Computing Fractional Laplacians on Complex-Geometry Domains: Algorithms and Simulations

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Abstract. We consider a fractional Laplacian defined in bounded domains by the eigen-decomposition of the integer-order Laplacian, and demonstrate how to compute very accurately (using the spectral element method) the eigenspectrum and corresponding eigenfunctions in two-dimensional prototype complex-geometry domains. We then employ these eigenfunctions as trial and test bases to first solve the fractional diffusion equation, and subsequently to simulate two-phase flow based on the Navier–Stokes equations combined with a fractional Allen–Cahn mass-preserving model. A key point to the effectiveness of an exponential convergence of this approach is the use of a weighted Gram–Schmidt orthonormalization of the eigenfunctions that guarantees accurate projection and recovery of spectral accuracy for smooth solutions. We demonstrate that even when only part of the eigenspectrum is computed accurately we can still obtain exponential convergence if we employ the complete set of the eigenvectors of the discrete Laplacian. Accuracy is also verified by computing the eigenfunctions on square, disk, and L-shaped domains and obtaining numerical solutions of the fractional diffusion equation for different fractional orders. This is accomplished without the need of solving any linear systems as the eigenfunction decomposition leads naturally to a system of ODEs, and hence no spatial discretization is employed during time stepping. In the second application of the method, we replace the integer-order Laplacian in the Allen–Cahn model with its fractional counterpart and a similar procedure is followed. However, for the Navier–Stokes equations we need to solve a linear system, which we invert using an efficient ADI scheme. We demonstrate the effectiveness of the fractional Navier–Stokes/Allen–Cahn solver for the rising bubble problem in a square domain, and compare the results with the integer-order system and also with results by a different treatment of the fractional diffusion model using one-dimensional fractional derivatives. The present model yields sharper interface thickness compared to the integer-order model for the same resolution while it preserves the isotropic diffusion, and hence it is a good candidate for phase-field modeling of multiphase fluid flows.

Key words. eigenvalue problem, spectral element method, fractional Laplacian, fractional diffusion, fractional phase-field equations

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1. Introduction. In the past two decades, fractional calculus has gained considerable popularity, mainly due to its potential application in numerous diverse fields, including control theory, biology, electrochemical processes, viscoelastic materials, polymers, finance, etc; see, e.g., [1, 2, 4, 5, 6, 18, 22, 26, 27, 30, 31, 32] and the references therein. In particular, fractional diffusion equations have been used to describe anomalous diffusion of particle spreading. Such anomalous behavior can be
represented by Lévy processes whose sample paths can be continuous, continuous with occasional discontinuities, or purely discontinuous. They appear in physics and biology and are also used in modeling finance, option pricing, and other financial instruments. Fractional differential equations have been the focus of many theoretical and numerical studies.

In this paper, we address two open questions in fractional PDEs.

• How to compute efficiently the numerical fractional Laplacian in multidimensions with focus on two-dimensional complex-geometry domains [8, 14].

• How to obtain exponential convergence for smooth solutions in two dimensions.

Considering the entire space $\mathbb{R}^n$, the fractional Laplacian $(-\Delta)^s$ with $0 < s < 1$ can be defined in many different but equivalent ways; see, for example, [9, 19, 28, 36, 39]. In this paper, we consider bounded domains, where defining and computing the fractional Laplacian is not well studied. In contrast to the case of the (standard, integer-order) Laplacian operator, where Dirichlet and Neumann boundary conditions are well understood and have simple interpretations at the particle or probabilistic level, physical or probabilistically motivated interpretations of the fractional Laplacian operator on bounded domains are not well established [11, 44]. The different representations of the fractional Laplacian may lead to different operators when restricted to a bounded domain, and this poses challenges for numerical methods, which naturally require truncation of the operator to a bounded domain. In the present work, we adopt the spectral decomposition approach to define fractional powers of such an operator through classical spectral theory [15] on a bounded domain $\Omega \subset \mathbb{R}^n$. Let $(\lambda_i, \phi_i)_{i=1}^{\infty}$ be the eigenpairs of the Laplacian operator $-\Delta$,

\begin{equation}
-\Delta \phi_i = \lambda_i \phi_i,
\end{equation}

subject to appropriate boundary conditions, which ensure that all the $\lambda_i$’s are non-negative and that $\{\phi_i\}$ is a complete orthonormal basis. Then, if $u$ has the expansion $u(x) = \sum_{i=1}^{\infty} c_i \phi_i(x)$, we formally define [19, 24]

\begin{equation}
(-\Delta)^s u(x) = \sum_{i=1}^{\infty} \lambda_i^s c_i \phi_i(x).
\end{equation}

Specifically, here we focus on eigenvalue problems on complex-geometry domains under Dirichlet and Neumann boundary conditions. We use the spectral/spectral element method [7, 10, 13, 23, 25, 33] for solving the integer-order eigenvalue problem. The numerical results in Appendix A show that the spectral element discrete Laplacian operator (matrix) is an efficient method for approximating the Laplacian operator. After we obtain the numerical eigenpairs of the eigenvalue problem on complex-geometry domains, we can compute the fractional Laplace based on the eigenvalue decomposition [24, 41] on these domains. Based on this construction, we then obtain numerical solutions of the fractional diffusion equation and the fractional Navier–Stokes/phase-field equations with the new fractional Laplace operator.

The rest of this paper is organized as follows. In section 2, we present the spectral element method for solving the eigenvalue problem on prototype complex-geometry domains. The fractional Laplace operator is defined in section 3. To demonstrate the use of the approximated fractional Laplacian for real applications, we introduce a fractional Navier–Stokes/phase-field equations system and present numerical results for the rising bubble problem. We provide a short summary in section 4. In Appendix A, we show the numerical eigenpairs of the problem with different boundary
conditions and demonstrate that the spectral element discrete eigenvalue decomposition is an efficient method to approximate the Laplacian operator. In Appendix B, we present a comparison of the present numerical results with a previous less accurate approximation of the fractional Laplacian.

2. Eigenvalues and eigenfunctions of Laplace operator. We consider the following eigenvalue problem (EVP) for the Laplacian operator:

\begin{align}
-\Delta u - \lambda u &= 0, \quad x \in \Omega, \\
\frac{\partial u}{\partial n} &= 0, \quad \text{or} \quad \partial u / \partial n = 0, \quad \text{or periodic boundary conditions},
\end{align}

where $\Omega \in \mathbb{R}^2$ is a bounded domain.

The spectral element method (SEM) is used for solving (2.1)–(2.2). Then, (2.1)–(2.2) can be written in the discretized form

\begin{equation}
A_N U - \lambda M_N U = 0,
\end{equation}

where $N$ represents the number of the degrees of freedom (DoF) of the linear system (2.3) for the given number of elements $El$ and the polynomial degree $N$ in each element. $A_N$ is the corresponding matrix of the Laplacian operator under certain boundary conditions, $M_N$ is the mass matrix, and $U$ is the numerical solution of $u$. Now the continuous EVP is approximated by the numerical solution of the eigenpairs $(\lambda_i, \phi_i)_{i=1}^N$ of the matrix $K = M_N^{-1} A_N$, and $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$.

In Appendix A, we present the numerical results of the EVP (one and two dimensions) with different boundary conditions. The numerical results show that even though the numerical eigenvalues of high frequency are not correct [43], we still obtain exponential convergence for smooth solutions represented in the complete eigenfunction space.

