A RIESZ BASIS GALERKIN METHOD FOR THE TEMPERED FRACTIONAL LAPLACIAN∗

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Abstract. The fractional Laplacian $\Delta^{\beta/2}$ is the generator of the $\beta$-stable Lévy process, which is the scaling limit of the Lévy flight. Due to the divergence of the second moment of the jump length of the Lévy flight, it may not be a suitable physical model in many practical applications. However, using a parameter $\lambda$ to exponentially temper the isotropic power law measure of the jump length leads to the tempered Lévy flight, which has finite second moment. For a short time the tempered Lévy flight exhibits the dynamics of Lévy flight, while after a sufficiently long time it turns to normal diffusion. The generator of the tempered $\beta$-stable Lévy process is the tempered fractional Laplacian $(\Delta + \lambda)^{\beta/2}$ [W. H. Deng et al., Multiscale Model. Simul., 16 (2018), pp. 125–149]. In the current work, we present new computational methods for the tempered fractional Laplacian equation, including the cases with the homogeneous and nonhomogeneous generalized Dirichlet type boundary conditions. We prove the well-posedness of the Galerkin weak formulation and provide convergence analysis of the single scaling B-spline and multiscale Riesz bases finite element methods. We propose a technique for efficiently generating the entries of the dense stiffness matrix and for solving the resulting algebraic equation by preconditioning. We also present several numerical experiments to verify the theoretical results.

Key words. tempered fractional Laplacian, Galerkin schemes, B-spline and Riesz basis, preconditioning

AMS subject classifications. 35R11, 65M60, 65M12, 65F08

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1. Introduction. Phenomena of anomalous diffusion are ubiquitous in nature [28]. Lévy flights with isotropic power law measure $|x|^{-n-\beta}$ of the jump length display superdiffusion, where $n$ is the dimension of space and $\beta \in (0, 2)$ is a parameter. The scaling limit of Lévy flight is the $\beta$-stable Lévy process, the generator of which is the fractional Laplacian $\Delta^{\beta/2}$. This topic has recently become popular in both pure and applied mathematical communities [30]. The divergence of the second moment of the Lévy flight is associated with the possible infinite speed of the motion of the particles, which contradicts their nonzero masses; i.e., the pure power law distribution of jump length sometimes makes the Lévy flight not a suitable physical model. Hence, tempering the distribution of the jump length becomes a natural idea [7]; namely, modify $|x|^{-n-\beta}$ as $e^{-\lambda|x|}|x|^{-n-\beta}$ with $\lambda$ being a small nonnegative real number, so that we can obtain the tempered Lévy flight.

For small $\lambda$, the tempered Lévy flight exhibits a slow transition of the dynamics from Lévy flight to normal diffusion, which may occur after a sufficiently long time.

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The scaling limit of the tempered Lévy flight is called the tempered Lévy process, the generator of which is the tempered fractional Laplacian \((\Delta + \lambda)^{\beta/2}\) [10]. In this paper, we mainly focus on developing numerical methods in the Riesz basis Galerkin framework for the tempered fractional Laplacian, i.e.,

\[
\begin{align*}
-(\Delta + \lambda)^{\beta/2} p(x) &= f(x), \quad x \in \Omega, \\
p(x) &= 0, \quad x \in \mathbb{R}\setminus\Omega,
\end{align*}
\]

which corresponds to the one-dimensional case of the initial and boundary value problem recently proposed in equation (49) in [10]. Here \(\beta \in (0, 2), \lambda \geq 0, \Omega = (a, b), f(x) \in H^{-\beta/2}(\Omega),\) and

\[
(\Delta + \lambda)^{\beta/2} p(x) := -c_\beta \text{ P.V.} \int_{\mathbb{R}} \frac{p(x) - p(y)}{|x - y|^{1+\beta}} dy
\]

with

\[
c_\beta = \begin{cases} 
\frac{\Gamma(\frac{\beta}{2})}{2\pi^{\frac{\beta}{2}}\Gamma(-\frac{\beta}{2})\beta\Gamma\left(\frac{1-\beta}{2}\right)} & \text{for } \lambda > 0 \text{ and } \beta \neq 1, \\
\frac{1}{\Gamma(1-\beta/2)} & \text{otherwise},
\end{cases}
\]

where \text{P.V.} denotes the Cauchy principle value, which is the limit of the integral over \(\mathbb{R}\setminus B_\epsilon(x)\) as \(\epsilon \to 0\); the definition of this form is indeed necessary when \(\beta \geq 1\).

Obviously, when \(\lambda = 0, (1.2)\) reduces to the fractional Laplacian

\[
(\Delta)^{\beta/2} p(x) := -c_\beta \text{ P.V.} \int_{\mathbb{R}} \frac{p(x) - p(y)}{|x - y|^{1+\beta}} dy,
\]

which has the Fourier transform (assuming that \(\mathcal{F}[\Delta^{\beta/2} p(x)](\xi)\) and \(\mathcal{F}[p(x)](\xi)\) exist)

\[
\mathcal{F}[\Delta^{\beta/2} p(x)](\xi) = -|\xi|^\beta \mathcal{F}[p(x)](\xi).
\]

Here, \(\mathcal{F}[w(x)](\xi), \xi \in \mathbb{R},\) is defined by \(\mathcal{F}[w(x)](\xi) = \int_{\mathbb{R}} w(x) e^{-ix\xi} dx,\) and for \(w_1, w_2 \in L^2(\mathbb{R}),\) the Parseval identity [19, p. 100] can be applied:

\[
\int_{\mathbb{R}} w_1(x) \overline{w_2(x)} dx = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}[w_1(x)](\xi) \overline{\mathcal{F}[w_2(x)](\xi)} d\xi.
\]

Recently, the fractional Laplacian has attracted a lot of attention, but even in the simplified context [1, 2, 13, 21] it is far from the well-developed status of the classical Laplacian. There are two major challenges to the numerical discretization of the fractional Laplacian—namely, the singular kernel and the integration in an unbounded region. For the finite difference method the convergence rate is even influenced by the regularity of the exact solution outside of the domain \(\Omega\) [21]. As for the tempered fractional differential equations [4, 7], there are some published works on numerical methods [4, 20, 25, 38], but no theoretical results under the variational framework exists. In the current paper we prove the well-posedness of the variational formulation of (1.1), where extra efforts must be made to obtain the \(H^{\beta/2}(\Omega)\)-coercivity. Subsequently, the convergence analysis and the effective implementation of the finite dimensional approximation with the single-scale or multiscale basis functions are presented, in which the properties of Riesz basis and multiresolution are used.

The rest of this paper is organized as follows. In section 2, we introduce the function spaces and the properties of the tempered fractional Laplacian to be used. The

\[
[7].
\]
2. Preliminaries. Throughout the paper by the notation \( A \lesssim B \) we mean that \( A \) can be bounded by a multiple of \( B \), independently of the parameters they may depend on, while the expression \( A \simeq B \) means that \( A \lesssim B \lesssim A \). Let \( E \) be an open set of \( \mathbb{R} \). If \( s \) is a nonnegative integer, we denote by \( H^s(E) \) the classical Sobolev space equipped with the norm

\[
\|w\|_{H^s(E)} := \left( \sum_{0 \leq k \leq s} \|w^{(k)}\|_{L^2(E)}^2 \right)^{1/2},
\]

where \( w^{(k)} \) stands for the \( k \)th distributional derivative, and \( H^0(E) := L^2(E) \). In the following, we define the fractional Sobolev spaces, where \( s \) is not an integer.

For a fixed \( s \in (0, 1) \), the Sobolev space \( H^s(E) \) is defined as

\[
H^s(E) := \{ w \in L^2(E) : |w|_{H^s(E)} < \infty \},
\]

where

\[
|w|_{H^s(E)} := \left( \int_E \int_E \frac{|w(x) - w(y)|^2}{|x - y|^{1+2s}} \, dx \, dy \right)^{1/2}
\]

is the Slobodeckii seminorm [27, p. 74] of \( w(x) \). The space \( H^s(E) \) is a Banach space endowed with the natural norm

\[
\|w\|_{H^s(E)} := \left( \|w\|_{L^2(E)}^2 + |w|_{H^s(E)}^2 \right)^{1/2}.
\]

Indeed, \( H^s(E) \) also is a Hilbert space [27, p. 75]. For \( s > 1 \) and \( s \notin \mathbb{N} \), we can define \( H^s(E) \) as follows:

\[
H^s(E) := \left\{ w \in H^{[s]}(E) : \left. |w|_{H^{[s]}(E)} \right|_{H^{-[s]}(E)} < \infty \right\},
\]

where \([s]\) is the biggest integer smaller than \( s \). In this case, \( H^s(E) \) is endowed with the norm

\[
\|w\|_{H^s(E)} := \left( \|w\|_{H^{[s]}(E)}^2 + \left. |w|_{H^{[s]}(E)} \right|^2_{H^{-[s]}(E)} \right)^{1/2}.
\]

We note that \( H^s(E) \) is a well-defined Banach space for every \( s \geq 0 \). Moreover, when \( E = \mathbb{R} \), for \( 0 < s < 1 \), it holds that [27, Lemma 3.15] \( \|w\|_{H^s(\mathbb{R})}^2 \simeq \int_{\mathbb{R}} |\xi|^{2s} |\mathcal{F}[w(x)](\xi)|^2 \, d\xi \), and for \( s \geq 0 \), by [27, Theorem 3.16] and [12, Lemma 2.1], we have

\[
\|w\|_{H^s(\mathbb{R})}^2 \simeq \int_{\mathbb{R}} (1 + |\xi|^{2s}) |\mathcal{F}[w(x)](\xi)|^2 \, d\xi \simeq \int_{\mathbb{R}} (1 + |\xi|^{2s}) |\mathcal{F}[w(x)](\xi)|^2 \, d\xi.
\]
In fact, the Sobolev space \( H^s(\mathbb{R}) \) can also be defined as
\[
H^s(\mathbb{R}) = \left\{ w \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (1 + |\xi|^{2s}) |\mathcal{F}[w(x)](\xi)|^2 d\xi < \infty \right\}.
\]

Let \( C_0^\infty(E) \) be the space of functions that are infinitely differentiable on \( E \) and have compact support in \( E \). Then \( C_0^\infty(\mathbb{R}) \) is dense in \( H^s(\mathbb{R}) \). However, if \( E \subset \mathbb{R} \) is strict, the space \( C_0^\infty(E) \) generally is not dense in \( H^s(\mathbb{R}) \). Hence, we denote by \( H^s_0(E) \) the closure of \( C_0^\infty(E) \) in \( H^s(\mathbb{R}) \). As usual, \( H^{-s}(E) \) is the dual space of \( H^s_0(E) \) [36]. In addition, let \( \Omega = (a, b) \) be a nonempty open interval of \( \mathbb{R} \). By \( C_0^\infty(\Omega) \) we denote the space of all infinitely differentiable functions on \( \mathbb{R} \) whose support is compact and contained in \( \Omega \). For \( s > 0 \), we use \( \bar{H}^s_0(\Omega) \) to denote the closure of \( C_0^\infty(\Omega) \) in \( H^s(\mathbb{R}) \). For \( s = 0 \), \( \bar{H}^s_0(\Omega) \) is interpreted as the closure of \( C_0^\infty(\Omega) \) in \( L^2(\mathbb{R}) \) and denoted as \( \bar{L}^2(\Omega) \). Obviously, \( \bar{H}^s_0(\Omega) \subset H^s_0(\Omega) \). Moreover, by [17], when \( s \in (0, 1) \), \( \bar{H}^s_0(\Omega) \) can also be defined by
\[
(2.8) \quad \bar{H}^s_0(\Omega) = \{ w(x) \in H^s(\mathbb{R}) : w(x) = 0 \text{ a.e. for } x \in \mathbb{R} \setminus \Omega \}.
\]

Here, the space \( \bar{C}_0^\infty(\Omega) \) is actually the space \( \bar{C}_c^\infty(a, b) \) in [23, p. 178] and the space \( C_0^\infty(a, b) \) in [17, p. 237]. The space \( \bar{H}^s_0(\Omega) \) is the space \( H^s_0(\Omega) \) with \( \mu = s \) in [23, p. 178] and the space \( X^s_0(a, b) \) with \( p = 2 \) in [17, pp. 236–237].

Next, we give some properties of the tempered fractional Laplacian.

**PROPOSITION 2.1.** For \( w \in C_0^\infty(\mathbb{R}) \) and \( \lambda > 0 \), we have
\[
\mathcal{F}[(\Delta + \lambda)^{\beta/2}w(x)](\xi) = (-1)^{|\beta|} \left( \lambda^\beta - \left( \lambda^2 + |\xi|^2 \right)^{\beta/2} \cos\left( \beta \arctan\left( \frac{|\xi|}{\lambda} \right) \right) \right) \mathcal{F}[w(x)](\xi)
\]
for \( \beta \in (0, 1) \cup (1, 2) \), and
\[
\mathcal{F}[(\Delta + \lambda)^{\beta/2}w(x)](\xi) = \frac{2}{\pi} \left( -|\xi| \arctan \left( \frac{|\xi|}{\lambda} \right) + \frac{\lambda}{2} \ln(\lambda^2 + |\xi|^2) - \lambda \ln(\lambda) \right) \mathcal{F}[w(x)](\xi)
\]
for \( \beta = 1 \), where \( \{ \beta \} := \{ z \in \mathbb{N} : 0 \leq \beta - z < 1 \} \).

