Fractional-order uniaxial visco-elasto-plastic models for structural analysis

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Highlights

• Two fractional-order visco-elasto-plastic models are developed, namely M1 and M2.
• Model M1 uses a rate-dependent yield function via time-fractional Caputo derivative.
• Model M2 uses a visco-plastic regularization via time-fractional Caputo derivative.
• A fractional return-mapping algorithm is developed for each model.
• Results show flexibility of the fractional-orders and recovery of classical models.

Abstract

We propose two fractional-order models for uniaxial large strains and visco-elasto-plastic behavior of materials in structural analysis. Fractional modeling seamlessly interpolates between the standard elasto-plastic and visco-elasto-plastic models, taking into account the history (memory) effects of the accumulated plastic strain to specify the state of stress. To this end, we develop two models, namely M1 and M2, corresponding to visco-elasto-plasticity considering a rate-dependent yield function and visco-plastic regularization, respectively. Specifically, we employ a fractional-order constitutive law that relates the Kirchhoff stress to the Caputo time-fractional derivative of the strain with order $\beta \in (0, 1)$. When $\beta \to 0$ the standard rate-independent elasto-plastic model with linear isotropic hardening is recovered by the models for general loading, and when $\beta \to 1$, the corresponding classical visco-plastic model of Duvaut–Lions (Perzyna) type is recovered by the model M2 for monotonic loading. Since the material behavior is path-dependent, the evolution of the plastic strain is achieved by fractional-order time integration of the plastic strain rate with respect to time. The plastic strain rate is then obtained by means of the corresponding plastic slip and proper consistency conditions. Finally, we develop the so called fractional return-mapping algorithm for solving the nonlinear system of the equilibrium equations developed for each model. This algorithm seamlessly generalizes the standard return-mapping algorithm to its fractional counterpart. We test both models for convergence subject to prescribed strain rates, and subsequently we implement the models in a finite element truss code and solve for a two-dimensional snap-through instability problem. The simulation results demonstrate the flexibility of fractional-order modeling using the Caputo derivative to account for rate-dependent hardening and
viscous dissipation, and its potential to effectively describe complex constitutive laws of engineering materials and especially biological tissues.

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1. Introduction

Fractional differential operators appear in many systems in science and engineering such as visco-elastic materials [1–3], electrochemical processes [4] and porous or fractured media [5]. For instance, it has been found that the transport dynamics in complex and/or disordered systems is governed by non-exponential relaxation patterns [6,7]. For such processes, a time-fractional equation, in which the time-derivative emerges as $D^\nu_t u(t)$, appears in the continuous limit. One interesting application of fractional calculus is to model complex elasto-plastic behavior of engineering materials (e.g. [8,9]). Recently, fractional calculus has been employed as an effective tool for modeling materials accounting for heterogeneity/multi-scale effects to the constitutive model [10–12], where the fractional visco-plasticity was introduced as a generalization of classical visco-plasticity of Perzyna type [13]. The fundamental role of the formulation is the definition of the flow rule by introducing a fractional gradient of the yield function. Also, a constitutive model for rate-independent plasticity based on a fractional continuum mechanics framework accounting for nonlocality in space was developed in [14].

Formulating fast and accurate numerical methods for solving the resulting system of fractional ODEs/PDEs in such problems is the key to incorporating such nonlocal/history-dependent models in engineering applications. Efficient discretization of the fractional operators is crucial. Lubich [15,16] pioneered the idea of discretized fractional calculus within the spirit of finite-difference method (FDM). Later, Sanz-Serna [17] adopted the idea of Lubich and presented a temporal semi-discrete algorithm for partial integro-differential equations, which was first order accurate. Sugimoto [18] also employed a FDM for approximating the fractional derivative emerging in Burgers’ equation. Later on, Gorenflo et al. [19] adopted a finite-difference scheme by which they could generate discrete models of random walk in such anomalous diffusion. Diethelm et al. proposed a predictor–corrector scheme in addition to a fractional Adams method [20,21]. After that, Langlands and Henry [22] considered the fractional diffusion equation, and analyzed the $L^1$ scheme for the time-fractional derivative. Sun and Wu [23] also constructed a difference scheme with $L^\infty$ approximation of the time-fractional derivative. Of particular interest, Lin and Xu [24] analyzed a FDM for the discretization of the time-fractional diffusion equation with order $(2 - \alpha)$. However, there are other classes of global methods (spectral and spectral element methods) for discretizing fractional ODEs/PDEs (e.g., [25–28]), which are efficient for low-to-high dimensional problems. Furthermore, Zayernouri and Matzavinos developed a fractional family of schemes, called fractional Adams–Bashforth and fractional Adams–Moulton method for high-order explicit and implicit treatment of nonlinear problems [29]. There were recent developments on meshless approaches applied to fractional-diffusion and space-fractional advection–dispersion problems [30,31]. Also, Chen [32] developed a new definition of fractional Laplacian and applied to three-dimensional, nonlocal heat conduction.

The main contribution of the present work is to propose and solve two fractional-order models, namely M1 and M2, for uniaxial large strains and visco-elasto-plastic behavior of materials. Both models account for fractional visco-elastic modeling by defining a stress–strain relationship involving the Caputo time derivative of fractional-order, but have distinct formulations to model the fractional visco-plasticity. For the model M1, visco-plasticity is achieved by including history effects in time for the internal hardening parameter in the yield function, making it rate-dependent. Differently from some works found in the literature [10–12], we do not modify the flow rule. The model M2 accounts for a rate-independent yield function without an internal hardening parameter, and the visco-plastic effect is achieved based on the approach of visco-plastic regularization used in the classical visco-plastic model of Duvaut–Lions type (which is equivalent to Perzyna’s model). Furthermore, the models consider different memory effects for visco-elasticity and visco-plasticity. Both models are used within the framework of a time-fractional backward-Euler integration procedure with a fractional return-mapping algorithm, based on the classical models in the literature [9,8]. The developed algorithm seamlessly generalizes the standard return-mapping algorithm to its fractional counterpart, making it amenable for path-dependent visco-elasto-plastic analyses in engineering and bio-
engineering applications. We also present the standard nonlinear finite element formulation for trusses and show that the only required modifications are in the stress update procedure, which will be described in constitutive boxes for each model.

Several numerical analyses are performed to investigate the behavior of the models. We test the algorithms presented in terms of convergence using a benchmark solution, then we perform tests with cyclic strains to account for hysteresis behavior. Both models are implemented in the context of finite element method (FEM) using an updated Lagrangian approach to solve a two-bar snap-through problem. Because no analytical solutions are derived, we implemented the classical one-dimensional models for elasto-plasticity with linear hardening and visco-plasticity of Duvaut–Lions type, and we recover these limit cases for verification. We verify that both fractional-order models recover the classical rate-independent elasto-plastic model for general loading/unloading conditions, and also interpolate between plastic/visco-plastic behavior with the variation of the fractional-order parameters. The obtained results show the flexibility of the fractional-order models to describe the rate-dependent hardening and viscous dissipation. This motivates the application of the models developed in this work to identify material parameters of complex constitutive laws of engineering materials and biological tissues.

2. Definitions of fractional calculus

We start with some preliminary definitions of fractional calculus [33]. The left-sided Riemann–Liouville integrals of order \( \mu \), when \( 0 < \mu < 1 \), are defined as

\[
(\mathcal{R}_L^\mu \mathcal{I}_x f)(x) = \frac{1}{\Gamma(\mu)} \int_{x_L}^x f(s) (x-s)^{1-\mu} \, ds, \quad x > x_L, \tag{1}
\]

where \( \Gamma \) represents the Euler gamma function and \( x_L \) denotes the lower integration limit. The corresponding inverse operator, i.e., the left-sided fractional derivatives of order \( \mu \), is then defined based on Eq. (1) as

\[
(\mathcal{R}_L^\mu \mathcal{D}_x f)(x) = \frac{d}{dx} (\mathcal{R}_L^{1-\mu} \mathcal{I}_x f)(x) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dx} \int_{x_L}^x f(s) (x-s)^{-\mu} \, ds, \quad x > x_L. \tag{2}
\]

Furthermore, the corresponding left-sided Caputo derivatives of order \( \mu \in (0, 1) \) are obtained as

\[
(\mathcal{C}_L^\mu \mathcal{D}_x f)(x) = \left( \mathcal{R}_L^{1-\mu} \frac{df}{dx} \right)(x) = \frac{1}{\Gamma(1-\mu)} \int_{x_L}^x f'(s) (x-s)^{-\mu} \, ds, \quad x > x_L. \tag{3}
\]

The definitions of Riemann–Liouville and Caputo derivatives are linked by the following relationship, which can be derived by a direct calculation

\[
(\mathcal{R}_L^\mu \mathcal{D}_x f)(x) = \frac{f(x_L)}{\Gamma(1-\mu)(x-x_L)^\mu} + (\mathcal{C}_L^\mu \mathcal{D}_x f)(x), \tag{4}
\]

which denotes that the definitions of the aforementioned derivatives coincide when dealing with homogeneous Dirichlet initial/boundary conditions.