2.1. SEM on square and L-shaped domains. In this subsection, we solve the EVP (2.1)–(2.2) with SEM on the unit square domain $\Omega = (-1, 1)^2$ and the L-shaped domain $\Omega = (-1, 1)^2/[0, 1)^2$. The domain is divided into $El$ nonoverlapping subdomains $\Omega_l$, $l = 1, \ldots, El$, and $\Omega = \bigcup_{l=1}^{El} \Omega_l$ (see Figure 1). Let $N$ be a nonnegative integer, and let $P_N(\Omega) = \{ p(x) \} \in P_N(\Omega_l), 1 \leq l \leq El$, where $P_N(\Omega_l)$ is the set of all polynomials of degree less than or equal to $N$ defined in $\Omega_l$.

We consider the problem (2.1) subject to homogeneous Dirichlet boundary conditions. Now, let $S_N(\Omega) = P_N(\Omega) \cap H_0^1(\Omega)$. Then we want to find $u_N \in S_N$ and real number $\lambda$ such that, $\forall v \in S_N$ we have

\begin{equation}
\int_{\Omega} \nabla u_N \cdot \nabla v dx - \int_{\Omega} \lambda u_N v dx = 0.
\end{equation}
We will employ spectral element discretization to compute the above formula 
(2.4). To this end, we introduce the affine mapping

\[ \Omega_l \rightarrow F^l; \bar{\Omega}, \]

where \( \bar{\Omega} \) stands for the reference square \((-1, 1)^2 \). Let \( \Omega_l = (a_l, a'_l) \times (b_l, b'_l) \), then \( F^l \)

is given by

\[ \xi = (\xi, \eta) = F^l(x, y) = (F_1^l(x), F_2^l(y)) = \left( \frac{x - a_l}{a'_l - a_l} - 1, \frac{y - b_l}{b'_l - b_l} - 1 \right). \]

Let \((\xi_l, \eta_j), i, j = 0, \ldots, N, \) are the Gauss–Lobatto–Legendre (GLL) points in the reference domain \( \bar{\Omega} \), then the corresponding GLL points in \( \Omega_l \), denoted by \( \xi^l_{ij} \equiv (\xi_i^l, \eta_j^l) \), are defined by \((F^l)^{-1}(\xi_i, \eta_j)\). Let \( \omega_i, i = 0, \ldots, N, \) be the weights of one-dimensional (1D) GLL quadrature in \((-1, 1)\), then the GLL quadrature weights in \( \Omega_l \) are given by \( \omega^l_{i, j} \),

\[ \omega^l_{i, j} = \omega_i h_1^l/2, \quad \omega^l_{2, j} = \omega_j h_2^l/2, \quad 0 \leq i, j \leq N, \]

where \( h_1^l, h_2^l \) are the length of the rectangle \( \Omega_l \) in the \( x \) and \( y \) directions, respectively, i.e., \( h_1^l = a'_l - a_l, \) \( h_2^l = b'_l - b_l \). Let \( G \) denote the set of the global GLL points, i.e., \( G = \{ \xi_{ij} \in \Omega_l, i, j = 0, \ldots, N; l = 1, \ldots, El \} \). Let \( G_D = \partial \Omega \cap G, \) \( G_0 = G/G_D \).

Now we consider the spectral element discretization problem as follows: find \( u_N \in S_N \) such that

\[ (\nabla u_N, \nabla v)_N - \lambda(u_N, v)_N = 0, \quad \forall v \in S_N, \]

where for all piecewise continuous functions \( \phi \) and \( \psi \),

\[ (\phi, \psi)_N = \sum_{l=1}^{El} \sum_{i,j=0}^{N} \phi(\xi_{ij}^l)\psi(\xi_{ij}^l)\omega^l_{i, j}, \]

Let \( \{ L_{ij}^l; i = 0, \ldots, N \} \) be the Lagrangian polynomials associated with GLL points \( \{ \xi_{ij}^l; i = 0, \ldots, N \} \), and \( \{ L_{ij}^l; j = 0, \ldots, N \} \) be associated with \( \{ \eta_{ij}^l; j = 0, \ldots, N \} \). Next we construct a nodal basis (Lagrangian basis) for the space \( S_N \), which are the following piecewise polynomials \( L_{ij}^l \), \( i, j \), such that \( \xi_{ij}^l \in G_0 \), where \( L_{ij}^l \) is defined by

\[ L_{ij}^l(x, y)|_{\Omega_k} = \begin{cases} L_{1,m}(x)L_{2,n}(y) & \text{if } \xi_{ij}^l = \xi_{mn}^k \in \Omega_k, k = 1, \ldots, El, \\ 0 & \text{otherwise}. \end{cases} \]

By choosing the test functions \( v \) to be the Lagrangian basis functions for \( S_N \) and expressing \( u_N \) in terms of these bases, we deduce from problem (2.8) a linear system (2.3). The spectral element stiffness matrix is defined as

\[ A_N = \{ (\nabla L_{ij}^k, \nabla L_{mn}^l)_N; \xi_{ij}^k, \xi_{mn}^l \in G_0 \}, \]

and the mass matrix is defined as

\[ M_N = \{ (L_{ij}^k, L_{mn}^l)_N; \xi_{ij}^k, \xi_{mn}^l \in G_0 \}; \]

\( U \) is the vector depending on nodal values of \( u_N \).

It is clear that \( A_N \) and \( M_N \) are \( N \times N \) symmetric positive definite and \( M_N \) is a diagonal matrix. In fact, it has been shown [7, 10] that the mass matrix of the standard spectral/spectral element method is always diagonal.
2.2. SEM on the unit disk domain. In this subsection, we propose an extension of SEM for solving (2.1) on the unit disk domain \( \Omega = \{(x, y) | x^2 + y^2 < 1\} \). In order to apply SEM in this case, we have to first transform the disk to a square domain (see Figure 2). Instead of introducing polar coordinates, we prefer (for accuracy reasons) the mapping of Gordon and Hall [20, 21, 23], who developed a fairly simple procedure for mapping a square \((\xi, \eta) \in S = (-1, 1)^2\) into a quadrilateral with curved boundaries.

The mapping from disk to square is given as follows:

\[
\xi = \frac{1}{2} \sqrt{2 + x^2 - y^2 + 2\sqrt{2}x} - \frac{1}{2} \sqrt{2 + x^2 - y^2 - 2\sqrt{2}x},
\]

\[
\eta = \frac{1}{2} \sqrt{2 - x^2 + y^2 + 2\sqrt{2}y} - \frac{1}{2} \sqrt{2 - x^2 + y^2 - 2\sqrt{2}y},
\]

and from square to disk the mapping is

\[
x = \xi \sqrt{1 - \frac{\eta^2}{2}}, \quad y = \eta \sqrt{1 - \frac{\xi^2}{2}}.
\]

The corresponding Jacobian matrix and determinant are computed from the above mappings:

\[
J = \begin{pmatrix}
\sqrt{1 - \frac{x^2}{2}} & -\frac{\xi x}{\sqrt{4 - 2\xi^2} - \sqrt{1 - \frac{x^2}{2}}} \\
-\frac{\eta y}{\sqrt{4 - 2\eta^2} - \sqrt{1 - \frac{\eta^2}{2}}} & \sqrt{1 - \frac{\eta^2}{2}}
\end{pmatrix}, \quad |J| = \sqrt{\left(1 - \frac{\xi^2}{2}\right)
\left(1 - \frac{\eta^2}{2}\right) - \frac{\xi^2\eta^2}{(4 - 2\xi^2)(4 - 2\eta^2)}}.
\]