**Proof.** For \( w \in C_0^\infty(\mathbb{R}) \), similar to [29, Lemma 3.2], we can get rid of P.V. in (1.2) and rewrite it as
\[
(2.11) \quad (\Delta + \lambda)^{\beta/2}w(x) = -c_\beta \frac{2}{\lambda} \int_{\mathbb{R}} \frac{2w(x) - w(x - y) - w(x + y)}{e^{\lambda |y|} |y|^{1+\beta}} dy.
\]
Then
\[
\mathcal{F}[(\Delta + \lambda)^{\beta/2}w(x)](\xi) = -c_\beta \int_{\mathbb{R}} \frac{2 - e^{-iy\xi} - e^{iy\xi}}{e^{\lambda |y|} |y|^{1+\beta}} dy \mathcal{F}[w(x)](\xi)
\]
\[
= -c_\beta \int_{0}^{\infty} (2 - 2\cos(y\xi)) e^{-\lambda y} y^{-1-\beta} dy \mathcal{F}[w(x)](\xi)
\]
\[
= \frac{c_\beta}{\beta} \int_{0}^{\infty} (2 - 2\cos(y\xi)) e^{-\lambda y} (y^{1-\beta}) \mathcal{F}[w(x)](\xi)
\]
\[
(2.12) \quad = \frac{2c_\beta}{\beta} \int_{0}^{\infty} y^{-\beta} e^{-\lambda y} \left( \xi \sin(y\xi) - \lambda (1 - \cos(y\xi)) \right) dy \mathcal{F}[w(x)](\xi),
\]
where we have used integration by parts in the last step. Since $\xi \sin(y\xi) - \lambda(1 - \cos(y\xi))$ is an even function w.r.t. $\xi$, in the following we assume $\xi \geq 0$.

If $0 < \beta < 1$, we have

$$
\int_{0}^{\infty} y^{-\beta} e^{-\lambda y} (\xi \sin(y\xi) - \lambda(1 - \cos(y\xi))) dy
= \xi \int_{0}^{\infty} y^{-\beta} e^{-\lambda y} \sin(y\xi) dy - \lambda \int_{0}^{\infty} y^{-\beta} e^{-\lambda y} dy + \lambda \int_{0}^{\infty} y^{-\beta} e^{-\lambda y} \cos(y\xi) dy
= \frac{\Gamma(1 - \beta)\xi}{(\lambda^2 + \xi^2)^{\frac{\beta}{2}}} \sin \left( (1 - \beta) \arctan \left( \frac{\xi}{\lambda} \right) \right) - \lambda^{\beta} \int_{0}^{\infty} y^{-\beta} e^{-y} dy
+ \frac{\Gamma(1 - \beta)\lambda}{(\lambda^2 + \xi^2)^{\frac{\beta}{2}}} \cos \left( (1 - \beta) \arctan \left( \frac{\xi}{\lambda} \right) \right)
$$

(2.13) = $\Gamma(1 - \beta)(\lambda^2 + \xi^2)^{\frac{\beta}{2}} \cos \left( \beta \arctan \left( \frac{\xi}{\lambda} \right) \right) - \lambda^{\beta} \Gamma(1 - \beta),$

where the formulae [18, eq. (3.944(5))] and [18, eq. (3.944(6))] have been used in the second step.

For $1 < \beta < 2$, using the integration by parts again, we have

$$
\int_{0}^{\infty} y^{-\beta} e^{-\lambda y} (\xi \sin(y\xi) - \lambda(1 - \cos(y\xi))) dy
= \frac{1}{1 - \beta} \int_{0}^{\infty} e^{-\lambda y} y^{1-\beta} \left( (\lambda^2 - \xi^2) \cos(y\xi) + 2\lambda\xi \sin(y\xi) - \lambda^2 \right) dy
= \frac{\Gamma(1 - \beta)}{(\lambda^2 + \xi^2)^{\frac{\beta}{2}}} \left( (\lambda^2 - \xi^2) \cos \left( (2 - \beta) \arctan \left( \frac{\xi}{\lambda} \right) \right) + 2\lambda\xi \sin \left( (2 - \beta) \arctan \left( \frac{\xi}{\lambda} \right) \right) \right)
- \Gamma(1 - \beta)\lambda^{\beta}
$$

(2.14) = $\Gamma(1 - \beta)(\lambda^2 + \xi^2)^{\frac{\beta}{2}} \cos \left( \beta \arctan \left( \frac{\xi}{\lambda} \right) \right) - \lambda^{\beta} \Gamma(1 - \beta)$.

For $\beta = 1$, using integration by parts, we have

$$
\int_{0}^{\infty} y^{-\beta} e^{-\lambda y} (\xi \sin(y\xi) - \lambda(1 - \cos(y\xi))) dy
= (\lambda^2 - \xi^2) \int_{0}^{\infty} \ln(y)e^{-\lambda y} \cos(y\xi) dy
+ 2\lambda\xi \int_{0}^{\infty} \ln(y)e^{-\lambda y} \sin(y\xi) dy - \lambda^2 \int_{0}^{\infty} \ln(y)e^{-\lambda y} dy
$$

$$
= \frac{2\lambda\xi}{\lambda^2 + \xi^2} \left( \lambda \arctan \left( \frac{\xi}{\lambda} \right) - \gamma \xi - \frac{\xi}{2} \ln(\lambda^2 + \xi^2) \right)
+ \frac{\xi^2 - \lambda^2}{\lambda^2 + \xi^2} \left( \frac{\lambda}{2} \ln(\lambda^2 + \xi^2) + \xi \arctan \left( \frac{\xi}{\lambda} \right) + \lambda \gamma \right) + \lambda (\gamma + \ln(\lambda))
$$

(2.15) = $\xi \arctan \left( \frac{\xi}{\lambda} \right) - \frac{\lambda}{2} \ln(\lambda^2 + \xi^2) + \lambda \ln(\lambda)$,

where $\gamma$ denotes the Euler constant and the formulae [18, eq. (4.441(1))–(4.441(2))] and [18, eq. (4.331(1))] have been used in the second step.
The following proposition is similar to Theorem 2.1 of [24].

**Proposition 2.2.** For $0 < \beta < 2$, the tempered fractional Laplacian can be extended to a continuous linear mapping from $H^\beta(\mathbb{R})$ to $L^2(\mathbb{R})$, and its Fourier transform remains the same.

**Proof.** First, assume $w \in C_0^\infty(\mathbb{R})$. For $\lambda = 0$, by (1.5), the Parseval identity (1.6), and (2.8), we have

\begin{equation}
\left| \Delta^{\beta/2} w \right|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |\xi|^{2\beta} |\hat{F}(w(x))(\xi)|^2 \, d\xi \lesssim \|w\|_{H^\beta(\mathbb{R})}^2.
\end{equation}

For $\lambda > 0$, by Proposition 2.1, the Parseval identity (1.6), and (2.8), we have

\begin{equation}
\left| \left(\Delta + \lambda\right)^{\beta/2} w \right|_{L^2(\mathbb{R})}^2 \leq \int_{\mathbb{R}} G^2(\lambda,\xi,\beta) |\hat{F}[w(x)](\xi)|^2 \, d\xi
\end{equation}

where

\begin{equation}
G(\lambda,\xi,\beta) := \begin{cases}
(1-1)^{\beta}/2 \left( (\lambda^2 + |\xi|^2)^{\frac{\beta}{2}} \cos(\beta \arctan(\frac{|\xi|}{\lambda})) - \lambda^\beta \right), & \beta \neq 1, \\
\frac{2}{\pi} \left( |\xi| \arctan(\frac{|\xi|}{\lambda}) - \frac{1}{2} \ln(\lambda^2 + |\xi|^2) + \lambda \ln(\lambda) \right), & \beta = 1;
\end{cases}
\end{equation}

and the inequalities $x_1^2 + x_2^2 + x_3^2 \leq 3 (x_1^2 + x_2^2 + x_3^2)$, $\cos(\beta \arctan(\frac{|\xi|}{\lambda})) \leq 1$, $\arctan(\frac{|\xi|}{\lambda}) \leq \frac{\pi}{2}$, $\ln(\lambda^2 + |\xi|^2) \leq \lambda^2 + |\xi|^2$, and $(\lambda^2 + |\xi|^2)^\beta \leq 2^\beta (\lambda^{2\beta} + |\xi|^{2\beta})$ have been used. Then by the density of $C_0^\infty(\mathbb{R})$ in $H^\beta(\mathbb{R})$, one can extend $(\Delta + \lambda)^{\beta/2}$ to a continuous operator from $H^\beta(\mathbb{R})$ to $L^2(\mathbb{R})$ with the same bounds as in (2.16) (for $\lambda = 0$) or in (2.17) (for $\lambda > 0$).

Second, let $w \in H^\beta(\mathbb{R})$. Then from the first part of the argument, we have $\hat{F}[\left(\Delta + \lambda\right)^{\beta/2} w(x)](\xi) \in L^2(\mathbb{R})$; in addition, there exists a sequence $\{w_k\} \in C_0^\infty(\mathbb{R})$ such that $\lim_{k \to \infty} \|w - w_k\|_{H^\beta(\mathbb{R})} = 0$. Therefore, for $\lambda > 0$, by (2.17) and the Parseval identity (1.6) we have

\begin{equation}
\left\| -G(\lambda,\xi,\beta) \hat{F}[w(x)](\xi) - \hat{F}[\left(\Delta + \lambda\right)^{\beta/2} w(x)](\xi) \right\|_{L^2(\mathbb{R})}
\leq \left\| -G(\lambda,\xi,\beta) \hat{F}[w(x)](\xi) \right\|_{L^2(\mathbb{R})}
+ \left\| \hat{F}[\left(\Delta + \lambda\right)^{\beta/2} (w_k - w)(x)](\xi) \right\|_{L^2(\mathbb{R})}
\lesssim 2 \|w - w_k\|_{H^\beta(\mathbb{R})} \to 0.
\end{equation}

Thus $\hat{F}[\left(\Delta + \lambda\right)^{\beta/2} w(x)](\xi) = -G(\lambda,\xi,\beta) \hat{F}[w(x)](\xi)$. The proof for $\lambda = 0$ is similar.

**Proposition 2.3.** If $w(x) \in C^3(\mathbb{R})$ and $w(x)$, $w^{(3)}(x) \in L^\infty(\mathbb{R})$, for $\lambda \geq 0$, we have

\begin{equation}
\lim_{\beta \to 2^-} (\Delta + \lambda)^{\beta/2} w(x) = \frac{d^2 w(x)}{dx^2}.
\end{equation}
Proof. Since \( w(x) \in C^3(\mathbb{R}) \), by the Taylor formula, we have

\[
\begin{align*}
    w(x+y) &= w(x) + w^{(1)}(x)y + \frac{w^{(2)}(x)}{2!} y^2 + \frac{w^{(3)}(x+\theta_1 y)}{3!} y^3, \\
    w(x-y) &= w(x) - w^{(1)}(x)y + \frac{w^{(2)}(x)}{2!} y^2 - \frac{w^{(3)}(x-\theta_2 y)}{3!} y^3,
\end{align*}
\]

where \( \theta_1, \theta_2 \in (0, 1) \).

For the case \( \lambda > 0 \), by (2.11), it holds that

\[
(\Delta + \lambda)^{3/2} w(x) = c_\beta \int_0^\infty \frac{w(x+y) - 2w(x) + w(x-y)}{e^{\lambda y y^{1+\beta}}} dy
\]

\[
= c_\beta \int_0^\infty \frac{w^{(2)}(x)y^2 + \frac{w^{(3)}(x+\theta_1 y)}{3!} y^3 - \frac{w^{(3)}(x-\theta_2 y)}{3!} y^3}{e^{\lambda y y^{1+\beta}}} dy.
\]

Note that

\[
\left| c_\beta \int_0^\infty \frac{w^{(3)}(x+\theta_1 y) y^3 - w^{(3)}(x-\theta_2 y) y^3}{3! e^{\lambda y y^{1+\beta}}} dy \right|
\]

\[
\leq \frac{c_\beta}{3} \| w^{(3)} \|_{L^\infty(\mathbb{R})} \int_0^\infty \frac{y^3}{e^{\lambda y y^{1+\beta}}} dy
\]

\[
= c_\beta \frac{\lambda^{3-\beta}}{3} \Gamma(3-\beta) \| w^{(3)} \|_{L^\infty(\mathbb{R})} \rightarrow 0, \quad \beta \rightarrow 2^+,
\]

where \( \lim_{\beta \rightarrow 2^-} c_\beta = \lim_{\beta \rightarrow 2^-} \frac{\Gamma(\frac{3}{2})}{2\pi\frac{1}{2} |\Gamma(-\beta)|} = 0 \) has been used in the last step. Therefore, we have

\[
\lim_{\beta \rightarrow 2^-} (\Delta + \lambda)^{3/2} w(x) = \lim_{\beta \rightarrow 2^-} c_\beta \frac{\lambda^{3-\beta} \Gamma(2-\beta) w^{(2)}(x)}{2} = w^{(2)}(x),
\]

where \( \Gamma(2-\beta) = (1-\beta)(-\beta)\Gamma(-\beta) \) and \( \Gamma(-\beta) > 0 \) for \( \beta \in (1, 2) \) have been used in the third step.

For the case \( \lambda = 0 \), it holds that

\[
(\Delta + \lambda)^{3/2} w(x) = c_\beta \int_0^\infty \frac{w(x+y) - 2w(x) + w(x-y)}{y^{1+\beta}} dy
\]

\[
+ c_\beta \int_0^\infty \frac{w(x+y) - 2w(x) + w(x-y)}{y^{1+\beta}} dy
\]

\[
= c_\beta \int_0^\infty \frac{w(x+y) - 2w(x) + w(x-y)}{y^{1+\beta}} dy
\]

\[
+ c_\beta \int_0^\infty \frac{w^{(2)}(x)y^2}{y^{1+\beta}} dy + c_\beta \int_0^\infty \frac{w^{(3)}(x+\theta_1 y) y^3 - w^{(3)}(x-\theta_2 y) y^3}{y^{1+\beta}} dy,
\]

\[
(\Delta + \lambda)^{3/2} w(x) = c_\beta \int_0^\infty \frac{w(x+y) - 2w(x) + w(x-y)}{y^{1+\beta}} dy
\]

\[
+ c_\beta \int_0^\infty \frac{w^{(2)}(x)y^2}{y^{1+\beta}} dy + c_\beta \int_0^\infty \frac{w^{(3)}(x+\theta_1 y) y^3 - w^{(3)}(x-\theta_2 y) y^3}{y^{1+\beta}} dy,
\]

where \( \theta_1, \theta_2 \in (0, 1) \).

