A Discontinuous Galerkin Method for Stochastic Conservation Laws

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Abstract

In this paper we present a discontinuous Galerkin (DG) method to approximate stochastic conservation laws, which is an efficient high-order scheme. We study the stability for the semi-discrete DG methods for fully nonlinear stochastic equations. Error estimates are obtained for smooth solutions of semi-linear stochastic equations with variable coefficients. We also establish a derivative-free second order time discretization scheme for matrix-valued stochastic ordinary differential equations. Numerical experiments are performed to confirm the analytical results.

Key Words. Discontinuous Galerkin method, Itô formula, multiplicative stochastic noise, stability analysis, error estimates, nonlinear stochastic conservation laws, stochastic Burgers equation.

1 Introduction

Many physical and engineering phenomena may contain some levels of stochastic influences such as random perturbation of forces and electromagnetic fields, which could be modeled by stochastic conservation laws. Thus the convergence of numerical methods for the discretization of stochastic conservation laws is a topic of high interest. In this paper we present a discontinuous Galerkin (DG) method for nonlinear stochastic hyperbolic scalar conservation

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laws with a periodic boundary condition and a multiplicative stochastic perturbation of the type:

\[
\begin{cases}
  du + f(u) \, dx + g(\omega, x, t, u) \, dW_t = 0 \\
  u(\omega, x, 0) = u_0(x),
\end{cases}
\in \Omega \times [0, 2\pi] \times (0, T),
\]

where the terminal time \( T > 0 \) is a fixed real number and \( \{W_t, 0 \leq t \leq T\} \) is a standard one-dimensional Brownian motion on a given probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with a filtration \( \{\mathcal{F}_t, 0 \leq t \leq T\} \) satisfying the usual conditions. We make the following hypotheses:

(H1) The initial condition \( u_0 \in L^2(0, 2\pi) \).

(H2) The function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is globally Lipschitz continuous.

(H3) The real scalar function \( g(\omega, x, t, u) \) is \( \mathcal{F} \otimes \mathcal{B}([0, 2\pi]) \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}) \)-measurable. There exist two constants \( L_g > 0 \) and \( C_g > 0 \) such that for any \( (\omega, x, t) \in \Omega \times [0, 2\pi] \times [0, T] \) and \( (u_1, u_2) \in \mathbb{R}^2 \),

\[
|g(\omega, x, t, u_1) - g(\omega, x, t, u_2)| \leq L_g |u_1 - u_2| \quad \text{and} \quad |g(\omega, x, t, 0)| \leq C_g.
\]

There are several papers on scalar conservation laws with a multiplicative stochastic forcing term involving a white noise in time. Feng and Nualart [14] discussed the spatially one-dimensional case, in which a notion of entropy solution is introduced to prove the existence and uniqueness of the solution. Later, much effort has been given to extend their results to the more general spatially multi-dimensional cases and to more extensive initial-boundary conditions. See e.g. [7, 13, 5, 6, 15]. In this article, we mainly consider the convergence of numerical methods for classical strong solutions with enough smoothness and integrability.

Concerning the study of numerical schemes for stochastic conservation laws with multiplicative noises, let us first mention that Bauzet, Charrier and Gallouët proposed several finite volume schemes. In [2], they studied the convergence of an explicit flux-splitting finite volume discretization, but with a more restrictive time step stability condition \( \Delta t \Delta x \rightarrow 0 \) as \( \Delta x \rightarrow 0 \). Then they investigated the case of a more general flux in [3]. In [4], they studied the convergence of the scheme when the stochastic conservation law is defined on a bounded domain with inhomogeneous Dirichlet boundary conditions. Let us also mention the convergence results of time-discretization of Holden and Risebro [16] and Bauzet [1] on a bounded domain of \( \mathbb{R}^d \), as well as the papers of Kröker [19], and Kroker and Rohde [20] of finite volume schemes in the one-dimensional case. But none of these articles gives the order of accuracy for numerical solutions. Also, there seems to be very little attention paid to the investigation of high-order approximate schemes for stochastic conservation laws.