3. Kinematics of large visco-elasto-plastic deformations

Consider the truss element with nodes 1, 2 illustrated in Fig. 1(a) for the initial configuration at time \( t = 0 \), with coordinate vectors \( \mathbf{X}_1, \mathbf{X}_2 \), length \( L \), area \( A \) and volume \( V \). The updated configuration at time \( t \) is denoted with the current coordinates \( \mathbf{x}_1, \mathbf{x}_2 \), area \( a \), volume \( v \) and normal vector \( \mathbf{n} \). The terms \( \mathbf{u}_1, \mathbf{u}_2 \) represent the nodal displacement vectors from the updated configuration at time \( t \) to a new configuration taking place at time \( t + dt \). Fig. 1(b) shows the multiplicative decomposition of the stretch \( \lambda \) into visco-elastic \( \lambda^{ue} \) and visco-plastic \( \lambda^{vp} \) parts, with the latter accounting for a visco-plastic updated length \( l^{vp} \).

The updated length \( l \) and normal unit vector \( \mathbf{n} \) are given by [34]

\[
l = \sqrt{(\mathbf{x}_2 - \mathbf{x}_1) \cdot (\mathbf{x}_2 - \mathbf{x}_1)}, \quad \mathbf{n} = \frac{\mathbf{x}_2 - \mathbf{x}_1}{l}, \tag{5}
\]
where the updated coordinates of the element nodes are denoted as
\[ \mathbf{x}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}. \] (6)

We consider the decomposition of the stretch illustrated in Fig. 1(b) for the truss element when subject to a change in configuration. The visco-elastic and visco-plastic stretches are given by
\[ \lambda^{ve} = \frac{l}{l^{vp}}, \quad \lambda^{vp} = \frac{l^{vp}}{L}. \] (7)
The total stretch is given by
\[ \lambda = \lambda^{ve} \lambda^{vp} = \frac{l}{L}. \] (8)

Applying the natural logarithm to the above equation, we obtain an additive decomposition given by [8]
\[ \ln(\lambda) = \ln(\lambda^{ve}) + \ln(\lambda^{vp}), \] (9)
which is called logarithmic strain and is usually denoted as
\[ \varepsilon = \varepsilon^{ve} + \varepsilon^{vp}. \] (10)
The logarithmic strain measure defined in Eq. (9) is thermodynamically conjugate to the Kirchhoff stress \( \tau \) [8], which will be addressed in the next section, along with the constitutive equations for the visco-elasto-plastic models.

4. Fractional-order visco-elasto-plastic models

We consider two visco-elasto-plastic models, called M1 and M2. The developed framework for both models incorporates memory effects for the evaluation of visco-elasto-plastic large strains. We present the mathematical formulation for each model and an efficient algorithm to solve the nonlinear system of fractional-order differential equations, and remark on how the models recover the classical local models.

To account for the memory effects in time, we modify the classical models presented in the literature [8,9]. The model M1 is a modification of classical elasto-plasticity with linear hardening and the model M2 is a modification of visco-plasticity of Duvaut–Lions type. For this purpose, we introduce Scott–Blair elements with fractional-order \( \beta \), which interpolate between linear spring when \( \beta \to 0 \) and viscous Newton element when \( \beta \to 1 \) [1].

The memory effects for both models will be presented in the stress–strain relationship regarding the visco-elastic part, which is evaluated for the entire time domain. To account for memory effects in visco-plasticity, we will consider the Caputo time-fractional derivative in the yield function for the model M1, while for the model M2 we will incorporate the memory effects via a separate equation that describes the visco-plastic regularization. The memory for the fractional derivatives describing visco-plasticity will be considered starting from the last attained yield stress.
4.1. Model M1

The model M1 is illustrated in Fig. 2(a), consisting of a Scott–Blair element with constant $E \{ Pa \cdot s^\beta_E \}$ and a fractional-order $\beta_E$ for the visco-elastic part with corresponding visco-elastic strain $\varepsilon^{ve}$. The visco-plastic device consists of a parallel combination of a Coulomb frictional element with yield stress $\tau_Y \{ Pa \}$, a linear hardening spring with constant $H \{ Pa \}$, a Scott–Blair element with constant $K \{ Pa \cdot s^\beta_K \}$ and fractional-order $\beta_K$. The corresponding visco-plastic strain is denoted by $\varepsilon^{vp}$. The term $\tau \{ Pa \}$ stands for the Kirchhoff stress.

We start by rewriting Eq. (10) in terms of the visco-elastic strain component:

$$\varepsilon^{ve} = \varepsilon - \varepsilon^{vp}. \quad (11)$$

The history-dependent constitutive law for this model is assumed to be of the form:

$$\tau = E \int_0^\tau D^\beta_E t (\varepsilon^{ve}) = E \int_0^\tau D^\beta_E t (\varepsilon - \varepsilon^{vp}), \quad 0 < \beta_E < 1. \quad (12)$$

To specify the homogeneous initial conditions for the Caputo time derivative, we assume the given point of the material to have no initial strains, that is, $\varepsilon(t = 0) = \varepsilon^{ve}(t = 0) = \varepsilon^{vp}(t = 0) = 0$. In this sense, we observe that the Riemann–Liouville definition (Eq. (2)) could also be employed, since we consider homogeneous initial conditions. To designate a set of admissible stresses, we define the following closed convex stress space:

$$E = \{ \tau \in \mathbb{R} | f(\tau, \alpha) \leq 0 \}, \quad (13)$$

where $f : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ represents the yield condition, defined by

$$f(\tau, \alpha) := |\tau| - \left[ \tau^{\gamma} + H \alpha \right], \quad (14)$$

where

$$\tau^{\gamma} = \tau^{\gamma} + K \int_0^\tau D^\beta_K (\alpha), \quad 0 < \beta_K < 1, \quad (15)$$

with $\alpha$ representing an internal hardening variable with initial condition $\alpha(t = 0) = 0$, that is, we assume no initial hardening. The term $\tau^{\gamma}$ denotes an yield stress which is updated according to unloading conditions ($\tau^{\gamma} = \tau^{\gamma}$ in the beginning of the process) and $\int_0^\tau$ denotes the time of onset of plasticity. The term $\tau^{\gamma}$ in Eq. (15) can be interpreted as an updated yield stress accounting for memory effects in visco-plasticity, while the term $H \alpha$ represents a local hardening parameter.

The time $\int_0^\tau$ for plasticity is reset when we unload the material from a visco-plastic state, and the yield stress is updated to a new value $\tau^{\gamma}$. Substituting Eq. (15) into Eq. (14), we obtain

$$f(\tau, \alpha) := |\tau| - \left[ \tau^{\gamma} + K \int_0^\tau D^\beta_K (\alpha) + H \alpha \right]. \quad (16)$$

We do not consider different time derivative limits for Eq. (12) because plastic strains inevitably take place in the visco-elastic range. Notice that we included the Caputo derivative of order $\beta_K$ inside $f(\tau, \alpha)$, thus making the yield
condition rate-dependent. The corresponding boundary of $E_{\tau}$ is the convex set denoted by $\partial E_{\tau}$, given as
$$\partial E_{\tau} = \{ \tau \in \mathbb{R} | f(\tau, \alpha) = 0 \},$$
where $f(\tau, \alpha) = 0$ is the so-called consistency condition in the classical elasto-plastic models. In the present model, we assume that the hardening is isotropic in the sense that at any state of loading the center of $E_{\tau}$ remains at the origin of the stress–strain domain. The expected stress versus strain response based on Eqs. (16) and (17) is presented in Fig. 2(b). The consistency condition (Eq. (17)) will be addressed in incremental form in time to derive the visco-plastic solutions. Moreover, similar to classical elasto-plasticity, the evolution of hardening is assumed to be linear in terms of the visco-plastic strain rate. Therefore
$$\dot{\alpha} = |\dot{\varepsilon}^{vp}|,$$
and the flow rule is not modified, and is given by
$$\dot{\varepsilon}^{vp} = \gamma \text{sign}(\tau),$$
where $\gamma$ denotes the amount of plastic slip, also with initial condition $\gamma(t = 0) = 0$, and the term $\text{sign}(\tau)$ represents the direction of the plastic flow. Recalling Eq. (16) and the definition of the fractional derivative, when $\beta_K \to 0$, we recover the limit case without rate effects (spring) and the constant $K$ accounts for the standard plastic modulus of rate-independent plasticity, with units of [Pa]. On the other hand, if $\beta_K \to 1$ we recover the limit case of a local integer-order derivative (dashpot), where $K$ would be equivalent to the material viscosity $\eta$, with corresponding units of [Pa · s].