For the case \( \lambda > 0 \), by (2.11), it holds that

\[
(\Delta + \lambda)^{3/2} w(x) = c_\beta \int_0^\infty \frac{w(x+y) - 2w(x) + w(x-y)}{e^{\lambda y y^{1+\beta}}} dy
\]

\[
= c_\beta \int_0^\infty \frac{w^{(2)}(x)y^2 + \frac{w^{(3)}(x+\theta_1 y)}{3!} y^3 - \frac{w^{(3)}(x-\theta_2 y)}{3!} y^3}{e^{\lambda y y^{1+\beta}}} dy.
\]
where $c$ is an arbitrary given positive constant. Note that

$$
\left| c \beta \int_{c}^{\infty} \frac{w(x + y) - 2w(x) + w(x - y)}{y^{1+\beta}} dy \right|
\leq c \beta \frac{4c^{-\beta}}{\beta} \|w\|_{L^\infty(R)} \to 0, \quad \beta \to 2^{-},
$$

$$
\left| c \beta \int_{0}^{\infty} \frac{w^{(3)}(x+\theta y) y^\beta - w^{(3)}(x-\theta y) y^\beta}{y^{1+\beta}} dy \right|
\leq c \beta \frac{3c^{-\beta}}{3(3 - \beta)} \|w^{(3)}\|_{L^\infty(R)} \to 0, \quad \beta \to 2^{-},
$$

where $\lim_{\beta \to 2^{-}} c = \lim_{\beta \to 2^{-}} \frac{\beta \Gamma(1 + \beta)}{2\pi \sin(\pi \beta)} = 0$ has been used. Therefore, we have

$$
\lim_{\beta \to 2^{-}} (\Delta + \lambda)^{\beta/2} w(x) = \lim_{\beta \to 2^{-}} c \beta \int_{0}^{\infty} \frac{w^{(2)}(x) y^\beta}{y^{1+\beta}} dy
= \lim_{\beta \to 2^{-}} \frac{c \beta}{2 - \beta} e^{2-\beta} w^{(2)}(x)
= \lim_{\beta \to 2^{-}} \sin \left( \frac{(2 - \beta)\pi}{2} \right) / \left( \frac{(2 - \beta)\pi}{\Gamma(1 + \beta)} \right) w^{(2)}(x)
= w^{(2)}(x),
$$

where the result (see the proof in Appendix A)

$$
c = \frac{1}{\pi} \Gamma(1 + \beta) \sin \left( \frac{(2 - \beta)\pi}{2} \right)
$$

has been used in the third step. \hfill \Box

Equation (2.20) shows that if $\beta \to 2^{-}$, the tempered fractional Laplacian reduces to the classical Laplacian. Finally, we give the concept of the Riesz basis, which will be used later.

**Definition 2.4** ([23, 34]). A countable collection of elements $\mathcal{E} := \{e_i\}_{i \in I} (I \subset \mathbb{Z})$ of a Hilbert space $H$ is called a Riesz basis of $H$ if each element in $H$ has an expansion in terms of $\mathcal{E}$ and there exist (Riesz) constants $0 < A \leq B < \infty$ such that the inequalities

$$
A \sum_{i \in I} |d_i|^2 \leq \left\| \sum_{i \in I} d_i e_i \right\|^2_H \leq B \sum_{i \in I} |d_i|^2
$$

hold true for every sequence $\{d_i\}_{i \in I}$ in $l^2(I)$, where $l^2(I)$ denotes the linear space of all sequences $\{d_i\}_{i \in I}$ such that $\left( \sum_{i \in I} |d_i|^2 \right)^{1/2} < \infty$.

### 3. Weak solution and well-posedness.

In this section, we first give the definition of the weak solution of (1.1), and then we discuss the well-posedness of the corresponding weak formulation. As in the usual approach to dealing with elliptic PDEs, multiplying both sides of (1.1) by $v \in \tilde{H}^{3/2}_0(\Omega)$ and integrating them over $\Omega$ leads to

$$
- \int_{\mathbb{R}} (\Delta + \lambda)^{\beta/2} p(x)v(x) dx = \int_{\Omega} f(x)v(x) dx.
$$
Since \(e^{-\lambda|x-y||x-y|^{-1-\beta}}\) is symmetric, inspired by [22, Lemma 2.1], we define the weak formulation of (1.1) as follows: find \(p \in \widehat{H}_0^{\beta/2}(\Omega)\) such that

\[
B(p, v) = (f, v)
\]

for all \(v \in \widehat{H}_0^{\beta/2}(\Omega)\), where the duality pairing \((f, v) := \int_\Omega f(x)v(x)dx\) and the bilinear form

\[
B(p, v) := \frac{c_\beta}{2} \int_\mathbb{R} \int_\mathbb{R} \frac{(p(x) - p(y))(v(x) - v(y))}{|x - y|^{1+\beta}} dy dx.
\]

When \(\lambda = 0\), [10] gives the weak formulation of (1.1) as follows: find \(p \in \widehat{H}_0^{\beta/2}(\Omega)\), such that

\[
\int_\mathbb{R} \Delta^{\beta/4} p \Delta^{\beta/4} v dx = \int_\Omega f v dx \quad \forall v \in \widehat{H}_0^{\beta/2}(\Omega),
\]

which is equivalent to (3.2) with \(\lambda = 0\). In fact, it can be simply verified as follows: for any \(p, v \in \widehat{H}_0^{\beta/2}(\Omega)\), by the Parseval identity (1.6) and \(\mathcal{F}[\Delta^{\beta/4} p(x)](\xi) = -|\xi|^{\beta/2} \mathcal{F}[p(x)](\xi)\),

\[
\frac{c_\beta}{2} \int_\mathbb{R} \int_\mathbb{R} \frac{(p(x) - p(y))(v(x) - v(y))}{|x - y|^{1+\beta}} dy dx
\]

\[
= \frac{c_\beta}{2} \int_\mathbb{R} \int_\mathbb{R} \frac{(p(x + y) - p(y))(v(x + y) - v(y))}{|x + y|^{1+\beta}} dy dx
\]

\[
= \frac{c_\beta}{4\pi} \int_\mathbb{R} \int_\mathbb{R} \left| e^{i\xi x - 1} \right|^2 dx \mathcal{F}[p(y)](\xi) \mathcal{F}[v(y)](\xi) d\xi
\]

\[
= \frac{1}{2\pi} \int_\mathbb{R} \left( -|\xi|^{\beta/2} \mathcal{F}[p(x)](\xi) \right) \left( -|\xi|^{\beta/2} \mathcal{F}[v(x)](\xi) \right) d\xi
\]

\[
= \int_\mathbb{R} \Delta^{\beta/4} p \Delta^{\beta/4} v dx,
\]

where the result

\[
\int_\mathbb{R} \left| e^{i\xi x - 1} \right|^2 dx = \int_\mathbb{R} \frac{2 - 2 \cos(\xi x)}{|x|^{1+\beta}} dx = 8|\xi|^{\beta} \int_0^\infty \frac{\sin^2 \frac{\omega}{2}}{|\omega|^{1+\beta}} d\omega = \frac{2|\xi|^{\beta}}{c_\beta}
\]

has been used in the second equality from below (note that the last “\(=\)” in (3.6) is derived from [30, eqs. (1.8.2), (1.8.3), and (1.8.14)]). However, when \(\lambda > 0\), (3.2) does not have the equivalent form like (3.4), which can be simply discovered by recalling the proof process of Proposition 2.1, i.e.,

\[
B(p, v) = \frac{c_\beta}{2\pi} \int_0^\infty \int_0^{\infty} \frac{2 - 2 \cos(\xi x)}{e^{\lambda|x|}|x|^{1+\beta}} dx \mathcal{F}[p(y)](\xi) \mathcal{F}[v(y)](\xi) d\xi
\]

\[
= \frac{1}{2\pi} \int_\mathbb{R} \mathcal{G}(\lambda, \xi, \beta) \mathcal{F}[p(y)](\xi) \mathcal{F}[v(y)](\xi) d\xi,
\]

where \(\mathcal{G}(\lambda, \xi, \beta)\) is as given in (2.18). On the contrary, for \(\beta \in (0, 1) \cup (1, 2)\), by introducing the left and right Riemann–Liouville tempered fractional derivatives \(-\infty D_x^{\alpha, \lambda}\)
and $x \mathbb{D}^{\alpha \lambda}_{\infty}(\alpha \in (k-1,k), k \in \mathbb{N}^+)$, given as [25, Definition 3]

$$
-\infty \mathbb{D}^\alpha_{x} u(x) = \frac{e^{-\lambda x}}{\Gamma(k-\alpha)} \frac{d^k}{dx^k} \int_{-\infty}^{x} \frac{e^{\lambda \xi} u(\xi)}{(x-\xi)^{\alpha-k+1}} d\xi
$$

and

$$
x \mathbb{D}^\alpha_{\infty} u(x) = \frac{e^{\lambda x}}{\Gamma(k-\alpha)} \frac{d^k}{dx^k} \int_{x}^{\infty} \frac{e^{-\lambda \xi} u(\xi)}{(\xi-x)^{\alpha-k+1}} d\xi,
$$

respectively, one has the following proposition.

**Proposition 3.1.** For $\beta \in (0,1) \cup (1,2)$, (3.2) has the following equivalent weak formulation: find $p \in \tilde{H}^{\beta/2} (\Omega)$ such that

$$
B(p,v) = (-1)^{\lfloor \beta \rfloor} B_1(p,v) = (f,v)
$$

for all $v \in \tilde{H}^{\beta/2} (\Omega)$, where $B_1(p,v)$ is given as

$$
\frac{1}{2} \int_{\mathbb{R}} -\infty \mathbb{D}^{\beta/2}_{x} p \cdot x \mathbb{D}^{\beta/2}_{\infty} v dx + \frac{1}{2} \int_{\mathbb{R}} x \mathbb{D}^{\beta/2}_{\infty} p \cdot \infty x \mathbb{D}^{\beta/2}_{x} v dx - \lambda^\beta \int_{\mathbb{R}} p v dx.
$$

Proof. According to (3.7), for $\beta \in (0,1) \cup (1,2)$, there exists

$$
B(p,v) = \frac{(-1)^{\lfloor \beta \rfloor}}{2\pi} \int_{\mathbb{R}} \left( \frac{1}{2}(\lambda + i\xi)^\beta + \frac{1}{2}(\lambda - i\xi)^\beta - \lambda^\beta \right) \mathcal{F}[p(y)](\xi) \mathcal{F}[v(y)](\xi) d\xi.
$$

Then the desired result is obtained by using the Parseval identity (1.6) and [25, Lemma 1]

$$
\mathcal{F} \left[ -\infty \mathbb{D}^{\beta}_{x} u(x) \right] = (\lambda + i\xi)^{\beta/2} \mathcal{F}[u](\xi),
$$

(3.8)

$$
\mathcal{F} \left[ x \mathbb{D}^{\beta}_{\infty} u(x) \right] = (\lambda - i\xi)^{\beta/2} \mathcal{F}[u](\xi).
$$

(3.9)

\[\square\]

**Remark 3.1.** Note that for $p \in H^\beta(\mathbb{R})$, by Proposition 2.2 and the Parseval identity (1.6), it holds that

$$
- \int_{\mathbb{R}} (\Delta + \lambda)^{\beta/2} p(x)v(x) dx = \frac{1}{2\pi} \left\{ \int_{\mathbb{R}} |\xi|^\beta \mathcal{F}[p(y)](\xi) \mathcal{F}[v(y)](\xi) d\xi, \right. \left. \int_{\mathbb{R}} \mathcal{G}(\lambda, \xi, \beta) \mathcal{F}[p(y)](\xi) \mathcal{F}[v(y)](\xi) d\xi \right\}, \lambda = 0,
$$

(3.10)

Then if $p \in H^\beta(\mathbb{R}) \cap \tilde{H}^{\beta/2} (\Omega)$, by (3.5) and (3.7), for any $v \in \tilde{H}^{\beta/2} (\Omega)$, we have

$$
- \int_{\mathbb{R}} (\Delta + \lambda)^{\beta/2} p(x)v(x) dx = B(u,v).
$$

(3.11)

Thus, the definition of the weak formulation (3.2) is reasonable.

**Proposition 3.2.** If $0 < s < 1$, then for any real number $\delta > 0$, there exists a positive constant $C = C(\Omega, \delta, s)$ such that

$$
\|w\|_{L^2(\Omega)} \leq C \|w\|_{H^s(\Omega)}, \forall w \in \tilde{H}^{\delta}_0(\Omega),
$$

where $\Omega^* = (a - \delta, b + \delta)$. In particular, if $\frac{3}{2} < s < 1$, the above result can be further improved as follows: there exists a positive constant $C = C(\Omega, s)$ such that

$$
\|w\|_{L^2(\Omega)} \leq C \|w\|_{H^s(\Omega)}, \forall w \in \tilde{H}^{\delta}_0(\Omega).
$$

(3.12)
Proof. From (2.3), we have
\[
|w|^2_{H^s(\Omega^*)} = \int_{\Omega^*} \int_{\Omega^*} \frac{|w(x) - w(y)|^2}{|x - y|^{1+2s}} \, dy \, dx \\
\geq \int_{\Omega^*} \left( \int_{a-\delta}^{a} \frac{|w(x) - w(y)|^2}{|x - y|^{1+2s}} \, dy + \int_{b}^{b+\delta} \frac{|w(x) - w(y)|^2}{|x - y|^{1+2s}} \, dy \right) \, dx \\
= \int_{\Omega} w^2(x) \left( \int_{a-\delta}^{a} \frac{1}{|x - y|^{1+2s}} \, dy + \int_{b}^{b+\delta} \frac{1}{|x - y|^{1+2s}} \, dy \right) \, dx \\
= \frac{1}{2s} \int_{\Omega} w^2(x) (h_1(x) + h_2(x)) \, dx,
\]
where
\[
h_1(x) = (x-a)^{-2s} - (x-a+\delta)^{-2s}, \\
h_2(x) = (b-x)^{-2s} - (b-x+\delta)^{-2s}.
\]
Note that
\[
h'_1(x) = -2s ((x-a)^{-2s-1} - (x-a+\delta)^{-2s-1}) < 0, \\
h'_2(x) = 2s ((b-x)^{-2s-1} - (b-x+\delta)^{-2s-1}) > 0
\]
for \(x \in \Omega\). Then
\[
|w|^2_{H^s(\Omega^*)} \geq \frac{h_1(b) + h_2(a)}{2s} \|w\|_{L^2(\Omega)}^2.
\]

For (3.12), we can assume \(w \in C_0^\infty(\Omega)\), concluding by density arguments (see [1, Corollary 2.6] and [12, Theorem 4.1]). For any \(w \in C_0^\infty(\Omega)\) and \(\frac{1}{2} < s < 1\), we have the fractional Hardy inequality (see [26, Theorem 2.6])
\[
\int_{\Omega} \frac{w^2(x)}{\text{dist}(x, \partial\Omega)^{2s}} \, dx \leq C(\Omega, s) \int_{\Omega} \int_{a}^{b} \frac{|w(x) - w(y)|^2}{|x - y|^{1+2s}} \, dy \, dx,
\]
where \(\text{dist}(x, \partial\Omega) := \min \{|(x-a), (b-x)|\}\). By (2.3) and (3.15), we have
\[
|w|^2_{H^s(\Omega)} \geq \frac{1}{C(\Omega, s)} \int_{\Omega} \frac{w^2(x)}{\text{dist}(x, \partial\Omega)^{2s}} \, dx \\
\geq \frac{4^s}{C(\Omega, s)(b-a)^{2s}} \|w\|_{L^2(\Omega)}^2,
\]
where \(\text{dist}(x, \partial\Omega) \leq \frac{b-a}{2}\) has been used.

