A PETROV-GALERKIN SPECTRAL METHOD OF LINEAR
COMPLEXITY FOR FRACTIONAL MULTI-TERM ODES ON THE
HALF LINE*

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Abstract. We present a new tunably-accurate Laguerre Petrov-Galerkin spectral method for
solving linear multi-term fractional initial value problems with derivative orders at most one and
constant coefficients on the half line. Our method results in a matrix equation of special structure
which can be solved in $O(N \log N)$ operations. We also take advantage of recurrence relations for
the entries of the stiffness and mass matrices, which can be factored into the product of a diagonal
matrix and a lower-triangular Toeplitz matrix. The resulting spectral method is efficient for solving
multi-term fractional differential equations with arbitrarily many terms, which we demonstrate by
solving a fifty-term example. We apply this method to a distributed order differential equation,
which is approximated by linear multi-term equations through the Gauss-Legendre quadrature rule.
We provide numerical examples demonstrating the spectral convergence and linear complexity of the
method.

Key words. spectral accuracy, linear complexity, singular solutions, tunable accuracy, dis-
tributed order

AMS subject classifications. 34L10, 58C40, 34K28, 65M70, 65M60

1. Introduction. While numerical methods for fractional differential equations
have been investigated for over two decades [1, 2, 3, 4], the main difficulty in developing
these methods, unlike their integer-order counterparts, is the large computational cost
due to the non-local nature of fractional differential operators. For example, in finite
difference [3, 5, 6, 7] or finite element methods [8, 9], data at all grid points or elements
are needed in order to achieve an accurate approximation to the fractional derivative
at a single grid point or element. This results in methods that are significantly more
complex in both implementation and computational cost than methods for integer-
order counterparts.

Recently, spectral methods have been applied to these problems, offering the
benefit of more natural non-local approximations in addition to high accuracy in the
case of smooth solutions. For non-smooth solutions with singularity of type $(x - a)^\alpha$
(where $a$ is the left endpoint of the approximation interval), we find that using an
approximation of the form $(x - a)^{\alpha}p(x)$, with $p(x)$ a polynomial approximation to
the smooth part of the solution, will also lead to numerical approximations with a
high order of accuracy. Zayernouri and Karniadakis derived functions of this type
as eigenfunctions of fractional Sturm-Liouville problems on a compact interval [10,
11, 12, 13, 14, 15]. Khosravian-Arab et al. extended this work to fractional Sturm-
Liouville problems on the half line and derived the generalized associated Laguerre

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1
Laguerre polynomials have been used for decades as basis functions in spectral methods for solving integer-order differential equations on the half line \([17]\). These polynomials are solutions of Laguerre’s equation:

\[
xy'' + (\alpha + 1 - x)y' + ny = 0,
\]

where \(\alpha \in \mathbb{R}\) is the parameter for the associated Laguerre polynomials, and \(y = L_n^{(\alpha)}(t)\). The Rodrigues formula for these polynomials is

\[
L_n^{(\alpha)}(t) = \frac{t^{-\alpha}e^t}{n!} \frac{d^n}{dx^n}(e^{-t}t^{n+\alpha}),
\]

and they satisfy the recurrence relation

\[
L_n^{(\alpha)}(x) = \sum_{i=0}^{n} \left( \frac{\alpha - \beta + n - i - 1}{n - i} \right) L_i^{(\beta)}(x),
\]

where \(\beta\) is another associated parameter in \(\mathbb{R}\). These polynomials are particularly useful in the context of solving equations defined in semi-infinite domains because they satisfy an orthogonality relation on the half line:

\[
\int_0^\infty L_n^{(\alpha)}(t)L_k^{(\alpha)}(t)e^{-t}t^\alpha dt = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)} \delta_{nk}.
\]

The two types of GALFs used in our work are discussed in Section 2.2 and are defined as

\[
\phi_n^{(1)}(t) := t^\alpha L_n^{(\alpha)}(t), \quad n = 0, 1, 2, \ldots, \quad \alpha > 0,
\]

\[
\phi_n^{(2)}(t) := e^{-t}L_n^{(\alpha)}(t), \quad n = 0, 1, 2, \ldots, \quad \alpha > -1.
\]

These functions were shown to be solutions of fractional Sturm-Liouville problems on the half line in \([16]\) and are utilized here as basis functions for our Laguerre Petrov-Galerkin method for multi-term fractional initial value problems.

Zhang et al. \([18]\) analyzed spectral methods on the half line for a single-term fractional initial value problem using a generalized version of the GALFs. The authors of \([18]\) derived both collocation and Petrov-Galerkin methods on bounded intervals and on the half line, including a Laguerre PG method for single-term fractional ODEs.

Our present work differs from the work by Zhang et al. in the following aspects.

- We apply our Laguerre PG method to multi-term equations, which increases the complexity of the problem significantly because the stiffness matrix becomes dense and therefore computationally difficult to assemble and invert. In the single-term case studied in \([18]\), all stiffness matrices are diagonal.
- We provide explicit expressions for the stiffness matrix entries in the multi-term case, thus avoiding the computational cost of applying quadrature rules to integrals of singular functions and assembling the stiffness and mass matrices exactly.
- In our work, we develop a method of factoring the linear system into a product of diagonal and Toeplitz matrices in order to invert the system with linear complexity. This is done by taking advantage of special recurrence properties of the Laguerre basis functions.
• The efficiency of our method is robust for multi-term equations that include a reaction term, which corresponds with a mass matrix in the discretized system. Although mass and stiffness matrices typically do not have similar structures in Petrov-Galerkin spectral methods, our method results in mass matrices which afford the same type of factorization as the stiffness matrices. Thus when our method is applied to multi-term equations with reaction terms, the resulting matrix equation can still be solved with linear complexity.

• Further, we have applied our efficient method to distributed order equations, which were not considered in [18].

Equations similar to those considered in our work have been considered by Baleanu, Bhrawy, and Taha in [19] and by Bhrawy, Baleanu, and Assas in [20], in which they apply spectral Tau and collocation methods based on Laguerre polynomials to solve multi-term fractional initial value problems on the half line. In [19] and [20], the authors use Newton’s iterative method and an LU decomposition respectively to solve the resulting discrete system of equations. Both methods require $O(N^3)$ operations, where $N$ represents the size of the system of equations. Additionally, the discrete system resulting from these methods is dense and ill-conditioned. In comparison, our method results in a well-conditioned system, which can be solved with $O(N \log N)$ operations. This improved efficiency offers the capability of solving multi-term equations with many terms at low computational cost, resulting in a computationally feasible approximation method for the solution of distributed order equations, as detailed below.

There has recently been significant work on multi-term time-fractional differential equations in bounded domains [21, 22, 23, 24]. In [21], Zheng et al. use a Galerkin space-time spectral method for a multi-term time-fractional diffusion equation on a bounded domain. They consider equations with both two- and three-term time-fractional derivatives in one spatial dimension, and Legendre polynomials are used as the basis functions for their space-time Galerkin method. The resulting temporal stiffness matrix is dense and stiff, requiring $O(N^3)$ operations to solve. To the best of our knowledge, all existing spectral methods for multi-term time-fractional differential equations require $O(N^3)$ operations to compute the numerical solution.