4.2. Model M2

The schematic diagram of the model M2 is illustrated in Fig. 3(a), which consists of the same elements as the model M1, except that we remove the linear hardening spring with constant $H$ in the visco-plastic part.

The stress–strain relation for this model is the same as Eq. (12). We consider the yield function of perfect plasticity given by
$$f(\tau) := |\tau| - \tau^{y'},$$
where we consider the yield stress $\tau^{y'} = \tau^{y'}(t = 0)$ and update it when the material is unloaded from visco-plastic behavior (more details will be addressed in Section 6). Because the model is based on visco-plasticity of Duvaut–Lions type, we use the idea of visco-plastic regularization in [9] to take into account the memory effect of the visco-plastic strain $\dot{\varepsilon}^{vp}$ when we obtain an over-stressed level $f(\tau) > 0$:
$$K_{tp} C \mathcal{D}_{\tau}^{\beta_k} (\dot{\varepsilon}^{vp}) = \tau - \tau_{\infty},$$
where $\tau_{\infty}$ is the relaxed stress when $t \to \infty$ (compared to the natural relaxation time of the material). Substituting the stress–strain relation from Eq. (12) into Eq. (21), and rearranging the visco-plastic strains to the left-hand-side, we obtain
$$E_{0} C \mathcal{D}_{\varepsilon}^{\beta_k} (\dot{\varepsilon}^{vp}) + K_{tp} C \mathcal{D}_{\tau}^{\beta_k} (\dot{\varepsilon}^{vp}) = E_{0} C \mathcal{D}_{\varepsilon}^{\beta_k} (\varepsilon) - \tau_{\infty}.$$
The solution for this model involves determination of the rate-independent stress $\tau_\infty$ by applying the consistency condition to Eq. (20) and substituting the result into Eq. (22) to determine the visco-plastic strains $\varepsilon^{\text{vp}}$. After that, the time-dependent stress can be determined from the constitutive relation (Eq. (12)). Fig. 3(b) presents the stress versus strain response in the relaxed state (path O-A-B) described by the yield function (Eq. (20)) and the regularized state (path O-A-B’) achieved with Eq. (22). More details about this procedure will be presented after the time discretization scheme. We note that when $\beta_k \to 1$ in Eq. (22) we recover the local first-order derivatives and therefore the classical Duvaut–Lions formulation.

4.2.1. Remark about parameter $H$

We note that when we set the linear hardening parameter $H = 0$ in the model M1, we obtain the same diagram for both models. However, the approaches are still different, since the model M1 considers the Caputo-time fractional derivative in the yield function (Eq. (16)) while the model M2 uses an yield function of visco-elastic perfectly plastic behavior and accounts for visco-plastic regularization with relaxation effects described by Eq. (21).

4.2.2. Remark about visco-elastic/plastic memory effects

The initial study of the presented models considered the entire time domain for the visco-plastic equations (16), (22) without updating the yield stress $\tau^\gamma$. However, for model M1 it was observed that due to long memory effects and no update in the yield stress, the visco-elastic range did not expand in an isotropic way when cyclic loads were applied. Furthermore, for the model M2, we obtained non-physical results for the visco-plastic part without updating the yield stress $\tau^\gamma$ due to lack of internal hardening combined with long memory on visco-plastic strains.

We consider the distinction between “visco-elastic time” and “visco-plastic time” a more natural way of treating the memory effects, since in a general problem the material will not be in a visco-plastic state (Eqs. (17) and (22)) at all times. On the other hand, the stress–strain relation (Eq. (12)) is used regardless of the stress state.

5. Time integration and discretization in space

For notation purposes, we denote variables at times $t_n$, $t_{n+1}$ by the lower-scripts $n$, $n+1$, respectively. The governing equations on the equilibrium of a truss are discretized in time and space to obtain (e.g., see [35])

$$\psi_{n+1} = M\left[b_1(u_{n+1} - u_n) - b_2v_n - b_3a_n\right] + R_{n+1} - P_{n+1} = 0,$$

(23)

where we denote $\psi_{n+1}$ as the residual force vector, $M$ as the global mass matrix for all nodes, $R_{n+1}$ as the global internal force vector dependent of the updated configuration with coordinates $x_{n+1}$, which in turn depend on the displacements $u_{n+1}$. The term $P_{n+1}$ represents the global external nodal force vector. The terms $a_n$ and $v_n$, respectively, denote the global acceleration and velocity vectors. More details regarding the Newmark scheme are presented in Appendix B with the description of the approximation coefficients $b_l$. We do not consider a linear damping matrix in Eq. (23) because the constitutive law will naturally introduce damping effects for both visco-elastic/plastic contributions.

We note that our approach can be employed in the context of any standard numerical method e.g., finite element method (FEM), spectral element methods, etc. This is particularly true because our history-dependent modeling results in a system of time-fractional equations. Hence, the spatial domain can be always treated using available standard discretizations. However, the computation of the incremental stresses needs special care as shown in the sequel.

The equilibrium system (Eq. (23)) is linearized by employing Newton’s method using incremental global displacements, defined as

$$u_{n+1} = u_n + \Delta u.$$  

(24)

Accordingly, the updated global coordinates are given by

$$x_{n+1} = x_n + u_{n+1} = x_n + u_{n+1},$$  

(25)

where the superscript $k + 1$ refers to the current iteration of the Newton’s method. The linearized form of Eq. (23) in the direction of a displacement increment $\Delta u$ is given by the following system of equations:

$$\left[b_1M + K_{I_{n+1}}\right] \Delta u = -M\left[b_1(u_{n+1} - u_n) - b_2v_n - b_3a_n\right] - R_{n+1} + P_{n+1}.$$  

(26)
The terms \( \mathbf{u}_n, v_n, \mathbf{a}_n \) are obtained from the last converged time step \( n \). The term \( \mathbf{K}_T \) is the tangent stiffness matrix and is updated at each iteration \( k \) along with the internal force vector. In the linear finite element spatial discretization, the element tangent stiffness for the current formulation is given by (see [8])

\[
\mathbf{K}_T^{(e)} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix}, \quad \text{with } \mathbf{K}_{11} = \frac{K_C}{l^2} \begin{bmatrix} (x_2 - x_1)^2 & (x_2 - x_1)(y_2 - y_1) \\ (x_2 - x_1)(y_2 - y_1) & (y_2 - y_1)^2 \end{bmatrix} + \frac{a \sigma}{l} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

(27)

where the superscript \( (e) \) denotes the local elemental operation associated with the \( e \)th element, and \( \mathbf{K}_{22} = \mathbf{K}_{11}, \mathbf{K}_{12} = \mathbf{K}_{21} = -\mathbf{K}_{11} \). The term \( l \) denotes the current element length, \( x_1, x_2, y_1, y_2 \) denote the element updated coordinates, \( \sigma \) denotes the Cauchy stress at the element, and \( K_C \) is given by

\[
K_C = \frac{V}{v} \frac{\partial \tau}{\partial \varepsilon} - 2a \sigma.
\]

(28)

The relation between the Kirchhoff and Cauchy stresses is given by \( \tau = \frac{v}{\nu} \sigma \). We note that there is no assumption in the constitutive behavior, therefore the formulation presented in this section is general. The local stress derivative in terms of strain in Eq. (28) is known as tangent modulus, and its computation will be addressed in the next section. Notice that this derivative is local in nature, and comes from the linearized kinematics of the problem. The elemental internal force vector and the corresponding mass matrix are given by

\[
\mathbf{R}^{(e)} = \frac{a \sigma}{l} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ x_1 - x_2 \\ y_1 - y_2 \end{bmatrix}, \quad \mathbf{M}^{(e)} = \frac{\rho AL}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix},
\]

(29)

where \( \rho \) is the material density. In the next section we determine the current stress \( \tau_{n+1} \) and tangent modulus \( \left( \frac{\partial \tau}{\partial \varepsilon} \right)_{n+1} \) for the fractional-order models.