Remark 3.2. For \(s \in (0, \frac{1}{2})\), the fractional Hardy inequality will be (see [15, eq. 17])
\[
\int_{\Omega} \frac{w^2(x)}{\text{dist}(x, \partial\Omega)^{2s}} \, dx \leq C(\Omega, s) \left( \int_{\Omega} \int_{\Omega} \frac{|w(x) - w(y)|^2}{|x - y|^{1+2s}} \, dy \, dx + \|w\|_{L^2((a,b))}^2 \right).
\]
Note that unlike (3.15), \(\|w\|_{L^2((a,b))}^2\) must appear at this time. By using reduction to absurdity, one can show that (3.12) does not hold.
Proposition 3.3. If \( 0 < s < 1, w \in \widetilde{H}_0^s(\Omega) \), then for any given real number \( \delta > 0, \)

\[
|w|_{H^s(\mathbb{R})} \approx |w|_{H^s(\Omega^*)}.
\]

Moreover, for \( \frac{1}{2} < s < 1 \), one actually has

\[
|w|_{H^s(\mathbb{R})} \approx |w|_{H^s(\Omega)}.
\]

Proof. The equivalence of \( |w|_{H^s(\mathbb{R})} \) and \( |w|_{H^s(\Omega)} \) comes from the facts that \( |w|_{H^s(\Omega^*)} \leq |w|_{H^s(\mathbb{R})} \) and

\[
|w|^2_{H^s(\mathbb{R})} = \int_{\Omega^*} \int_{\Omega^*} \frac{|w(x) - w(y)|^2}{|x - y|^{1 + 2s}} \, dx \, dy + 2 \int_{\Omega^*} \int_{\Omega^*} \frac{w^2(x)}{|x - y|^{1 + 2s}} \, dx \, dy
\]

\[
\leq |w|^2_{H^s(\Omega^*)} + 2 \left( \int_{-\infty}^{a-\delta} \frac{1}{|x - y|^{1 + 2s}} + \int_{b+\delta}^{\infty} \frac{1}{|x - y|^{1 + 2s}} \right) \|w\|^2_{L^2(\Omega)}
\]

\[
\leq |w|^2_{H^s(\Omega^*)} + \frac{2\delta^{2s}}{s} \|w\|^2_{L^2(\Omega)}
\]

(3.20)

\[
\leq C \|w\|^2_{H^s(\Omega^*)},
\]

where (3.11) has been used in the last step.

When \( \frac{1}{2} < s < 1 \), the equivalence of \( |w|_{H^s(\mathbb{R})} \) and \( |w|_{H^s(\Omega)} \) comes from the facts that \( |w|_{H^s(\Omega^*)} \leq |w|_{H^s(\mathbb{R})} \) and

\[
|w|^2_{H^s(\mathbb{R})} \leq \int_{\Omega} \int_{\Omega} \frac{|w(x) - w(y)|^2}{|x - y|^{1 + 2s}} \, dx \, dy + 2 \int_{\Omega} \int_{\Omega \setminus \Omega} \frac{w^2(x)}{|x - y|^{1 + 2s}} \, dx \, dy
\]

\[
\leq |w|^2_{H^s(\Omega)} + \frac{2}{s} \int_{\Omega} \frac{w^2(x)}{\text{dist}(x, \partial \Omega)^{2s}} \, dx
\]

(3.21)

\[
\leq C \|w\|^2_{H^s(\Omega)}
\]

where (3.16) has been used in the last step.

Theorem 3.4. The weak formulation (3.2) is well-posed, and \( \|u\|_{H^{s/2}(\mathbb{R})} \lesssim \|f\|_{H^{-s/2}(\mathbb{R})} \).

Proof. For any \( p, v \in \tilde{H}^{\beta/2}_0(\Omega) \), combining the bilinear form (3.3), the Cauchy–Schwarz inequality, and \( 0 < e^{-\lambda|y|} \leq 1 \), we have

\[
|B(p, v)| \leq \frac{c_2}{2} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} e^{\lambda|x-y|}|x - y|^{1 + \beta} \, dx \, dy \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} e^{\lambda|x-y|}|x - y|^{1 + \beta} \, dx \, dy \right)^{\frac{1}{2}}
\]

(3.22)

\[
\leq \frac{c_2}{2} \|p\|_{H^{\beta/2}(\mathbb{R})} \|v\|_{H^{\beta/2}(\mathbb{R})}.
\]

By Propositions 3.2 and 3.3, we have

\[
B(p, p) \geq \frac{c_2}{2} \int_{\Omega^*} \int_{\Omega^*} e^{\lambda|x-y|}|x - y|^{1 + \beta} \, dx \, dy
\]

\[
\geq \frac{c_2}{2} e^{-\lambda(b-a+2\delta)} \|p\|^2_{H^{\beta/2}(\Omega^*)}
\]

(3.23)

\[
\geq C \|p\|^2_{H^{\beta/2}(\mathbb{R})},
\]
where (3.18) has been used in the last step. In addition,

\[(3.24) \quad |f, v| \leq \|f\|_{H^{\beta/2}(\Omega)} \|v\|_{H^{\beta/2}(\Omega)} \leq \|f\|_{H^{\beta/2}(\Omega)} \|v\|_{H^{\beta/2}(\mathbb{R})}.\]

Therefore, by the Lax–Milgram theorem, problem (3.2) has a unique solution. \(\Box\)

4. Riesz basis Galerkin approximation. In this section, we propose the Galerkin approximation of (3.2) with error analysis. Without loss of generality, in the following sections of this paper, we take \(\Omega := (0, 1)\).

4.1. Single scaling B-spline and multiscale Reisz basis functions. To develop the numerical approximation of (3.2), we need to choose the appropriate finite dimensional subspace of \(\overline{H}^{\beta/2}_{0} (\Omega)\). Here, we use the spline spaces introduced in [23]. Some important facts about them are as follows. Let \(M_m (m \in \mathbb{N}^+)\) be the B-spline of order \(m\); i.e., for \(x \in \mathbb{R}\),

\[M_1(x) = \chi_{[0, 1]} = \begin{cases} 1 & \text{if } x \in [0, 1], \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } M_m(x) = \int_0^1 M_{m-1}(x-t)dt,\]

where \(m \geq 2\). Then \(M_m\) is supported on \([0, m]\), \(M_m > 0\) for \(x \in (0, m)\), and \(M_m\) satisfies the following refinement equation [34]:

\[M_m(x) = 2^{1-m} \sum_{k=0}^{m} \binom{m}{k} M_m(2x-k).\]

Moreover, \(\mathcal{F}[M_m(x)](\xi) = (\frac{1-\xi}{2})^m\) and \(M_m \in \overline{H}^{\mu}_{0}(0, m)\) for \(0 < \mu < m - \frac{1}{2}\). In this paper, we focus on the cases of \(m = 1\) and \(m = 2\).

Let \(r = 1\) or \(r = 2\), and let \(n_0\) be the least integer such that \(2^{n_0} \geq 2r\). For \(j \in \mathbb{Z}\), denote

\[\phi_{n,j}^r(x) := 2^{n/2} M_r(2^n x - j), \quad x \in \mathbb{R}.\]

If \(n \geq n_0\) and \(j \in I_n := \{0, 1, \ldots, 2^n - r\}\), then \(\phi_{n,j}^r(x) = 0\) for \(x \in \mathbb{R}/[0, 1]\), and \(V_n = \text{span} \{\phi_{n,j}^r\}\) with \(\Phi^r_n := \{\phi_{n,j}^r : j \in I_n\}\) (because of (4.3), in the rest of this paper, we say \(V_n\) is generated by \(M_r(x)\)) is a subspace of \(\overline{H}^{\mu}_{0}(\Omega)\) for \(\mu \in (0, r - \frac{1}{2})\).

Moreover, the sequence \(\{V_n\}_{n \geq n_0}\) is a multiresolution analysis (MRA) of \(L^2(\Omega)\); i.e.,

- \(V_{n-1} \subset V_n\) for all \(n \geq n_0\);
- \(\bigcup_{n=n_0}^{\infty} V_n\) is dense in \(L^2(\Omega)\) (in fact, by [23, Theorem 5], \(\bigcup_{n=n_0}^{\infty} V_n\) is also dense in \(H^{\mu}(\Omega)\) for \(\mu \in (0, r - \frac{1}{2})\));
- for all \(n \geq n_0\) there exist constants \(0 < c_1 \leq c_2 < \infty\) independent of \(n\), such that the set \(\Phi^r_n\) forms a Riesz basis of \(V_n\); i.e., for all sequences \(d_n = \{d_{n,0}, d_{n,1}, \ldots, d_{n,2^n-r}\}\) we have

\[c_1 \sum_{j \in I_n} |d_{n,j}|^2 \leq \|d_{n,j} \phi_{n,j}^r\|_{L^2(\mathbb{R})}^2 \leq c_2 \sum_{j \in I_n} |d_{n,j}|^2.\]

For \(n \geq n_0\), the nest property of \(V_n\) allows one to construct the spaces \(W_n := V_{n+1} \cap V_n^\perp\) satisfying \(V_{n+1} = V_n \oplus W_n\). More precisely, let \(J_n := \{1, \ldots, 2^n\}\); for \(r = 1\), defining

\[\psi(x) := \frac{1}{2} (M_1(2x) - M_1(2x - 1)), \quad \psi_{n,j}^1(x) = 2^{n/2} \psi(2^n x - j + 1),\]
and \( \Psi_n^1 = \{ \psi_{n,j}^1(x), j \in J_n \} \), then \( W_n = \text{span} \{ \Psi_n^1 \} \); for \( r = 2 \), defining

\[
\psi(x) = \frac{1}{24} M_2(2x) - \frac{1}{4} M_2(2x - 1) + \frac{5}{12} M_2(2x - 2) - \frac{1}{4} M_2(2x - 3) + \frac{1}{24} M_2(2x - 4),
\]

(4.6)

\[
\psi_1(x) = \frac{3}{8} M_2(2x) - \frac{1}{4} M_2(2x - 1) + \frac{1}{24} M_2(2x - 2),
\]

(4.7)

\[
\psi_{n,j}^2(x) = \begin{cases} 
2^{n/2} \psi_1(2^n x), & j = 1, \\
2^{n/2} \psi_1(2^n x - j + 2), & j = 2, \ldots, 2^n - 1, \\
2^{n/2} \psi_1(2^n (1 - x)), & j = 2^n,
\end{cases}
\]

(4.8)

and \( \Psi_n^2 = \{ \psi_{n,j}^2(x), j \in J_n \} \), then \( W_n = \text{span} \{ \Psi_n^2 \} \).

Remark 4.1. Here, the cases for \( r = 1 \) and \( r = 2 \) are obtained by letting \( r = s = 1 \) and \( r = s = 2 \) in [23, pp. 179–181], respectively.

Because of the property of MRA, we have \( \widetilde{L}^2(\Omega) = V_{n_0} \oplus \oplus_{n=n_0} W_n \). Therefore, \( \Phi_{n_0} \cup \cup_{n} \Psi_n \) is a new basis of \( \widetilde{L}^2(\Omega) \) called multiscale basis.

Lemma 4.1 (Theorems 1 and 2 of [23]). For \( n \geq n_0 \), let \( \phi_{n,j}^r, j \in J_n \), be the functions as constructed above. Then

\[
\{ 2^{-n_0 \mu} \phi_{n_0,j}^r : j \in I_{n_0} \} \cup \cup_{l=n_0} \{ 2^{-l \mu} \psi_{l,j}^r : j \in J_l \}
\]

(4.9)

forms a Riesz basis of \( \widetilde{H}^\mu_0(\Omega) \) for any \( 0 \leq \mu < r - \frac{1}{2} \).

By Lemma 4.1, for \( \mu \in [0, r - \frac{1}{2}] \), it holds that

\[
\| \phi_{n_0,j}^r \|_{H^r(\Omega)} \simeq 2^{n_0 \mu}, \quad j \in I_{n_0}; \quad \| \psi_{l,j}^r \|_\Omega \simeq 2^{l \mu}, \quad j \in J_l, \ l \geq n_0.
\]

(4.10)

Then for \( l \geq n_0 \), combining (3.22) and (3.23), we know that the set

\[
\left\{ \phi_{n_0,j}^r : j \in I_{n_0} \right\} \cup \cup_{l=n_0} \left\{ \psi_{l,j}^r, \psi_{l,j}^r : \left| 2^{-l-1} \psi_{l,j}^r \right| \right\}
\]

(4.11)

also forms a Riesz basis of \( \widetilde{H}^{\beta/2}_0(\Omega) \). Here \( \beta \in (0, 1) \) for \( r = 1 \), and \( \beta \in (0, 2) \) for \( r = 2 \).

We take the subspace \( V_n \) as the approximation space of \( \widetilde{H}^{\beta/2}_0(\Omega) \), that is, find \( p_n \in V_n \) such that

\[
B(p_n, v_n) = (f, v_n) \quad \forall v_n \in V_n.
\]

(4.12)

Note that the space \( V_n \) generated by \( M_1(x) \) is a subspace of \( \widetilde{H}^{\beta/2}_0(\Omega) \) only for \( 0 < \beta < 1 \).

