\alpha > 0$, the volume of the bubble is also preserved for the cases $\alpha = 0, 0.1, 0.5$. However, the rate at which the solution approaches the steady asymptotic state varies with the fractional order $\alpha$. This is to be expected since, as predicted in Lemma 2, the smaller the value of $\alpha$, the slower the energy decays.
Figure 2. Evolution of solutions of FCHE with initial condition (36) for different values of fractional order $\alpha$. From top to bottom $t = 0, 12, 72, 100, 200$.

Example 3. Coarsening dynamics of FCHE. Coarsening is a widely observed phenomenon in materials systems involving microstructures. The coarsening rates for the Cahn-Hilliard equation with phase-dependent diffusion mobility have been studied theoretically [6, 7] and numerically [8, 17]. We study the coarsening dynamics of FCHE as the fractional order $\alpha$ is varied, in the case $\varepsilon^2 = 16/10000$ and $N = 512$. The initial condition is set to be uniform state $\bar{u} = 2\phi - 1$ augmented with a random perturbation uniformly distributed in $[-0.2, 0.2]$. We apply a stabilized second order semi-implicit time scheme [22], which reads, $\forall v, \varphi \in X_N$,

$$
\frac{1}{\delta t}(3u_N^{n+1} - 4u_N^{n} + u_N^{n-1}, v) + \langle (-\triangle)^{\alpha/2} w^{n+1}, (-\triangle)^{\alpha/2} v \rangle = 0,
$$

$$
\varepsilon^2 (\nabla u_N^{n+1}, \nabla \varphi) + S(u_N^{n+1} - 2u_N^{n} + u_N^{n-1}, \varphi) + (2f(u_N^{n}) - f(u_N^{n-1}), \varphi) = (w^{n+1}, \varphi),
$$

where $S$ is a constant. It is known [22] that this scheme is stable in the case $\alpha = 1$ and has the advantage of allowing a large time step. Here the time step is set to be $\delta t = 10^{-3}$. 

(A) $\alpha = 0$  (B) $\alpha = 0.1$  (C) $\alpha = 0.5$  (D) $\alpha = 1.0$
Figure 3. Configurations at time $T = 20$ with random initial condition for different values of fractional order $\alpha$ and $\phi$. From top to bottom $\alpha = 0.1, 0.5, 1.0$.

The configurations at time $T = 20$ with the same random initial condition for different $\alpha = 0.1, 0.5, 1.0$ and volume fractions $\phi = 0.75, 0.5, 0.25$ are shown in Figure 3. To study the coarsening behavior of FCHE, we plot the $\log_2 E(t)$ vs $\log_2 t$ for $t \in [1, 250]$ with two sets of initial data in Figure 4-6. We can see that, the coarsening dynamics for $\phi = 0.25$ and $\phi = 0.75$ are almost the same. Moreover, while the coarsening is initially slower for small values of $\alpha$, the asymptotic rates are roughly the same in all cases, and agree with that of the standard Cahn-Hilliard equation ($\alpha = 1$).

Appendix

Appendix A: Technical Results. We collect a number of elementary technical results that were used earlier in the article.

Lemma 8. Let $\mu, \nu \geq 0$ then for any $u, v \in H^{\mu+\nu}_{\text{per}}(\Omega)$, it holds

$$((-\Delta)^{\mu+\nu} u, v) = ((-\Delta)^{\mu} u, (-\Delta)^{\nu} v).$$

(37)
Figure 4. Coarsening dynamics for different values of fractional order $\alpha$, $\phi = 0.75$ (left: initial data 1, right: initial data 2).

Figure 5. Coarsening dynamics for different values of fractional order $\alpha$, $\phi = 0.5$ (left: initial data 1, right: initial data 2).

Figure 6. Coarsening dynamics for different values of fractional order $\alpha$, $\phi = 0.25$ (left: initial data 1, right: initial data 2).
Proof. Since \( u, v \in H^{\mu+\nu}_{\text{per}}(\Omega) \), then \( u \) can be expanded by Fourier series (3), and similarly, \( v(x) \) can be expanded as
\[
v(x) = \sum_{k,l \in \mathbb{Z}} \hat{v}_{kl} e^{ikx_1 + ilx_2}.
\]
Then, with the help of (4), we have
\[
(-\triangle)^{\mu+\nu} u = \sum_{k,l \in \mathbb{Z}} \hat{u}_{kl} \lambda_{kl}^{\mu+\nu} e^{ikx_1 + ilx_2},
\]
where \( \lambda_{kl} = k^2 + l^2 \), thus,
\[
((-\triangle)^{\mu+\nu} u, v) = \left( \sum_{k,l \in \mathbb{Z}} \hat{u}_{kl} \lambda_{kl}^{\mu+\nu} e^{ikx_1 + ilx_2}, \sum_{k,l \in \mathbb{Z}} \hat{v}_{kl} e^{ikx_1 + ilx_2} \right)
\]
\[
= \left( \sum_{k,l \in \mathbb{Z}} \hat{u}_{kl} \lambda_{kl}^{\mu+\nu} \hat{v}_{kl}, \sum_{k,l \in \mathbb{Z}} \hat{v}_{kl} \lambda_{kl}^{\mu+\nu} \hat{u}_{kl} \right)
\]
\[
= \left( \sum_{k,l \in \mathbb{Z}} \hat{u}_{kl} \lambda_{kl}^{\mu+\nu} e^{ikx_1 + ilx_2}, \sum_{k,l \in \mathbb{Z}} \hat{v}_{kl} \lambda_{kl}^{\mu+\nu} e^{ikx_1 + ilx_2} \right)
\]
\[
= ((-\triangle)^{\mu} u, (-\triangle)^{\nu} v).
\]

