Nonlinear random vibrations of plates endowed with fractional derivative elements

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ABSTRACT

This paper deals with the problem of determining the nonlinear response of a plate endowed with fractional derivative elements and exposed to random loads. It shows that an approximate solution of the nonlinear fractional partial differential equation governing the plate vibrations can be obtained via a statistical linearization based approach. The approach is implemented by considering a time-dependent representation of the response involving the eigen-functions of the linear problem. This representation allows deriving a nonlinear fractional ordinary differential equation governing the variation of the time-dependent part of the response, which is linearized in a mean square sense. Then, an iterative procedure provides the response statistics and power spectral density functions.

Next, a Boundary Element Method is proposed for conducting relevant Monte Carlo data. The method is developed in conjunction with a Newmark integration scheme for estimating the response in the time domain given spectrum compatible realizations of the excitation. Monte Carlo data and statistical linearization solutions are calculated for square plates with simply supported stress-free edges, but problems involving other boundary conditions can be solved by the proposed approach.

1. Introduction

Nowadays, fractional operators are investigated in a number of disciplines as diverse as electrical engineering, chemistry and biology [1]. Structural mechanics has taken advantage of fractional calculus, as well. Indeed, it became a quite established tool for describing the viscoelastic behaviour of materials. In this context, the pioneering works of Nutting [2] and Gemant [3] demonstrated that the stiffness and damping of viscoelastic materials involve fractional powers of frequency and that this tool overcomes the limitations associated with the traditional mechanical models of Maxwell and Kelvin–Voigt, while the works of Bagley and Torvik [4–7] gave a theoretical framework to the use of fractional operators in viscoelasticity. Two review articles proposed by Rossikhin and Shitikova [8,9] gave a broad view on the use of fractional calculus in solid mechanics by considering a number of problems involving single-degree-of-freedom (SFOA) oscillators, multi-degree of freedom systems, beams and plates excited by deterministic loads.

The problem of determining the response of a system comprising fractional derivative terms under random excitations was investigated, among others, by Spanos and Zeldin [10], Agrawal [11,12] and Di Paola, Failla and Pirrotta [13] in a linear setting, while the problem of determining the response of a nonlinear SDOF system was analysed by Spanos and Evangelatos [14], Evangelatos and Spanos [15] and Huang and Jin [16]. In these papers, analytical approaches were developed for accommodating an efficient calculation of the response statistics via techniques such as statistical linearization and stochastic averaging. More recently, Di Matteo, Kougioumtzoglou, Pirrotta, Spanos and Di Paola [17] proposed a Weiner path integral method for determining the non-stationary response of both linear and nonlinear fractional systems. Kougioumtzoglou and Spanos [18] developed a harmonic wavelet based statistical linearization technique for the response determination of non-linear SDOF oscillators exposed to a non-stationary excitation. A stochastic averaging procedure was proposed also by Yang, Xu, Jia and Han [19] for a system endowed with a Caputo fractional operator.

The nonlinear response of continua endowed with fractional derivative elements was investigated, as well. For instance, Agrawal [20] considered a continuous beam with a fractional damping term for which a fractional Green function is used for determining the system response.
Failla [21] showed that an exact frequency response function depending solely on four integration constants can be derived for a beam with an arbitrary number of fractional dampers. Sazova and Failla [22] derived a statistical linearization solution for the nonlinear vibration of a beam exposed to a stationary load. A finite element method was developed by Di Paola and Fileccia Scimemi [23] for visco-elastic frames.

In this paper, the problem of determining the large displacements of a nonlinear plate endowed with a fractional derivative is addressed. The plate vibration problem was investigated also by Rossikhin, Shitikova and co-authors. Indeed, Rossikhin and Shitikova [24] studied the transient problem pertaining to an impact mass acting on a visco-elastic plate and the method of multiple scales was developed for studying the plate dynamics in case of internal resonances [25–28]. In this paper, an approximate statistical linearization solution is developed for estimating the response statistics and its reliability is assessed against relevant Monte Carlo data. For this purpose, a Boundary Element Method approach is proposed in conjunction with a Newmark numerical integration scheme for calculating the system response.