6. Fractional return-mapping algorithms

We present the time-fractional backward-Euler integration procedure for both fractional models, where a trial state is defined by freezing the internal variables and a fractional return-mapping algorithm is obtained enforcing the proper conditions. For the model M1, the solution for the plastic slip is given by a fractional-order differential equation. For the model M2 we solve a fractional-order differential equation for the visco-plastic strain instead, using the idea of visco-plastic regularization.

The backward-Euler procedure is implicit in time, unconditionally stable and is first-order accurate. We assume that at time \( t_{n+1} \), with \( t \in [0, T] \) all variables for the previous time step \( t_n \) are known. We consider a strain increment \( \Delta \varepsilon_n \), which in the context of the standard finite element method, can be obtained using Eqs. (8) and (9) from an increase in element length \( \Delta l \), calculated from the displacement increments \( \Delta \mathbf{u} \) (Eq. (26)). From the constitutive model point of view, we just consider this increment to be known, regardless of being prescribed or obtained by the equilibrium of the system. The strain for time \( t_{n+1} \) is given by

\[
\varepsilon_{n+1} = \varepsilon_n + \Delta \varepsilon_n.
\]

(30)

The stress–strain relation is given by

\[
\tau_{n+1} = E \int_0^c D_{\varepsilon}^{\beta_E} \left( \varepsilon - \varepsilon_{\text{up}} \right) \bigg|_{t=t_{n+1}}.
\]

(31)

The incremental flow rule and evolution of the hardening parameter are, respectively, given by

\[
\varepsilon_{\text{up}}^{n+1} = \varepsilon_{\text{up}}^n + \Delta \gamma \, \text{sign} \left( \tau_{n+1} \right),
\]

(32)

\[
\alpha_{n+1} = \alpha_n + \Delta \gamma,
\]

(33)

where \( \Delta \gamma \) denotes the plastic slip for the time interval \([t_n, t_{n+1}]\) under consideration. In our fractional return-mapping algorithm, we make use of the so called trial state, where we freeze the internal variables of the model at time \( t_{n+1} \) in
the following way:
\[
\varepsilon^{\text{trial}}_{n+1} = \varepsilon_{n}^{p}, \quad \alpha_{n+1}^{\text{trial}} = \alpha_n.
\] (34)

Having the trial visco-plastic strain defined, we perform a trial visco-elastic stress given by
\[
\tau_{n+1}^{\text{trial}} = E_0 C D_t^\beta E \left( \varepsilon - \varepsilon^{\text{trial}}_{n+1} \right) \bigg|_{t=t_{n+1}},
\] (35)

where we will keep the term \( \varepsilon^{\text{trial}}_{n+1} \) instead of \( \varepsilon_n^{p} \) for notation purposes, since the time-fractional Caputo derivative is evaluated at time \( t_{n+1} \). The trial state defined in Eq. (34) will be substituted in the discrete form of the fractional Caputo derivatives, which is presented in Section 6.1. The result of Eq. (35) is applied in a trial yield function \( f_{n+1}^{\text{trial}} \) in order to check if the stress state lies within the visco-elastic or over the visco-plastic ranges, and perform the return-mapping procedure if necessary.

The current visco-plastic reference time is denoted here as \( t_{p_{n+1}} \), and is updated when a new yield stress is achieved from cyclic behavior. We introduce an auxiliary notation to track this visco-plastic time by using an incremental variable \( p_{n+1} \). The initial value is considered to be \( p_0 = 0 \). In the incremental procedure, we account for the current time step \( p_{n+1} = 0 \) if the state is visco-elastic. The value \( p_{n+1} = n + 1 \) is set when the stress state exceeds the yield stress, coming from a visco-elastic state. When the stress state is an increasing visco-plastic state (without change of load direction), the visco-plastic time reference is the same as the previous step, that is, \( p_{n+1} = p_n \).

### 6.1. Algorithm for the model M1

The yield function (Eq. (16)) at time \( t_{n+1} \) is given by
\[
f_{n+1} = |\tau_{n+1}| - \left[ \tau^y + H \alpha_{n+1} + K_C^{\text{trial}} D_t^\beta K (\alpha) \bigg|_{t=t_{n+1}} \right].
\] (36)

Considering the definition of the trial state, we obtain
\[
f_{n+1}^{\text{trial}} = |\tau_{n+1}^{\text{trial}}| - \left[ \tau^y + H \alpha_n + K_C^{\text{trial}} D_t^\beta K (\alpha^{\text{trial}}) \bigg|_{t=t_{n+1}} \right],
\] (37)

where the Caputo time-fractional derivative of \( \alpha^{\text{trial}} \) is taken starting from time \( t_{p_{n}} \), because it is the available information about the last known yield stress \( \tau^y \). If \( f_{n+1}^{\text{trial}} \leq 0 \) we are within the visco-elastic range. Otherwise, we have an inadmissible stress indicating the onset of visco-plasticity. We enforce the discrete consistency condition \( f_{n+1} = 0 \) to obtain the solution for \( \Delta \gamma \) and then to perform a projection of the trial stress onto the yield surface, as illustrated in Fig. 4. Substituting Eq. (32) into Eq. (31), and recalling Eq. (35), we obtain
\[
\tau_{n+1} = \tau_{n+1}^{\text{trial}} - E_0 C D_t^\beta E \left( \Delta \gamma \right) \bigg|_{t=t_{n+1}} \text{sign} (\tau_{n+1}).
\] (38)

We can rewrite the above equation as
\[
|\tau_{n+1}| \text{sign}(\tau_{n+1}) = |\tau_{n+1}^{\text{trial}}| \text{sign}(\tau_{n+1}^{\text{trial}}) - E_0 C D_t^\beta E \left( \Delta \gamma \right) \bigg|_{t=t_{n+1}} \text{sign} (\tau_{n+1}),
\] (39)

hence,
\[
|\tau_{n+1}| + E_0 C D_t^\beta E \left( \Delta \gamma \right) \bigg|_{t=t_{n+1}} \text{sign}(\tau_{n+1}) = |\tau_{n+1}^{\text{trial}}| \text{sign}(\tau_{n+1}^{\text{trial}}).
\] (40)

The terms inside the brackets must be positive since \( \Delta \gamma > 0 \) and \( E > 0 \), and therefore
\[
\text{sign}(\tau_{n+1}) = \text{sign}(\tau_{n+1}^{\text{trial}}).
\] (41)

Substituting Eqs. (38) and (33) into Eq. (36) and recalling Eqs. (37) and (41), yields
\[
f_{n+1} = f_{n+1}^{\text{trial}} - H \Delta \gamma - E_0 C D_t^\beta E \left( \Delta \gamma \right) \bigg|_{t=t_{n+1}} - K_C^{\text{trial}} D_t^\beta K (\Delta \gamma) \bigg|_{t=t_{n+1}}.
\] (42)
Fig. 4. Schematic of stress update in the fractional return-mapping algorithm. The trial stress $\tau_{n+1}^{\text{trial}}$ is projected to $\tau_{n+1}$. When unloading is performed from $\tau_{n+1}$, the new yield stress is $\tau^Y$ instead of $\tau_{n+1}^Y$.

i - Database at $x \in \Omega$: $\{\varepsilon, \varepsilon^{\text{vp}}, \alpha, \Delta \gamma, \tau^Y, \tau'^Y, p_n\}$

ii - Enforce a strain increment $\Delta \varepsilon_n$:
$$\varepsilon_{n+1} = \varepsilon_n + \Delta \varepsilon_n$$

iii - Trial state (freezing the visco-plastic state):
$$\tau_{n+1}^{\text{trial}} = E_0^C D_t^\beta E (\varepsilon - \varepsilon^{\text{vp}}_n) \bigg|_{t=t_{n+1}}$$

If $p_n > 0$ Then last step was visco-plastic
$$\tau'^Y = \tau^Y + K^C \int_{t_{n+1}}^{t_n} (\alpha')^{\text{trial}} dt$$

End

If $f_{n+1}^{\text{trial}} = 0$ Then visco-elastic step:
$$\varepsilon_{n+1}^{\text{vp}} = \varepsilon_n^{\text{vp}}, \quad \alpha_{n+1} = \alpha_n, \quad \tau_{n+1} = \tau_{n+1}^{\text{trial}}, \quad p_{n+1} = 0$$

Else visco-plastic step: (return-mapping)
If $p_n > 0$ Then increasing visco-plasticity
$$p_{n+1} = p_n$$

Else update yield stress, reset visco-plastic time reference
$$\tau_{n+1} = \tau'^Y$$
$$p_{n+1} = n + 1$$

End

Solve for $\Delta \gamma$:
$$E_0^C D_t^\beta E (\Delta \gamma) \bigg|_{t=t_{n+1}} + K^C \int_{t_{n+1}}^{t_n} (\alpha')^{\text{trial}} dt$$
$$\tau_{n+1} = \tau_{n+1}^{\text{trial}} - E_0^C D_t^\beta E (\Delta \gamma) \bigg|_{t=t_{n+1}} + H \Delta \gamma = f_{n+1}^{\text{trial}}$$
$$\varepsilon_{n+1}^{\text{vp}} = \varepsilon_n^{\text{vp}} + \Delta \gamma \text{ sign}(\tau_{n+1}^{\text{trial}})$$
$$\alpha_{n+1} = \alpha_n + \Delta \gamma$$

End

Box I. Fractional return-mapping algorithm for the model M1.