□

Another important property of fractional Laplacian is the semigroup property which follows easily using the definition (4):

**Lemma 9.** For any \( u \in H^{\mu+\nu}_{\text{per}}(\Omega) \), it holds
\[
((-\triangle)^{\mu+\nu} u, v) = ((-\triangle)^{\mu} (-\triangle)^{\nu} u, v) = ((-\triangle)^{\nu} (-\triangle)^{\mu} u, v).
\]

Likewise, using the definition (4), it is easy to see that the following Sobolev inequality holds:

**Lemma 10.** If \( \mu \leq \nu \) and \( v \in H^{\nu}_{\text{per}}(\Omega) \), then
\[
\|((-\triangle)^{\mu} v)^2\| \leq \|((-\triangle)^{\nu} v)^2\|.
\]

To prove Theorem 2, the following lemmas will be needed.

**Lemma 11.** If \( w, v \in H^{2\nu}_{\text{per}}(\Omega) \) for all \( \nu \geq 0 \), then
\[
\|((-\triangle)^{\nu} (wv))^2\| \leq C(\nu) \left( \|v\|_{\infty}^2 \|((-\triangle)^{\nu} w)^2\| + \|w\|_{\infty}^2 \|((-\triangle)^{\nu} v)^2\| \right),
\]
and
\[
\|((-\triangle)^{\nu} v^3)^2\| \leq 3C(\nu) \|v\|_{\infty}^4 \|((-\triangle)^{\nu} v)^2\|,
\]
where
\[
C(\nu) = \max(1, 2^{2\nu-1}).
\]

Proof. Let \( w, v \in H^{2\nu}_{\text{per}}(\Omega) \), so that
\[
w = \sum_{k,l \in \mathbb{Z}} \hat{w}_{kl} e^{ikx_1 + ilx_2}, \quad v = \sum_{k,l \in \mathbb{Z}} \hat{v}_{kl} e^{ikx_1 + ilx_2}.
\]
Then the product of \( w \) and \( v \) can be written as a discrete convolution
\[
wv = \sum_{p,q \in \mathbb{Z}} \left( \sum_{k,l \in \mathbb{Z}} \hat{w}_{kl} \hat{v}_{p-k,q-l} \right) e^{ipx_1 + ipx_2},
\]
and by (4), we obtain
\[
(-\Delta)^\nu (wv) = \sum_{p,q \in \mathbb{Z}} (p^2 + q^2)^\nu \left( \sum_{k,l \in \mathbb{Z}} \hat{w}_{kl} \hat{v}_{p-k,q-l} \right) e^{ipx + iqy},
\]
and hence,
\[
\|(-\Delta)^\nu (wv)\|^2 = \sum_{p,q \in \mathbb{Z}} \left[ (p^2 + q^2)^\nu \left( \sum_{k,l \in \mathbb{Z}} \hat{w}_{kl} \hat{v}_{p-k,q-l} \right) \right]^2.
\]
The following inequality holds for all \(\nu \geq 0\),
\[
|a + b|^{2\nu} \leq C(\nu)(|a|^{2\nu} + |b|^{2\nu}), \quad \forall \ a, b \in \mathbb{R}^2,
\]
where \(C(\nu)\) is given in (42). Choosing \(a = (k, l)\) and \(b = (p - k, q - l)\), we obtain
\[
(p^2 + q^2)^\nu \leq C(\nu)\left( (k^2 + l^2)\nu + ((p - k)^2 + (q - l)^2)\nu \right).
\]
Hence, we obtain
\[
\|(-\Delta)^\nu (wv)\|^2 \leq C(\nu) \left[ \sum_{p,q \in \mathbb{Z}} \left( \sum_{k,l \in \mathbb{Z}} ((k^2 + l^2)\nu \hat{w}_{kl} \hat{v}_{p-k,q-l}) \right)^2 \right.
\]
\[
+ \sum_{p,q \in \mathbb{Z}} \left( \sum_{k,l \in \mathbb{Z}} ((p - k)^2 + (q - l)^2)\nu \hat{w}_{kl} \hat{v}_{p-k,q-l}) \right)^2 \bigg] \bigg]
\leq C(\nu) \left[ \|v(-\Delta)^\nu w\|^2 + \|w(-\Delta)^\nu v\|^2 \right]
\leq C(\nu) \left[ \|v\|_\infty^2 \|(-\Delta)^\nu w\|^2 + \|w\|_\infty^2 \|(-\Delta)^\nu v\|^2 \right],
\]
and the estimate (40) follows.