2. Preliminary remarks on fractional operators

Fractional operators generalize the concept of differentiation and of integration. They are commonly denoted by the symbols \(\partial_{t}^{\gamma}\) or \(\partial_{x}^{\gamma}\), valid, respectively, for total or partial differentiation. The subscripts denote the “terminals”, that are the two limits associated with the operation of differentiation, while \(\gamma\) denotes the order of the fractional derivative taking integer, real or complex values. For positive integer values of \(\gamma\) classical differentiation is recovered, while for negative integer values \(\gamma = -n\) the \(n\)-fold integration is obtained. In this last case, the symbol \(\partial_{t}^{\gamma}\) \((\gamma < 0)\) is also defined as fractional integral.

A number of representations, all generalizing the operators of differentiation and of integration, is available in the open literature [1,29,30]. In this paper, we consider the representations of Riemann–Liouville (RL) differentiation and of integration, is available in the open literature [1,29,30].

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\[\Gamma(\gamma)\] being the Gamma function [31]. It is seen that for integer values \(n\) of the power \(\gamma\) the Gamma function renders the factorial of the integer number, thus Eq. (1) provides the classical \(n\)-fold integral. The RL fractional derivative is constructed by differentiating Eq. (1) \(m\) times. That is

\[\frac{D_{t}^{\gamma}u}{\partial t^{\gamma}} = \frac{1}{\Gamma(\gamma)} \int_{0}^{t} \frac{u(t)}{(t - \tau)^{\gamma + 1}} \, d\tau, \quad \gamma > 0, \quad \Gamma(\gamma) \text{ being the Gamma function.} \]  
(1)

\[\Gamma(m - \gamma)\] d\(m\) \int_{0}^{t} \frac{u(t)}{(t - \tau)^{\gamma + 1}} \, d\tau, \quad m - 1 \leq \gamma < m. \]  
(2)

Therefore, the fractional derivative is calculated by first integrating the function \(m - \gamma\) times, and then by differentiating the result \(m\) times.

The GL representation [32] is given by the equation

\[G^{1/0}_{t}\frac{D_{t}^{\gamma}u}{\partial t^{\gamma}} = \frac{1}{\Gamma(\gamma)} \int_{0}^{t} \frac{u(t)}{(t - \tau)^{\gamma + 1}} \, d\tau, \quad m - 1 \leq \gamma < m. \]  
(3)

Such a representation accommodates the implementation of algorithms for the numerical computation of fractional derivatives. Indeed, the series in Eq. (3) can be expanded and the following series representation of the GL derivative can be derived:

\[\frac{G^{1/0}_{t}\frac{D_{t}^{\gamma}u}{\partial t^{\gamma}}}{\partial t^{\gamma}} = \lim_{\Delta t \to 0} \sum_{k=0}^{n} G_{L_{k}} u(t - k\Delta t), \]  
(4)

where \(G_{L_{k}}\) are calculated recursively by the relationship [30]

\[G_{L_{k}} = \frac{k - \gamma - 1}{k} G_{L_{k+1}}, \quad G_{L_{1}} = 0. \]  
(5)

Eq. (4) provides the G1-algorithm that is used in this paper for conducting the numerical integrations. Such an algorithm renders the fading memory property of the fractional derivative through the quantity \((k - \gamma - 1)/k < 1\).

A common property of fractional operators relates to their Fourier Transform (FT). Specifically, the FT of the fractional derivative of \(u(t)\) is connected to the FT of \(u(t)\) by the equation

\[F_{\gamma}[D_{t}^{\gamma}u(t)] = (i\omega)^{\gamma} F[u(t)]. \]  
(6)

where \(i\) is the imaginary unit and \(F[\cdot]\) denotes the FT operator. In this context, it is seen that the Eq. (6) generalizes the relation between the FT of a function and its integer order derivatives.