4.2. Convergence analysis. For any \( w \in \widetilde{L}^2(\Omega) \) and \( l \geq n_0 \), let \( P_l f \) be the orthogonal projection from \( \widetilde{L}^2(\Omega) \) to \( V_l \), i.e.,

\[
\langle P_l w, \phi_{l,j}^r \rangle = \langle w, \phi_{l,j}^r \rangle \quad \forall j \in I_l.
\]

(4.13)

We have the following proposition.
PROPOSITION 4.2. For \( w \in H^\alpha(\Omega) \cap \widetilde{H}^{\beta/2}(\Omega) \) (\( \alpha \geq \beta/2 \)) and the orthogonal projection operator \( P_n \) from \( \widetilde{L}^2(\Omega) \) to \( V_n \), it holds that

\[
\|w - P_n w\|_{H^{\beta/2}(\mathbb{R})} \lesssim 2^{-n(\alpha - \beta/2)} \|w\|_{H^\alpha(\Omega)},
\]

where \( 0 < \beta < 1 \) and \( \beta/2 \leq \alpha \leq 1 \) if \( V_n \) is generated from \( M_1(x) \), and \( 0 < \beta < 2 \) and \( \beta/2 \leq \alpha \leq 2 \) if \( V_n \) is generated from \( M_2(x) \).

Proof. For any \( n \geq n_0 \), by Remark 4.1, \( P_l \) actually is the projector \( P_n \) defined in [23, p. 197] with \( n = l \). Then it holds that [23, p. 197]

- \( \|P_l\| := \sup_{w \in L^2(\Omega), \|w\|_{L^2(\Omega)} \leq 1} \|P_l w\|_{L^2(\mathbb{R})} \) is bounded by a constant independent of \( l \);
- \( P_{l+1} w - P_l w \) lies in \( V_{l+1} \cap V_l^\perp = W_l = \text{span}\{\psi_{l,j}^r : j \in J_l\} \);
- \( \lim_{l \to \infty} \|P_l w - w\|_{L^2(\mathbb{R})} = 0 \) for any \( w \in \widetilde{L}^2(\Omega) \).

Thus, for any \( w \in \widetilde{L}^2(\Omega) \), there exist coefficients \( d_{n_0,j} \) (\( j \in I_{n_0} \)) and \( c_{l,j} \) (\( j \in J_l, l = n_0, n_0 + 1, \ldots \)) such that

\[
w = P_{n_0} w + \sum_{l=n_0}^\infty (P_{l+1} w - P_l w)
\]

(4.15)

\[
= \sum_{j \in I_{n_0}} d_{n_0,j} \phi_{n_0,j}^r + \sum_{l=n_0}^\infty \sum_{j \in J_l} c_{l,j} \psi_{l,j}^r.
\]

In addition, if \( w \in \widetilde{H}^\mu_0(\Omega), \mu \in [0, r - \frac{1}{2}) \), further, by (4.15), Lemma 4.1, and Definition 2.4, we have

\[
\|w\|^2_{H^\nu(\mathbb{R})} = \left\| \sum_{j \in I_{n_0}} d_{n_0,j} \phi_{n_0,j}^r + \sum_{l=n_0}^\infty \sum_{j \in J_l} c_{l,j} \psi_{l,j}^r \right\|_{H^\nu(\mathbb{R})}^2
\]

\[
\lesssim \sum_{j \in I_{n_0}} |2^{n_0 \mu} d_{n_0,j}|^2 + \sum_{l=n_0}^\infty \sum_{j \in J_l} |2^{l \mu} c_{l,j}|^2
\]

(4.16)

\[
\lesssim 2^{2n_0 \mu} \|P_{n_0} w\|^2_{L^2(\mathbb{R})} + \sum_{l=n_0}^\infty 2^{2l \mu} \|(P_{l+1} - P_l) w\|^2_{L^2(\mathbb{R})},
\]

where the fact that \( \{2^{-n_0 \mu} \phi_{n_0,j}^r : j \in I_{n_0}\} \cup \bigcup_{l=n_0}^\infty \{2^{-l \mu} \psi_{l,j}^r : j \in J_l\} \) forms a Riesz basis of \( \widetilde{H}_0^\mu(\Omega) \) has been used in the second step and the fact that \( \{\phi_{n_0,j}^r : j \in I_{n_0}\} \cup \bigcup_{l=n_0}^\infty \{\psi_{l,j}^r : j \in J_l\} \) forms a Riesz basis of \( \widetilde{L}_2(\Omega) \) has been used in the third step.

In the following, we will prove Proposition 4.2 in three steps.

First, for any \( w_m \in V_m \) with \( m \geq n_0 \), we show the inverse estimate

\[
\|w_m\|^2_{H^\nu(\mathbb{R})} \lesssim 2^{2m \mu} \|w_m\|^2_{L^2(\Omega)} \quad \forall \mu \in \left[0, r - \frac{1}{2}\right).
\]

(4.17)
In fact, since \( w_m = P_m w_m = P_{n_0} w_m + \sum_{l=n_0}^{m-1} (P_l - P_{l-1}) w_m \), we have
\[
\|w_m\|_{H^\alpha(\mathbb{R})}^2 \simeq 2^{2n_0\mu} \| P_{n_0} w_m \|_{L^2(\mathbb{R})}^2 + \sum_{l=n_0}^{m-1} 2^{2l\mu} \| (P_{l+1} - P_l) w_m \|_{L^2(\mathbb{R})}^2
\]
\[
\lesssim 2^{2n_0\mu} \| P_{n_0} w_m \|_{L^2(\mathbb{R})}^2 + \sum_{l=n_0}^{m-1} 2^{2l\mu} \left( \|P_{l+1} w_m\|^2 + \|P_l w_m\|^2 \right)
\]
\[
\lesssim \left( \sum_{l=n_0}^{m} 2^{2l\mu} \right) \|w_m\|_{L^2(\mathbb{R})}^2
\]
\[
\lesssim 2^{2m\mu} \|w_m\|_{L^2(\Omega)}^2,
\]
where (4.16) has been used in the first step, the uniform boundedness of \( \|P_l\| \) has been used in the third step, and \( \|w_m\|_{L^2(\mathbb{R})} = \|w_m\|_{L^2(\Omega)} \) has been used in the last step.

Second, we give the \( L^2 \) direct estimates. For any \( w \in H^{\alpha}(\Omega) \cap H^{\beta/2}_0(\Omega) \) \((\alpha \geq \beta/2)\) and \( m \geq n_0 \), if \( r = 1 \), it holds that [34, pp. 13–16]
\[
\|w - P_m w\|_{L^2(\Omega)} \leq 2^{-m\alpha} \|w\|_{H^{\alpha}(\Omega)}, \quad 0 \leq \alpha \leq 1;
\]
if \( r = 2 \), since the restriction \( V_m|\Omega \) actually is the space \( S_{d-j} \) with \( d = 2 \) and \( j = m \) in [39, Lemma 5], we have
\[
\|w - P_m w\|_{L^2(\Omega)} \leq \inf_{g \in V_m(\Omega)} \left( \|w - g\|_{L^2(\Omega)} + \|g - P_m w\|_{L^2(\Omega)} \right)
\]
\[
\leq \left( 1 + \sup_{l \geq n_0} \|P_l\| \right) \inf_{g \in V_m(\Omega)} \|w - g\|_{L^2(\Omega)}
\]
\[
\lesssim 2^{-m\alpha} \|w\|_{H^{\alpha}(\Omega)}, \quad 0 \leq \alpha \leq 2,
\]
\[
(4.19)
\]
where \( g = P_m g \) has been used in the second step.

Finally, we show the result in (4.14). By (4.15), for any \( w \in H^{\alpha}(\Omega) \cap H^{\beta/2}_0(\Omega) \), it holds that
\[
w = P_{n_0} w + \sum_{l=n_0}^{\infty} (P_{l+1} - P_l) w = P_{n_0} w + \sum_{l \geq n} (P_{l+1} - P_l) w.
\]
Then
\[
\|w - P_n w\|_{H^{\beta/2}(\mathbb{R})} \leq \sum_{l \geq n} \|P_{l+1} w - P_l w\|_{H^{\beta/2}(\mathbb{R})}
\]
\[
\lesssim \sum_{l \geq n} 2^{\beta/2} \|P_{l+1} w - P_l w\|_{L^2(\Omega)}
\]
\[
\lesssim \sum_{l \geq n} 2^{\beta/2} \left( \|P_{l+1} w - w\|_{L^2(\Omega)} + \|w - P_l w\|_{L^2(\Omega)} \right)
\]
\[
\lesssim \sum_{l \geq n} 2^{(\beta/2 - \alpha)l} \|w\|_{H^{\alpha}(\Omega)}
\]
\[
\lesssim 2^{-n(\alpha - \beta/2)} \|w\|_{H^{\alpha}(\Omega)},
\]
\[
(4.20)
\]
where \( P_{l+1} w - P_l w \in V_{l+1} \) and (4.17) has been used in the second step, and (4.18) (for \( r = 1 \)) and (4.19) (for \( r = 2 \)) has been used in the fourth step.
Theorem 4.3. Let $p \in H^\mu(\Omega) \cap \tilde{H}^{\beta/2}(\Omega) (\mu \geq \beta/2)$ be the exact solution of (3.2) and $p_n \in V_n$ be the approximation solution of (4.12). Then

$$\|p - p_n\|_{H^{\beta/2}(\mathbb{R})} \lesssim 2^{-n(\min\{\mu, r\} - \beta/2)} \|p\|_{H^\mu(\Omega)},$$

where $\beta \in (0, 1)$ if $V_n$ is generated from $M_1(x)$, and $\beta \in (0, 2)$ if $V_n$ is generated from $M_2(x)$.

Proof. Using the standard argument technique for Céa's lemma (see Theorem (2.8.1) of [6]), we have

$$\|p - p_n\|_{H^{\beta/2}(\mathbb{R})} \lesssim \inf_{v \in V_n} \|p - v\|_{H^{\beta/2}(\mathbb{R})}.$$  

Then the desired result is a direct conclusion of Proposition 4.2.

5. Implementation details. It is easy to check that $V_n = V_n \oplus \bigoplus_{j=0}^{n-1} W_j$. Then $V_n$ has two types of basis functions: the single scaling B-spline basis functions $\Phi^r_n$ and the multiscale Reisz basis functions

$$\bar{\Psi}^r_n := \left\{ \frac{\phi_{n,j}}{\sqrt{B(\phi_{n,j})}} : j \in I_{n_0} \right\} \cup \bigcup_{n_0 = 0}^{n-1} \left\{ \frac{\psi_{n,j}^1}{\sqrt{B(\psi_{n,j}^1, \psi_{n,j}^2)}} \left| j \right|^{\frac{n}{2}} , \frac{\psi_{n,j}^2}{\sqrt{B(\psi_{n,j}^1, \psi_{n,j}^2)}} \right\}.$$  

5.1. Computing the stiffness matrix. This subsection focuses on generating the stiffness matrix under the single scaling basis functions $\Phi^r_n$. Making use of the fact that $\Phi^r_n$ are obtained from the translations of a single function $2^{n/2} M_r(2^n, x)$, we have the following proposition.

Proposition 5.1. Matrix $A := B(\Phi^r_n, \Phi^r_n)$ is a symmetric Toeplitz matrix.

Proof. By (4.3) and the properties of Fourier transform, it holds that

$$\mathcal{F} \left[ \phi^r_{n,j_1} \right] (\xi) = 2^{-\frac{n}{2}} e^{-i \frac{j_1}{2^n} \xi} \mathcal{F} \left[ M_r(x) \right] \left( \frac{\xi}{2^n} \right),$$

$$\mathcal{F} \left[ \phi^r_{n,j_2} \right] (\xi) = 2^{-\frac{n}{2}} e^{-i \frac{j_2}{2^n} \xi} \mathcal{F} \left[ M_r(x) \right] \left( \frac{\xi}{2^n} \right).$$

Thus, for $\lambda = 0$, by (3.5) we have

$$B \left( \phi^r_{n,j_1}, \phi^r_{n,j_2} \right) = \int_{\mathbb{R}} \left| \xi \right|^{\beta} \left| \mathcal{F} \left[ \phi_{n,j_1}(x) \right] \mathcal{F} \left[ \phi_{n,j_2}(x) \right] \right| d\xi$$

$$= \frac{1}{2^n} \int_{\mathbb{R}} \left| \xi \right|^{\beta} e^{i \frac{j_2 - j_1}{2^n} \xi} \left| \mathcal{F} \left[ M_r(x) \right] \left( \frac{\xi}{2^n} \right) \right|^2 d\xi$$

$$= \frac{2^{r+1}}{2^n} \int_{0}^{\infty} \xi^{\beta} \cos \left( \frac{j_2 - j_1}{2^n} \xi \right) \frac{1 - \cos(\xi/2^n)}{(\xi/2^n)^2} \left( \frac{1 - \cos(\xi/2^n)}{(\xi/2^n)^2} \right)^{r} d\xi;$$

similarly, for $\lambda > 0$, by (3.7) we have

$$B \left( \phi^r_{n,j_1}, \phi^r_{n,j_2} \right) = \int_{\mathbb{R}} G(\lambda, \xi, \beta) e^{i \frac{j_2 - j_1}{2^n} \xi} \left| \mathcal{F} \left[ M_r(x) \right] \left( \frac{\xi}{2^n} \right) \right|^2 d\xi$$

$$= \frac{2^{r+1}}{2^n} \int_{0}^{\infty} G(\lambda, \xi, \beta) \cos \left( \frac{j_2 - j_1}{2^n} \xi \right) \left( \frac{1 - \cos(\xi/2^n)}{(\xi/2^n)^2} \right)^{r} d\xi.$$
where in the last step the fact that $G(\lambda, \xi, \beta)$ is an even function w.r.t. $\xi$ has been used. The desired result follows from that $B(\Phi_{n,j_1}^r, \Phi_{n,j_2}^r)$ remains constant if $|j_1 - j_2|$ is a constant.