The DG method we discuss is a class of high-order finite element methods using completely discontinuous piecewise polynomial space for the numerical solution and the test functions in the spatial variables, coupled with an explicit and nonlinear stable high-order time discretization. It was first introduced in 1973 by Reed and Hill [24], in the framework of neutron transport, which is a deterministic time-independent linear hyperbolic equation. It was later developed for nonlinear hyperbolic conservation laws containing first derivatives.
by Cockburn, Shu, et al. in a series of papers [9, 10, 11, 12], in which a framework is given to efficiently solve deterministic nonlinear time-dependent equations. Since the basis functions can be discontinuous, the DG methods have certain advantage and flexibility which are not shared by typical finite element methods such as: (1) it is easy to design high order approximations, thus allowing for efficient $p$ adaptivity; (2) it is flexible on complicated geometries, thus allowing for efficient $h$ adaptivity; (3) it is local in data communications, thus allowing for efficient parallel implementations. In this paper, we shall consider stochastic counterparts of these works and propose a DG scheme for stochastic conservation laws (1.1) and (3.10), respectively.

Jiang and Shu [17] proved a cell entropy inequality for the semidiscrete DG method to possibly nonsmooth solutions of nonlinear conservation laws, which gives the stability result for the numerical solutions. We shall consider possibly nonsmooth solutions of nonlinear stochastic equations, and prove that the numerical solutions of the DG scheme are stable. By similar method, we could also prove the stability of approximate solutions for semilinear variable-coefficient stochastic conservation laws.

Following the ideas for the deterministic case [25], we give the optimal error estimates ($O(h^{k+1})$ for the one-dimensional case) for the semilinear stochastic conservation laws with variable coefficients. Zhang and Shu [26] presented a priori error estimates for fully discrete Runge-Kutta DG methods with smooth solutions of scalar nonlinear conservation laws. Unfortunately, the unboundedness nature of the stochastic process driven by a Brownian motion prevents us from applying their method to get the error estimates for the fully nonlinear stochastic equation.

The DG method is a scheme for spatial discretization, which needs to be coupled with a high-order time discretization. Unlike the deterministic case, there is no simple heuristic generalizations of deterministic Runge-Kutta schemes to stochastic differential equations (SDEs). Kloeden and Platen [18] presented an explicit order 1.5 strong scheme. Milstein and Tretyakov [23] gave an implementable way to model Itô integrals, which is essential to construct an order 2.0 (second order) scheme. Combining these methods, in this paper we establish an explicit order 2.0 strong scheme for SDEs, which seems to be new.

Our effective computational methods for stochastic partial differential equations (SPDEs) have to face new difficulties. The solutions of SPDEs, when they do exist, are not naturally time-differentiable, and are not bounded in the path variable. These new features complicate the calculation and analysis in our stochastic context.

The paper is organized as follows. In Section 2, we introduce notations, definitions and auxiliary results used in the paper. In Section 3, we present the DG schemes for (1.1) and (3.10) respectively, and investigate the stability and error estimates of the schemes. In Section 4, we establish a derivative-free second order time discretization to collaborate with the semi-discrete scheme presented before. Finally the paper ends with a series of numerical experiments on some model problems in Section 5 which confirm the analytical results.
2 Notations, definitions and auxiliary results

In this section, we introduce notations, definitions, and also some auxiliary results.

2.1 Notations

We denote the mesh by \( I_j = \left[ x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}} \right] \), for \( j = 1, \ldots, N \). The center of the cell is \( x_j = \frac{1}{2} \left( x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}} \right) \) and the mesh size is denoted by \( h_j = x_{j-\frac{1}{2}} - x_{j+\frac{1}{2}} \), with \( h = \max_{1 \leq j \leq N} h_j \) being the maximum mesh size. We assume that the mesh is regular, namely the ratio between the maximum and the minimum mesh sizes stays bounded during mesh refinements. We define the piecewise-polynomial space \( V_h \) as the space of polynomials of the degree up to \( k \) in each cell \( I_j \), i.e.

\[
V_h = \{ v : v \in P^k(I_j) \text{ for } x \in I_j, \quad j = 1, \ldots, N \}.
\]

Note that functions in \( V_h \) are allowed to have discontinuities across element interfaces.