Applying the discrete consistency condition ($f_{n+1} = 0$), we obtain the following fractional-order differential equation for $\Delta \gamma$
$$E_0^C D_t^\beta E (\Delta \gamma) \bigg|_{t=t_{n+1}} + K^C \int_{t_{n+1}}^{t_n} (\alpha')^{\text{trial}} dt + H \Delta \gamma = f_{n+1}^{\text{trial}}$$
(43)

which is a Volterra integral equation of second kind. The fractional return-mapping algorithm is summarized in Box I for this model.
The fractional derivatives present in the fractional return-mapping procedure are computed implicitly using the finite difference method (FDM) developed in [24], where \( D^\nu_t u(t) \) is discretized as

\[
D^\nu_t \left( u(t) \right) \bigg|_{t=\Delta t^{-1}} = \frac{1}{\Gamma(2-\nu)} \sum_{j=0}^n b_j \frac{u(t_{n+1-j}) - u(t_{n-j})}{(\Delta t)^\nu} + r^\nu_{\Delta t},
\]

where \( r^\nu_{\Delta t} \leq C_u (\Delta t)^{2-\nu} \) and \( b_j := (j+1)^{1-\nu} - j^{1-\nu}, j = 0, \ldots, n \). The term \( \Delta t \) denotes the time increment size \( \Delta t = T/N \), where \( T \) represents the total time and \( N \) denotes the number of increments. Taking the first term \( j = 0 \) outside of the sum sign, we obtain

\[
D^\nu_t \left( u(t) \right) \bigg|_{t=\Delta t^{-1}} = \frac{1}{(\Delta t)^\nu \Gamma(2-\nu)} \left[ u(t_{n+1}) - u(t_n) + \mathcal{H}^\nu u \right],
\]

where

\[
\mathcal{H}^\nu u = \sum_{j=1}^n b_j \left[ u(t_{n+1-j}) - u(t_{n-j}) \right].
\]

Therefore, the use of the above equations allows us to write explicit expressions for \( u(t_{n+1}) \). The use of this scheme does not cause any loss of accuracy for this framework, because the backward-Euler procedure is already first-order accurate. The discretization for the variables subject to the trial state \( u^{\text{trial}}(t_{n+1}) \) is given by

\[
D^\nu_t \left( u^{\text{trial}}(t) \right) \bigg|_{t=\Delta t^{-1}} = \frac{1}{(\Delta t)^\nu \Gamma(2-\nu)} \left[ u^{\text{trial}}(t_{n+1}) - u(t_n) + \mathcal{H}^\nu u \right],
\]

recalling that \( u^{\text{trial}}(t_{n+1}) = u(t_n) \), we obtain

\[
D^\nu_t \left( u^{\text{trial}}(t) \right) \bigg|_{t=\Delta t^{-1}} = \mathcal{H}^\nu u.
\]

Therefore, only the history term \( \mathcal{H}^\nu u \) is taken into account. For the time-fractional derivatives in the visco-plastic time reference, we consider

\[
t_{p_n+1} D^\nu_t \left( u(t) \right) \bigg|_{t=\Delta t^{-1}} = \frac{1}{\Gamma(2-\nu)} \left[ u(t_{n+1}) - u(t_n) + \mathcal{H}^\nu_{t_{p_n+1}} u \right],
\]

with

\[
\mathcal{H}^\nu_{t_{p_n+1}} u = \sum_{j=1}^{n-p_{n+1}} b_j \left[ u(t_{n+1-j}) - u(t_{n-j}) \right],
\]

where only the \( n - p_{n+1} \) terms from the beginning of the current visco-plastic time reference are taken into consideration. The trial state for this derivative is considered with the visco-plastic time reference from the last configuration \( p_n \), and is denoted by

\[
t_{p_n} D^\nu_t \left( u^{\text{trial}}(t) \right) \bigg|_{t=\Delta t^{-1}} = \mathcal{H}^\nu_{t_{p_n}} u
\]

with

\[
\mathcal{H}^\nu_{t_{p_n}} u = \sum_{j=1}^{n-p_n} b_j \left[ u(t_{n+1-j}) - u(t_{n-j}) \right],
\]

leading to a complexity of mathematical operations of \( \mathcal{O}(N^2) \).

6.1.1. Remark

Although Eq. (16) indicates that when \( \beta_K \to 1 \) we recover a local derivative operator, this is not what happens due to the adopted algorithmic procedure. Applying Eq. (51) for the trial state of the internal hardening parameter,
we obtain
\[
\left. t_{\beta_k} D_t^\beta_k \left( \alpha_{\text{trial}}^{(\text{trial})} \right) \right|_{t=t_n+1} = \frac{\mathcal{H}_{t_{\beta_k}} \alpha}{(\Delta t)^\beta_k \Gamma(2 - \beta_k)}.
\]
(53)

Recalling Eq. (50), when the fractional-order \( \nu \to 1 \), the associated weights \( b_j \to 0 \), and therefore the above equation for the fractional derivative of \( \alpha_n \) vanish. This asymptotic case, in fact, leads to the following trial yield function:
\[
f_{n+1}^{\text{trial}} = \left| \tau_{n+1}^{\text{trial}} \right| - \left[ \tau^\gamma + H \alpha_n \right].
\]
(54)

Therefore, the only remaining hardening effect relies on the rate-independent linear hardening term \( H \alpha_n \). If we combine both effects \( \beta_K \to 1 \) and \( H = 0 \), we obtain a limit case of asymptotic perfect visco-plasticity, which will be observed in results of Section 7.2.

The above discussion shows that due to the employment of a trial state, there is no reason in using a local first order derivative of the hardening parameter \( \alpha \) in the definition of \( f(\tau, \alpha) \), since the term would vanish regardless of the material parameters. However, the inclusion of a fractional derivative introduces memory effects that do not vanish, except when \( \beta_K \to 1 \).

6.2. Algorithm for the model M2

We consider the same stress–strain relationship as the model M1, given by Eq. (31). The incremental yield function is given by
\[
f_{n+1} = \left| \tau_{n+1} \right| - \tau^\gamma,
\]
(55)

with the corresponding trial function
\[
f_{n+1}^{\text{trial}} = \left| \tau_{n+1}^{\text{trial}} \right| - \tau^\gamma.
\]
(56)

Substituting Eq. (31) into Eq. (55), and recalling Eqs. (32) and (56), we obtain
\[
f_{n+1} = f_{n+1}^{\text{trial}} - E_0 C D_t^\beta_E (\Delta \gamma) \bigg|_{t=t_{n+1}}.
\]
(57)

With the discrete consistency condition \( (f_{n+1} = 0) \), we obtain the following fractional-order differential equation:
\[
E_0 C D_t^\beta_E (\Delta \gamma) \bigg|_{t=t_{n+1}} = f_{n+1}^{\text{trial}}.
\]
(58)

To calculate the rate-independent solution for stress, we use Eq. (38) replacing \( \tau_{n+1}^{\text{trial}} \) with \( \tau_\infty \):
\[
\tau_\infty = \left. \tau_{n+1}^{\text{trial}} - E_0 C D_t^\beta_E (\Delta \gamma) \right|_{t=t_{n+1}} \text{sign} \left( \tau_{n+1}^{\text{trial}} \right).
\]
(59)

Comparing Eqs. (58) and (59), we can rewrite Eq. (59) as
\[
\tau_\infty = \tau_{n+1}^{\text{trial}} - \text{sign}(\tau_{n+1}^{\text{trial}}) f_{n+1}^{\text{trial}} = \text{sign}(\tau_{n+1}^{\text{trial}}) \tau^\gamma.
\]
(60)

Therefore, the expression for \( \tau_\infty \) is given in a closed form. The algorithm to be used is of same type as the model M1, but we consider an additional equation when solving for the visco-plastic step, which is the incremental form of Eq. (22), given by
\[
E_0 C D_t^\beta_E (\varepsilon_{\text{vp}}) \bigg|_{t=t_{n+1}} + K C_{t_{\beta_k}} D_t^\beta_k (\varepsilon_{\text{vp}}) \bigg|_{t=t_{n+1}} = E_0 C D_t^\beta_E (\varepsilon) \bigg|_{t=t_{n+1}} - \text{sign}(\tau_{n+1}^{\text{trial}}) \tau^\gamma.
\]
(61)

Also, we use an auxiliary incremental loading function denoted by \( f_{n+1}^* \) to check for unloading conditions in the visco-plastic range, in order to perform an update in the yield stress in case of unloading:
\[
f_{n+1}^* = \left| \tau_{n+1}^{\text{trial}} \right| - \tau_{\alpha},
\]
(62)
Box II. Fractional return-mapping algorithm with visco-plastic regularization for the model M2.

where \( \tau_n \) is the stress from the previous time step \( n \). The fractional return-mapping algorithm for the model M2 is summarized in Box II, and the solution for the incremental fractional-order equations for both models are presented in Appendix A.