3. Large plate displacements

This section discusses the fundamental equations of motion describing the large displacements of a plate and proposes both analytical and numerical approaches for determining its response. The equations are derived following the classical approach of von Kármán [33,34] and include a fractional derivative element representing an external force acting on the plate. Next, an approximate analytical solution of the equation of motion is sought via a statistical linearization approach. Finally, a numerical algorithm involving the GL fractional derivative representation and a Newmark integration algorithm is derived.

3.1. Equations governing the large plate displacements

Consider a rectangular plate of sides \(a\) and \(b\), with mass density \(\rho\), thickness \(h\), Young modulus \(E\), and flexural stiffness \(D\). The plate is exposed to a transverse load \(q = q(x,y,t)\) dependent on the space coordinates \((x,y)\) and on the time variable \(t\) and is endowed with a fractional derivative element of order \(\gamma\) and constant damping \(\mu\). Its transverse displacement \(u = u(x,y,t)\) is governed by the equation of motion

\[\frac{\partial^{2\gamma} u}{\partial t^{\gamma}} + \mu_0 \frac{D_{t}^{\gamma} u}{\partial t^{\gamma}} + D V^{\gamma} u = \rho \left(\frac{\partial^{2\gamma} \phi}{\partial x^{\gamma}} + \frac{\partial^{2\gamma} \phi}{\partial y^{\gamma}} + 2D^{\gamma/2} \frac{\partial^{2\gamma} \phi}{\partial x^{\gamma/2} \partial y^{\gamma/2}} - 2D^{\gamma/2} \frac{\partial^{2\gamma} \phi}{\partial x^{\gamma} \partial y^{\gamma/2}}\right), \]  
(7)

where \(V^{\gamma} = (\frac{\partial^{\gamma}}{\partial x^{\gamma}} + \frac{\partial^{\gamma}}{\partial y^{\gamma}}) + 2D^{\gamma/2} \frac{\partial^{\gamma/2}}{\partial x^{\gamma/2} \partial y^{\gamma/2}}\) is the biharmonic operator, and \(\phi = \phi(x,y,t)\) is the Airy stress function that is governed by the equation

\[V^{\gamma}\phi = E \left(\frac{\partial^{2\gamma} \phi}{\partial x^{\gamma}} - \frac{\partial^{2\gamma} \phi}{\partial x^{\gamma}}\right). \]  
(8)

The load is assumed of a separable type, so that

\[q(x,y,t) = p(x,y)f(t), \]  
(9)

where \(p(x,y)\) is a deterministic function and \(f(t)\) is a random process of a given power spectral density function \(S(\omega)\) and with autocorrelation function

\[\langle f(t - t_1)f(t - t_2) \rangle = \int_{-\infty}^{\infty} S(\omega) \exp(i\omega(t_2 - t_1)) \, d\omega. \]  
(10)

Eq. (7) allows investigating the vibrations of an elastic plate into a viscous medium or on a viscoelastic foundation [9]. In this context, the fractional derivative operator allows introducing simultaneously distributed stiffness and damping elements so that \(\gamma = 0\) and \(\gamma = 1\) represent the case, respectively, of a linear spring and of a viscous damper acting on the plate.

3.2. Approximate plate response determined by a statistical linearization-based approach

Currently, an exact solution of Eq. (7) is unavailable. Therefore, an approximate solution is sought via a statistical linearization based
approach [35] in order to estimate approximately the second order statistics and the power spectral density function of the response.

For this purpose, the response of the system is represented by Galerkin expansions of the vertical displacement and of the stress function having time-dependent amplitudes. Specifically,

\[ u = \sum_{m,n} w_{mn}(t) U_{MN}(x,y), \]  

(11)

and

\[ \phi = \frac{P_y y^2}{2bh} + \frac{P_x x^2}{2bh} + \sum_{m,n} w_{mn}(t) \varphi_{mn}(x,y), \]  

(12)

where \( P_y \) and \( P_x \) are total tension loads applied, respectively, on the sides \( x = 0, a \) and \( y = 0, b \) of the plate and \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \).