We denote by \( \| \cdot \| \) the usual \( L^2(0, 2\pi) \) norm with respect to the spatial variable \( x \). The solution of the numerical scheme is denoted by \( u_h \), which belongs to the finite element space \( V_h \). We denote by \( (u_h)^+_{j+\frac{1}{2}} \) and \( (u_h)^-_{j+\frac{1}{2}} \) the values of \( u_h \) at \( x_{j+\frac{1}{2}} \), from the right cell \( I_{j+1} \), and from the left cell \( I_j \), respectively. We use the conventional notation \( [u_h] := u_h^+ - u_h^- \) to denote the jump of the function \( u_h \) at each element boundary point.

Denote by \( |x| \) the Euclidean norm of \( x \in \mathbb{R}^k \) and by \( \langle x_1, x_2 \rangle \) the inner product of the vectors \( x_1 \) and \( x_2 \) in \( \mathbb{R}^k \). An element of \( \mathbb{R}^{k \times d} \) is a \( k \times d \) matrix, and its Euclidean norm is given by \( |y| := \sqrt{\text{trace}(yy^*)} \) and the inner product \( \langle y_1, y_2 \rangle := \text{trace}(y_1y_2^*) \) for \( y_1, y_2 \in \mathbb{R}^{k \times d} \).

By \( C > 0 \), we denote a generic constant, which in particular does not depend on the discretization width \( h \) and possibly changes from line to line. Since the It\’o integral is not defined path-wisely, the argument \( \omega \) of the integrand as a stochastic process will be omitted in the rest of this paper if there is no danger of confusion.

2.2 The numerical flux

For notational convenience we would like to introduce the following numerical flux related to the DG spatial discretization. The given monotone numerical flux \( \hat{f}(q^-, q^+) \) depends on the two values of the function \( q \) at the discontinuity point \( x_{j+\frac{1}{2}} \), namely \( q^\pm_{j+\frac{1}{2}} = q \left( x^\pm_{j+\frac{1}{2}} \right) \).

The numerical flux \( \hat{f}(q^-, q^+) \) satisfies the following conditions:

(a) it is globally Lipschitz continuous;
(b) it is consistent with the physical flux \( f(q) \), i.e., \( \hat{f}(q,q) = f(q) \);
(c) it is nondecreasing in the first argument, and nonincreasing in the second argument.
2.3 Inverse property

Finally we list an inverse property of the finite element space $V_h$ that will play a basic role in our error analysis. There exists a positive constant $C$ such that for any $q \in V_h,$

$$\left\| \frac{\partial q}{\partial x} \right\| \leq C h^{-1} \|q\|,$$  \hspace{1cm} (2.1)

where $C$ is independent of $q$ and $h.$ For more details, see Ciarlet [8].

3 The DG method for stochastic conservation laws and the stability analysis and error estimates

3.1 The DG method for fully nonlinear stochastic conservation laws

We present the DG method to approximate equation (1.1). For any $(\omega,t) \in \Omega \times [0,T],$ find $u_h(\omega,\cdot,t) \in V_h$ such that for any $v \in V_h,$

$$\int_{I_j} (du_h(\omega,x,t)) \cdot v(x) dx = \left( \int_{I_j} f(u_h(\omega,x,t)) v_x(x) dx - \hat{f}_{j+\frac{1}{2}} v_{j+\frac{1}{2}}^- + \hat{f}_{j-\frac{1}{2}} v_{j-\frac{1}{2}}^+ \right) dt$$