Now, we are able to apply the spectral method for the transformed equation (2.1) in \((\xi, \eta)\) coordinates. Let us consider the problem with homogeneous Dirichlet boundary conditions. We denote by \(P_N(S)\) the set of all polynomials of degree less than or equal to \(N\) defined in \(S\). For a function \(v(x, y)\) defined in the unit disk, we denote \(\tilde{v}(\xi, \eta) = v(x(\xi, \eta), y(\xi, \eta))\). Let \(S_N = P_N(S) \cap H^1_0(S)\). We then find \(u_N\) such that for \(\tilde{u}_N \in S_N\), and real number \(\lambda\), \(\forall \tilde{v} \in S_N\) we have

\[
\int_{\Omega} (\nabla u_N \cdot \nabla v - \lambda u_N v) dx = \int_S \left((\tilde{u}_N, \xi_x + \tilde{u}_N, \eta_x)(\tilde{v}_x + \tilde{v}_y) + (\tilde{u}_N, \xi_y + \tilde{u}_N, \eta_y)(\tilde{v}_x + \tilde{v}_y) - \lambda \tilde{u}_N \tilde{v}\right) |J| d\xi d\eta = 0.
\]
where $\xi_x, \eta_x, \xi_y, \eta_y$ can be derived from the mappings (2.13) and (2.14):

$$
\begin{align*}
\xi_x &= \frac{(2 - \xi^2)\sqrt{2 - \eta^2}}{\sqrt{2(2 - \eta^2 - \xi^2)}}, \\
\eta_x &= \frac{\xi\eta\sqrt{2 - \eta^2}}{\sqrt{2(2 - \eta^2 - \xi^2)}}, \\
\xi_y &= \frac{\xi\eta\sqrt{2 - \xi^2}}{\sqrt{2(2 - \eta^2 - \xi^2)}}, \\
\eta_y &= \frac{(2 - \eta^2)\sqrt{2 - \xi^2}}{\sqrt{2(2 - \eta^2 - \xi^2)}}.
\end{align*}
$$

**Remark 2.1.** We know that $\tilde{u}_N(\xi(x, y), \eta(x, y))$ is a polynomial in $(\xi, \eta)$ coordinates. However, $u_N(x, y) = \tilde{u}_N(\xi(x, y), \eta(x, y))$ will no longer be a polynomial in coordinates $(x, y)$ as usual. This means that $u(x, y)$ is approximated with a fractional polynomial $u_N(x, y)$ in $(x, y)$ coordinates. This method is similar to the triangle SEM developed in [13, 25]. Here, we use Legendre polynomials as the basis of the polynomial space $S_N$. Finally, we obtain the discretization form with Gauss quadrature in the $(\xi, \eta)$ coordinates. The discretization formula can still be written as in (2.3).

### 3. Fractional Laplace operator.

In this section we will show how to compute the fractional Laplace based on the eigenvalue decomposition. Then, we will present numerical solutions of the fractional diffusion equation and the fractional Navier–Stokes/phase-field equations with the new fractional Laplace operator.

Usually, the numerical eigenfunctions associated with a repeated eigenvalue are not perfectly orthogonal with each other, which causes a loss of accuracy if we want to approximate a function with this set of eigenfunctions as trial basis. To deal with this problem, we introduce a weighted Gram–Schmidt (W-GS) method to orthonormalize the numerical eigenfunctions $\{\phi_1, \phi_2, \ldots, \phi_N\}$ of the Laplace operator, and we denote the new basis as $\{\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_N\}$. Then we expand the solution with the new orthogonalized basis. We will see that there is a significant improvement in accuracy with W-GS as reflected in the numerical results below.

**Remark 3.1.** The W-GS process is employed here to orthonormalize a set of vectors $\{\phi_i\}$ in an inner product space with a weight vector $w$, such that

$$
(\tilde{\phi}_i, \tilde{\phi}_j)_w = \delta_{ij},
$$

where $\delta_{ij}$ is the Kronecker delta and $(v_1, v_2)_w = \sum_{k=1}^n v_{1,k}v_{2,k}w_k$ with $v_1, v_2$ $n$-dimensional vectors and the weight $w = \text{diag}(M_N)$ for our numerical computation. Here $M_N$ is the diagonal mass matrix from the spectral element discretization as demonstrated in section 2.

### 3.1. Fractional diffusion equation.

We now consider the fractional diffusion equation

$$
\begin{align*}
\partial_t u &= -\mu(-\Delta)^s u, \ x \in \Omega, \\
u(x, t)|_{\partial \Omega} &= 0, \\
u(x, 0) &= u_0(x), \ x \in \Omega,
\end{align*}
$$

where $\Omega$ is a two-dimensional (2D) bounded domain, which can be any shape discussed in section 2. Set $u(x, t) = \sum_{n=0}^\infty \sum_{m=0}^\infty c_{m,n}(t)\phi_{m,n}$, where $\phi_{m,n}$ are the orthogonal eigenfunctions of the Laplace operator with the same boundary conditions as in (3.2) and $\lambda_{m,n}$ are the corresponding eigenvalues. Then from (3.2) we have

$$
\sum_{n=0}^\infty \sum_{m=0}^\infty \partial_t c_{m,n}(t)\phi_{m,n} = -\mu \sum_{n=0}^\infty \sum_{m=0}^\infty c_{m,n}(t)\lambda_{m,n}^s \phi_{m,n}.
$$
By the orthogonality of the eigenfunctions we obtain
\begin{equation}
\partial_t c_{m,n}(t) = -\mu c_{m,n}(t) \lambda_{m,n}^s.
\end{equation}
Solving the above equations with initial conditions \(c_{m,n}(0)\) gives
\begin{equation}
c_{m,n}(t) = e^{-\mu \lambda_{m,n}^s t} c_{m,n}(0),
\end{equation}
where \(c_{m,n}(0) = \int_{\Omega} u_0(x) \phi_{m,n} dx\). Thus we obtain the analytic solution of (3.2)
\begin{equation}
u(x, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} e^{-\mu \lambda_{m,n}^s t} c_{m,n}(0) \phi_{m,n}.
\end{equation}

Using the numerical eigenpairs \((\lambda_i, \phi_i)\) of the Laplace operator \(-\Delta\) from the SEM solution obtained in section 2, we can approximate the fractional Laplace operator as
\begin{equation}
(-\Delta)^s u \approx \sum_{i=1}^{\mathcal{N}} c_i \lambda_i^s \phi_i, \quad 0 < s \leq 1,
\end{equation}
where \(c_i = (\phi_i, u)_{\mathcal{N}}\), and \(\mathcal{N}\) is the number of the eigenpairs (i.e., DoF). Then, we get the approximate solution (Ap) of (3.2) as follows:
\begin{equation}
u_N(x, t) = \sum_{i=1}^{\mathcal{N}} e^{-\mu \lambda_i^s t} c_i(0) \phi_i,
\end{equation}
where \(c_i(0) = (u_0, \phi_i)_{\mathcal{N}}\). In our calculation, the eigenfunctions \(\{\phi_i\}\) are replaced by their orthogonalized counterparts \(\{\tilde{\phi}_i\}\) via the W-GS method. Figure 3 shows the \(L^2\)-errors of the numerical solution to diffusion equation (3.2) on the unit square domain \(\Omega = (-1,1)^2\) under Dirichlet boundary conditions. The numerical solution \(u_N(x, t)\) is given by (3.7) with different polynomial degree \(\mathcal{N}\); here \(\mu = 1\), \(El = 1\), \(t = 0.2\), and the initial condition is given by \(u_0(x) = 10(x - x^2) \sin(\pi x) \sin(\pi y)\). The analytical solution is given by (3.5) with \(m = n\) truncated after 32. The approximation (3.7) is exact in time, so all of the error in this scheme is associated with the spatial discretization. The lower bound of the error also depends on the truncation error of the analytical solution.