The eigen-functions \( U_{MN} \) and \( \varphi_{mn} \) depend upon the specific boundary conditions and are orthogonal to each other. That is,

\[ \int_A U_{MN} U_{KL} dA = \sum_{m,n} \int_A \varphi_{mn} \varphi_{kl} dA = \frac{4}{ab} \delta_{mn} \delta_{kl}, \]  

(13)

\( \delta_{mn} \) denoting the Kronecker delta (1 for \( m = n \), 0 otherwise) and \( A \) denoting the plate surface. Further, the following quantities are defined:

\[ R_{xk}(M,N,m,n) = \int_A \partial^2 U_{MN} \partial x^2 U_{MN} dA, \]  

(14)

and

\[ R_{yp}(M,N,m,n) = \int_A \partial^2 U_{MN} \partial y^2 U_{MN} dA. \]  

(15)

By substituting Eqs. (11) and (12) into Eqs. (7) and (8), considering algebraic manipulations reflecting error projection in the space of eigen-functions, and by observing that the stress function amplitudes \( w_{mn}^{(2)} \) can be expressed in terms of \( w_{mn} \), a nonlinear fractional ordinary differential equation for the time-dependent amplitudes \( w_{mn} \) is found. Specifically,

\[ \ddot{w}_{MN} + \frac{\mu}{pb} D^2_y w_{MN} + \omega^2_{MN} w_{MN} \]

\[ = - \frac{4}{abpb} \left( P_x \sum_{m,n} w_{mn} R_{xk}(M,N,m,n) + P_y \sum_{m,n} w_{mn} R_{yp}(M,N,m,n) \right) \]

\[ - \frac{4}{ab} \int_A I \left( M,N,m,n,k,l,p,q \right) dA, \]  

(16)

where

\[ R_{xk}(M,N,m,n) = \int_A \partial^2 U_{MN} \partial x^2 U_{MN} dA, \]  

(14)

and \( R_{yp}(M,N,m,n) = \int_A \partial^2 U_{MN} \partial y^2 U_{MN} dA. \)  

(15)

The average values included into Eq. (23) are determined under the stipulation that the time-dependent amplitudes \( w_{mn} \) are Gaussian random processes. In this context, it is shown that [36]

\[ \langle w_{mn} \rangle = \frac{4}{abpb} P_x R_{xk}(M,N,m,n) + P_y R_{yp}(M,N,m,n) \]

\[ + \frac{4}{ab} \int_A I \left( M,N,m,n,k,l,p,q \right) dA, \]  

(24)

and

\[ \langle w_{MN}^2 \rangle = \frac{4}{abpb} P_{MN} P_{MN} S_{MN,mn} \]  

(25)

where

\[ S_{MN,mn} = \int_0^\infty \left( 2 \rho h M_N(\omega^2) + \rho h M_N(\omega)^2 \right) \]  

\[ \frac{1}{\omega^2 + \frac{1}{\rho h} M_N(\omega)} d\omega, \]  

(26)

and \( H_{MN}(\omega) \) is the transfer function associated with Eq. (20). That is,

\[ H_{MN}(\omega) = \frac{1}{\omega^2 + \frac{1}{\rho h} M_N(\omega)^2} + \omega^2_{eq, MN}, \]  

(27)

Therefore, it is seen that,

\[ \omega^2_{eq, MN} = \frac{4}{abpb} P_{MN} S_{MN,mn} \sum_{m,n} P_{MN,mn} \]

\[ \times \left( P_x R_{xk}(M,N,m,n) + P_y R_{yp}(M,N,m,n) \right) \]

\[ + \frac{4}{ab} \int_A I \left( M,N,m,n,k,l,p,q \right) dA, \]  

(28)

where \( \omega_{eq, MN} \) starting iterations from \( \omega^2_{eq, MN} = \omega^2_{MN} \) until convergence to a certain value is achieved.