Therefore, we only need to calculate and store the first row of matrix $B(\Phi_{n,1}^r, \Phi_{n,j}^r)$. Let $h = 2^{-n}, \zeta(y) = e^{-\lambda y}y^{1-\beta}$, and $\kappa = \frac{\lambda^2}{e\beta}$. By using the Fubini theorem, for the entries of $B(\Phi_{n,1}^r, \Phi_{n,j}^r)$ we have

\begin{equation}
B(\Phi_{n,1}^r, \Phi_{n,j}^r)_{0,0} = \frac{2c_2}{h} \int_0^h y\zeta(y)dy + 2c_3 \int_h^\infty \zeta(y)dy,
\end{equation}

\begin{equation}
B(\Phi_{n,1}^r, \Phi_{n,j}^r)_{0,j} = \frac{c_3}{h} \int_{(j-1)h}^{jh} ((j-1)h - y) \zeta(y)dy
\end{equation}

\begin{equation}
+ \frac{c_3}{h} \int_{jh}^{(j+1)h} (y - (j+1)h) \zeta(y)dy,
\end{equation}

where $j = 1, 2, \ldots, 2^n - 1$; for the entries of $B(\Phi_{n,1}^r, \Phi_{n,j}^r)$ we have

\begin{equation}
kB(\Phi_{n,1}^r, \Phi_{n,j}^r)_{0,0} = \int_0^h (2h - y)y^2 \zeta(y)dy + \frac{4h^3}{3} \int_{2h}^\infty \zeta(y)dy
\end{equation}

\begin{equation}
+ \int_h^{2h} \left( \frac{y^3}{3} - 2hy^2 + 4h^2y - \frac{4h^3}{3} \right) \zeta(y)dy,
\end{equation}

\begin{equation}
kB(\Phi_{n,1}^r, \Phi_{n,j}^r)_{0,1} = \int_0^h 2y - 3h y^2 \zeta(y)dy + \frac{3h^3}{6} \int_{2h}^{3h} (y - 3h)^3 \zeta(y)dy
\end{equation}

\begin{equation}
- \int_h^{2h} \left( \frac{y^3}{2} - \frac{5hy^2}{2} + \frac{7h^2y}{2} - \frac{7h^3}{6} \right) \zeta(y)dy + \frac{h^3}{3} \int_{2h}^\infty \zeta(y)dy,
\end{equation}

\begin{equation}
kB(\Phi_{n,1}^r, \Phi_{n,j}^r)_{0,j} = \frac{1}{6} \int_{jh}^{(j+1)h} b_2(y) \zeta(y)dy + \frac{1}{6} \int_{(j+1)h}^{(j+2)h} b_2(y) \zeta(y)dy
\end{equation}

\begin{equation}
+ \frac{1}{6} \int_{(j+1)h}^{(j+2)h} b_1(y) \zeta(y)dy
\end{equation}

for $j = 2, 3, \ldots, 2^n - 2$, where

\begin{align*}
b_1(y) &= 3y^3 + (6 - 9j) hy^2 + 3j (3j - 4) h^2y - (3j^3 - 6j^2 + 4) h^3,
b_2(y) &= -3y^3 + (6 + 9j) hy^2 - 3j (3j + 4) h^2y + (3j^3 + 6j^2 - 4) h^3.
\end{align*}

If $\lambda = 0$, all the integrals above can be calculated exactly. When $\lambda \neq 0$, we can calculate them numerically with some regularization techniques. For example, for $\beta \neq 1$, using integration by parts, we can first rewrite the $\int_0^h y\zeta(y)dy$ in (5.4) as

\begin{equation}
\Gamma(1 - \beta) \left( \sum_{l=1}^{K-1} e^{-\lambda h} \lambda^{l-1} h^{l-\beta} / \Gamma(l + 1 - \beta) + \frac{\lambda^{K-1}}{\Gamma(K - \beta)} \int_0^h e^{-\lambda y} y^{K-1-\beta} dy \right)
\end{equation}

and then calculate

\begin{equation}
\int_0^h e^{-\lambda y} y^{K-1-\beta} dy = \left( \frac{h}{2} \right)^{K-\beta} \int_{-1}^1 e^{-\lambda h (1+\eta)} (1 + \eta)^{K-1-\beta} d\eta
\end{equation}
by the Gauss–Jacobi quadrature with the weight function \((1 - \eta)^p(1 + \eta)^{1-\beta}\) [32]; we can first rewrite the \(\int_{\eta}^{\infty} \zeta(y)dy\) in (5.4) as
\[
\sum_{l=1}^{2} \frac{-\Gamma(-\beta)}{e^{\lambda h \beta} \Gamma(1-\beta)} \left( \frac{(\lambda h)^{1-\beta}}{\Gamma(l-\beta)} \right) + \lambda^2 \Gamma(-\beta) - \frac{\lambda^2 \Gamma(-\beta)}{\Gamma(2-\beta)} \int_{0}^{h} y^{1-\beta} e^{-\lambda y}dy,
\]
and then calculate \(\int_{0}^{h} y^{1-\beta} e^{-\lambda y}dy\) with techniques similar to (5.9). For \(\beta = 1\), using integration by parts, we can first rewrite the \(\int_{\eta}^{\infty} \zeta(y)dy\) in (5.6) as
\[
\frac{e^{-2\lambda h}}{2h} - \lambda \int_{2h}^{\infty} e^{-y} y^{-1}dy
\]
and then calculate the exponential integral \(\int_{2h}^{\infty} e^{-y} y^{-1}dy\) with the series expansion representation in [3, eq. 5.1.11].

### 5.2. Condition number and preconditioning.
This subsection focuses on reducing the condition number by using the multiscale basis. We first examine the condition number of the matrix \(A\). Let \(\lambda_{\max}(A)\) and \(\lambda_{\min}(A)\) be the maximal and minimal eigenvalues of \(A\), respectively. Then \(\text{Cond}(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}\). Let \(d_n = (d_{n,0}, d_{n,1}, \ldots, d_{n,2^n-r})^T\). By Theorem 1.2 of [8], it holds that
\[
\lambda_{\min}(A) = \inf_{d_n \neq 0} \frac{(d_n, Ad_n)}{(d_n, d_n)}, \quad \lambda_{\max}(A) = \sup_{d_n \neq 0} \frac{(d_n, Ad_n)}{(d_n, d_n)}.
\]
First, it holds that \(d_n^T Ad_n = B(\Phi_n, \Phi_n) \simeq \|\Phi_n\|^2_{H^{\beta/2}(\Omega)}\), where the latter is because of (3.22) and (3.23). By (4.4) and (4.17), we have
\[
1 \leq \frac{B(\Phi_n^T, \Phi_n)}{d_n^T d_n} \simeq \frac{\|\Phi_n\|^2}{\|\Phi_n\|^2_{L^2(\Omega)}} \simeq 2^{n \beta}.
\]
Hence, \(\text{Cond}(A) \lesssim 2^{n \beta}\) for \(r = 1\) and \(r = 2\).

Second, if \(0 < \beta < 1\) and \(r = 1\), we have (5.4). Then by choosing \(d_n = (1, 0, 0, \ldots, 0)\), it holds that
\[
\lambda_{\max}(A) \geq B(\Phi_{\eta,0}^1, \Phi_{\eta,0}^1) = B(\Phi_n^1, \Phi_n^1) = \frac{c_\beta e^{-\beta \eta}}{\beta \eta h^\beta}.
\]
In addition, \(\chi_{[0,1]} = \Phi_n^1 d \in V_n\) with \(d = (\frac{1}{2^{n/2}}, \frac{1}{2^{n/2}}, \ldots, \frac{1}{2^{n/2}})^T\) and
\[
B(\chi_{[0,1]}, \chi_{[0,1]}) \leq 2 \int_{\Omega} \int_{\Omega} \frac{1}{|x - y|^{1+\beta}}dydx = \frac{4}{\beta(1-\beta)}.
\]
Thus, \(\lambda_{\min}(A) \leq \frac{B(\chi_{[0,1]}, \chi_{[0,1]})}{d_n^T d_n} \lesssim \frac{4}{\beta(1-\beta)}\). Hence, \(h^{-\beta} = 2^\beta \lesssim \text{Cond}(A)\).

Finally, if \(0 < \beta < 2\) and \(r = 2\), we have (5.6). Then
\[
\lambda_{\max}(A) \geq B(\Phi_n^2, \Phi_n^2) = B(\Phi_n^2, \Phi_n^2)_{\eta,0} \geq \frac{1}{\kappa} \int_{0}^{h} (2h-y)^{\beta} \zeta(y)dy
\]
\[
\geq \frac{h e^{-\lambda}}{\kappa} \int_{0}^{h} y^{1-\beta}dy = \frac{c_\beta e^{-\lambda}}{2-\beta} h^{-\beta}.
\]

\begin{align*}
\text{Proposition 5.2.} & \quad \text{The condition number of matrix } A \text{ satisfies } \text{Cond}(A) \simeq 2^{n \beta}. \\
\text{Proof.} & \quad \text{Let } \lambda_{\max}(A) \text{ and } \lambda_{\min}(A) \text{ be the maximal and minimal eigenvalues of } A, \\
& \quad \text{respectively. Then } \text{Cond}(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}. \text{ Let } d_n = (d_{n,0}, d_{n,1}, \ldots, d_{n,2^n-r})^T. \text{ By Theorem 1.2 of [8], it holds that }
\end{align*}
where \( \frac{h^2}{4} - 2hy^2 + 4h^2y - \frac{4h^3}{3} > 0 \) with \( y \in [h, 2h] \) has been used in the second step. In addition, note that

\[
\theta(x) = \begin{cases} 
2x, & x \in \left[0, \frac{1}{2}\right] \\
2 - 2x, & x \in \left[\frac{1}{2}, 1\right] \\
0, & \text{else}
\end{cases} = \sum_{j \in I_n} \frac{\theta((j + 1)/2^n)}{2^{n/2}} \phi_{n,j}^2 \in V_n;
\]

and by (3.22) and (3.18), we have

\[
B(\theta(x), \theta(x)) \lesssim |\theta(x)|_{H^{\beta/2}(\Omega)} \lesssim \|\theta(x)\|_{H^1(\Omega)} \lesssim 1.
\]

Let

\[
d = \frac{1}{2^n} \left( \theta \left( \frac{1}{2^n} \right), \theta \left( \frac{2}{2^n} \right), \ldots, \theta \left( \frac{2^n - 1}{2^n} \right) \right)^T.
\]

By (4.4), it holds that \( d^T d \simeq 1 \). Thus, \( \lambda_{\min}(A) \leq \frac{B(\theta(x), \theta(x))}{d^T d} \lesssim 1 \). Hence, \( 2^{\beta}\simeq \text{Cond}(A) \).

In the following, we consider the stiffness matrix \( \tilde{B} := B(\tilde{\Psi}_n^r, \tilde{\Psi}_n^r) \), where \( \tilde{\Psi}_n^r \) is the multiscale basis functions given in (5.1).

**Proposition 5.3.** The condition number of matrix \( \tilde{B} \) satisfies \( \text{Cond}(\tilde{B}) \lesssim 1 \).

**Proof.** Let

\[
\tilde{c}_n := (d_{n_0}^T, c_{n_0}^T, c_{n_0+1}^T, \ldots, c_l^T, \ldots, c_{n-1}^T)^T
\]

with \( c_l = (c_l,1, c_l,2, \ldots, c_l,n_l)^T \) for \( l = n_0, \ldots, n - 1 \). It holds that

\[
\tilde{c}_n^T \tilde{B} \tilde{c}_n = B(\tilde{\Psi}_n^r, \tilde{\Psi}_n^r) \simeq \|\tilde{\Psi}_n^r \tilde{c}_n\|_{H^{\beta/2}(\Omega)}^2 \simeq \tilde{c}_n^T \tilde{c}_n,
\]

where the second term comes from (3.22) and (3.23), and the last term comes from the fact that (4.11) forms a Riesz basis of \( H_0^{\beta/2}(\Omega) \). Thus, we complete the proof.

Unlike the single scaling basis functions \( \Psi_n^r \), it can be seen from (5.1) that \( \tilde{\Psi}_n^r \) cannot be obtained by the translation of a function; thus matrix \( \tilde{B} \) does not satisfy Proposition 5.1. Calculating \( \tilde{B} \) directly is quite tedious, and so we examine its relationship with \( A \) instead. In fact, since both \( \Phi_n^r \) and \( \tilde{\Psi}_n^r \) are the basis functions of \( V_n \), there exists a matrix \( \tilde{M}_n^r \) such that

\[
\tilde{\Psi}_n^r = \Phi_n^r \tilde{M}_n^r.
\]

Note that here \( \Phi_n^r \) and \( \tilde{\Psi}_n^r \) have been regarded as row vectors. We can give \( \tilde{M}_n^r \) explicitly. First, for \( l \geq n_0 \), by (4.2), (4.3), and (4.5)–(4.8), there exist matrices \( M_{l,0}^r \) and \( M_{l,1}^r \) such that

\[
\Phi_l^r = \Phi_{l+1}^r M_{l,0}^r, \quad \Psi_l^r = \Phi_{l+1}^r M_{l,1}^r.
\]