Similarly, Figure 4 shows the \(L^2\)-errors for solutions to the diffusion equation (3.2) on the unit square domain \(\Omega = (-1,1)^2\) under periodic boundary conditions. The numerical solution \(u_N(x, t)\) is given by (3.7) with different polynomial degree \(\mathcal{N}\); here \(\mu = 0.5\), \(El = 4\), \(t = 0.1\), and the initial condition is given by \(u_0(x) = \frac{16}{(2 - \sin(2\pi x))(2 - \sin(2\pi y))}\). The analytical solution is given by (3.5) with \(m = n\) truncated after 48. The approximation (3.7) is exact in time.

Finally, Figure 5 shows the \(L^2\)-errors of solutions to the diffusion equation (3.2) on the unit disk domain \(\Omega = \{(x, y) | x^2 + y^2 < 1\}\) under Dirichlet boundary conditions. The numerical solution \(u_N(x, t)\) is given by (3.7) with different polynomial degree \(\mathcal{N}\); here \(\mu = 1\), \(El = 1\), \(t = 0.1\), and the initial condition is given by \(u_0(x) = (e - e^{x^2 + y^2}) \sin(\pi(x - y + 0.3))\). The analytical solution is given as (3.5) with \(m = n\) truncated after 30.

To summarize these tests, we conclude that the W-GS orthogonalization significantly improved the accuracy of the numerical solutions.
3.2. A new formulation of the fractional phase-field equation system.
Previously, we have derived a new fractional phase-field equation from the fractional mixing energy [38]. The fractional Laplace operator was defined by combining the Caputo and Riemann–Liouville derivatives, leading to a system that admits the energy...
law. The advantage of the fractional model is that it can yield sharper interfaces than the standard (integer-order) phase-field model without increasing the resolution. In this subsection we introduce the new fractional Laplace operator (3.6) into the fractional Allen–Cahn equation.

### 3.2.1. Fractional phase-field equation system

We consider a low density ratio mixture of two immiscible and incompressible fluids with densities $\rho_1$, $\rho_2$ and viscosities $\mu_1$, $\mu_2$. The fractional phase-field system is given by

\begin{equation}
\phi_t + (u \cdot \nabla)\phi = -\gamma \left( (-\Delta)^s \phi + f(\phi) - \xi(t) \right),
\end{equation}

\begin{equation}
\rho(\phi) = \frac{\rho_1 - \rho_2}{2} \phi + \frac{\rho_1 + \rho_2}{2}, \quad \mu(\phi) = \frac{\mu_1 - \mu_2}{2} \phi + \frac{\mu_1 + \mu_2}{2},
\end{equation}

\begin{equation}
\rho(u_t + (u \cdot \nabla)u) = \nabla \cdot \mu D(u) - \nabla p + \lambda \left( (-\Delta)^s \phi + f(\phi) \right) \nabla \phi + g(\rho),
\end{equation}

\begin{equation}
\nabla \cdot u = 0,
\end{equation}

where $\phi$ is a phase-field function which identifies the regions occupied by the two fluids, such that

\begin{equation}
\phi(x,t) = \begin{cases} 1, & \text{fluid 1}, \\ -1, & \text{fluid 2} \end{cases}
\end{equation}

with the smooth transition layer of thickness $\eta$ connecting the two fluids; the interface of the mixture can be described by $\Gamma_t = \{x : \phi(x,t) = 0\}$; and $\xi(t) = \frac{1}{|\Omega|} \int_{\Omega} f(\phi) dx$ is a nonlocal Lagrange multiplier to ensure that the mass is conserved. Let $F(\phi) = \frac{1}{4\eta^2}(\phi^2 - 1)^2$ be the Ginzburg–Landau double-well potential, and $f(\phi) = F'(\phi)$. In the momentum equation (3.10), where $D(u) = \nabla u + \nabla u^T$, $\lambda$ is the mixing energy density, and $g(\rho)$ is an additional gravitational force to account for the density difference.

To be specific, we consider the homogeneous Dirichlet boundary conditions for $u$:

\begin{equation}
u|_{\partial \Omega} = 0,
\end{equation}

and homogeneous Neumann boundary conditions for $\phi$:

\begin{equation}
\nabla \phi \cdot n|_{\partial \Omega} = 0.
\end{equation}

The above fractional Laplacian is defined by

\begin{equation}
(-\Delta)^s v = \sum_{i=1}^{N} \hat{v}_i \lambda_i^s \phi_i,
\end{equation}

where $(\lambda_i, \phi_i)$ are the eigenpairs of the Laplace operator $-\Delta$ with homogeneous Neumann boundary condition. $\{\phi_i\}$ forms a complete set of orthonormal eigenfunctions, and $\hat{v}_i$ is the spectrum of $v$ in the $\phi_i$-expansion, i.e.,

\begin{equation}
v = \sum_{i=1}^{N} \hat{v}_i \phi_i,
\end{equation}

\begin{equation}
\hat{v}_i = (v, \phi_i)^N.
\end{equation}
3.2.2. A proof of the energy law. We give a proof of the energy law of our new phase-field system (3.8)–(3.11). First it is readily seen from (3.15) that, for all functions $u, v \in H^s(\Omega)$, it holds

$$( -\Delta)^s u, v = \sum_{i=1}^{\infty} \hat{\alpha}_i \lambda_i^s = (u, (-\Delta)^s v) = ((-\Delta)^{s/2} u, (-\Delta)^{s/2} v).$$

We denote the norm $\| \cdot \|_s$ by

$$(3.16) \quad \|v\|_s = \left( \sum_{i=1}^{\infty} |\hat{\alpha}_i|^2 \lambda_i^2 \right)^{1/2}.$$

Then,

$$(3.17) \quad \|v\|_s^2 = ((-\Delta)^s u, v) = ((-\Delta)^{s/2} v, (-\Delta)^{s/2} u) = \|(-\Delta)^{s/2} v\|_0^2.$$

The energy law of the new Allen–Cahn phase-field model is then given in the following theorem.

**Theorem 3.1.** If $u, \phi, \xi$ are solutions of (3.8)–(3.10)–(3.11) subject to suitable boundary condition, then

$$(3.18) \quad \frac{d}{dt} \left( \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \lambda F(\phi) \right) dx + \frac{\lambda}{2} \|\phi\|_s^2 \right) = \int_{\Omega} g \cdot u dx - \int_{\Omega} \left( \frac{\mu}{2} |D(u)|^2 + \lambda \gamma |(-\Delta)^s \phi - f(\phi) + \xi(t)|^2 \right) dx,$$

where $\|\phi\|_s$ is defined in (3.16).