In [22], Zhao et al. consider a two-dimensional multi-term time-fractional diffusion equation in a bounded, convex, polygonal domain, where the spatial derivative is the standard integer-order Laplacian. They applied a finite element method in space and a classical $L^1$ approximation in time, achieving a convergence rate of order $O(h^2+\tau^{2-\alpha})$, where $\alpha$ corresponds to the largest order of the time derivatives. The errors for this method are evaluated at times less than $t = 1$, hence the problem of long-time integration is still of interest.

In [23], Liu et al. derive numerical methods for multi-term time-fractional wave-diffusion equations in bounded domains using finite difference methods. Two implicit finite difference schemes and two predictor-corrector Adams-Bashforth-Moulton methods for the two-term time-fractional wave-diffusion equation in one dimension are proposed. The authors use a central difference scheme for the second-order spatial derivative and thus achieve second order convergence in space and $1 + \min(\alpha_i)$ order convergence in time (where $\{\alpha_i\}$ are the orders of the time-fractional derivatives). Our present work offers the advantage of high-order, efficient approximations in the case where the solutions are smooth.

There has also been recent work on deriving analytical solutions of multi-term time-fractional differential equations [24, 25]. In [25], Ming et al. consider analytical solutions for unsteady flows of non-Newtonian fluids on a moving, infinite plate. The
authors expanded the solution in space using a sine series and found the solutions of the resulting system of time-fractional ordinary differential equations as a finite sum of multivariate Mittag-Leffler functions. In [24], a similar technique was used to derive solutions for equations defined in two- and three-dimensional bounded domains. In this case, the authors consider a multi-term time- and space-fractional equations, where the two terms with different orders of the space-fractional Laplacian appear.

The multi-term equations considered in this work are motivated by the approximation of distributed order differential equations using a quadrature rule as in the paper by Diethelm and Ford [26]. This type of equation arises in many physical and biological applications: for example, in applications to viscoelastic oscillators [27], distributed order membranes in the ear [28], dielectric induction [29], and anomalous diffusion [30, 31]. In their paper, Diethelm and Ford considered distributed order equations of the form

\[
\int_0^m a(r) D_r^\alpha u(t) \, dr = f(t),
\]

to which they applied the trapezoid quadrature rule to derive a multi-term fractional differential equation on a bounded interval. To improve the quality of the approximation, many terms in the resulting multi-term equation may be needed. The efficiency and high order of accuracy of our proposed method offers the capability of accurately solving equations with many terms with low computational cost.

The multi-term FIVPs considered in this work have fractional order at most one. There is reason to consider this problem important, as it is possible to reduce any linear multi-term fractional equation to a system of multi-term fractional equations with order at most one [4].

In the Petrov-Galerkin method presented in the following sections, we also introduce a tuning parameter enabling us to “speed up” the rate of convergence of the method when the order of the singularity is known. In cases where the optimal value of the tuning parameter is not known a priori, our numerical results demonstrate that the method exhibits spectral accuracy for any valid choice of the tuning parameter. Determining the optimal value of the tuning parameter for a given equation and forcing function is the subject of ongoing research. Additionally, our method is very efficient, as we are able to solve the resulting matrix equation in \(O(N \log N)\) operations for any value of the tuning parameter.

One of the key aspects of our method is the choice of approximation basis functions, which are the eigenfunctions of a fractional singular Sturm-Liouville problem [16]. We present some analysis that shows how we can use the fractional Sturm-Liouville operator to determine the decay rate of the coefficients of the Petrov-Galerkin approximation. We also use recurrence relations for Laguerre polynomials to derive explicit expressions for the entries of the stiffness matrices in the Petrov-Galerkin method. This offers savings in the cost of assembling these matrices since we avoid using quadrature, in addition to avoiding the (potentially large) Gauss-Laguerre quadrature error in stiffness matrix entries.

Another key aspect of the derivation of the Petrov-Galerkin method is fractional integration by parts, which we perform in such a way as to offer flexibility in what order of the derivative is transferred from the trial function to the test function in the variational form. We demonstrate how this flexibility translates into a tunably-accurate method through the derivation of the method and with numerical experiments.

The remainder of the paper is organized as follows. In Section 2, we introduce the multi-term fractional initial value problem along with the notation and definitions
used throughout the paper. In Section 3, we introduce our Petrov-Galerkin spectral method and discuss its computational cost. In Section 4, we present numerical examples using fabricated solutions as well as a short analysis of the decay rates of the coefficients of the Galerkin projection. In Section 5, we introduce distributed order fractional initial value problems as an application of our PG method for multi-term equations, with numerical examples in Section 6. Finally, Section 7 offers a summary of our results and directions for future research.

2. Preliminaries.

2.1. Notation and definitions. We are interested in solving the multi-term fractional initial value problem (FIVP) with constant coefficients \( \{b_i\}_{i=1}^{K} \), on the interval \( t \in (0, \infty) \):

\[
\sum_{i=1}^{K} b_i \, \mathcal{D}_t^{\nu_i} u(t) = f(t),
\]

\[
u_0 = 0,
\]

where \( \mathcal{D}_t^{\nu_i} \) represents the Riemann-Liouville fractional derivative of order \( \nu_i \in (0, 1) \) for all \( i = 1, 2, \ldots, K \). Notice that if the initial condition \( u(0) \) is not equal to zero, then we can simply apply the same method to solving the modified FIVP

\[
\sum_{i=1}^{K} b_i \, \mathcal{D}_t^{\nu_i}(u - u_0)(t) = f(t),
\]

\[
u_0 = u_0.
\]

**Definition 1.** [32] Let \( \alpha > 0 \). The left- and right-sided Riemann-Liouville fractional integrals of order \( \alpha \) on the semi-infinite interval \( (0, \infty) \) are defined as

\[
\mathcal{I}_t^\alpha u(t) := \frac{1}{\Gamma(\alpha)} \int_0^t u(s)(t-s)^{\alpha-1} \, ds, \quad t > 0,
\]

\[
\mathcal{I}_t^\infty u(t) := \frac{1}{\Gamma(\alpha)} \int_t^\infty u(s)(s-t)^{\alpha-1} \, ds, \quad t > 0,
\]

where \( \Gamma(\cdot) \) denotes the Euler Gamma function.

Note that as the right-sided integral is defined on the interval \( (t, \infty) \), \( u \) must be a function with suitable decay properties as \( t \to \infty \) so that this integral is well-defined.