$$+ \left( \int_{I_j} g(\omega,x,t,u_h(\omega,x,t)) v(x) dx \right) dW_t,$$ \hspace{1cm} (3.1)

where $\hat{f}_{j+\frac{1}{2}} := \hat{f} \left( u_h(\omega,x_{j+\frac{1}{2}},t), u_h(\omega,x_{j+\frac{1}{2}},t) \right)$ and $v_{j+\frac{1}{2}}^+ := v(x_{j+\frac{1}{2}})$ for $j = 0,1,\ldots,N,$ and $\hat{f}$ is a numerical flux related to the physical flux $f.$

For $x \in I_j,$ the approximating solution should have the form

$$u_h(\omega,x,t) = \sum_{l=0}^{k} c_l^j(\omega,t) \varphi_l^j(x),$$

where $\{\varphi_l^j, l = 0,1,\ldots,k\}$ is a arbitrary basis of $P^k(I_j).$

Our aim is to solve (3.1) to get $\{c_l^j(\omega,t) : l = 0,1,\ldots,k, j = 0,1,\ldots,N+1\}.$ Taking $v := \varphi_m^j,$ $m = 0,1,\ldots,k,$ we have

$$\sum_{l=0}^{k} \left( \int_{I_j} \varphi_m^j(x) \varphi_l^j(x) dx \right) dc_l^j(\omega,t)$$

$$= \left( \int_{I_j} f \left( \sum_{l=0}^{k} c_l^j(\omega,t) \varphi_l^j(x) \right) \varphi_m^j(x) dx - \hat{f}_{j+\frac{1}{2}} \varphi_m^j(x_{j+\frac{1}{2}}) + \hat{f}_{j-\frac{1}{2}} \varphi_m^j(x_{j-\frac{1}{2}}) \right) dt$$

$$+ \left( \int_{I_j} g(\omega,x,t,\sum_{l=0}^{k} c_l^j(\omega,t) \varphi_l^j(x)) \varphi_m^j(x) dx \right) dW_t.$$
The mass matrix $A_i^j := [A^j_{ml}]$ with

$$A^j_{ml} := \int_{I_j} \varphi^j_m(x) \varphi^j_l(x) \, dx$$

is invertible, and its inverse is denoted by $A^{j,-1}$. Then the problem is reduced to solve the following $(k+1) \times (N+2)$-dimensional stochastic differential equation (SDE):

$$dc(\omega, t) = F(c(\omega, t)) \, dt + G(\omega, t, c(\omega, t)) \, dW_t, \quad (3.2)$$

where $c_{lj}(\omega, t) := c^j_l(\omega, t)$,

$$F_{lj}(c) := \int_{I_j} f \left( \sum_{n=0}^{k} c_{nj} \varphi^j_n(x) \right) \sum_{m=0}^{k} A^{j-1}_{lm} \varphi^j_m(x) \, dx$$

and

$$G_{lj}(\omega, t, c) := \int_{I_j} g \left( \omega, x, t, \sum_{n=0}^{k} c_{nj} \varphi^j_n(x) \right) \sum_{m=0}^{k} A^{j-1}_{lm} \varphi^j_m(x) \, dx.$$

The initial value of $c$ is determined by $u_0$ as follows:

$$c^j_l(0) := \sum_{m=0}^{k} A^{j-1}_{lm} \int_{I_j} u_0(x) \varphi^j_m(x) \, dx. \quad (3.3)$$

**Lemma 3.1.** Let Assumptions (H2) and (H3) hold. Then for any $N \in \mathbb{N}_+$, $F$ and $G$ are uniformly Lipschitz continuous in the variable $c$.

**Proof.** We only show the uniformly Lipschitz continuity of $F$ for fixed $N \in \mathbb{N}$, and that of $G$ can be proved in a similar way.