**Proof.** Taking the inner product of (3.8) with $\lambda(-\Delta)^s \phi - f(\phi) + \xi(t)$, we obtain

$$(3.19) \quad (\phi_t + (u \cdot \nabla) \phi, \lambda(-\Delta)^s \phi - f(\phi) + \xi(t)) = \gamma \lambda \|(-\Delta)^s \phi - f(\phi) + \xi(t)\|_0^2.$$

For the terms on the left-hand side, we have the following basic facts:

$$(\phi_t, f(\phi)) = \frac{d}{dt} \int_{\Omega} F(\phi) dx,$$

$$(u \cdot \nabla) \phi, f(\phi) = -(\phi f(\phi), \nabla \cdot u) + \int_{\partial \Omega} \phi f(\phi) u \cdot n d\sigma = 0,$$

$$(\phi_t, \xi(t)) = \xi(t) \frac{d}{dt} \int_{\Omega} \phi dx = 0,$$

$$(u \cdot \nabla) \phi, \xi(t) = -\xi(t)(\phi, \nabla \cdot u) + \xi(t) \int_{\partial \Omega} \phi u \cdot n d\sigma = 0.$$

Bringing all these into (3.19), we get

$$(3.20) \quad \lambda \phi_t + (u \cdot \nabla) \phi, -(-\Delta)^s \phi - \frac{d}{dt} \int_{\Omega} \lambda F(\phi) dx = \gamma \lambda \|(-\Delta)^s \phi - f(\phi) + \xi(t)\|_0^2.$$

Next, taking the inner product of (3.10) with $u$, and using the boundary conditions for $u$, we get

$$\rho(u_t + (u \cdot \nabla) u, u) = (\nabla \cdot \mu D(u), u) - (\nabla p, u) + \lambda ((-\Delta)^s \nabla \phi + \nabla F(\phi), u) + (g, u).$$
Taking into account the fact that
\begin{equation}
\rho(u_t + (u \cdot \nabla)u) = \frac{d}{dt} \int_{\Omega} \frac{1}{2} |u|^2 \, dx,
\end{equation}
and
\begin{equation}
(\nabla \cdot \mu D(u), u) = -\int_{\Omega} \mu |D(u)|^2 \, dx,
\end{equation}
we obtain
\begin{equation}
-(\nabla p, u) + (\nabla F(\phi), u) = 0,
\end{equation}
we obtain
\begin{equation}
\frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho |u|^2 \, dx = -\int_{\Omega} \mu |D(u)|^2 \, dx + \lambda (-\Delta)^s \phi \nabla \phi, u) + (g, u).
\end{equation}
Subtracting (3.22) from (3.20) gives
\begin{equation}
\frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho |u|^2 \, dx + \lambda (\phi_t, (-\Delta)^s \phi) + \frac{d}{dt} \int_{\Omega} \lambda F(\phi) \, dx
\end{equation}
\begin{equation}
= -\int_{\Omega} \mu |D(u)|^2 \, dx + (g, u) - \gamma \|\Delta^s \phi - f(\phi) + \xi(t)\|^2.
\end{equation}
Finally, we derive from (3.17)
\begin{equation}
\lambda (\phi_t, (-\Delta)^s \phi) = \frac{\lambda}{2} \frac{d}{dt} \|\phi\|^2_s.
\end{equation}
Bringing (3.24) into (3.23) gives (3.18).

### 3.2.3. Time discretization scheme for the phase field in two dimensions.
We reformulate the system (3.8)–(3.11) into an equivalent form, which is more convenient for numerical approximation. Using the phase-field equation (3.8), and the fact that 
\begin{equation}
\xi(t) \nabla \phi = \nabla (\xi(t) \phi),
\end{equation}
we have
\begin{equation}
\nabla p = \lambda ((-\Delta)^s \phi) \nabla \phi + f(\phi) \nabla \phi = \nabla (p - \lambda \xi(t) \phi) + \gamma \nabla ((\phi_t + u \cdot \nabla) \phi).
\end{equation}
Therefore, if we define the modified pressure as $\Pi = p - \lambda \xi(t) \phi$, we can rewrite the system (3.8)–(3.11) as
\begin{equation}
\phi_t + (u \cdot \nabla) \phi = \gamma ((-\Delta)^s \phi - f(\phi) + \xi(t)),
\end{equation}
\begin{equation}
\rho(\phi) = \frac{\rho_1 - \rho_2}{2} \phi + \frac{\rho_1 + \rho_2}{2},
\end{equation}
\begin{equation}
\mu(\phi) = \frac{\mu_1 - \mu_2}{2} \phi + \frac{\mu_1 + \mu_2}{2},
\end{equation}
\begin{equation}
\rho(u_t + (u \cdot \nabla)u) = \nabla \cdot \mu D(u) - \nabla \Pi - \lambda (\phi_t + u \cdot \nabla) \phi \nabla \phi + g(\rho),
\end{equation}
\begin{equation}
\nabla \cdot u = 0.
\end{equation}
Then, let $L$ be the number of time steps to integrate up to final time $T$, with $\Delta t = T/L$. We denote by superscripts the time levels and set initial conditions $\phi^0 = \phi(x, 0)$,
as a Legendre expansion of order \( N \) with boundary conditions defined in (3.14). Here, \( S \) satisfies (3.30)

\[ \rho^{n+\frac{1}{2}} = \frac{\rho^n + \rho_{\ast}}{2}, \quad \rho^{*,n+\frac{1}{2}} = \frac{1}{2}(3\rho^n - \rho^{n-1}), \quad \mu^{n+\frac{1}{2}} = \frac{\mu^{n+1} + \mu^n}{2}, \]

\[ u^{n+\frac{1}{2}} = \frac{1}{2}(u^{n+1} + u^n), \quad u^{*,n+\frac{1}{2}} = \frac{1}{2}(3u^n - u^{n-1}), \]

\[ \eta^{n+\frac{1}{2}} = \frac{\eta^n + \eta_{\ast}}{2}, \quad \eta^{*,n+\frac{1}{2}} = \frac{1}{2}(3\eta^n - \eta^{n-1}), \]

\[ \frac{\partial}{\partial t} \phi^{n+1} - \gamma((\nabla)\phi^{n+\frac{1}{2}} + f(\phi^{*,n+\frac{1}{2}}) - \xi(t^{*,n+\frac{1}{2}})) = S t^{*,n+\frac{1}{2}} = 0, \]

with boundary conditions defined in (3.14). Here, \( S \Delta t \leq \frac{l^2}{4T} \), then the above scheme is unconditionally stable, where \( l \) is the upper bound of \( |f'(\phi)| \) with \( F(\phi) \) modified as [35, (4.9)].