**Definition 2.** [32] Let \( \nu \in \mathbb{R}_+ \) be the order of differentiation on the semi-infinite interval \( (0, \infty) \), and define \( m \) such that \( m-1 \leq \nu \leq m \). Then the left- and right-sided Riemann-Liouville derivatives are given by

\[
\mathcal{D}_t^{\nu} u(t) = \frac{1}{\Gamma(m-\nu)} \frac{d^m}{dt^m} \int_0^t u(s)(t-s)^{m-\nu-1} \, ds, \quad t > 0,
\]

\[
\mathcal{D}_t^{\infty} u(t) = \frac{1}{\Gamma(m-\nu)} \frac{(-d)^m}{dt^m} \int_t^\infty u(s)(s-t)^{m-\nu-1} \, ds, \quad t > 0.
\]

2.2. Fractional Sturm-Liouville problem on the half line. Following [16], we consider the fractional Sturm-Liouville problem of the first kind (FSLP-1) on the half line, and we use the following theorem.
Theorem 3. [16] The exact eigenfunctions of the following FSLP-1

\[ L_{\alpha,\beta}^1[\phi] := t D_{\infty}^\alpha p_1(t) D_t^\beta \phi(t) - \lambda_1^1 \omega_1^\beta \phi(t) = 0, \]

where \( \alpha \in (0,1) \), and

\[ p_1(t) = t^{\alpha-\beta} e^{-t}, \quad \omega_1^\beta(t) = t^{-\beta} e^{-t} \]

subject to the boundary values

\[ \phi(0) = 0, \quad \mathcal{I}_{\infty}^{1-\alpha}(p_1(t) D_t^\alpha y(t)) \bigg|_{t=\infty} = 0, \]

are given as

\[ \phi_{n}^{1}(t) = t^{\beta} L_{n}^{(\beta)}(t), \quad n = 0, 1, 2, \ldots, \]

where \( \beta > 0 \) and the corresponding distinct eigenvalues are

\[ \lambda_1^n = \frac{\Gamma(n + \beta + 1)}{\Gamma(n + \beta - \alpha + 1)}, \quad n = 0, 1, 2, \ldots \]

We also have from [16] the solution to the fractional Sturm-Liouville problem of the second kind (FSLP-2) on the half line.

Theorem 4. [16] The exact eigenfunctions of the following FSLPs-2

\[ L_{\alpha,\beta}^2[\phi] := \theta D_t^\alpha p_2(t) D_{\infty}^\beta \phi(t) - \lambda_2^\beta \omega_2^\beta(t) y(t) = 0, \]

where \( \alpha \in (0,1) \) and

\[ p_2(t) = t^{\beta+\alpha} e^t, \quad \omega_2^\beta(t) = t^\beta e^t, \]

subject to the boundary values

\[ \lim_{t \to +\infty} y(t) = 0, \quad \mathcal{I}_t^{1-\alpha}(p_2(t) D_{\infty}^\alpha \phi(t)) \bigg|_{t=0} = 0, \]

are given as

\[ \phi_{n}^{2}(t) = t^{-\beta} L_{n}^{(\beta)}(t), \quad n = 0, 1, 2, \ldots, \]

where \( \beta > -1 \) and the corresponding distinct eigenvalues are

\[ \lambda_2^n = \frac{\Gamma(n + \beta + \alpha + 1)}{\Gamma(n + \beta + 1)}, \quad n = 0, 1, 2, \ldots \]

We will make use of the fact that our trial basis functions are the eigenfunctions of the FSLP-1 in Section 3.4 below, where we discuss the rate of decay of the coefficients of our Galerkin expansion.
2.3. Useful properties of Laguerre polynomials. The left- and right-sided Riemann-Liouville derivatives of the generalized associated Laguerre functions (GALFs) are given by (from [16])

\[\begin{align}
0D_t^{\nu} \phi_m^{1,2}(t) &= \frac{\Gamma(m + \alpha_1)}{\Gamma(m + \alpha_1 - \nu)} t^{\alpha_1 - \nu} L_{m-1}^{(\alpha_1 - \nu)}(t) = \frac{\Gamma(m + \alpha_1)}{\Gamma(m + \alpha_1 - \nu)} \phi_m^{1,2}(t), \\
\end{align}\]

where \(\nu > 0\) and \(\alpha_1, \alpha_2 > -1\).

**Lemma 5.** The GALFs satisfy the following orthogonality property.

\[\begin{align}
\int_0^{\infty} \phi_n^{1,2}(t) \phi_k^{1,2}(t) \, dt &= \int_0^{\infty} t^{\beta} e^{-t} L_{n-1}^{(\beta)}(t)L_{k-1}^{(\beta)}(t) \, dt = \gamma_n^\beta \delta_{kn}, \\
\gamma_n^\beta := \frac{\Gamma(n + \beta)}{\Gamma(n)},
\end{align}\]

Notice that when \(\beta = 0\), the resulting matrix is the identity.

2.4. Fractional integration by parts. In order to develop the Petrov-Galerkin method, we will need to employ fractional integration by parts on the half line involving the GALFs. We will prove Lemma 6 following the technique presented in [33].

**Lemma 6.** For real \(\nu, 0 < \nu < 1\), if \(\Omega := (0, +\infty)\), \(\phi_n^{1,2}(t)\) is the GALF of the first kind, and \(\phi_k^{1,2}(t)\) is the GALF of the second kind, and \(\alpha, \beta > -1\), then

\[\begin{align}
\left(0D_t^{\nu} \phi_n^{1}(t), \phi_k^{1,2}(t) \right)_\Omega = \left(\phi_n^{1}(t), tD_t^{\nu} \phi_k^{1,2}(t) \right)_\Omega.
\end{align}\]

**Proof.** Using integration by parts,

\[\begin{align}
\left(0D_t^{\nu} \phi_n^{1}(t), \phi_k^{1,2}(t) \right)_\Omega &= \int_0^{\infty} 0D_t^{\nu} \left(t^\alpha L_{n-1}^{(\alpha)}(t)\right) e^{-t} L_{k-1}^{(\beta)}(t) \, dt \\
&= \int_0^{\infty} \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int_0^t \frac{s^\alpha L_{n-1}^{(\alpha)}(s)}{(t-s)^{\nu}} \, ds \ e^{-t} L_{k-1}^{(\beta)}(t) \, dt \\
&= \frac{e^{-t} L_{k-1}^{(\beta)}(t)}{\Gamma(1-\nu)} \int_0^t \frac{s^\alpha L_{n-1}^{(\alpha)}(s)}{(t-s)^{\nu}} \, ds \ \bigg|_0^{\infty} - \frac{1}{\Gamma(1-\nu)} \int_0^{\infty} \int_0^t \frac{s^\alpha L_{n-1}^{(\alpha)}(s)}{(t-s)^{\nu}} \, ds \ \frac{d}{dt} \left\{e^{-t} L_{k-1}^{(\beta)}(t)\right\} \, dt \\
&= -\frac{1}{\Gamma(1-\nu)} \int_0^{\infty} \int_0^t \frac{s^\alpha L_{n-1}^{(\alpha)}(s)}{(t-s)^{\nu}} \, ds \ \frac{d}{dt} \left\{e^{-t} L_{k-1}^{(\beta)}(t)\right\} \, dt.
\end{align}\]
Now we use integration by parts again.

\[
\frac{d}{dt} \int_t^\infty \frac{e^{-s} L_k^{(\beta)}(s) (s-t)^\nu}{(s-t)^\nu} \, ds = \frac{d}{dt} \left[ \frac{e^{-s} L_k^{(\beta)}(s) (s-t)^{1-\nu}}{1-\nu} \right]_t^\infty - \frac{1}{1-\nu} \int_t^\infty \frac{d}{ds} \left\{ e^{-s} L_k^{(\beta)}(s) \right\} (s-t)^{1-\nu} \, ds \\
= -\frac{1}{1-\nu} \int_t^\infty \frac{d}{ds} \left\{ e^{-s} L_k^{(\beta)}(s) \right\} (s-t)^{1-\nu} \, ds \\
= \int_t^\infty \frac{d}{ds} \left\{ e^{-s} L_k^{(\beta)}(s) \right\} (s-t)^{1-\nu} \, ds.
\]