Fix $c, d \in \mathbb{R}^{(k+1) \times (N+2)}$, $l = 0, 1, \ldots, k$, and $j = 0, 1, \ldots, N + 1$. We have

$$F_{lj}(c) - F_{lj}(d) = E_{lj} + J_{lj} + K_{lj},$$

where

$$E_{lj} := \int_{I_j} \left\{ f \left( \sum_{n=0}^{k} c_{nj} \varphi^j_n(x) \right) - f \left( \sum_{n=0}^{k} d_{nj} \varphi^j_n(x) \right) \right\} \sum_{m=0}^{k} A^{j-1}_{lm} \varphi^j_m(x) \, dx,$$
\[ J_{l,j} := - \left\{ \hat{f} \left( \sum_{n=0}^{k} c_{n,j} \varphi_n^l(x_{j+\frac{1}{2}}) \right) \sum_{n=0}^{k} c_{n,j+1} \varphi_n^{l+1}(x_{j+\frac{1}{2}}) \right\} \]

\[ \hat{f} \left( \sum_{n=0}^{k} d_{n,j} \varphi_n^l(x_{j+\frac{1}{2}}) \right) \sum_{n=0}^{k} d_{n,j+1} \varphi_n^{l+1}(x_{j+\frac{1}{2}}) \right\} \sum_{m=0}^{k} A_{lm} \varphi_m(x_{j+\frac{1}{2}}), \]

\[ K_{l,j} := \left\{ \hat{f} \left( \sum_{n=0}^{k} c_{n,j-1} \varphi_n^{l-1}(x_{j-\frac{1}{2}}) \right) \sum_{n=0}^{k} c_{n,j} \varphi_n^l(x_{j-\frac{1}{2}}) \right\} \]

\[ \hat{f} \left( \sum_{n=0}^{k} d_{n,j-1} \varphi_n^{l-1}(x_{j-\frac{1}{2}}) \right) \sum_{n=0}^{k} d_{n,j} \varphi_n^l(x_{j-\frac{1}{2}}) \right\} \sum_{m=0}^{k} A_{lm} \varphi_m(x_{j-\frac{1}{2}}). \]

Since \( f \) is globally Lipschitz continuous in the variable \( c \), we have

\[
|E_{l,j}| \leq \int_{I_j} L_f \left| \sum_{n=0}^{k} (c_{n,j} - d_{n,j}) \varphi_n^l(x) \right| \| A^{l-1} \|_\infty \sum_{m=0}^{k} \varphi^l_{mx}(x) \, dx
\]

\[
\leq \left( \sum_{n=0}^{k} (c_{n,j} - d_{n,j})^2 \right)^{\frac{1}{2}} \| A^{l-1} \|_\infty \int_{I_j} \left( \sum_{n=0}^{k} \varphi_n^l(x) \right)^{\frac{1}{2}} \sum_{m=0}^{k} \varphi^l_{mx}(x) \, dx
\]

\[
\leq C(h) \left( \sum_{n=0}^{k} (c_{n,j} - d_{n,j})^2 \right)^{\frac{1}{2}},
\]

where \( L_f \) is the globally Lipschitz constant of \( f \), and \( C(h) \) is a positive constant which depends on \( h \). Then we have

\[
|E|^2 = \sum_{l=0}^{N+1} \sum_{j=0}^{N+1} |E_{l,j}|^2 \leq \sum_{l=0}^{N+1} \sum_{j=0}^{N+1} C(h)^2 \sum_{n=0}^{k} (c_{n,j} - d_{n,j})^2 = (k + 1)C(h)^2(c - d)^2.
\]