### 6.3. Incremental tangent modulus

The computation of the tangent modulus present in the tangent stiffness matrix shown in Eq. (28) is fundamental to achieve quadratic convergence for Newton’s method. In classical local models it is obtained by differentiating the incremental equations of Boxes I and II in terms of \( \epsilon_{n+1} \) to achieve an expression in closed form [9]. However, we did not obtain explicit expressions for such derivatives of the fractional-order equations presented for the models. Moreover, the tangent modulus is local in nature, and therefore we compute it using a simple backward finite-difference procedure given by

\[
\left( \frac{\partial \tau}{\partial \epsilon} \right)_{n+1} = \frac{\tau_{n+1} - \tau_n}{\epsilon_{n+1} - \epsilon_n},
\]

where \( \tau_{n+1} \) is obtained from the fractional return-mapping procedure of Boxes I and II. We note that the first-order accuracy of such finite-difference approximation does not affect the overall accuracy since our algorithm, in which the backward-Euler method is employed, is also first-order accurate.

Regarding the stability of the proposed algorithms, we have employed an unconditionally stable finite-difference method, developed in [24], for the discretization of time-fractional ODEs in the fractional return-mapping algorithms. Moreover, the governing equations of motion are integrated using a stable Newmark scheme, where the tangent modulus is computed using an implicit backward-Euler method, see e.g., [9].
Fig. 5. Stress versus total strain curves for the benchmark solutions. The number of data points for the model M2 is truncated for better visualization, since the results overlap with the model M1. Both models provide the same qualitative results for monotonic strains for the considered values of $\beta_E, \beta_K$.

7. Results and discussion

Three different analyses were performed to examine the developed models and algorithms. The first one consists of a convergence analysis using a benchmark solution, since there exist no available analytical solutions. The second test investigates the stress versus strain response of the models for prescribed monotonic and cyclic strains at constant rates. The last one solves a two-member truss with a snap-through instability with large strains and high strain rates.

7.1. Convergence analysis

A convergence analysis is performed for both models considering a reference benchmark solution for the stress, denoted by $\tau_b$. We consider the solution of Boxes I and II using prescribed strains. The error for the approximate solution denoted by $\tau_{ap}$ is calculated using the $L^\infty$ norm as

$$E_{L^\infty} = \frac{\|\tau_{ap} - \tau_b\|_{L^\infty}}{\|\tau_b\|_{L^\infty}}.$$ (64)

The material properties used are $E = 50 \text{ Pa} \cdot s^{\beta_E}, K = 50 \text{ Pa} \cdot s^{\beta_K}, \beta_E = 0.5, \beta_K = \{0.3, 0.7\}, H = 0 \text{ Pa}$ and $\tau^Y = 1 \text{ Pa}$. The material parameters are chosen to give a nonlinear response for both visco-elastic/plastic ranges for the given strain rate, as shown in Fig. 5. The final time considered is $T = 0.08 \text{ s}$ with total monotonic strain $\varepsilon = 0.015$, which gives a strain rate of $\dot{\varepsilon} = 0.1875 \text{ s}^{-1}$. The benchmark solution uses $N = 40960$ time steps, which is equivalent to a time increment size $\Delta t = 1.95 \times 10^{-6} \text{ s}$.

Fig. 6 shows the $L^\infty$-norm error of the approximate stress versus the time increment size $\Delta t$. The error levels obtained for the model M2 are significantly smaller because this model does not account for the integration of the internal variables $\alpha$ and $\Delta \gamma$, as presented in Box II. This also impacts the convergence rate, which is approximately linear for the model M1 but with slightly higher rate for the model M2.

7.2. Stress vs. strain response for prescribed strains

In order to show the influence of the fractional-order parameters $\beta_E$ and $\beta_K$ over the stress response, we performed monotonic and cyclic tests for both models. The purpose is to show the rate-dependency of the models according to the choice of the fractional-order parameters. For validation purposes, we also show the expected recovery to the limit case of rate-independent linear elasto-plasticity when $\beta_E$ and $\beta_K$ are close to zero (e.g. $\beta = \beta_E = \beta_K = 0.01$), as well as the behavior when $\beta \to 1$. 
Fig. 6. $L^\infty$-norm errors in $\tau_{ap}$ versus time step size $\Delta t$ in both models M1 and M2, where, $\beta_E = 0.5$.

Fig. 7. Stress versus strain responses for both models using $\beta_E = 0.01$. We observe that both models recover the limit case of rate-independent elasto-plasticity (EP) when $\beta_K \to 0$. However, the classical visco-plasticity of Duvaut–Lions type (EVP-DL) is not recovered for the model M1 when $\beta_K \to 1$. An asymptotic perfect visco-plastic solution is obtained instead.

7.2.1. Monotonic strains

The first step of this test consists of showing the recovery of the classical models of linear elasto-plasticity and visco-plasticity of Duvaut–Lions type. We consider the application of strain increments up to $\varepsilon = 0.4$ with rate $\dot{\varepsilon} = 0.05 \, s^{-1}$, where $T = 8 \, s$, and $N = 500$. The material properties used are $E = 50 \, Pa \cdot s^{\beta_E}$, $K = 5 \, Pa \cdot s^{\beta_K}$. For classical linear elasto-plasticity we use $H = 5 \, Pa$ and for the classical Duvaut–Lions model we use $\eta = 5 \, Pa \cdot s$.

Fig. 7 shows the obtained results. We observe that the classical elasto-plasticity with linear hardening is recovered when $\beta_E$, $\beta_K \to 0$. The consideration of $\beta_K \to 1$ did not recover the classical visco-plasticity for the model M1, as expected from the algorithmic discussion presented in Section 6.1.1.

The results for intermediate values of the fractional derivatives are shown in Fig. 8.

The effect of the linear hardening parameter $H$ for constant values of $K$, $\beta_K$ is presented in Fig. 9 for the model M1. Generally, this parameter $H$ contributes to less nonlinearity in this example, but may be used in conjunction with higher values of $\beta_K$ to provide more hardening when convenient.

7.2.2. Cyclic strains

To compute the loading/unloading response of the models and analyze the cyclic hardening behavior, we performed a cyclic test comprising three loading cycles. We start with a traction cycle from $\varepsilon = 0$ to $\varepsilon = 0.4$, followed by unloading and compression cycle until $\varepsilon = -0.4$, from where we increment the strains again up to $\varepsilon = 0.4$. We consider the material parameters $E = 100 \, Pa \cdot s^{\beta_E}$, $K = 50 \, Pa \cdot s^{\beta_K}$, $H = 0 \, Pa$. The prescribed strain rate is $\dot{\varepsilon} = 0.005 \, s^{-1}$, with time parameters $T = 400 \, s$ and $N = 10,000$. 
Fig. 8. Stress versus strain response for both models with $\beta_E = \beta_K$. We observe the same response for monotonic strains. The variation of the fractional-orders affects the rate-dependent behavior for both visco-elastic and visco-plastic ranges. In this case, the increase of the fractional-order led to higher stress levels in the visco-elastic range and also more hardening.

Fig. 9. Stress versus total strain curves for the model M1 with $\beta_E = \beta_K = 0.5$, $K = 5 \text{ Pa/s}^{\beta_K}$ and different values of $H$. We observe less nonlinearity and more dominant linear hardening as the numerical value of $H$ approaches $K$.

Fig. 10. Cyclic stress versus strain response for the models. We observe that due to the lack of internal hardening for the model M2, the stress amplitudes are smaller. Also, the response of the models after the first loading cycle is distinct.

The results for the cyclic tests are presented in Fig. 10. We observe that both models give the same qualitative results for the entire process for the considered material properties and strain rate.
\( K/E = 0.1 \, s^{0.799} \).