Thus, letting \( M_l^r = (M_{l,0}^r, M_{l,1}^r) \), we have

\[
(\Phi_{n_0}^r, \Psi_{n_0}^r, \phi_{n_0+1}^r, \Psi_{n_0+2}^r, \ldots, \phi_{n-2}^r, \Psi_{n-1}^r)
\]

\[
= (\phi_{n_0+1}^r, \Psi_{n_0+1}^r, \phi_{n_0+2}^r, \Psi_{n_0+2}^r, \ldots, \phi_{n-2}^r, \Psi_{n-1}^r)
\]

\[
\begin{pmatrix}
M_{n_0}^r & 0 \\
0 & I_{n_0}^r
\end{pmatrix}
\]

\[
= \Phi_n^r M_r^r,
\]
where

\begin{equation}
M_r = M_{n-1}^r \begin{pmatrix} M_{n-2}^r & \mathbf{0} \\ \mathbf{0} & I_{n-2}^r \end{pmatrix} \begin{pmatrix} M_{n-3}^r & 0 \\ 0 & I_{n-3}^r \end{pmatrix} \cdots \begin{pmatrix} M_{n_0}^r & 0 \\ 0 & I_{n_0}^r \end{pmatrix}
\end{equation}

with $I_r^n$, $l = n_0, n_1, \ldots, n - 2$, being $(2^n - 2^l + 1)$st order identity matrices. Second, define a diagonal matrix $\tilde{D}_n^r$ as

\[
\text{diag} \left( a_{n0}^r, \ldots, a_{n0}^r, b_{n0,1}^r, b_{n0,2}^r, \ldots, b_{n0,1}^r, b_{n0,1}^r \right),
\]

\[
b_{n0+1,1}^r, b_{n0+1,2}^r, \ldots, b_{n0+1,2}^r, b_{n0+1,1}^r, \ldots, b_{n0+1,2}^r, b_{n0+1,1}^r
\]

with $a_{n0}^r = B(\Phi_{n0}^r, \Phi_{n0}^r)^{-\frac{1}{2}}$, and $b_{n0}^r = B(\Phi_{n0}^r, \Phi_{n0}^r)^{-\frac{1}{2}}$, $b_{n0}^r = B(\Phi_{n0}^r, \Phi_{n0}^r)^{-\frac{1}{2}}$ for $l = n_0, n_0 + 1, \ldots, n - 1$. Note that $a_{n0}^r$ and $b_{n0}^r$ can be calculated by the relations (4.5)–(4.8), (4.2), Proposition 5.1, and the formulae (5.4)–(5.8). For example, by

\[
\psi_{i,2} = \frac{1}{2} \left( 2^{n/2} L_1(2^{n+1} x - 2) - 2^{n/2} M_1(2^n x - 3) \right) \frac{1}{2 \sqrt{2}} (\phi_{i+1,2}^1(x) - \phi_{i+1,3}^1(x)),
\]

we obtain

\[
b_{i,2} = \frac{1}{\sqrt{\left( \frac{1}{2 \sqrt{2}} \right)^2 - \frac{1}{2 \sqrt{2}}} \left( B \left( \Phi_{i+1,1}^1, \Phi_{i+1}^1 \right)_{2,2} + B \left( \Phi_{i+1,1}^1, \Phi_{i+1}^1 \right)_{2,3} \right) \left( B \left( \Phi_{i+1,1}^1, \Phi_{i+1}^1 \right)_{3,2} + B \left( \Phi_{i+1,1}^1, \Phi_{i+1}^1 \right)_{3,3} \right)}{\sqrt{\left( \frac{1}{2 \sqrt{2}} \right)^2 - \frac{1}{2 \sqrt{2}}} \left( B \left( \Phi_{i+1,1}^1, \Phi_{i+1}^1 \right)_{0,0} + B \left( \Phi_{i+1,1}^1, \Phi_{i+1}^1 \right)_{0,1} \right) \left( B \left( \Phi_{i+1,1}^1, \Phi_{i+1}^1 \right)_{0,0} + B \left( \Phi_{i+1,1}^1, \Phi_{i+1}^1 \right)_{0,1} \right)}
\]

Then, combining this with the definition of $\tilde{\Psi}_n^r$ in (5.1), we have

\begin{equation}
\tilde{M}_n^r = M^r \tilde{D}_n^r.
\end{equation}

Thus

\begin{equation}
\tilde{B} = \left( \Phi_n^r \tilde{M}_n^r, \Phi_n^r \tilde{M}_n^r \right) = \left( \tilde{M}_n^r \right)^T B(\Phi_n^r, \Phi_n^r) \tilde{M}_n^r = \tilde{D}_n^r (M^r)^T A M^r \tilde{D}_n^r.
\end{equation}

In practice, we do not need to generate the stiffness matrix $\tilde{B}$ explicitly. The purpose of introducing the multiscale basis functions $\tilde{\Psi}_n^r$ usually is to obtain the preconditioning matrix of $A$, due to its increasing condition number. More clearly, by denoting $p_n = \Phi_n^r d_n$ and $f_n = (f_n, (\Phi_n^r)^T)$, the matrix equation for (4.12) is

\begin{equation}
A d_n = f_n,
\end{equation}

which can be equivalently written as

\begin{equation}
\left( \tilde{D}_n^r (M^r)^T A M^r \tilde{D}_n^r \right) (M^r \tilde{D}_n^r)^{-1} d_n = \tilde{D}_n^r (M^r)^T f_n.
\end{equation}
We call (5.27) the preconditioned form of the system (5.26). Since the condition number of matrix $\mathbf{B}$ is uniformly bounded, if the conjugate gradient (CG) method is used, the iteration number will be independent of the size of $\mathbf{d}_n$ [8]. The CG method for (5.27) can be performed like the programs provided in [8], where in each iteration the matrix vector products like $\mathbf{M}^\top \mathbf{e}, (\mathbf{M}^\top)^2, \mathbf{D}^\top \mathbf{e},$ and $\mathbf{A} \mathbf{e}$ are needed (here $\mathbf{e} \in \mathbb{R}^N$ represents an arbitrary vector of length $N = 2^n - r + 1$), but in fact, they can be performed effectively with the total cost $O(N \log N)$. More specifically,

- $\mathbf{D}^\top \mathbf{e}$ is a diagonal matrix, which can be generated with the cost $O(\log_2(N))$ and stored with the cost $O(N)$;
- $\mathbf{M}^\top$ and $(\mathbf{M}^\top)^2$ are usually called the fast wavelet transform (FWT) matrices; they do not need to be prestored or assembled, and $\mathbf{M}^\top \mathbf{e}$ and $(\mathbf{M}^\top)^2 \mathbf{e}$ can be implemented following a process like [9, pp. 431], with the cost $O(N)$;
- $\mathbf{A}$ is a Toeplitz matrix, so the storage cost is $O(N)$, and by the discrete FFT, the computation cost for $\mathbf{A} \mathbf{e}$ is $O(N \log N)$ [8, pp. 11–12, 35].

6. Weak solutions for problems with generalized Dirichlet type boundary condition. Like the existing literature on variational numerical methods for non-local diffusion problems [2, 14, 13, 16, 37, 33], we have discussed numerical methods for (1.1) with the homogeneous boundary condition in the previous sections. In this section, we consider the problem with generalized Dirichlet type boundary condition, i.e.,

\[
\begin{align*}
-(\Delta + \lambda)^{\beta/2} p(x) &= f(x), & x \in \Omega, \\
p(x) &= g(x), & x \in \mathbb{R} \setminus \Omega.
\end{align*}
\]

Introducing a function $\eta(x)$ defined in $\mathbb{R}$ such that $\eta(x) = g(x)$ in $\mathbb{R} \setminus \Omega$, the weak solution (6.1) can be defined as follows: find $p = u + \eta$ such that $u \in \tilde{H}_0^\beta/2(\Omega)$ and

\[
B(u, v) = (f, v) - B(\eta, v) \quad \forall v \in \tilde{H}_0^\beta/2(\Omega).
\]

**Theorem 6.1.** Assume that $f \in H^{-\beta/2}(\Omega)$ and there exists a function $\eta(x)$ satisfying $\int_{\Omega} \int_{\mathbb{R}} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{1+\beta}} \, dx \, dy < \infty$. Then (6.1) has a unique weak solution.

**Proof.** According to the proof of Theorem 3.4, it remains to show that $B(\eta, \cdot)$ is a bounded linear functional on $\tilde{H}_0^\beta/2(\Omega)$. In fact, for any $v \in \tilde{H}_0^\beta/2(\Omega)$, it holds that

\[
\begin{align*}
|B(\eta, v)| &= \frac{c_2}{2} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\eta(x) - \eta(y))(v(x) - v(y))}{e^{\lambda|x-y|}|x-y|^{1+\beta}} \, dx \, dy \right| \\
&= \frac{c_2}{2} \left( \int_{\mathbb{R}} \int_{\Omega} \frac{(\eta(x) - \eta(y))(v(x) - v(y))}{e^{\lambda|x-y|}|x-y|^{1+\beta}} \, dx \, dy \\
&\quad + \int_{\Omega} \int_{\mathbb{R} \setminus \Omega} \frac{(\eta(x) - \eta(y))(v(x) - v(y))}{e^{\lambda|x-y|}|x-y|^{1+\beta}} \, dx \, dy \right).
\end{align*}
\]

by the triangle inequality and the Cauchy–Schwarz inequality we have

\[
|B(\eta, v)| \leq \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(|\eta(x) - \eta(y)|^2)}{|x-y|^{1+\beta}} \, dx \, dy \right)^{1/2} \|v\|_{H^{\beta/2}(\mathbb{R})}.
\]

Thus (6.2) has a unique solution $u(x)$. 
Further, let $\eta, \tilde{\eta}$ be two functions satisfying $\eta = \tilde{\eta} = g$ in $\mathbb{R}\setminus\Omega$, and $p$ and $\tilde{p}$ are the corresponding weak solutions. Then
\begin{equation}
B(p - \tilde{p}, v) = 0 \quad \forall v \in H^3_0(\Omega).
\end{equation}
Choosing $v = p - \tilde{p}$ in (6.5) yields that $p = \tilde{p}$, which means that the weak solution actually depends only on the values of $g$ in $\mathbb{R}\setminus\Omega$. Therefore, (6.1) has a unique weak solution $p(x) = u(x) + \eta(x)$.

For the second order elliptic problem, since $\eta(x) = p(0)(1 - x) + p(1)x$ satisfies $\eta(0) = p(0), \eta(1) = p(1)$, and $p(x) - \eta(x) \in H^3_0(\Omega)$, one can easily translate the problem with the general Dirichlet boundary condition to the problem with homogeneous Dirichlet type boundary condition (the existence of $\eta(x)$ can also be ensured by the trace theorem). However, for the nonlocal problems with nonlocal boundary conditions, to the best of our knowledge, there are no general methods to find the suitable $\eta(x)$ and no general theory to ensure the existence of $\eta(x)$. Here, we point out that if $g(x) \in L^\infty(\mathbb{R}\setminus\Omega)$, one can take $\eta(x)$ by the following ways to ensure $\int_\Omega \int_R \frac{(\eta(x) - \eta(y))^2}{|x - y|^{1+\beta}} \, dx \, dy < \infty$:
1. If $0 < \beta < 1$, one only needs to extend $g(x)$ such that $\eta(0) = g(0), \eta(1) = g(1)$, $\eta(x) \in H^{3/2}(\Omega)$, and $\|\eta\|_{L^\infty(\Omega)} < \infty$. In particular, the function $S_1(x) = g(0)(1 - x) + g(1)x$ can be used as the $\eta(x)$ for $x \in \Omega$.
2. If there exist $a_1 < b_1$ such that $g(x)$ is one-time continuously differentiable on $[a_1, 0]$ and $[1, b_1]$, one only needs to extend $g(x)$ such that $\eta(x)$ is one-time continuously differentiable on $[0, 1]$. In particular, the spline polynomial $S_3(x)$ satisfying $S_3(0) = g(0), S_3'(0) = g'(0), S_3(1) = g(1), S_3'(1) = g'(1)$ can be used as the $\eta(x)$ for $x \in \Omega$.

In fact, for case 1,
\begin{equation}
\int_\Omega \int_R \frac{(\eta(x) - \eta(y))^2}{|x - y|^{1+\beta}} \, dx \, dy \leq \int_\Omega \int_R \frac{(\eta(x) - \eta(y))^2}{|x - y|^{1+\beta}} \, dx \, dy
+ 2 \left( \|g(x)\|_{L^\infty(\mathbb{R}\setminus\Omega)}^2 + \|\eta(x)\|_{L^\infty(\Omega)}^2 \right) \int_\Omega \int_R \frac{1}{|x - y|^{1+\beta}} \, dx \, dy < \infty.
\end{equation}

For case 2,
\begin{align*}
\int_\Omega \int_R \frac{(\eta(x) - \eta(y))^2}{|x - y|^{1+\beta}} \, dx \, dy &\leq \int_\Omega \int_{a_1}^{b_1} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{1+\beta}} \, dx \, dy \\
+ 2 \left( \|g(x)\|_{L^\infty(\mathbb{R}\setminus\Omega)}^2 + \|\eta(x)\|_{L^\infty(\Omega)}^2 \right) \int_\Omega \int_R \frac{1}{|x - y|^{1+\beta}} \, dx \, dy;
\end{align*}
and by the mean value theorem
\begin{align*}
\int_\Omega \int_{a_1}^{b_1} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{1+\beta}} \, dx \, dy &\leq \|\eta\|_{L^\infty(\mathbb{R}\setminus\Omega)}^2 \int_\Omega \int_{a_1}^{b_1} \frac{1}{|x - y|^{1+\beta}} \, dx \, dy < \infty.
\end{align*}

Thus
\begin{equation}
\int_\Omega \int_R \frac{(\eta(x) - \eta(y))^2}{|x - y|^{1+\beta}} \, dx \, dy < \infty.
\end{equation}

Remark 6.1. In particular, if $f(x) = 0$ and $g(x) = 1$ in (6.1), then $S_1(x)$ and $S_3(x)$ will be 1. Thus, one can choose $\eta(x) = 1$ for $x \in \mathbb{R}$, and the weak formulation (6.2) reduces to $B(u, v) = 0$, which admits a unique solution $u(x) = 0$. Therefore, (6.1) has a unique solution $p = \eta(x) + u(x) = 1$. 

Theorem 6.2. Let \( u = p - \eta \) with \( u \in H^\alpha(\Omega) \cap \tilde{H}^{\beta/2}(\Omega) \) (\( \mu \geq \beta/2 \)) be the exact solution of (6.2) and \( u_n = p_n - \eta \in V_n \) be the Galerkin approximation solution. Then
\[
\| p - p_n \|_{H^{\beta/2}(\mathbb{R})} \leq 2^{-n(\min\{\mu, r\} - \beta/2)} \| p - \eta \|_{H^{\alpha}(\Omega)},
\]
where \( \beta \in (0, 1) \) if \( V_n \) is generated from \( M_1(x) \), and \( \beta \in (0, 2) \) if \( V_n \) is generated from \( M_2(x) \).

Proof. Note that \( p - p_n = u - u_n \). Then the desired result is a direct conclusion of Theorem 4.3.