In order to get estimate (41), by virtue of (40), we obtain
\[
\|(-\Delta)^\nu v^2\|^2 \leq C(\nu) \left[ \|v\|_\infty^2 \|(-\Delta)^\nu v\|^2 + \|v\|_\infty^2 \|(-\Delta)^\nu v\|^2 \right].
\]
Using (40) again, we have
\[
\|(-\Delta)^\nu v^2\|^2 \leq 2C(\nu) \|v\|_\infty^2 \|(-\Delta)^\nu v\|^2.
\]
Combining above two equations, we can get the estimate (41). \(\square \)

**Appendix B: Uniform Gronwall Lemma.** To establish the \(L_\infty\) estimate, we use the following uniform Gronwall lemma which is similar to the results obtained in [13, 21].

**Lemma 12.** Let \(Q(t)\) be a locally integrable positive functions on \((0, \infty)\) satisfying
\[
\frac{d}{dt} Q(t) \leq (aQ(t) + b)(cQ(t) + d), \quad \forall \ t > 0,
\]
and
\[
\int_t^{t+1} Q(\tau) d\tau \leq e, \quad \forall \ t > 0,
\]
where \(a, b, c, d\) and \(e\) are non-negative constants, \(c \neq 0\). Then we have
\[
Q(t) \leq \tilde{C} e^{\tilde{C}} \quad \forall \ t \geq 0,
\]
where \(\tilde{C}\) and \(\tilde{\tilde{C}}\) are positive constants that are independent of \(t\).
Proof. Let

\[ g(t) = c(aQ(t) + b), \]

and

\[ G(r, t) = \int_r^t g(\tau) d\tau \]

for \( 0 \leq r \leq t \leq r + 1 \), then we have

\[ \frac{\partial G}{\partial t}(r, t) = g(t); \quad (44) \]

where \( C_1 \) is a positive constant which is independent of \( t \);

\[ G(r, s) - G(r, t) = \int_t^sg(\tau) d\tau \geq 0, \quad \text{for } r \leq t \leq s \leq 1. \quad (46) \]

Then by (44), we get

\[ \frac{d}{ds} Q(s) e^{-G(r, s)} = e^{-G(r, s)} \left( \frac{dQ(s)}{ds} - g(s)Q(s) \right) \leq e^{-G(r, s)} \frac{d}{ds} g(s). \quad (47) \]

Integrate over \( s : r \leq s \leq r + 1 \) to get

\[ Q(r + 1)e^{-G(r, r+1)} \leq Q(t)e^{-G(r, t)} + \frac{d}{c} \int_t^{r+1} e^{-G(r, s)} g(s) ds, \]

which gives

\[ Q(r + 1)e^{G(r, t) - G(r, r+1)} \leq Q(t) + \frac{d}{c} \int_t^{r+1} e^{G(r, s) - G(r, r+1)} g(s) ds. \]

Hence, with the help of (45)-(46), for \( r \leq t \), we obtain

\[ Q(r + 1)e^{-C_1} \leq Q(t) + \frac{d}{c} \int_t^{r+1} g(s) ds \leq Q(t) + \frac{d}{c} C_2, \]

where \( C_2 \) is a positive constant that is independent of \( t \). Integrate over \( (t, t+1) \) to get

\[ Q(r + 1) \leq C_3 e^{C_1}, \quad \forall r \geq 0. \quad (48) \]

This indicate that

\[ Q(t) \leq C_3 e^{C_1}, \quad \forall t \geq 1. \]

We want the result holds true for \( 0 \leq t < 1 \) as well. Integrate (47) from \( 0 \leq s \leq t \) and put \( r = 0 \), we have

\[ Q(t) \leq e^{G(0, t)} \left\{ Q(0) + \frac{d}{c} \int_0^t e^{-G(0, s)} g(s) ds \right\} \]

for \( 0 \leq t \leq 1 \), where

\[ G(0, t) = \int_0^t g(s) ds \leq \int_0^1 g(s) ds \leq C_4 \]

and \(-G(0, s) \leq 0\). Therefore,

\[ Q(t) \leq C_5 e^{C_1}, \quad \forall 0 \leq t \leq 1. \quad (49) \]