After determining the equivalent stiffness, the response statistics are readily determined via the equivalent linear system. Specifically, the variance of the transverse displacement is given by the equation:

\[ \sigma^2(x,y) = \langle \hat{w}^2(x,y) \rangle = \frac{4}{abpb} \sum_{m,n} \sum_{k,l} P_{MN,kl} U_{mn} U_{kl} S_{mn,kl}, \]  

(29)
and the frequency spectrum of the transverse displacement at a certain point is
\[ S_u(x, y, \omega) = \left( \frac{4}{\pi b h^2} \right) \sum_{n \in N} \sum_{k \in K} P_{mn} x_{mn} u_{nk} (\omega) H_{mn} (-\omega) H_{nk} (\omega). \tag{30} \]

### 3.3. Plate response estimated by Monte Carlo simulation

The numerical solution of the nonlinear fractional partial differential equations (7)-(8) is determined by a Boundary Element Method (BEM) based algorithm in the formulation proposed by Katsikadelis and Nerantzaki [37]. Such a formulation was utilized for calculating large displacements of circular and of square plates and comparisons with analytical and different BEM approaches demonstrated that it is a reliable approach for estimating the system response. The key concept is to estimate the plate response via a time-varying representation of the solution of a classical linear BEM problem. Specifically, considering that the nonlinear problem (7)-(8) involves space derivatives of the fourth order, the approach is developed by considering the solution of the linear problems
\[ \nabla^4 u = b_1(x, y, t), \tag{31} \]
and
\[ \nabla^4 \varphi = b_2(x, y, t). \tag{32} \]

In this context, \( b_1(x, y, t) \) and \( b_2(x, y, t) \) are space–time dependent fictitious loads, which are identified by BEM.

The solution of the problem (31) has the following integral representation [37]:
\[ \varepsilon u(P) = \int_A A_1 b_1 dA - \int_f A_1 u + A_2 u_x + A_3 \nabla^2 u + A_4 (\nabla^2 u)_y \, ds, \tag{33} \]
where \( \varepsilon = 2\pi \kappa \) or \( \kappa \) if the point \( P \) is inside the domain \( A \) or on the boundary \( f \) respectively, and the other quantities are given by the equations:
\[ A_1 = -\frac{\cos \theta}{r}, \tag{34} \]
\[ A_2 = \ln r + 1, \tag{35} \]
\[ A_3 = \frac{1}{r^2} (2r \ln r + r) \cos \theta, \tag{36} \]
\[ A_4 = \frac{1}{r^2} \ln r. \tag{37} \]

\( r \) being the distance between point \( P \) and a point inside the domain or on the boundary, and \( \theta \) being the angle between \( r \) and the outgoing normal to the boundary. It is worth-mentioning that the integral representation (33) pertains to classical linear BEM applications and it is implemented here in conjunction with a nonlinear problem by considering time-varying values of the coefficients \( b_1(x, y, t) \).

Further, the following equation holds:
\[ \varepsilon \nabla^2 u(P) = \int_A A_1 b_1 dA - \int_f A_1 \nabla^2 u + A_3 (\nabla^2 u)_y \, ds. \tag{38} \]

Eqs. (33) and (38) can be used for estimating the unknown boundary quantities by introducing the associated boundary conditions. For this purpose, the plate domain and boundary are discretized and Eqs. (33) and (38) are collocated on the boundary points. By doing so, the following linear system of equations is derived:
\[
\begin{bmatrix}
A_{11} & A_{12} & 0 & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
0 & 0 & A_{42} & A_{44}
\end{bmatrix}
\begin{bmatrix}
u \\
u_x \\
\nabla^2 u \\
(\nabla^2 u)_y
\end{bmatrix}
= \begin{bmatrix}
\{b_1\} \\
\{b_2\} \\
0 \\
0
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
C_1 \\
C_1
\end{bmatrix}
\begin{bmatrix}
h_1 \\
\end{bmatrix}. \tag{39}
\]

The sub-matrices composing the first two rows are determined by the boundary conditions. In this context, it is seen that in case of inhomogeneous boundary conditions constants are on the right hand side. The sub-matrices composing the last two rows are estimated from the discretized counterparts of Eqs. (33) and (38) via Gaussian integration over the domain and the boundary. The linear system (39) allows determining the boundary quantities in terms of the fictitious load \( b_1 \). Thus, the response of the plate can be calculated by the equation:
\[ u = G \cdot b_1. \tag{40} \]

where \( G \) is a known matrix, \( b_1 \) is a vector containing the values of the fictitious load at each point of the domain and \( u \) is a vector containing the response at that points.