**Velocity:**

\[ \frac{\rho^{n+1} - \rho^n}{\Delta t} - \Delta \phi^{n+\frac{1}{2}} + \frac{\rho^{1+\frac{1}{2}}}{2} \nabla \cdot \varepsilon u^{n+\frac{1}{2}} = \nabla \cdot \mu^{n+\frac{1}{2}} D(u^{n+\frac{1}{2}}) + \rho^{*,n+\frac{1}{2}} u^{*,n+\frac{1}{2}} \cdot \nabla u^{*,n+\frac{1}{2}} \]

\[ \phi^{n+\frac{1}{2}} + \frac{\lambda}{\gamma} \frac{\phi^{n+\frac{1}{2}} - \phi^n}{\Delta t} + u^{*,n+\frac{1}{2}} \cdot \nabla \phi^{*,n+\frac{1}{2}} = g^{n+\frac{1}{2}}, \]

\[ u^{n+1}|_{\partial \Omega} = 0. \]

**Pressure correction:**

\[ -\nabla^2 \psi^{n+\frac{1}{2}} = -\frac{\rho}{\Delta t} \nabla \cdot u^{n+1}, \]

\[ \nabla \psi^{n+\frac{1}{2}} \cdot n_{\partial \Omega} = 0, \]

\[ \Pi^{n+\frac{1}{2}} = \Pi^{n+\frac{1}{2}} + \psi^{n+\frac{1}{2}} - \chi \mu^{n+\frac{1}{2}} \nabla \cdot u^{n+\frac{1}{2}}, \]

where \( \chi \in [0, 1] \) is a user-defined coefficient; the choice \( \chi = 0 \) yields the standard form of the algorithm, whereas \( \chi = 1 \) yields the rotational form [42].

In order to solve the incompressible Navier–Stokes equations, we will employ the spectral direction splitting method [12] based on pressure stabilization. The velocity is represented as Legendre expansions of order \( N \) whereas the pressure is represented as a Legendre expansion of order \( N - 2 \).
3.2.4. Simulations. In order to investigate if the fractional phase-field equation is converging to the integer-order model as the fractional order $2s$ is approaching 2, we consider the system in $\Omega = [-1,1]^2$ with $\mu = 1$. We also set the forcing term of the phase-field equation equal to the one derived from the exact solutions corresponding to $s = 1$ such that the exact solutions of phase field $\phi$, density $\rho$, velocity $u$, and pressure $p$ are given by

$$
\phi(x,t) = \sin(t) \cos(\pi x) \cos(\pi y),
$$
$$
\rho(x,t) = 1 + 2,
$$
$$
u(x,t) = \pi \sin(t) \left( \sin(2\pi y) \sin^2(\pi x) - \sin(2\pi x) \sin^2(\pi y) \right),
$$
$$
p(x,t) = \sin(t) \cos(\pi x) \sin(\pi y),
$$

and the densities of the two fluids are $\rho_1 = 3$, $\rho_2 = 1$, while the viscosities are $\mu_1 = \mu_2 = 1$. Here we employ the fixed parameter $\chi = 0.5$. We used $65^2$ GLL points so the spatial discretization errors are negligible compared with the time discretization error. This corresponds to $N = 65^2$ eigenfunctions computed as shown in Appendix A for the fractional phase-field equation under Neumann boundary conditions. In Figure 6 we see that the solutions of the fractional phase-field model are approaching the integer-order phase-field solutions with linear convergence with respect to the parameter $(1 - s)$ at time $T = 1.0$.

Next, we add the gravity term in the Navier–Stokes equation to conduct numerical simulations of a rising bubble for a low density ratio case. We also want to compare the new results with our previous fractional Laplacian model. In this case all four boundaries are walls with a no-slip flow velocity as boundary condition. The densities are $\rho_1 = 0.5$, $\rho_2 = 3\rho_1$, and the viscosities are $\mu_1 = \mu_2 = 0.1$, $g = 10$, $\lambda = 0.1$, $\gamma = 1$, $\Delta t = 10^{-3}$, $T = 1.8$, $\eta = 0.04$, $El = 1$, $N = 256$, while the external body force $g$ is...
Fig. 7. Phase-field evolution from left to right at \( t = 0.1, 0.7, 1.1, 1.2, 1.6, 1.8, 2.0 \), from bottom to top for fractional orders \( s = 0.75, 0.85, 0.90, 1.00 \). (PM) corresponds to the fractional Laplacian defined in Appendix B.

Initially the bubble is described by \( \phi(x,0) = \tanh(\frac{\sqrt{x^2 + (y+0.4)^2} - 0.3}{\eta}) \). It starts as a circular bubble near the bottom of the domain and then it rises as shown in Figure 7. Specifically, Figure 7 shows the bubble rising for fractional orders \( s = 0.75, 0.85, 0.90, 1.00 \). The interface between the bubble and the background fluid (solvent) is sharper for smaller fractional order, but we also observe that the rising speed has a weak dependence on the fractional order. However, the previous model (PM) [38] had a much stronger dependence on the fractional order \( s \) resulting in changes of the rising speed but also of the shape of the bubble (See Figure 7(g)). Figures 7(a), (b), (d), and (f) show that our new model does not affect the shape of the bubble and only yields sharper interfaces. In addition, Figure 8 shows the profiles for the buoyancy force. This corresponds to \( 257^2 \) eigenfunctions computed as shown in Appendix A for Neumann boundary conditions.
Fig. 8. Phase-field profiles along the lines (left) $y = 1$ (upper wall) and (right) $x = 0$ (centerline) at $t = 1.8$. The inset shows a zoom in around the bubble.

Fig. 9. Phase-field profiles along the lines (left) $y = 0.1947$ (bubble centerline) and (right) $x = 0$ (centerline), $s = 0.75$ at $t = 0.7$. The inset of the left figure shows the details around side wall ($x = -1$) and the insert of the right figure shows the details around the bottom ($y = -1$).

4. Summary. Fractional calculus has provided us with a new powerful approach for modeling nonlocal phenomena in space time but significant new progress is required to properly formulate the basic elements of multidimensional vector fractional calculus. A fundamental such issue is the definition and numerical approximation of the fractional Laplacian in more than one dimension, with multiple definitions currently in the literature, which unfortunately do not seem to be equivalent on bounded domains. In the current work, we adopted a natural definition through the eigenfunction decomposition, also used in previous works by Liu and collaborators [24, 41].

In numerical computations, the success of this definition is determined largely by our ability to compute very accurately the eigenfunctions of the integer-order Lapla-
cian on complex geometry domains. This can be accomplished by the SEM and h-p refinement for nonsmooth domains, as we demonstrated here, but there is an additional issue, which we encounter also for the eigenfunctions of the singular Sturm–Liouville problem even when we obtain them analytically, namely, the loss of orthogonality as the number of eigenfunctions increases. Hence, simply computing these eigenfunctions and employing them in a Galerkin projection (as Liu and collaborators have done in the past) would lead to a severe loss of accuracy, especially for realistic applications in nontrivial domains where high numerical resolution is required. To this end, we have introduced a W-GS orthonormalization that leads to exponential accuracy in the numerical approximation of (smooth) functions and solutions to fractional PDEs for any order of the fractional Laplacian.