Fig. 11. Cyclic stress versus strain response for the models considering different ratios \( K/E \), \( \beta_E = 1.0 \times 10^{-3}, \beta_K = 0.8 \). We observe higher stress values for the model M2 as we increase \( K/E \), which become more pronounced with higher strain rates.

7.3. Two-member truss with snap-through effect

We consider the solution of the two-member truss presented in Fig. 12(a). The material properties are \( E = 2.1 \times 10^{11} \, Pa \cdot s^{\beta_E}, K = 1.0 \times 10^{11} \, Pa \cdot s^{\beta_K}, \tau_Y = 7.0 \times 10^8 \, Pa, \rho = 7.85 \times 10^{-6} \, kg/mm^3, \nu = 0.5 \) and \( A = 7.0 \, mm^2 \). A vertical force \( P(t) \) is applied at node 2, with the behavior over time illustrated in Fig. 12(b). We consider total time \( T = \{0.3, 1.0\} \, s \) and we analyze the recovery of classical elasto-plasticity for the models. Then, we will present the behavior for both models considering the variation of intermediate values of the fractional-orders.

7.3.1. Behavior of the models when \( \beta \to 0 \)

Fig. 13 shows the recovery of rate-independent elasto-plasticity with linear hardening for the models, by setting small values of \( \beta_E = 1.0 \times 10^{-5} \) and \( \beta_K = 1.0 \times 10^{-4} \), so that both Scott–Blair elements in the visco-elastic/plastic devices recover linear springs. We do not try the recovery of Duvaut–Lions elasto-visco-plasticity in this test for the models because of the previous observations in Section 6.1.1, when \( \beta_K \to 1 \) for the model M1, and the fact that the model M2 only recovers the classical model for monotonic loading (because of the update of the yield stress when unloading).
Fig. 13. Recovery of the models to rate-independent elasto-plasticity, $\Delta t = 6.0 \times 10^{-5}$ s. The snap-through effect occurs at $t = 0.089$ s.

Fig. 14. Results for the model M1 considering $\beta_E = 1.0 \times 10^{-4}$ and variation of $\beta_K$. We observe the increased visco-plastic hardening and damping with the increase of $\beta_K$.

7.3.2. Variation of fractional-orders for the model M1

We consider a constant value of $\beta_E = 1.0 \times 10^{-4}$ with variations of $\beta_K$ (fractional elasto-visco-plasticity order) to investigate the strain-hardening effect due to the Scott–Blair element in the visco-plastic device. Fig. 14 shows the results for displacement, stress and internal force. The increasing hardening with the increase of $\beta_K$ is observed from Fig. 14(a) with the late occurrence of the snap-through for $\beta_K = 0.9$ and larger amplitudes in displacement. The late snap-through can be justified by the higher peak in the internal force in Fig. 14(c) at approximately 60 mm of vertical displacement. The hardening is also seen in Fig. 14(b) with the larger visco-elastic range.

The results for longer time integration $T = 1$ s using $\beta_E = 1.0 \times 10^{-4}$ and time increment $\Delta t = 1.0 \times 10^{-4}$ s are presented in Fig. 15. We observe that due to the high strain rates, the elastic domain expands significantly more with higher values of $\beta_K$, and no dissipation is observed for the oscillations because the value of $\beta_E$ is sufficiently small.

We also considered the use of $\beta_E = 0.1$ to account for visco-elastic dissipation, with $\Delta t = 1.0 \times 10^{-5}$ s. Fig. 16(a) shows a very significant reduction in the displacement amplitudes for the short time interval, where increasing the value of $\beta_K$ led to more hardening. However, the use of a fractional-order value $\beta_E = 0.1$ is high enough to suppress the oscillations for the considered time domain. Figs. 16(b) and 16(c) also show a more pronounced visco-elastic relaxation behavior for $\beta_K = 0.5, 0.9$.

7.3.3. Variation of fractional-orders for the model M2

In the same way as the previous section, we tested the response of the truss for the model M2 using variation of the fractional-order parameters. Fig. 17 shows the results obtained for $\beta_E = 1.0 \times 10^{-4}$ and variation of $\beta_K$. We observe the same strain-hardening behavior as in the model M1 before the snap-through. Moreover, the snap-through
Fig. 15. Displacement versus time and stress versus strain for the model M1, $T = 1$ s, $\beta_E = 1.0 \times 10^{-4}$ and variation of $\beta_K$.

Fig. 16. Displacement versus time, stress versus strain and internal force versus displacement for the model M1, $\beta_E = 0.1$ and different values for $\beta_K$.

phenomenon for $\beta_K = 0.9$ occurs at $t \approx 0.2$ s, which is later than observed for the model M1 (Fig. 14(a)). This is compatible with the cyclic results presented in Section 7.2.2 with more rate-dependent hardening for the model M2 when using higher relaxation times and strain rates combined with higher values for $\beta_K$.

The behavior for $T = 1$ s with $\Delta t = 1.0 \times 10^{-4}$ s is shown in Fig. 18, which is qualitatively similar to the model M1, except for $\beta_K = 0.9$, where more hardening is observed for this case.

Fig. 19 shows that the use of a fractional-order $\beta_E = 0.1$ increased the visco-elastic dissipation similarly to the model M1. The time increment considered for this test was $\Delta t = 1.0 \times 10^{-4}$ s. We observe that the combination of
(a) Displacement versus time. (b) Stress versus strain.

Fig. 17. Displacement versus time, stress versus strain and internal force versus displacement for the model M2, $\beta_E = 1.0 \times 10^{-4}$ and variation of $\beta_K$.

(c) Internal force versus displacement.

Fig. 18. Displacement versus time and stress versus strain for the model M2, $\beta_K = 0.1$.

this fractional-order for visco-elasticity with $\beta_K = 0.9$ for the model M2 increased even more the hardening before the snap-through, occurring at $t \approx 0.23$ s (Fig. 19(a)), compared to $t \approx 0.2$ s (Fig. 16(a)) for the model M1.

The behavior of the presented models for the snap-through problem is qualitatively similar to another application in the literature for integer order visco-elasto-plastic formulation [36]. Also, the frequency response of the structure observed in Figs. 14(a) and 17(a) due to higher effective stiffness when using lower fractional-order $\beta$ (Scott–Blair element with lower viscosity) is also observed for linear single and multiple degree-of-freedom oscillators using fractional derivatives [37–39].
8. Conclusions

The main contribution of the present work is the development of two new fractional-order models for uniaxial large strains and visco-elasto-plastic behavior of materials in structural analysis. This generalized framework is amenable to modeling nonlinear and more complex effects namely visco-elasto-plastic response of materials. Two models, namely M1 and M2, were introduced with the following approaches for visco-plastic behavior:

- The model M1 was developed by modifying the classical rate-independent elasto-plasticity, with a rate-dependent yield function via the application of the time-fractional Caputo derivative on the accumulated plastic strain.
- The model M2 was developed as a fractional-order extension of the classical Duvaut–Lions elasto-visco-plastic model. In this sense, the time-fractional Caputo derivative was introduced in a visco-plastic regularization equation that solved for the visco-plastic strain when an over-stress level was achieved.

Based on the adopted yield stress update procedure, both models assumed a fast transition from visco-plastic to visco-elastic regime when unloading the material. Nevertheless we performed cyclic strain tests with different relaxation times and strain rates for both models and determined that:

- The model M2 showed more rate-dependency in the visco-plastic range, and is suitable for materials with general relaxation times and strain rates, since it was naturally defined based on the classical Duvaut–Lions formulation.
- The model M1 proved to be more suitable for materials with lower relaxation times (less dissipative) and lower strain rates, since the enforcement of the consistency condition assumed a fast relaxation occurring during the time interval $\Delta t$ under consideration.

An algorithm, called fractional return-mapping, was proposed to solve the nonlinear system of equilibrium equations resulted from the models. This algorithm seamlessly generalizes the standard return-mapping algorithm to its fractional counterpart, leading to an efficient framework for treating engineering applications involving visco-elasto-plastic materials. Most of the existing numerical methods for standard (integer-order) cases are not better than first-order accurate in time. Therefore, in our future work, we will also focus on developing further efficient numerical methods in terms of accuracy and computational costs.

A benchmark test performed with prescribed strains verified a linear convergence rate for the model M1. We observed a superlinear convergence rate for the model M2, since this model had no time integration of the plastic slip. We implemented the classical models of rate-independent elasto-plasticity and elasto-visco-plasticity of Duvaut–Lions type and verified that:

- Both models recover the classical elasto-plastic model with the choice of $\beta_E \to 0$ and $\beta_K \to 0$, even for the snap-through problem where a large number of tension/compression cycles was observed.
- The model M2 recovered the classical Duvaut–Lions model for monotonic strains by setting $\beta_E \to 0$ and $\beta_K \to 1$.