Since \( \hat{f} \) is globally Lipschitz continuous, we have

\[
|J_{l,j}| \leq (k + 1) \| A^{l-1} \|_\infty \max_{m,i} \| \varphi^l_m \|_\infty L_{\hat{f}} \]

\[
\times \left\{ \sum_{n=0}^{k} (c_{n,j} - d_{n,j}) \varphi_n^l(x_{j+\frac{1}{2}}) \right\} \left\{ \sum_{n=0}^{k} (c_{n,j+1} - d_{n,j+1}) \varphi_n^{l+1}(x_{j+\frac{1}{2}}) \right\}
\]

\[
\leq (k + 1)^{\frac{3}{2}} \| A^{l-1} \|_\infty \left( \max_{m,i} \| \varphi^l_m \|_\infty \right)^{\frac{1}{2}} L_{\hat{f}} \]

\[
\times \left\{ \left( \sum_{n=0}^{k} |c_{n,j} - d_{n,j}| \right)^{\frac{1}{2}} \right\} \left\{ \left( \sum_{n=0}^{k} |c_{n,j+1} - d_{n,j+1}| \right)^{\frac{1}{2}} \right\}
\]

\[
\leq C(h) \left\{ \left( \sum_{n=0}^{k} |c_{n,j} - d_{n,j}| \right)^{\frac{1}{2}} \right\} \left\{ \left( \sum_{n=0}^{k} |c_{n,j+1} - d_{n,j+1}| \right)^{\frac{1}{2}} \right\}.
\]

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where $L_f$ is the global Lipschitz constant of $\hat{f}$. Then we have

$$|J|^2 = \sum_{l=0}^{k} \sum_{j=0}^{N+1} |J_{l,j}|^2 \leq \sum_{l=0}^{k} \sum_{j=0}^{N+1} 2C(h)^2 \left( \sum_{n=0}^{k} |c_{n,j} - d_{n,j}|^2 + \sum_{n=0}^{k} |c_{n,j+1} - d_{n,j+1}|^2 \right)$$

$$= 4(k + 1)C(h)^2 |c - d|^2.$$ 

By similar calculation, we could get that

$$|K|^2 \leq 4(k + 1)C(h)^2 |c - d|^2.$$ 

Thus

$$|F(c) - F(d)|^2 \leq 3 \left( |E|^2 + |J|^2 + |K|^2 \right) \leq C(h) |c - d|^2.$$ 

Since $u_0$ is deterministic, by (3.3) we get that $c(0)$ is a deterministic matrix, which is $L^p(\Omega)$-integrable for any $p \geq 1$. Thus, SDE (3.2) has a unique solution \( \{c(t)\}_{0\leq t\leq T} \) such that for any $p \geq 1$,

$$E \left[ \sup_{0\leq t\leq T} |c(t)|^p \right] < \infty. \quad (3.4)$$

We have the following stability result for the numerical solutions.

**Theorem 3.1.** If the assumptions (H1)-(H3) hold, then there exists a constant $C > 0$ which is independent of $h$, such that for any $t \in [0, T]$,

$$E \left[ \|u_h(\cdot, t)\|^2 \right] \leq (C + \|u_h(\cdot, 0)\|^2) e^{Ct}.$$ 

**Proof.** For any $N \in \mathbb{N}_+$ and $(\omega, t) \in \Omega \times [0, T)$, take $v = u_h(\omega, \cdot, t)$ in (3.1),

$$\int_{I_j} (du_h(x,t)) \cdot u_h(x,t) \, dx = \left( \int_{I_j} f(u_h(x,t)) u_{h_1}(x,t) \, dx - \hat{f}_{j+\frac{1}{2}} u_{h,j+\frac{1}{2}} + \hat{f}_{j-\frac{1}{2}} u_{h,j-\frac{1}{2}} \right) \, dt$$

$$+ \int_{I_j} g(x,t,u_h(x,t)) \, u_h(x,t) \, dW_t. \quad (3.5)$$

In order to simplify the proof, we use the scaled Legendre basis of $\mathbb{P}^k(I_j)$ which satisfies that

$$A^j_{ml} = \int_{I_j} \varphi^j_m(x) \varphi^j_l(x) \, dx = A^j_{il} \delta_{ml}.$$ 