7. Numerical experiments. In this part, we set \( \Omega = (0, 1) \). The data under "\( H^{3/2}\)-Err" and "\( L^2\)-Err" are the errors in the norms \( \| \cdot \|_{H^{3/2}(\mathbb{R})} \) and \( \| \cdot \|_{L^2(\mathbb{R})} \), respectively. If the true solution is unknown, the "\( H^{3/2}\)-Err" and "\( L^2\)-Err" are, respectively, replaced by "\( \tilde{H}^{3/2}\)-Err" and "\( \tilde{L}^2\)-Err," where the errors at level \( n \) are defined by
\[
\| p_{n+1}(x) - p_n(x) \|_{H^{3/2}(\mathbb{R})} \quad \text{and} \quad \| p_{n+1}(x) - p_n(x) \|_{L^2(\mathbb{R})},
\]
respectively, which are similar to [11, Example 5.2]. We will examine if the computed convergence rates reflect their counterparts in the \( \| \cdot \|_{H^{3/2}(\mathbb{R})} \) and \( \| \cdot \|_{L^2(\mathbb{R})} \) norms, respectively. The convergence rates (i.e., the data under "\# rate") at level \( n \) are calculated by
\[
\text{Rate} = \log_2 \left( \frac{\text{the error with solution approximated in } V_{n-1}}{\text{the error with solution approximated in } V_n} \right).
\]
Note that it is different from the data under "\# rate" in Table 7.2, where the results at level \( n \) are calculated by
\[
\# \text{rate} = \log_2 \left( \frac{\text{the condition number at } n}{\text{the condition number at } n - 1} \right).
\]

Example 7.1. Consider model (1.1) with the right-hand side source term \( f(x) \) derived from the exact solution \( p(x) = x^2(1 - x) \chi_{[0,1]} \).

If \( \lambda = 0 \), the right-hand side term \( f(x) \) can be explicitly given as
\[
f(x) = \frac{1}{\pi} \left( 3x - 1/2 + (3x^2 - 2x) \log \left( \frac{1 - x}{x} \right) \right)
\]
for \( \beta = 1 \), and
\[
f(x) = -\frac{c_\beta \Gamma(-\beta)}{\Gamma(4 - \beta)} \left( 2(3 - \beta)x^{2-\beta} - 6x^{3-\beta} + 6(1 - x)^{3-\beta} \right)
\]
\[
- 4(3 - \beta)(1 - x)^{2-\beta} + (3 - \beta)(2 - \beta)(1 - x)^{1-\beta}
\]
for \( \beta \in (0, 1) \cup (1, 2) \). If \( \lambda \neq 0 \), the term \( f(x) \) is obtained numerically. For different \( \lambda \) and \( \beta \), the numerical results are listed in Table 7.1, where in the case \( r = 2 \), the \( \| \cdot \|_{H^{3/2}(\mathbb{R})} \) errors for \( \lambda = 0 \) and \( \lambda = 3 \) are almost the same, and both the \( \| \cdot \|_{H^{3/2}(\mathbb{R})} \) convergence rates of \( r = 1 \) and \( r = 2 \) indeed confirm the theoretical predictions in Theorem 4.3 (since the exact solution \( p(x) \) is smooth enough on \( \Omega \), the theoretical rate is \( r - \frac{\beta}{2} \)).
The condition numbers of systems (5.26) and (5.27) and the corresponding iterations of the CG methods (run in MATLAB 7.0) are presented in Table 7.2, where “Gauss” denotes the Gaussian elimination method, and the “CG” and “PCG” denote the CG iterations for solving systems (5.26) and (5.27), respectively. The stopping criterion for the iteration methods is

$$\frac{\|R(k)\|_{L_2}}{\|R(0)\|_{L_2}} \leq 1e^{-9},$$

with $R(k)$ being the residual vector of linear systems after $k$ iterations. The comparisons for the three methods are made almost with the same $L_2$ approximation errors (not listed in the table). One can see that without preconditioning, the condition number (see the data under “Cond”) of the stiffness matrix behaves like $O(2^{n^β})$, and the iteration numbers (see the data under “iter”) increase with $n$, especially when $β$ is big. After preconditioning, uniformly bounded condition numbers are obtained, and the iteration numbers of the CG method are essentially independent of $n$. We also display the eigenvalue distributions of the stiffness matrices for $(β, r) = (0.3, 1), (β, r) = (0.8, 1), (β, r) = (1, 2)$, and $(β, r) = (1.8, 2)$ in Figure 7.1; they show the preconditioning benefits of a more concentrated eigenvalue distribution.

**Example 7.2.** We now take $f(x) = 1$ in model (1.1).

If $λ = 0$, the exact solution is $p(x) = \frac{(x-x^2)^{\beta/2}}{1+(1+β)} \chi_{[0,1]}$. Although the right-hand side is smooth in $Ω$, $p(x)$ just belongs to $H^{p}(Ω)$ with $μ = 1/2 + 1/2 - ϵ$ for any $ϵ > 0$ [1, Theorem 2.2]. The numerical results are listed in Table 7.3, where the predicted $1/2 - ϵ$ order of convergence in the $\|\cdot\|_{H^{p/2}(R)}$ norm by Theorem 4.3 is obtained. The $L^2$ convergence orders $(1 + \beta)/2$ for $β \in (0, 1)$ and 1 for $β \in (1, 2)$ confirm the result given in [5, Proposition 4.3] for $λ = 0$. When $λ ≠ 0$, $p(x)$ cannot be obtained explicitly, so we list the $\hat{H}^{\beta/2}$ and $\hat{L}^2$ errors instead and examine if the convergence rates reflect the convergence rates in the $\|\cdot\|_{H^{p/2}(R)}$ and $\|\cdot\|_{L^2(R)}$ norms, respectively. The numerical results are presented in Table 7.4, suggesting that the exact solution has a low regularity, but this needs to be confirmed by more in-depth analysis.
The condition numbers and iteration performances of the conjugate gradient method for Example 7.1 with \( \lambda = 3 \).

<table>
<thead>
<tr>
<th>( (\beta, r) )</th>
<th>( n )</th>
<th>CG</th>
<th>PCG</th>
<th>Gauss</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>#Cond</td>
<td>rate</td>
<td># iter</td>
<td>CPU(s)</td>
</tr>
<tr>
<td>(0.3, 1)</td>
<td>11</td>
<td>1.5869e+02</td>
<td>– 60</td>
<td>0.0290</td>
</tr>
<tr>
<td>(0.5, 1)</td>
<td>12</td>
<td>1.9934e+02</td>
<td>0.33</td>
<td>67</td>
</tr>
<tr>
<td>(1.0, 2)</td>
<td>13</td>
<td>2.4939e+02</td>
<td>0.32</td>
<td>75</td>
</tr>
<tr>
<td>(0.8, 1)</td>
<td>11</td>
<td>5.7500e+02</td>
<td>– 107</td>
<td>0.1120</td>
</tr>
<tr>
<td>(1.5, 2)</td>
<td>12</td>
<td>8.1873e+02</td>
<td>0.33</td>
<td>128</td>
</tr>
<tr>
<td>(2.0, 2)</td>
<td>13</td>
<td>1.1634e+03</td>
<td>0.32</td>
<td>152</td>
</tr>
<tr>
<td>(0.5, 2)</td>
<td>10</td>
<td>6.5001e+03</td>
<td>– 318</td>
<td>0.1535</td>
</tr>
<tr>
<td>(1.0, 2)</td>
<td>11</td>
<td>1.1355e+04</td>
<td>0.60</td>
<td>128</td>
</tr>
<tr>
<td>(1.5, 2)</td>
<td>12</td>
<td>1.9803e+04</td>
<td>0.50</td>
<td>152</td>
</tr>
</tbody>
</table>

Example 7.3. In this example, model (6.1) is considered in two cases.

For the first case, let \( g(x) \) and \( f(x) \) in (6.1) be the functions derived from the postulated exact solution \( p(x) = e^{-x^2} \). Note that \( \mathcal{F}[e^{-x^2}](\xi) = \sqrt{\pi} e^{-\xi^2/4} \). Then \( p(x) \in H^\mu(\mathbb{R}) \) for any \( \mu \geq 0 \). The numerical results for \( r = 1 \) and \( \eta(x) = S_1(x) = (e^{-1} - 1) x + 1 \) are presented in Figure 7.2 and show \( \frac{2}{2} \)th order convergence in the
TABLE 7.3
Numerical results for Example 7.2 with \( r = 2 \) and \( \lambda = 0 \).

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( H^{\beta/2})-Err Rate</th>
<th>( L^2)-Err Rate</th>
<th>( H^{\beta/2})-Err Rate</th>
<th>( L^2)-Err Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>9.0822e-02</td>
<td>7.8260e-03</td>
<td>6.4207e-02</td>
<td>5.3755e-03</td>
</tr>
<tr>
<td>0.5</td>
<td>9</td>
<td>6.4156e-02</td>
<td>4.6533e-03</td>
<td>0.75</td>
</tr>
<tr>
<td>10</td>
<td>4.5340e-02</td>
<td>2.7668e-03</td>
<td>0.75</td>
<td>3.2042e-02</td>
</tr>
<tr>
<td>0.5</td>
<td>8</td>
<td>4.7148e-02</td>
<td>1.1967e-03</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>3.3235e-02</td>
<td>6.1260e-04</td>
<td>0.97</td>
<td>2.3567e-02</td>
</tr>
<tr>
<td>1.0</td>
<td>2.3554e-02</td>
<td>3.1339e-04</td>
<td>0.97</td>
<td>1.6657e-02</td>
</tr>
<tr>
<td>0.5</td>
<td>8</td>
<td>2.2024e-02</td>
<td>1.7078e-04</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
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<td>8.3518e-05</td>
<td>1.03</td>
<td>1.1297e-02</td>
</tr>
<tr>
<td>1.5</td>
<td>1.1268e-02</td>
<td>4.1119e-05</td>
<td>1.02</td>
<td>7.9727e-03</td>
</tr>
</tbody>
</table>

TABLE 7.4
Numerical results for Example 7.2 with \( r = 2 \) and \( \lambda \neq 0 \).

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( n )</th>
<th>( \beta/2)-Err Rate</th>
<th>( \chi_2)-Err Rate</th>
<th>( \beta/2)-Err Rate</th>
<th>( \chi_2)-Err Rate</th>
<th>( \beta/2)-Err Rate</th>
<th>( \chi_2)-Err Rate</th>
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\( \|\cdot\|_{H^{\beta/2}(\mathbb{R})} \) norm and first order convergence in the \( \|\cdot\|_{L^2(\mathbb{R})} \) norm.

![Image](image1.png)

![Image](image2.png)

FIG. 7.2. Numerical results for the first case of Example 7.3 with \( p(x) = e^{-x^2} \), \( \eta(x) = S_1(x) \), and \( r = 1 \). The left one is for \( \beta = 0.3 \) and \( \lambda = 1.5 \), and the right one is for \( \beta = 0.7 \) and \( \lambda = 3 \).

For the second case, consider model (6.1) with the generalized Dirichlet type boundary condition \( g(x) = (-2x)\chi_{[-1,0]} + (2x - 2)\chi_{[1,2]} \) and the source term \( f(x) \) derived from the exact solution \( p(x) = (-2x)\chi_{[-1,0]} + (x - x^2)^2\chi_{[0,1]} + (2x - 2)\chi_{[1,2]} \). The image of \( p(x) \) is given in Figure 7.3.

Obviously, \( p(x) \) does not belong to \( H^\mu(\mathbb{R}) \) for \( \mu > 1/2 \) because of its discontinuity at \( x = 3/2 \) and \( x = -1/2 \). We consider two different \( \eta(x) \), i.e., the \( \eta(x) = S_2(x) = 0 \) for \( x \in \Omega \), and the \( \eta(x) = S_3(x) = 2x(x - 1) \) for \( x \in \Omega \). Note that both of them satisfy
Fig. 7.3. Image of the $p(x)$ in the second case of Example 7.3.

Table 7.5
Numerical results for the second case of Example 7.3 with $r = 2$.

<table>
<thead>
<tr>
<th>$(\eta, \beta)$</th>
<th>$n$</th>
<th>$\lambda = 0$</th>
<th>$\lambda = 3$</th>
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<td></td>
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<td>$H^{\beta/2}$-Err</td>
<td>Rate</td>
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\[
\int_{\Omega} \int_{\mathbb{R}} \frac{(\eta(x) - \eta(y))^2}{|x-y|^{1+\beta}} \, dx \, dy < \infty,
\]

as required in Theorem 6.1, but do not belong to $H^{\beta/2}(\mathbb{R})$ (the requirement in [10, subsection 4.1]) for $\beta > 1$, which implies that the condition in Theorem 6.1 is weaker than the one of [10, subsection 4.1]. The numerical results are presented in Table 7.5 and also confirm the theoretical prediction of Theorem 6.2.

8. Conclusions. We have presented Riesz basis Galerkin methods for effectively solving the tempered fractional Laplacian equation, where the operator is the generator of the tempered $\beta$-stable Lévy process. The well-posedness of the equation and convergence of the scheme were theoretically proved. When $\lambda = 0$, the model reduces to a fractional Laplacian equation, and the present theoretical framework is still valid. We also discussed efficient implementations of our methods, including the generation of a stiff matrix and the effectiveness of multiscale preconditioning. We performed several numerical simulations to confirm the theoretical results and to demonstrate the high efficiency of the schemes. The present work is confined to one dimensional problems with basis functions on uniform meshes. The generalization to
higher dimensions and the approximation with locally refined basis functions are very important topics and will be considered in future work.

Appendix A. The proof of (2.21).

Proof. By the properties [19, pp. 106]

\[ \Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}, \quad \Gamma(z + 1) = z\Gamma(z), \]

and

\[ \Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = \sqrt{\pi}2^{1-2z}\Gamma(2z) \]

of the Gamma function, we have

\[ c_\beta = \frac{\beta\Gamma\left(1+\frac{\beta}{2}\right)\Gamma\left(\frac{\beta}{2}\right)}{2^{1-\beta}\pi^{1/2}\Gamma(1-\frac{\beta}{2})\Gamma\left(\frac{\beta}{2}\right)} = \frac{1}{\pi}\Gamma(1+\beta)\sin\left(\frac{\beta\pi}{2}\right), \]

which leads to the desired result by using \( \sin\left(\frac{\beta\pi}{2}\right) = \sin\left(\frac{(2-\beta)\pi}{2}\right) \).

REFERENCES