Using (30), the right hand side of (29) can be written as

\[
-\frac{1}{\Gamma(1-\nu)} \int_0^t \int_0^s \frac{s^\alpha L_n^{(\alpha)}(s-t)^\nu}{(s-t)^\nu} \, ds \frac{d}{dt} \left\{ e^{-t} L_k^{(\beta)}(t) \right\} \, dt \\
= -\frac{1}{\Gamma(1-\nu)} \int_0^t \int_0^s \frac{d}{ds} \left\{ e^{-s} L_k^{(\beta)}(s) \right\} (s-t)^\nu \, ds \, t^\alpha L_n^{(\alpha)}(t) \, dt \\
= -\frac{1}{\Gamma(1-\nu)} \int_0^t \left( \frac{d}{dt} \int_0^t \frac{e^{-s} L_k^{(\beta)}(s)}{s} \, ds \right) t^\alpha L_n^{(\alpha)}(t) \, dt \quad \text{(by (30))} \\
= \left( t^\alpha L_n^{(\alpha)}(t), tD_\infty^\nu \left\{ e^{-t} L_k^{(\beta)}(t) \right\} \right)_\Omega. \\
\]

The combination of (29) and (31) gives the desired result.

\[\text{Remark 7. Using the property of Riemann-Liouville fractional derivatives from \cite{3} that if } 0 < p < 1, 0 < q < 1, v(0) = 0, \text{ and } t > 0, \]

\[
g_0 D_t^{\nu+q} v(t) = g_0 D_t^p g_0 D_t^q v(t) = g_0 D_t^p v(t),
\]

we can infer from Lemma 6 that

\[
g_0 D_t^{\nu+q} \phi_n^{1,2}(t) = \left( g_0 D_t^p \phi_n^{1}(t), tD_\infty^\nu \phi_k^{2}(t) \right)_\Omega.
\]

We will use formula (33) in the variational form for the derivation of our Petrov-Galerkin method in the following section.

3. Petrov-Galerkin spectral method. As an example problem, we consider the case \( K = 2 \), with \( b_1 = b_2 = 1 \):

\[
g_0 D_t^{\nu_1} u(t) + g_0 D_t^{\nu_2} u(t) = f(t), \quad t \in (0, +\infty), \\
u_1(0) = 0,
\]

where \( \nu_1, \nu_2 \in (0, 1) \). We use the GALFs of the first kind to approximate the solution:

\[
u(t) \approx u_N(t) = \sum_{n=1}^N a_n \phi_n^{\nu_1,1}(t),
\]
with \( \{a_n\}_{n=1}^N \) the unknown coefficients. The trial and test functions are defined as the eigenfunctions of the singular Sturm-Liouville problems of the first and second kinds, respectively:

\[
\begin{align*}
\phi_n^{0,1}(t) &= t^{\alpha_1} L_{n-1}^{(\alpha_1)}(t), \\
\phi_k^{\alpha_2}(t) &= e^{-t} L_{k-1}^{(\alpha_2)}(t).
\end{align*}
\]

Then the variational form for the PG spectral method is

\[
\begin{align*}
\sum_{n=1}^{N} a_n \int_{0}^{\infty} \phi_k^{\alpha_2}(t) \phi_n^{0,1}(t) dt + \sum_{n=1}^{N} a_n \int_{0}^{\infty} \phi_k^{\alpha_2}(t) \phi_n^{0,1}(t) dt &= \int_{0}^{\infty} f(t) \phi_k^{\alpha_2}(t) dt =: \hat{f}_k.
\end{align*}
\]

Next, we apply Lemma 6 to the variational form:

\[
\begin{align*}
\sum_{n=1}^{N} a_n \int_{0}^{\infty} \partial_t^{\alpha_1} \phi_n^{0,1}(t) \partial_t^{\nu_1-\alpha_1} \phi_k^{\alpha_2}(t) dt + \\
\sum_{n=1}^{N} a_n \int_{0}^{\infty} \partial_t^{\alpha_1} \phi_n^{0,1}(t) \partial_t^{\nu_2-\alpha_1} \phi_k^{\alpha_2}(t) dt &= \hat{f}_k,
\end{align*}
\]

where we keep the left-sided derivative of order \( \alpha_1 \) applied to the trial basis functions and transfer the rest of the derivative to the test functions. We tune \( \alpha_1 \) to optimize the convergence of the spectral method, and \( \alpha_2 = \alpha_1 - \nu_1 \).

Using the parameters defined above and Lemma 5, the variational form reduces to:

\[
\begin{align*}
\sum_{n=1}^{N} a_n \int_{0}^{\infty} \Gamma(n + \alpha_1) \phi_n^{0,1}(t) \phi_k^{0,2}(t) dt + \\
\sum_{n=1}^{N} a_n \int_{0}^{\infty} \Gamma(n + \alpha_1) \phi_n^{0,1}(t) \phi_k^{0,2}(t) dt &= \sum_{n=1}^{N} \Gamma(n + \alpha_1) \phi_n^{0,1}(t) \phi_k^{0,2}(t) dt \\
&= \sum_{n=1}^{N} \frac{\Gamma(n + \alpha_1)}{\Gamma(n)} \left[ \delta_{kn} + \int_{0}^{\infty} \phi_n^{0,1}(t) \phi_k^{0,2}(t) dt \right] \\
&= \sum_{n=1}^{N} \frac{\Gamma(n + \alpha_1)}{\Gamma(n)} \left[ \delta_{kn} + \int_{0}^{\infty} e^{-t} L_{n-1}(t) L_{k-1}(t) dt \right].
\end{align*}
\]

Then it remains to solve the linear system

\[
\begin{align*}
S\hat{a} &= \hat{f}.
\end{align*}
\]
where the coefficient matrix $S$ is defined

$$S_{kn} = \frac{\Gamma(n + \alpha_1)}{\Gamma(n)} \left[ \delta_{kn} + \int_0^\infty \phi_n^{0,1}(t) \phi_k^{\nu_2 - \nu_1, 2}(t) \, dt \right],$$

and $\hat{f}_k$ is defined by the integral

$$\hat{f}_k := \int_0^\infty f(t) \phi_k^{\nu_2 - \nu_1, 2}(t) \, dt = \int_0^\infty f(t) e^{-t} L_{n-1}^{(\nu_2 - \nu_1)}(t) \, dt.$$ We compute this integral using Gauss-Laguerre quadrature.

### 3.1. Factorization of the linear system.

The integral in (42) has the form

$$Q_{kn} := \int_0^\infty \phi_n^{0,1}(t) \phi_k^{\nu_2 - \nu_1, 2}(t) \, dt = \int_0^\infty f(t) e^{-t} L_{n-1}^{(\nu_2 - \nu_1)}(t) \, dt.$$ The matrix $Q$ is a lower-triangular Toeplitz matrix, i.e.

$$Q = \begin{bmatrix} q_1 & 0 & 0 & \cdots & 0 \\ q_2 & q_1 & 0 & \cdots & 0 \\ q_3 & q_2 & q_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ q_N & q_{N-1} & q_{N-2} & \cdots & q_1 \end{bmatrix},$$

where the entries are given by the formula

$$q_{k-n+1} = \prod_{i=1}^{k-n} \frac{\nu_2 - \nu_1 + i - 1}{\alpha_i},$$

with $k$ as the row index and $n$ as the column index of $Q$. We can use the formula (46) to assemble the stiffness matrix with explicit expressions for each entry instead of using quadrature, which offers significant savings in the cost of assembling the stiffness matrix as well as eliminating any approximation error for these entries. The Toeplitz structure offers additional savings in storage and makes the process of p-refinement efficient since we can store the values of the stiffness matrix from the previous approximation. Indeed, going from the $N^{th}$ order expansion to the $(N+1)^{th}$ requires that we add one row and one column to $Q$ (hence $S$), but as this matrix will also be Toeplitz and the entries $q_m$ of $Q$ only depend on the orders of the fractional derivatives and the number of their diagonal ($m$), the only new entry that we will need to compute is $q_{N+1}$.