We did not obtain the same result for the model M1 due to the adopted algorithmic procedure.

The application to a two-member truss showed the influence of dissipation and strain hardening effects on the structure according to the value of the fractional-orders for the visco-elastic and visco-plastic parts.
The developed models can be fitted to experimental data from uniaxial stress/strain tests at constant or varying strain rates, as well as creep and relaxation tests, in order to identify the corresponding material coefficients and fractional-orders. Although the visco-elastic portion of the models consisted of a single Scott–Blair element, a more sophisticated fractional-order model can be incorporated (e.g. Kelvin–Voigt, Zener [40]).

In the case of very large visco-elastic strains (e.g. rubber), a hyper-visco-elastic behavior should be incorporated to the models to provide an accurate description. Moreover, the accuracy of the developed models can be improved by using a higher-order time integration method (e.g. fast convolution [41]). However, more theoretical developments would be necessary to derive the constitutive equations in convolution form. Also, a higher-order method to estimate the algorithmic tangent modulus would also be required (e.g. complex-step derivative [42]), since the finite-difference method adopted here was first-order accurate. Moreover, an extension of the developed models can be done in a straightforward way to account for kinematic hardening effects, as well as a continuum damage model (e.g. Lemaitre’s model [43]).

Therefore, fractional-order constitutive relations like the models developed in this work may be suitable to describe the constitutive behavior of a range of applications like: polymers, glass, metal alloys at higher temperatures [44], biological tissues (e.g. skin, bone, respiratory tissue, tendons [45–49]), as well as other relaxation phenomena in chemistry, electronics, magnetic systems [50,44,51] and social sciences (e.g. memory phenomena in psychology [52]).

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**Appendix A. Solution of the fractional-order differential equations in incremental form**

In this section we present the explicit expressions for the variables at time \( t_{n+1} \) after applying the finite-difference scheme for the Caputo time-fractional derivatives for both models. The trial stress is given by

\[
\tau_{n+1}^{\text{trial}} = E^* \left( \varepsilon_{n+1} - \varepsilon_n + \mathcal{H}^{\beta_E} \varepsilon_n + \varepsilon_{n+1}^{\text{vp}} + \varepsilon_n^{\text{vp}} - \mathcal{H}^{\beta_E} \varepsilon_n^{\text{vp}} \right),
\]

where

\[
E^* = \frac{E}{\Delta t^{\beta_E} \Gamma(2 - \beta_E)},
\]

where \( E^* \) has units of [Pa]. Recalling that \( \varepsilon_{n+1}^{\text{vp}} = \varepsilon_n^{\text{vp}} \), we can rewrite Eq. (A.1) as

\[
\tau_{n+1}^{\text{trial}} = E^* \left( \varepsilon_{n+1} - \varepsilon_n + \mathcal{H}^{\beta_E} \varepsilon_n - \mathcal{H}^{\beta_E} \varepsilon_n^{\text{vp}} \right).
\]

The associated trial yield function for the model M1 is explicitly given by

\[
f_{n+1}^{\text{trial}} = \left| \tau_{n+1}^{\text{trial}} \right| - \left[ \tau^Y + H \alpha_{n+1}^{\text{trial}} + K^* (\alpha_{n+1}^{\text{trial}} - \alpha_n + \mathcal{H}_{t_{n+1}}^{\beta_k} \alpha) \right],
\]

with

\[
K^* = \frac{K}{\Delta t^{\beta_k} \Gamma(2 - \beta_k)},
\]

where \( K^* \) has units of [Pa]. Recalling that \( \alpha_{n+1}^{\text{trial}} = \alpha_n \), we obtain

\[
f_{n+1}^{\text{trial}} = \left| \tau_{n+1}^{\text{trial}} \right| - \left( \tau^Y + H \alpha_n + K^* \mathcal{H}_{t_{n+1}}^{\beta_k} \alpha \right).
\]

Now we consider the solution for the plastic slip \( \Delta \gamma_{n+1} \). Applying Eqs. (44) and (49) to Eq. (43) for the model M1, we obtain

\[
\frac{E}{\Delta t^{\beta_E} \Gamma(2 - \beta_E)} \left( \Delta \gamma_{n+1} - \Delta \gamma_n + \mathcal{H}^{\beta_E} \Delta \gamma \right) + \frac{K}{\Delta t^{\beta_k} \Gamma(2 - \beta_k)} \left( \Delta \gamma_{n+1} - \Delta \gamma_n + \mathcal{H}_{t_{n+1}}^{\beta_k} \Delta \gamma \right)
\]
with the following Newmark coefficients

\[ \dot{u}_{n+1} = \ddot{u}_{n+1} = \dddot{u}_{n+1} = 0. \]

Solving the above equation for the plastic slip at current time \( t_{n+1} \), we obtain

\[ \Delta \gamma_{n+1} = \frac{E^* (\Delta \gamma_n - \mathcal{H}^{\beta_E} \Delta \gamma) + K^* (\Delta \gamma_n - \mathcal{H}^{\beta_E}_{\tau_{n+1}} \Delta \gamma) + f_{trial}^{t_{n+1}}}{E^* + K^* + H}. \]

(A.8)

Applying the same procedure to Eq. (38), we obtain the updated stress \( \tau_{n+1} \) for the model M1 given by

\[ \tau_{n+1} = \frac{E^* (\Delta \gamma_n - \Delta \gamma_n + \mathcal{H}^{\beta_E} \Delta \gamma \text{ sign} (\tau_{trial}^{t_{n+1}})). \]

(A.9)

Now we present the explicit expression for \( \varepsilon_{n+1}^{vp} \) from Eq. (61) for the model M2. Therefore, we have

\[ E^* (\varepsilon_{n+1}^{vp} - \varepsilon_n^{vp} + \mathcal{H}^{\beta_E} \varepsilon_{n+1}^{vp}) + K^* (\varepsilon_{n+1}^{vp} - \varepsilon_n^{vp} + \mathcal{H}^{\beta_E}_{\tau_{n+1}} \varepsilon_{n+1}^{vp}) = E^* (\varepsilon_{n+1} - \varepsilon_n + \mathcal{H}^{\beta_E} \varepsilon - \text{sign} (\tau_{trial}^{t_{n+1}}) \tau). \]

(A.10)

Solving for \( \varepsilon_{n+1}^{vp} \), we obtain

\[ \varepsilon_{n+1}^{vp} = \varepsilon_n^{vp} + \frac{E^* (\varepsilon_{n+1} - \varepsilon_n + \mathcal{H}^{\beta_E} \varepsilon - \mathcal{H}^{\beta_E}_{\tau_{n+1}} \varepsilon - \text{sign} (\tau_{trial}^{t_{n+1}}) \tau)}{E^* + K^*}. \]

(A.11)

Appendix B. Newmark integration scheme

We consider the equation of conservation of linear momentum in the discrete implicit form using a Newmark integration scheme without damping effects [35],

\[ \mathbf{M} \mathbf{a}_{n+1} + \mathbf{R}_{n+1} = \mathbf{P}_{n+1}, \]

(B.1)

with the above terms already described in Section 5. The initial conditions at \( t = 0 \) are given by

\[ \mathbf{u}_0 = \bar{\mathbf{u}}, \quad \mathbf{v}_0 = \bar{\mathbf{v}}. \]

(B.2)

The global accelerations and velocities are approximated as

\[ \mathbf{a}_{n+1} = b_1 (\mathbf{u}_{n+1} - \mathbf{u}_n) - b_2 \mathbf{v}_n - b_3 \mathbf{a}_n \]

\[ \mathbf{v}_{n+1} = b_4 (\mathbf{u}_{n+1} - \mathbf{u}_n) - b_5 \mathbf{v}_n - b_6 \mathbf{a}_n, \]

(B.3)

(B.4)

with the following Newmark coefficients

\[ b_1 = \frac{1}{g_1 \Delta t^2}, \quad b_2 = \frac{1}{g_1 \Delta t}, \quad b_3 = \frac{1 - 2g_1}{2g_1} \]

\[ b_4 = \frac{g_2}{g_1 \Delta t^2}, \quad b_5 = \left(1 - \frac{g_2}{g_1}\right), \quad b_6 = \left(1 - \frac{g_2}{2g_1}\right) \Delta t, \]

where it is usual to choose \( g_1 = 0.5, g_2 = 0.25 \) for unconditional stability.

References