Combining (48) and (49) gives the estimate (43). \qed
Appendix C: Stability of Scheme (32). The scheme (32) can be alternatively formulated in the following mixed form:

\[
\frac{1}{\delta t}(u_{N+1}^n - u_n^0, v) + ((-\Delta)^{\alpha/2} u_{n+1}^0, (-\Delta)^{\alpha/2} v) = 0, \quad \forall v \in \mathcal{X}_N,
\]

(50)

Under the condition

\[
\frac{\varepsilon^2}{2} (|\nabla u_{n+1}^n|^2 - |\nabla u_n^n|^2) + (f(u_{n+1}^n), u_{n+1}^n - u_n^n) 
+ (f(u_n^n), u_{n+1}^n - u_n^n) = 0,
\]

(53)

The next result is similar to the result proved for the standard Cahn-Hilliard equation in [22], and shows that the scheme (50) is stable provided the step size is sufficiently small:

Lemma 13. Under the condition

\[
\delta t \leq \frac{4\varepsilon^2}{L^2},
\]

(51)

where \(L = \max\{|f'(s)|\}\), the solution of (50) satisfies

\[
E(u_{n+1}^n) \leq E(u_n^n), \quad \forall n \geq 0.
\]

(52)

Proof. Taking \(v = \delta t u_{n+1}^n\) and \(\phi = u_{n+1}^n - u_n^n\) in (50) and using the identity

\[
(a-b, 2a) = |a|^2 - |b|^2 + |a-b|^2,
\]

we obtain

\[
(u_{n+1}^n - u_n^n, w_{n+1}^n) + \delta t \|(-\Delta)^{\alpha/2} w_{n+1}^n\|^2 = 0,
\]

(53)

and

\[
\frac{\varepsilon^2}{2} (|\nabla u_{n+1}^n|^2 - |\nabla u_n^n|^2) + (f(u_{n+1}^n), u_{n+1}^n - u_n^n) + (f(u_n^n), u_{n+1}^n - u_n^n) = (w_{n+1}^n, u_{n+1}^n - u_n^n).
\]

(54)

For the last term of the left side in (54), by Taylor expansion

\[
F(u_{n+1}^n) - F(u_n^n) = f(u_n^n)(u_{n+1}^n - u_n^n) + \frac{f'(\zeta^n)}{2}(u_{n+1}^n - u_n^n)^2,
\]

where \(\zeta^n\) is between \(u_n^n\) and \(u_{n+1}^n\). Therefore, we can get

\[
\frac{\varepsilon^2}{2} (|\nabla u_{n+1}^n|^2 - |\nabla u_n^n|^2) + \frac{f'(\zeta^n)}{2}(u_{n+1}^n - u_n^n)^2 + (F(u_{n+1}^n) - F(u_n^n), 1)
\]

\[
= \frac{f'(\zeta^n)}{2}(u_{n+1}^n - u_n^n)^2 + (w_{n+1}^n, u_{n+1}^n - u_n^n).
\]

(55)

On the other hand, taking \(v = \sqrt{\delta t} \varepsilon (u_{n+1}^n - u_n^n)\) in (50), we have

\[
\frac{\varepsilon}{\sqrt{\delta t}} ||u_{n+1}^n - u_n^n||^2 = - \sqrt{\delta t} \varepsilon ((-\Delta)^{\alpha/2} w_{n+1}^n, (-\Delta)^{\alpha/2} (u_{n+1}^n - u_n^n))
\]

\[
\leq \frac{\delta t}{2} \|(-\Delta)^{\alpha/2} w_{n+1}^n\|^2 + \frac{\varepsilon^2}{2} \|(-\Delta)^{\alpha/2} (u_{n+1}^n - u_n^n)\|^2
\]

\[
\leq \frac{\delta t}{2} \|(-\Delta)^{\alpha/2} w_{n+1}^n\|^2 + \frac{\varepsilon^2}{2} \|\nabla (u_{n+1}^n - u_n^n)\|^2.
\]

(56)

Then by (53), (55) and (56), we arrive at

\[
\frac{\varepsilon^2}{2} (|\nabla u_{n+1}^n|^2 - |\nabla u_n^n|^2) + \frac{\delta t}{2} \|(-\Delta)^{\alpha/2} w_{n+1}^n\|^2 + \frac{\delta t}{2} \|\nabla (u_{n+1}^n - u_n^n)\|^2 + (F(u_{n+1}^n) - F(u_n^n), 1) = \frac{f'(\zeta^n)}{2} ||u_{n+1}^n - u_n^n||^2 \leq \frac{L^2}{2} ||u_{n+1}^n - u_n^n||^2.
\]

We then conclude that the desired result holds true under the condition (51). \(\square\)
References


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