We can derive formula (46) using the recurrence identity

$$L_{\alpha}^{(\alpha)}(t) = \sum_{i=0}^{n} \binom{\alpha - \beta + n - i - 1}{n - i} L_{i}^{(\beta)}(t).$$

Consider again the matrix entry $Q_{kn}$:

$$Q_{kn} = \int_0^\infty e^{-t} L_{n-1}^{(\nu)}(t) L_{k-1}^{(\nu_2 - \nu_1)}(t) \, dt.$$
We plug in the recurrence identity (47) to expand the Laguerre polynomial $L_{k-1}^{(\nu_2-\nu_1)}(t)$ in terms of standard Laguerre polynomials:

\begin{align}
Q_{kn} &= \int_0^\infty e^{-t} L_{n-1}(t) \sum_{i=0}^{k-1} \left( \frac{\nu_2 - \nu_1 + k - i - 2}{k - i - 1} \right) L_i(t) \, dt \\
&= \sum_{i=0}^{k-1} \left( \frac{\nu_2 - \nu_1 + k - i - 2}{k - i - 1} \right) \int_0^\infty e^{-t} L_{n-1}(t) L_i(t) \, dt \\
&= \sum_{i=1}^{k} \left( \frac{\nu_2 - \nu_1 + k - i - 1}{k - i} \right) \int_0^\infty e^{-t} L_{n-1}(t) L_{i-1}(t) \, dt \\
&= \sum_{i=1}^{k} \left( \frac{\nu_2 - \nu_1 + k - i - 1}{k - i} \right) \delta_{ni} \\
&= \left\{ \begin{array}{ll} (\nu_2 - \nu_1 + k - n - 1), & n \leq k \\ 0, & n > k. \end{array} \right.
\end{align}

This implies that $Q$ is lower triangular. Using the product formula to compute the binomial coefficient, we find that

\begin{equation}
Q_{kn} = \begin{cases} \frac{1}{(k-n)!} \Pi_{\ell=1}^{k-n} (\nu_2 - \nu_1 + \ell - 1), & k \geq n, \\ 0, & k < n. \end{cases}
\end{equation}

Hence we can construct the coefficient matrix $S$ exactly, and we will show that $S$ is a lower-triangular matrix which can be factored in a way that reduces the complexity of solving the linear system to $O(N \log N)$ operations.

To demonstrate how this is done, we define the matrix $\tilde{S}$ by

\begin{equation}
\tilde{S}_{kn} = \delta_{kn} + \int_0^\infty e^{-t} L_{n-1}(t) L_{k-1}^{(\nu_2-\nu_1)}(t) \, dt.
\end{equation}

Then

\begin{equation}
S_{kn} = \frac{\Gamma(n + \alpha_1)}{\Gamma(n)} \tilde{S}_{kn},
\end{equation}

where $n$ is the column index. If $\{\tilde{s}_n\}_{n=1}^N$ are the column vectors of $S$, and $\{\tilde{s}_n\}_{n=1}^N$ are the column vectors of $\tilde{S}$, then for the solution vector $a = [a_1 \ a_2 \ \cdots \ a_N]$, we have

\begin{equation}
Sa = \tilde{s}_1 a_1 + \tilde{s}_2 a_2 + \cdots + \tilde{s}_N a_N
= \tilde{s}_1 \frac{\Gamma(1 + \alpha_1)}{\Gamma(1)} a_1 + \cdots + \tilde{s}_N \frac{\Gamma(N + \alpha_1)}{\Gamma(N)} a_N
= \tilde{s}_1 \tilde{a}_1 + \cdots + \tilde{s}_N \tilde{a}_N
= \tilde{S} \tilde{a} = \hat{f},
\end{equation}

where $\hat{f}$ is known, $\tilde{S}$ is a lower-triangular Toeplitz matrix, and

\begin{equation}
\tilde{a}_k := \frac{\Gamma(k + \alpha_1)}{\Gamma(k)} a_k.
\end{equation}
This procedure is equivalent to factoring the stiffness matrix into a Toeplitz matrix \( \tilde{S} \) and a diagonal matrix \( D \), resulting in the linear system with the form

\[
Sa = \tilde{S}a = \tilde{S}Da = \hat{f},
\]

where

\[
\hat{f} = \begin{bmatrix}
q_1 + 1 & 0 & 0 & \cdots & 0 \\
q_2 & q_1 + 1 & 0 & \cdots & 0 \\
q_3 & q_2 & q_1 + 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
q_N & q_{N-1} & q_{N-2} & \cdots & q_1 + 1
\end{bmatrix},
\]

\[
\theta = \begin{bmatrix}
\Gamma(1+\alpha_1) & 0 & 0 & \cdots & 0 \\
0 & \Gamma(2+\alpha_1) & 0 & \cdots & 0 \\
0 & 0 & \Gamma(3+\alpha_1) & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \Gamma(N+\alpha_1) \\
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
\vdots \\
a_N
\end{bmatrix}.
\]

Hence we can solve for \( \hat{a} \) in \( O(N \log N) \) operations using the algorithm in [34] and compute \( a \) from \( \hat{a} \) in another \( O(N) \) operations. Since the matrix \( D \) only depends on parameter \( \alpha_1 \), which comes from the approximation \( u_N \) itself, the stiffness matrix will have this structure in the case where the number of terms in the FIVP, \( K \), is greater than 2. In fact, we can solve multi-term FIVPs with any number of terms with \( O(N \log N) \) operations, as discussed in Section 3.3 below.

It is interesting to note here that mass matrices will have a similar form using this approximation method. Mass matrices appear in the approximation when \( \nu_i = 0 \) for some \( i \leq K \).