A similar procedure can be used for representing the stress function:
\[ \varphi = G \cdot b_2, \tag{41} \]
where it is observed that the only difference with the determination of \( u \) relates to the different boundary conditions.

The representation obtained in this manner is used for collocating the displacements and stress values into the original equations (7)-(8) in order to derive a set of fractional nonlinear ordinary differential equations for the fictitious loads \( b_1 \) and \( b_2 \). That is,
\[ \rho \dot{G} \cdot b_1 + \mu \dot{G} \cdot \nabla \cdot b_1 + \dot{G} \cdot b = F_1 (G), \tag{42} \]
and
\[ b_2 = G \cdot F_2 (G), \tag{43} \]
where \( F_1 \) and \( F_2 \) are nonlinear functions encapsulating the nonlinear elements of the original system.

The numerical solution of this fractional differential equation is obtained by a Newmark based algorithm implemented in conjunction with the G1-algorithm of the GL fractional derivative. Specifically, the incremental equation of motion associated with Eq. (42) is
\[ \rho \ddot{G} \cdot \Delta b_1 (t) + \mu \ddot{G} \cdot \nabla \cdot \Delta b_1 (t) + \dot{G} \cdot \Delta b_2 (t) - \dot{G} \cdot F_1 (G) = \Delta F_1 (G), \tag{44} \]
\[ \Delta b_2 (t) = G \cdot F_2 (G) \tag{45} \]
and \( GL \) are coefficients arising from the application of the G1-algorithm for the calculation of the fractional derivative [14]. Considering the fact that the fractional derivative calculation involves both present and past values of the response, such equation is recast as follows:
\[ \rho \ddot{G} \cdot \Delta b_1 (t) + \mu \ddot{G} \cdot \nabla \cdot \Delta b_1 (t) + \dot{G} \cdot \Delta b_2 (t) - \dot{G} \cdot F_1 (G) \]
\[ = \Delta F_1 (G) - \mu \Delta \cdot (G \cdot F_2 (G)) \cdot (\sum_{k=1}^n GL \cdot \Delta b_1 (t - k \Delta t) + GL \cdot b_1 (0)). \tag{46} \]

This form of the incremental equation of motion can be used with a classical Newmark algorithm for calculating the response of a system. The crucial element of the formulation relates to the fact that the influence of the past history of the response is incorporated in the excitation at each time step. Obviously, calculations are pursued by incorporating only a limited number of past value terms. In this regard, numerical studies have been undertaken for identifying a reliable number of components and the results confirmed that including about 200 past values is sufficient for obtaining a reliable estimate as mentioned by Spanos and Malara [36].

### 4. Numerical results

In this section, numerical results are presented considering the case of a square plate excited by a uniform random load. The first part
introduces the particular case study. Next, numerical results are presented with the objective of investigating the reliability of the proposed approximate solution.

4.1. Case study: square plate excited by a uniform random load

The computations are pursued considering the large vibrations of a square plate exposed to a uniform random load. In this context, the time-dependent part of the load is compatible with a coloured white noise spectrum given by the equation

\[ \dot{S}\left(\omega\right) = \frac{C m^4}{\left[(\omega^2 - k_1)^2 + (c_1 \omega)^2\right] \left[(\omega^2 - k_2)^2 + (c_2 \omega)^2\right]} \]  \hspace{1cm} (47)

\( \omega = \omega/\omega_p \), being a normalized frequency spectrum \( \omega_p \) (the peak frequency of the spectrum), and \( C, k_1, c_1, k_2, c_2 \) being shape parameters. This particular spectrum was proposed by Spanos [38] and can be regarded as the output of a cascade of two linear filters.