We demonstrated our numerical results on square, disk, and L-shaped domains by computing the solution of fractional diffusion and comparing with analytical solutions wherever possible. In addition to the accuracy, the approach we developed is also efficient as it does not require the solution of any linear systems since the eigenfunction decomposition leads naturally to a system of ODEs, which can be readily solved with standard methods. As a second application of the method, we also considered multiphase flow modeling and fractional energy laws, by replacing the integer-order Laplacian in the Allen–Cahn model with its fractional counterpart and following a similar procedure as in the numerical approximation of the fractional diffusion equation. However, for the Navier–Stokes equations we needed to solve a linear system, which we inverted using an efficient ADI scheme. Specifically, we demonstrated the effectiveness of the fractional Navier–Stokes/Allen–Cahn solver for the rising bubble problem in a square domain, and compared the results with the integer-order system and also with results by a different treatment of the fractional diffusion model using 1D fractional derivatives. The surprising advantage of the fractional Allen–Cahn model is that it yields sharper interfaces compared to the numerical results obtained by the integer-order models for the same resolution while it preserves the isotropic diffusion. To the best of our knowledge, this is the first application of fractional Laplacians to multiphase flows, and it would be interesting to expand such studies in the future to document the modeling advantages of fractional descriptions of diffusion in diverse applications in multidimensions.

Appendix A. Eigenvalue problems. In this appendix, we will present some numerical results to demonstrate that the approximations (3.6) and (3.7) are efficient
and highly accurate. Zhang [43] has shown that when using hp methods to approximate eigenvalues of 2m-order elliptic problems, the number of reliable numerical eigenvalues can be estimated in terms of the total number of DoF $N$ in resulting discrete systems. However, we found that in our numerical experiments this estimate is rather optimistic, and the actual number of correctly computed eigenvalues is less.

Here, we present eigenvalues and eigenfunctions computed on different shape domains in 2 dimensions under different boundary conditions.

A.1. Numerical results. In this subsection, we first show the accuracy of the eigenvalues in a 1D domain. Then we show the spectral convergence of the approximation (3.6) and (3.7) for given functions.

A.1.1. Numerical accuracy in 1 dimension. We first consider the 1D EVP (2.1) under Neumann boundary conditions. Figure 11 plots the exact eigenvalues and numerical eigenvalues with different polynomial degree $N$. We observe that only the first few leading eigenvalues are accurate for both spectral method (SM, $El = 1$) and SEM (SEM, $El = 4, 16, 32$). In the following numerical examples, we want to demonstrate that every numerical function plays an important role for our approximation scheme (3.7), even though the accuracy of the high frequencies is worsening with index number. We first introduce an incomplete approximation (IAp) as follows:

$$u_n(x) = \sum_{i=1}^{n} c_i \phi_i, \quad n \leq N,$$

where $c_i = (u, \phi_i)_N$. The fractional Laplacian operator is approximated as

$$-(-\Delta)^{\frac{\alpha}{2}} u \approx \sum_{i=1}^{n} c_i \lambda_i^{\alpha/2} \phi_i, \quad 1 < \alpha \leq 2, \quad n \leq N.$$
Then let us consider the Helmholtz equation as follows:

\[(A.3) \quad u + \mu(\Delta)^{\frac{\alpha}{2}} u = f, \quad \alpha = 2.0,\]

where \(\Omega = [-1, 1]\) and \(\mu = 1\). We test the accuracy for the following three analytic solutions:

\[(A.4) \quad u(x) = \frac{5}{4 + \sin(\pi x)} - \frac{5}{4},\]

\[(A.5) \quad u(x) = 1 - x^2,\]

with homogeneous Dirichlet boundary conditions \(u(\pm 1) = 0\), and

\[(A.6) \quad u(x) = \sum_{i=1}^{15} \cos(i\pi x)\]

with homogeneous Neumann boundary conditions \(u_x(\pm 1) = 0\).

Figure 12 shows that the approximation accuracy can be significantly improved with W-GS. It also shows that every numerical eigenfunction is critical to obtain high accuracy. In particular, exponential convergence is achieved if and only if the approximation (IAp) is complete (i.e., \(n = N\)). Figure 13 shows that the complete approximation can even achieve spectral convergence for a highly oscillatory function. Finally, we test the convergence of (IAp) with the eigenmodes obtain from the spectral element discretization, where the number of DoF \(N\) is constant but we change the element number \(El\) and polynomial order \(N\). The numerical results in Figure 14 show the same property as the one element case.

A.1.2. Numerical eigenvalues and eigenfunctions on different shaped domains. Here, we will present the numerical eigenvalues and eigenfunctions on different shaped domains under different boundary conditions, and compare the results to the analytic solutions.

If we consider (2.1) on the unit square domain \(\Omega = (-1, 1)^2\) under homogeneous Neumann boundary conditions, we have the following analytical solutions for the eigenvalues:

\[(A.7) \quad \lambda_{m,n} = \frac{\pi^2}{4}(n^2 + m^2), \quad m, n = 0, 1, 2, 3, \ldots,\]
(a) The profile of solution (A.6).
(b) The error to solution (A.6).

Fig. 13. The $L^2$-error as a function of $n$ with different polynomial degree $N$, $El = 1$ and $\alpha = 2.0$. The inset shows the spectral convergence for the complete interpolation.

(a) The error to (A.4) with different $El$.
(b) The error to (A.5) with different $El$.

Fig. 14. The $L^2$-error as a function of $n$ with different element number $El$ and $\alpha = 2.0$. We fix the DoF $N = El \times N - 1 = 63$.

and for the eigenfunctions

$$\phi_{m,n}(x) = \cos \left(\frac{m \pi (x + 1)}{2}\right) \cos \left(\frac{n \pi (y + 1)}{2}\right).$$

Table 1 shows the leading 20 eigenvalues error $|\lambda_i - \lambda_{m,n}|$ with different polynomial degree $N$ under homogeneous Neumann boundary conditions; here we have set $El = 16$ for computing the EVP.

Next, we show the leading 20 eigenvalues error $|\lambda_i - \lambda_{m,n}|$ with different polynomial degree $N$ on a unit disk domain under homogeneous Dirichlet boundary conditions in Table 2; here we have set $El = 1$ in the computations. For the sake of comparison, we also list the results presented first in [34], which were obtained by the Legendre–Galerkin method in polar coordinates. The analytic eigenvalues $\lambda_{m,n} = \tau_{m,n}$, where $\tau_{m,n}$ are the roots of the first kind Bessel functions [40].

The results in Tables 1 and 2 show that we can obtain spectral convergence under different boundary conditions both on the unit square and the unit disk domains. This implies that SEM is an efficient method for solving EVPs on regular domains. However, we need higher polynomial degree $N$ for improving the accuracy of the higher eigenmodes.
Table 1

The leading 20 eigenvalues error $|\lambda_i - \lambda_{m,n}|$ with different polynomial degree $N$ on unit square domain under homogeneous Neumann boundary condition. Here, $N = (4N + 1)^2$.

<table>
<thead>
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<th>$N = 4$</th>
<th>$N = 5$</th>
<th>$N = 7$</th>
<th>$N = 9$</th>
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<tr>
<td>9</td>
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</tbody>
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Table 2

The leading 20 eigenvalues error $|\lambda_i - \lambda_{m,n}|$ with different polynomial degree $N$ on the unit disk domain under homogeneous Dirichlet boundary condition. Here, $N = (N - 1)^2$.

<table>
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<td>1.112e-08</td>
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Finally, we solve the EVP on the L-shaped domain $\Omega = (-1,1)^2/\{0,1\}$ under Dirichlet boundary conditions. Since, the domain is non-smooth at the corner, the rate of convergence of the solution from the Galerkin method is restricted by the vertex singularities [3]. In order to increase the convergence, we refine geometrically the mesh near the corner as in Figure 15. The mesh sizes near the singular corner in Figure 15 are $h_a = 0.5$, $h_b = h_a^2$, and $h_c = h_b^2$.