### 3.2. Arbitrary number of terms in the FIVP.

In the development of the method above, we have assumed that \( K = 2 \), yielding the two-term equation

\[
\sum_{i=1}^{K} b_i \int_{t}^{t} d^{\nu_i} u(t) = b_{10} \int_{t}^{t} d^{\nu_i} u(t) + b_{20} \int_{t}^{t} d^{\nu_i} u(t) = f(t).
\]

The next natural question is whether we achieve a similar structure of the stiffness matrix if the number of terms on the left hand side, \( K \), is greater than 2. If we follow the same derivation of the stiffness matrix as above in the case where \( K = 3 \) with \( b_1 = b_2 = b_3 = 1 \), for example, we find that

\[
S_{nk} = \frac{\Gamma(n + \alpha_1)}{\Gamma(n)} \left[ \delta_{nk} + \int_{0}^{\infty} \phi^{0,1}_{n}(t) \phi^{0,1}_{k} d^{\nu_1} u(t) dt + \int_{0}^{\infty} \phi^{0,1}_{n}(t) \phi^{0,1}_{k} d^{\nu_1} u(t) dt \right]
\]
Then we define matrices $Q_1$ and $Q_2$ as

\[
(Q_1)_{kn} := \int_0^\infty \phi_n^{\nu_1}(t) \phi_k^{\nu_2-\nu_1}(t) \, dt = \int_0^\infty e^{-t} L_{n-1}(t) L_{k-1}^{(\nu_2-\nu_1)}(t) \, dt
\]

\[
= \begin{cases} 
\frac{1}{(k-n)!} \prod_{\ell=1}^{k-n} (\nu_2 - \nu_1 + \ell - 1), & k \geq n \\
0, & k < n,
\end{cases}
\]

(63)

\[
(Q_2)_{kn} := \int_0^\infty \phi_n^{\nu_1}(t) \phi_k^{\nu_3-\nu_1}(t) \, dt = \int_0^\infty e^{-t} L_{n-1}(t) L_{k-1}^{(\nu_3-\nu_1)}(t) \, dt
\]

\[
= \begin{cases} 
\frac{1}{(k-n)!} \prod_{\ell=1}^{k-n} (\nu_3 - \nu_1 + \ell - 1), & k \geq n \\
0, & k < n.
\end{cases}
\]

If we represent the diagonal entries of $Q_1$ and $Q_2$ by $q_m^{(1)}$ and $q_m^{(2)}$, respectively, with $m = k - n + 1$, the resulting stiffness matrix is

\[
S_{kn} = \frac{\Gamma(n + \alpha_1)}{\Gamma(n)} S_{kn},
\]

(64)

\[
\tilde{S}_{kn} = \begin{bmatrix} 
q_1^{(1)} + q_1^{(2)} + 1 & 0 & \cdots & 0 \\
q_2^{(1)} + q_2^{(2)} & q_1^{(1)} + q_1^{(2)} + 1 & \cdots & 0 \\
q_3^{(1)} + q_3^{(2)} & q_2^{(1)} + q_2^{(2)} & q_1^{(1)} + q_1^{(2)} + 1 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
q_N^{(1)} + q_N^{(2)} & q_{N-1}^{(1)} + q_{N-1}^{(2)} & \cdots & q_1^{(1)} + q_1^{(2)} + 1
\end{bmatrix}
\]

Hence $S$ can again be factored into $\tilde{S}D$ with $D$ defined as in (61). Then we define $\tilde{a}_k := \frac{\Gamma(k+\alpha_1)}{\Gamma(k)} a_k$ in the same way as before, and follow the same procedure as in the $K = 2$ case to invert $\tilde{S}$ and $D$. We continue in this way for any value of $K \in \mathbb{N}$ to see that we can solve the resulting linear system for any number of terms using $O(N \log N)$ operations.

### 3.3. Spectral decay of coefficients in Galerkin projection.

In this section, we are mainly interested in the rate of decay of the coefficients of the Galerkin expansion. Given the weight function $w(t) = t^{-\beta} e^{-t}$, we expand a function $u(t) \in L_w^2(0, \infty)$ by

\[
u(t) \approx u_N(t) := \sum_{n=1}^N a_n t^\beta L_{n-1}^\beta(t).
\]

(65)

Then following [35], since $t^\beta L_{n-1}^\beta(t)$ is an eigenfunction for the FSLP-1, we have from (27) and Lemma 6,

\[
\|u\|_{L_w^2}^2 = \sum_{n=1}^N \gamma_n^\beta |a_n|^2.
\]

(66)
Therefore

\[
\begin{align*}
    a_n &= \frac{1}{\gamma_n} (u, \phi_n^{\beta, 1})_{L^2_w} \\
    &= \frac{1}{\gamma_n} \int_0^\infty u(t) \phi_n^{\beta, 1}(t) w(t) \, dt \\
    &= \frac{\lambda_n}{\gamma_n} \int_0^\infty u(t) \mathcal{L}_1^{\alpha, \beta} [\phi_n^{\beta, 1}(t)] \, dt \\
    &= \frac{\lambda_n}{\gamma_n} \int_0^\infty (u \mathcal{D}_1^\alpha u(t)) e^{-\alpha t} \, dt \\
    &= \frac{\lambda_n}{\gamma_n} \int_0^\infty \mathcal{L}_1^{\alpha, \beta} [u(t)] \phi_n^{\beta, 1}(t) \, dt \\
    &= \frac{1}{\gamma_n} \int_0^\infty (u(1), \phi_n^{\beta, 1})_{L^2_w}.
\end{align*}
\] (67)

We have defined \( u_m \) as in [36]

\[
\begin{align*}
    u_m(t) &= \frac{1}{w(t)} \mathcal{L}_1 u_{m-1}(t) = \left( \frac{\mathcal{L}}{w(t)} \right)^m u(t).
\end{align*}
\] (68)

Then

\[
\begin{align*}
    \frac{1}{\gamma_n} (u(1), \phi_n^{\beta, 1})_{L^2_w} &= \frac{1}{\gamma_n} (u(2), \phi_n^{\beta, 1})_{L^2_w} \\
    &= \cdots \\
    &= \frac{1}{\gamma_n} (u(m), \phi_n^{\beta, 1})_{L^2_w}.
\end{align*}
\] (69)

We know from [16] that the eigenvalues have the asymptotic similarity

\[
\lambda_n \sim n^\alpha.
\] (70)

So the coefficients of the approximation decay at the rate:

\[
|a_n| \sim C \frac{1}{(\lambda_n^1)^{\alpha}} \|u_m\|_{L^2_w} \sim C n^{-\alpha m} \|u_m\|_{L^2_w}.
\] (71)

If \( u \in C^\infty(0, \infty) \), we expect exponential convergence of the approximation.

\textbf{4. Numerical Results.} In this section, we present numerical examples which demonstrate the validity of our proposed method. We plot relative errors computed using Gauss Laguerre quadrature for various values of \( N \), which represents the number of terms in the Galerkin expansion. The formula for the relative errors, represented by \( e_N \), is given by

\[
\begin{align*}
    e_N &= \frac{\|u^{\text{ext}} - u_N\|_{L^2(0, \infty)}}{\|u^{\text{ext}}\|_{L^2(0, \infty)}}.
\end{align*}
\] (72)

where the weight function is \( \omega(t) = e^{-t} \).
4.1. Example 1. In this example, we solve the multi-term FIVP

\[
\partial_t^{2/3} u(t) + \partial_t^{1/10} u(t) = f(t),
\]
\[
u(0) = 0.
\]

(73)

We test the method using the fabricated solution \(\hat{u}^{\text{ext}}(t) = t^{5+1/2}\).

In Figure 1, we plot the numerical solutions using six different values of the tuning parameter \(\alpha_1\). Recall that the basis functions used in the Galerkin expansion for this method have the form

\[
\phi_n^{\alpha_1,1}(t) = t^{\alpha_1} L_n^{(\alpha_1)}(t),
\]

(74)

so adjusting this tunable parameter requires an entirely new approximation. Since the fabricated solution has a fractional singularity of order 1/2, we expect that the method will return the exact solution when \(\alpha_1 = 1/2\). We can see that this is consistent with Figure 1, where the relative errors corresponding to these values of \(\alpha_1\) drop to machine precision after three and four terms are used in the expansion, respectively.