The geometric and material properties of the plate are summarized in Table 1, while the spectral parameters are shown in Table 2. In this regard, note that the excitation has a peak spectral period of 5 s and a standard deviation of 50 kN/m, while the quantity \( \rho(x, y) = 1 \). The numerical computations consider the case of simply supported stress-free edges investigated also by Katsikadelis and Nerantzaki [37].

4.2. Statistical linearization solution vis-à-vis Monte Carlo data

Monte Carlo data and statistical linearization solutions are calculated for the proposed case study in conjunction with different values of the fractional derivative order.

The Monte Carlo data are obtained by synthesizing spectrum compatible realizations of the plate load and then implementing the BEM method with the Newmark algorithm. The plate load is generated according to the Spectral Method in conjunction with a Fast Fourier Transform algorithm [39]. Each case study is investigated under the assumption of ergodicity and stationarity, so that one realization of the excitation and, thus, of the response is used for estimating the relevant statistics. In this regard, note that 10^6 samples of the response are used for determining the response statistics [35]. The BEM is implemented considering a uniform discretization of the boundary in 31 elements per side and a uniform discretization of the domain in 121 elements (Fig. 1). The Newmark integration is pursued via the constant average acceleration method [40]. The algorithm is initiated starting from quiescent initial conditions and by padding with zeros the unknown past values of the response. Such an approximation affects the transient part of the system response, but it has no influence on the stationary part of the response which is considered during the post-processing. The numerical data are used for estimating the standard deviation of the plate response along the mid-span of the plate \( x = a/2, 0 \leq y \leq b \) and the power spectral density function of response at the centre of the plate \( x = a/2, y = b/2 \) by the Welch method [41].

The statistical linearization solution is estimated considering the appropriate eigen-functions associated with the boundary conditions. In this context, the response is calculated via the linear modes of plate vibration [34]. Each figure shows results obtained by considering different fractional derivative orders. It is seen that the approximate solution is in good agreement with the numerical computation. Indeed, the crucial features of the response statistics are predicted reliably over the entire plate and the spectral content of the response is captured successfully. Further, the agreement is irrespective of the fractional derivative order. A second set of analyses considers the influence of the plate thickness \( h \). This parameter plays the role of a nonlinearity parameter into Eq. (7), as the nonlinear contribution to the system response vanishes as \( h \rightarrow 0 \). Specifically, Fig. 6 shows the standard deviation of the vertical plate displacement at the centre of the plate. All calculations involve identical plates with different thicknesses \( h \). In this context, it is seen that the approximate solution provides a quite good estimate of the response statistics even in conjunction with large values of the nonlinearity parameter \( h \).

5. Concluding remarks

This paper has discussed an approximate method for determining the nonlinear response of a plate endowed with a fractional derivative element and excited by a random load.
The solution of the nonlinear fractional partial differential equation governing the large plate displacement is sought via a statistical linearization based technique. The technique is implemented by expanding the unknown system response in terms of eigen-functions of
the linear plate vibration problem and by retaining the nonlinear term in its time-dependent part. In this context, the statistical linearization approach is pursued in the time domain by replacing each nonlinear fractional differential equation with a surrogate linear equation. This equation involves an arbitrary stiffness determined by minimizing the difference between the linear equation and the nonlinear one in a mean square sense.

The paper has developed also a numerical algorithm for calculating the system response in the time domain. The algorithm relies on a time-dependent integral representation of the solution implemented through a Newmark scheme, for calculating the response time history.

The reliability of the statistical linearization based approach has been assessed vis-à-vis Monte Carlo data. The numerical results have shown that the proposed approximate approach provides a quite good estimate of the response statistics and captures well the spectral content of the response. The quality of the approximation is irrespective of the fractional derivative order and of the magnitude to the nonlinearity parameter $h$.

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