Here, we do not know the exact solutions of the EVP on the L-shaped domain, however, for the sake of comparison, we list available results from the references [17, 29] in Table 3. We fix the polynomial degree $N = 8$ in each element for this computation. We list four cases corresponding to different element number $El = 27, 75, 300, 675$. The first leading 10 numerical eigenvalues are all accurate up to the ninth significant digit (i.e., all the eigenvalues are within the bounds of references [17, 29] when $El = 675$). We also list the number of DoF ($N$) for each computation.

We plot the first 9 leading eigenfunctions on different shaped domains at the end of this subsection. In Figures 16 (a), (b), (c), and (d), we plot the contours of the numerical eigenfunctions $\phi_i$, $i = 1, 2, \ldots, 9$ corresponding to the eigenvalues $\lambda_i$. We
FIG. 15. Geometric h-refinement in the L-shaped domain around the 270 degrees angle.

Table 3
The leading 10 eigenvalues $\lambda_i$ with different element number $El$ on the L-shaped domain under homogeneous Dirichlet boundary conditions. The polynomial degree is $N = 8$ in each element. The third exact eigenpair is known: $\{\lambda_3 = 2\pi^2, \phi_3(x) = \sin(\pi x)\sin(\pi y)\}$.

<table>
<thead>
<tr>
<th>i</th>
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<th>$El = 75$</th>
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<th>$El = 675$</th>
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<td>15.1972519</td>
<td>15.1725$^{170}_{166}$</td>
<td>15.225$^{170}_{166}$</td>
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<td>29.5214811</td>
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<td>29.52148$^{170}_{166}$</td>
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<td>31.9161352</td>
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<td>31.9161352</td>
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<tr>
<td>6</td>
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<td>41.4771420</td>
<td>41.4771420</td>
<td>41.4771420</td>
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<td>41.477142$^{170}_{166}$</td>
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<td>44.9485007</td>
<td>44.9485007</td>
<td>44.9485007</td>
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<tr>
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<td>49.3480220</td>
<td>49.3480220</td>
<td>49.3480220</td>
<td>49.348022$^{170}_{166}$</td>
<td>49.348022$^{170}_{166}$</td>
</tr>
<tr>
<td>9</td>
<td>56.7125467</td>
<td>56.7125467</td>
<td>56.7125467</td>
<td>56.7125467</td>
<td>56.712546$^{170}_{166}$</td>
<td>56.712546$^{170}_{166}$</td>
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<tr>
<td>$N$</td>
<td>1633</td>
<td>4641</td>
<td>18881</td>
<td>42721</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

normalize the eigenfunctions $(\phi_i, \phi_i)_N = 1$ in each figure. In Figure 16(b) the first eigenvalue is equal to 0 and the eigenfunction is a constant $c = 0.5$.

Appendix B. Comparison of different eigenvalue models. In this appendix, we recall our previous model (PM) of the nonlocal fractional Laplace operator defined by combining the Caputo derivative and Riemann–Liouville derivative. The fractional EVP is given as follows:

\begin{equation}
- D^{2s}u - \lambda u = 0, \quad x \in \Omega = (-1, 1)^2,
\end{equation}

the boundary condition is $u|_{\partial \Omega} = 0$, and the fractional Laplace operator is defined as $D^{2s} := \frac{1}{2}(RL_D^2 x + RL_D^2 y + RL_D^2 x + RL_D^2 y)$, where $RL_D^2 x$ and $RL_D^2 y$ are the left and right Riemann–Liouville fractional derivatives, respectively.

We still use the spectral method for solving the above fractional EVP. The details of the discretization are given in reference [37]. We set the polynomial degree $N = 64$ for solving problem (B.1). Following the definition of the fractional Laplace in (3.6), we know analytically the eigenvalues of (3.6) under homogeneous Dirichlet boundary conditions, given as $\lambda_{m,n}^s = (m^2 + n^2)^{\frac{s}{2}}(\pi^2)^{\frac{-s}{2}}$. Table 4 and Figures 17 and 18 show that the eigenvalues of the problem (B.1) are the same as in (3.6) if and only if $s = 1$, but different for $0 < s < 1$. Here, we denote the difference $E_{i,s} = |\lambda_{i,s} - \lambda_{m,n}^s|/\lambda_{m,n}^s$ in the table,
Fig. 16. The first 9 leading eigenfunctions of EVP (2.1): (a) on the unit square domain with Dirichlet boundary condition. $E \ell = 100$, $N = 20$; (b) on the unit square domain with Neumann boundary condition. $E \ell = 100$, $N = 20$; (c) on the unit disk domain with Dirichlet boundary condition. $E \ell = 1$, $N = 62$; (d) on the L-shaped domain with Dirichlet boundary condition. $E \ell = 75$, $N = 20$.

Table 4
Comparison of eigenvalues for the two different fractional Laplacian models.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\lambda_{1,s}$</th>
<th>$E_{1,s}$</th>
<th>$\lambda_{1,s}$</th>
<th>$E_{1,s}$</th>
<th>$\lambda_{1,s}$</th>
<th>$E_{1,s}$</th>
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</thead>
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<tr>
<td>1</td>
<td>4.9348</td>
<td>2.8437e-14</td>
<td>2.2593</td>
<td>3.1762e-01</td>
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<td>6.9237e-01</td>
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<tr>
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<td>4.7076</td>
<td>2.8486e-01</td>
<td>1.4784</td>
<td>6.7261e-01</td>
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<tr>
<td>5,6</td>
<td>24.674</td>
<td>6.7673e-14</td>
<td>7.9142</td>
<td>2.8513e-01</td>
<td>2.2278</td>
<td>6.7451e-01</td>
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<td>2.2285e-13</td>
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<td>2.3118e-01</td>
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<tr>
<td>9,10</td>
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<td>6.7928e-14</td>
<td>11.750</td>
<td>2.8711e-01</td>
<td>3.0384</td>
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<td>2.1130e-01</td>
<td>3.6540</td>
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<td>6.4018e-14</td>
<td>17.405</td>
<td>2.3218e-01</td>
<td>4.4645</td>
<td>6.3234e-01</td>
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<tr>
<td>18,19</td>
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<td>18.562</td>
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<td>20.954</td>
<td>2.0890e-01</td>
<td>4.7812</td>
<td>6.5238e-01</td>
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</table>
Fig. 17. Comparison of eigenvalues using two different definitions of the Laplacian for $s = 1.00, 0.75, 0.60$ on the unit square domain with Dirichlet boundary conditions. The lines correspond to analytic solutions given by (3.6) for $s = 1.0$ (−), $s = 0.75$ (·), and $s = 0.60$ (○), whereas the symbols correspond to eigenvalues of the fractional Laplacian defined in (B.1) for $s = 1.0$ (▽), $s = 0.75$ (□), and $s = 0.60$ (♦).

Fig. 18. The first 9 leading eigenfunctions of the fractional EVP with Dirichlet boundary conditions on the unit square domain and the fractional orders $s = 1.00, 0.90, 0.75, 0.60$. 
where $(\lambda_i,s)_{i=1}^p$ are the numerical solutions of the problem (B.1). This means that the two Laplace operator definitions (B.1) and (3.6) are equivalent for $s = 1$ but different for $0 < s < 1$ on bounded domains.

REFERENCES