We achieve algebraic convergence in this example since the solution has finite regularity. The rates of convergence printed in the legend of Figure 1 are computed by taking the slope in the log-log scale of the line between the last two computed relative errors. In view of the regularity of the fabricated solution, the results in Figure 1 demonstrate that the method converges optimally for this example.

Further, the tunable accuracy of the method is demonstrated in that the smallest perturbation from the optimal \(\alpha_1\)-values results in the fastest rate of convergence (apart from the case where the solution is achieved exactly). We note that finding the optimal value of the tuning parameter \(\alpha_1\) is non-trivial, and we do not propose a method for discovering the optimal value in this work. However, our numerical examples demonstrate that even when the optimal value of \(\alpha_1\) is not known, the method still converges with the correct rate corresponding to the smoothness of the true solution.
4.2. Example 2. We again solve a two-term FIVP:

\[
0D_t^{1/4}u(t) + 0D_t^{1/5}u(t) = f(t), \quad t \in (0, +\infty),
\]

\[
u_1 = 2/3, \quad \nu_2 = 1/10
\]

\[u(0) = 0.\]

We use the fabricated solution \(u^{\text{ext}}(t) = t^{1/2} \sin(t)\), and we observe exponential convergence of the method for this example as shown in Figure 3.

Fig. 1: Weighted Relative \(L^2\)-error for Example 1 in a log-log scale.
Fig. 2: Weighted relative $L^2$-error for Example 2 in a log-log scale.

Fig. 3: Weighted relative $L^2$-error for Example 2 in a log-linear scale.
4.3. Example 3. We repeat Example 2 in the case where only the forcing function \( f(t) \) is given, and we have chosen \( f(t) = t^5 \) so that \( u \) satisfies the equation

\[
0D^{1/4}_tu(t) + 0D^{1/5}_tu(t) = t^5, \\
u(0) = 0.
\]

The solutions for \( N = \{3, 4, 5, 6\} \) are plotted below in addition to the reference solution computed with \( N = 128 \). We use this reference solution to compute the weighted relative \( L^2 \) error in Figure 4.

![Fig. 4: Plots of computed solutions with \( N = \{3, 4, 5, 6\}, \alpha_1 = 1/2, \) and \( N = 128 \) as the reference solution. In this range, the solution for \( N = 6 \) coincides with the reference solution.](image)

4.4. Example 4. In Example 4, we solve the two-term FIVP

\[
0D^{1/5}_tu(t) + 0D^{1/2}_tu(t) = f(t), \quad t \in (0, +\infty) \\
u(0) = 0.
\]

We use the fabricated solution \( u^{\text{ext}}(t) = 5t^{7/2} + 4t^2 + t^{5/3} \). We believe this to be an interesting example because the optimal value of \( \alpha_1 \) is not clear. Using our set of basis functions to approximate this solution will not allow us to capture the result exactly in only a few terms as before, since there are two terms with different order fractional singularities at \( t = 0 \).

As shown in Figure 6, the approximation using \( \alpha_1 = 1/2 \) seems to give the best approximation to the fabricated solution after the first few values of \( N \), although the asymptotic convergence rate is slower than for the other tested values. The \( \alpha_1 \) with the fastest convergence rate of those tested is \( \alpha_1 = 1/6 \).
Fig. 5: Weighted relative $L^2$-error for Example 3 with respect to the reference solution computed using $N = 128$ and $\alpha_1 = 1/2$.

Fig. 6: Weighted Relative $L^2$-error for Example 4 in a log-log scale.
4.5. Example 5. To demonstrate that we can also solve equations with a larger number of terms with high accuracy, we solve the fifty-term FIVP:

\[
\sum_{i=1}^{50} \phi_i D_i^{\nu_i} u(t) = f(t), \quad t \in (0, +\infty)
\]

\[u(0) = 0,
\]

(78)

where each \( \nu_i \in [0, m] \), with \( m \leq 1 \). We choose the parameters as

\[
\nu_i = \frac{(i - 1)m}{K - 1}, \quad K = 50, \quad m = \frac{11}{12}.
\]

We use the fabricated solution \( u^{ext}(t) = t^{2+1/4} \) to plot the weighted relative \( L^2 \) error in Figure 7.

![Fig. 7: Weighted Relative \( L^2 \)-error for Example 5 in a log-log scale.](image)

For this example, we also computed the condition numbers of the stiffness matrices resulting from the different values of \( \alpha_1 \). In Table 1, we observe that the condition numbers in our examples are approximately \( O(N) \).
Table 1: Condition numbers of the stiffness matrices $S$ in the fifty-term equation for different values of the tuning parameter $\alpha_1$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\alpha_1 = \frac{1}{4}$</th>
<th>$\alpha_1 = \frac{1}{2} - \frac{1}{10}$</th>
<th>$\alpha_1 = \frac{1}{3} - \frac{1}{10}$</th>
<th>$\alpha_1 = \frac{1}{4}$</th>
<th>$\alpha_1 = \frac{1}{2}$</th>
</tr>
</thead>
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<tr>
<td>4</td>
<td>2.4325</td>
<td>2.2840</td>
<td>2.4152</td>
<td>2.9990</td>
<td>3.4963</td>
</tr>
<tr>
<td>8</td>
<td>3.9999</td>
<td>3.5478</td>
<td>3.9490</td>
<td>5.6683</td>
<td>7.2349</td>
</tr>
<tr>
<td>16</td>
<td>6.9501</td>
<td>5.7990</td>
<td>6.8199</td>
<td>11.4541</td>
<td>16.1454</td>
</tr>
<tr>
<td>32</td>
<td>12.5912</td>
<td>9.8565</td>
<td>12.2769</td>
<td>24.3153</td>
<td>38.0687</td>
</tr>
<tr>
<td>64</td>
<td>23.5561</td>
<td>17.2719</td>
<td>22.8183</td>
<td>53.5526</td>
<td>93.4535</td>
</tr>
<tr>
<td>128</td>
<td>45.1956</td>
<td>31.0068</td>
<td>43.4901</td>
<td>121.3111</td>
<td>236.4697</td>
</tr>
</tbody>
</table>

In order to compare timings of the method for different values of $K$, we timed our PG method solving the equation in Example 5 for $K = 2, 10,$ and 50, where the orders $\nu_i$ are defined using the formula in (79). In Figure 8, we show the timings in actual seconds for $N = 1, 2, 3, \ldots, 30$, along with a best-fit line. The timings include the computation of the load vector $\tilde{f}$ and inverting the linear system to solve for the coefficients $\tilde{a}$. As $N$ increases, we also increase the number of quadrature points used for computing $\tilde{f}$ to maintain the desired level of accuracy. These timings were collected with Mathematica using a 3 GHz Intel Core i7 processor.

5. Application to distributed order equations. Multi-term fractional differential equations have been used in combination with a quadrature rule to solve distributed order differential equations of the form

$$
\int_0^m g(r)_{0D^r_t}u(t) \, dr = f(t),
$$

where the integral on the left hand side is called the distributed order derivative. The function $g(r)$ that appears in the integrand is a distribution where the argument $r$ corresponds to the order of the fractional derivative. This function must be integrable on $[0,m]$ and satisfy the property $g(r) \geq 0$ for all $r \in [0,m]$.

5.1. Discretized distributed order equation. The idea for solving this equation using multi-term fractional differential equations was proposed by Diethelm and Ford [26], where they applied trapezoidal quadrature to the integral in (80) to derive a linear multi-term equation in a bounded interval with constant coefficients, and then applied a finite difference method to solve the distributed order equation. This application highlights the usefulness of algorithms, which can efficiently solve multi-term equations with a high number of terms, as may be necessary to decrease the error due to the quadrature.

For our numerical experiments, we applied the Gauss-Legendre quadrature rule, which has been shown to be spectrally accurate for this setting in the paper by Kharazmi et al. [37]. This is demonstrated in Figures 9 and 10.

We are interested in solving the distributed order fractional differential equation on the half line:

$$
\int_0^m g(r)_{0D^r_t}u(t) \, dr = f(t), \quad t \in (0, +\infty)
$$

$$
u(0) = 0,
$$
where $m \in [0, 1]$ and $\_0^{D^m_r} \cdot$ represents a Riemann-Liouville fractional derivative.

We apply Gauss-Legendre quadrature to the left hand side of (81) side to get the multi-term FIVP:

$$
\sum_{i=1}^{K} w_i g(\nu_i) \_0^{D^m_r} u(t) \approx f(t), \quad t \in (0, +\infty)
$$

(82)

$$
u(0) = 0,
$$

where $K$ is the number of quadrature nodes $\{\nu_i\}$ and the weights of the quadrature rule are represented by $\{w_i\}_{i=1}^{K}$. Recall that we approximate the solution to the multi-term equation as

$$
\approx u_N(t) = \sum_{n=1}^{N} a_n \phi_n^{\alpha_1,1}(t),
$$

(83)

where

$$
\phi_n^{\alpha_1,1}(t) := t^{\alpha_1} L_{n-1}^{(\alpha_1)}(t),
$$

(84)
where \( L_{n-1}^{(\alpha_1)}(t) \) is the associated Laguerre polynomial of order \( n-1 \).

We integrate against the test functions
\[
\phi_k^{\alpha_2} (t) := e^{-t} L_{k-1}^{(\alpha_2)}(t)
\]
where \( \alpha_2 = \alpha_1 - \nu_1 \). Then the variational form for the Petrov-Galerkin method is given by
\[
\int_0^\infty \phi_k^{\alpha_2} (t) \sum_{i=1}^K w_i g(\nu_i) D_i^{\nu_i} \left( \sum_{n=1}^N a_n \phi_n^{\alpha_1} (t) \right) dt = \int_0^\infty f(t) \phi_k^{\alpha_2} (t) dt =: \hat{f}_k.
\]

Next, we apply fractional integration by parts and the properties of the GALFs as described above:
\[
\sum_{n=1}^N a_n \frac{\Gamma(n + \alpha_1)}{\Gamma(n)} \left[ w_1 g(\nu_1) \delta_{kn} + \sum_{i=2}^K w_i g(\nu_i) \int_0^\infty e^{-t} L_{n-1}^{(\nu_1)}(t) L_{k-1}^{(\nu_i - \nu_1)}(t) dt \right] = \hat{f}_k.
\]

It remains to solve the linear system
\[
S \tilde{a} = \tilde{f}
\]
for the vector of coefficients \( \tilde{a} \) using the factorization methods as described above, where the stiffness matrix \( S \) is given by
\[
S_{kn} = w_1 g(\nu_1) \delta_{kn} + \sum_{i=2}^K w_i g(\nu_i) \int_0^\infty e^{-t} L_{n-1}^{(\nu_1)}(t) L_{k-1}^{(\nu_i - \nu_1)}(t) dt.
\]

5.2. Numerical results for distributed order equations. We present convergence results of our PG method and Gauss-Legendre quadrature applied to the distributed order equation (81). The distribution functions \( g(r) \) are chosen to be smooth on the interval \([0,m]\) where \( m < 1 \).

5.2.1. Example 6. In this example, we choose the fabricated solution to be the smooth function \( u^{\text{ext}}(t) = t^3 \) and the distribution function to be \( g(r) = \Gamma(4 - r) \sinh(r) \). Given these choices, we find that the right hand side function \( f(t) \) is
\[
\int_0^m g(r) D_r^{\nu} u^{\text{ext}}(t) dr = \frac{t^3 - m(t^m - \cosh(m) - \log(t) \sinh(m))}{(\log(t))^2 - 1} =: f(t).
\]

The plateaus in the error in Figure 9 correspond to the error for the Gauss-Legendre rule, and we reach machine precision with (a) \( K = 10 \) and (b) \( K = 6 \) quadrature points, where in (a) we use \( m = 9/10 \) and in (b) we use \( m = 1/10 \). We choose the tuning parameter for the PG method to be \( \nu_1 = 1 \). The weighted relative \( L^2 \) error for both quadrature rules is plotted in Figure 9.
5.2.2. Example 7. Now we test a non-smooth example, where the fabricated solution is \( u^{\text{ext}}(t) = t^\lambda \) with \( \lambda = 2 + 1/3 \) and the distribution function is \( g(r) = (1+\frac{r}{3})^2 \). Then the right hand side function \( f(t) \) is

\[
\int_0^m g(r) D_t^r u^{\text{ext}}(t) \, dr = \frac{\lambda^m (t^m - 1)}{\log(t)} =: f(t).
\]

We choose the tuning parameter for the PG method to be \( \alpha_1 = 1/3 \).

The weighted relative \( L^2 \) error for \( m = 9/10, 1/2 \) and \( 1/10 \) is plotted in Figure 10.

Fig. 9: (a) Weighted relative \( L^2 \)-error for Gauss-Legendre quadrature and our PG method applied to Example 6 with \( m = 9/10 \), where \( K \) is the number of quadrature points used. (b) Weighted relative \( L^2 \)-error for Example 6 with \( m = 1/10 \).

Fig. 10: (a) Weighted relative \( L^2 \) error for Example 7 with \( m = 9/10 \). (b) Weighted relative \( L^2 \) error for Example 7 with \( m = 1/10 \).
6. Summary and Conclusion. We have presented a Laguerre Petrov-Galerkin spectral method for solving multi-term fractional initial value problems on the half line with linear complexity, which is a significant reduction from the cubic complexity required for existing spectral methods. The key element of this work that resulted in the reduced complexity was our derivation of a factorization of the discretized system of equations, in which we took advantage of the special structure of the generalized associated Laguerre functions. Our method also has the unusual advantage that it is applicable without modification to multi-term equations that include a reaction term, as the resulting mass matrix has the same special structure that allows us to invert the stiffness matrices efficiently. We have demonstrated the effectiveness of our method by solving a 50-term equation and distributed order equations, which were discretized by a Gauss-Legendre quadrature rule. Our numerical results highlight the sensitivity of the convergence rate to the tunable parameter $\alpha_1$, while we were able to achieve spectral convergence whether or not the parameter was chosen optimally.

In the future, we will examine methods of analyzing our PG method and derive error estimates in the weighted relative $L^2$-norm to prove the spectral convergence of the method demonstrated in the numerical results sections.

REFERENCES