Then we have

$$A^j_{ml}^{-1} = \frac{1}{A^j_{il}} \delta_{ml}.$$ 

By the orthogonality of the Legendre basis, we get

$$G_{l,j}(t,c(t)) = A^j_{il} \int_{I_j} g(x,t,u_h(x,t)) \varphi^j_l(x) \, dx.$$ 

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Then for any \( l \in \{0, 1, \ldots, k\} \),

\[
dc^l_j(t) \cdot dc^l_j(t) = |G_{l,j}(t, c(t))|^2 dt
\]

\[
= \left\{ A_{l}^{j-1} \int_{I_j} g(x, t, u_h(x, t)) \varphi^j_l(x) dx \right\}^2 dt
\]

\[
\leq (A_{l}^{j-1})^2 \int_{I_j} |g(x, t, u_h(x, t))|^2 dx \int_{I_j} |\varphi^j_l(x)|^2 dx dt
\]

\[
\leq A_{l}^{j-1} \int_{I_j} (C_g + L_g|u_h(x, t)|)^2 dx dt.
\]

Thus we have

\[
\int_{I_j} du_h(x, t) \cdot du_h(x, t) dx = \int_{I_j} \left( \sum_{m=0}^{k} \varphi^2_m(x) dc^j_m(t) \right)^2 dx
\]

\[
= \sum_{l=0}^{k} A_{l}^{j} dc^l_j(t) \cdot dc^l_j(t) \leq C \int_{I_j} (1 + |u_h(x, t)|^2) dx dt. \tag{3.6}
\]

According to Itô formula, we have

\[
d |u_h(x, t)|^2 = 2u_h(x, t) du_h(x, t) + du_h(x, t) \cdot du_h(x, t). \tag{3.7}
\]

Combining (3.5), (3.6) and (3.7), we have

\[
\int_{I_j} (d |u_h(x, t)|^2) \ dx
\]

\[
\leq 2 \left( \int_{I_j} f(u_h(x, t)) u_{hx}(x, t) dx - \hat{f}_{j+\frac{1}{2}} u_{h,j+\frac{1}{2}}^- + \hat{f}_{j-\frac{1}{2}} u_{h,j-\frac{1}{2}}^+ \right) dt
\]

\[
+ 2 \int_{I_j} g(x, t, u_h(x, t)) u_h(x, t) dx dW_t + C \int_{I_j} (1 + |u_h(x, t)|^2) dx dt.
\]

Summarizing over \( j \) from 1 to \( N \), we have for \( t \in [0, T] \),

\[
\|u_h(\cdot, t)\|^2 - \|u_h(\cdot, 0)\|^2
\]

\[
\leq 2 \int_0^t \sum_{j=1}^N \left( \int_{I_j} f(u_h(x, s)) u_{hx}(x, s) dx - \hat{f}_{j+\frac{1}{2}} u_{h,j+\frac{1}{2}}^- + \hat{f}_{j-\frac{1}{2}} u_{h,j-\frac{1}{2}}^+ \right) ds
\]

\[
+ 2 \int_0^t \int_0^{2\pi} g(x, s, u_h(x, s)) u_h(x, s) dx dW_s + C + C \int_0^t \|u_h(\cdot, s)\|^2 ds. \tag{3.8}
\]

From (3.4), we have that for any \( p \geq 1 \),

\[
\mathbb{E} \left[ \int_0^T \int_0^{2\pi} |u_h(x, s)|^p \ dx \ ds \right] < \infty.
\]
and thus that the process
\[
\left\{ \int_0^t \int_0^{2\pi} g(x, s, u_h(x, s)) \, dx \, dW_s, \quad 0 \leq t \leq T \right\}
\]
is a martingale. Taking expectation on both sides of inequality (3.8), we have
\[
\mathbb{E} \left[ \| u_h(\cdot, t) \|^2 \right] \leq \| u_h(\cdot, 0) \|^2 + C \int_0^t \mathbb{E} \left[ \| u_h(\cdot, s) \|^2 \right] \, ds
\]
\[
+ 2\mathbb{E} \left[ \int_0^t \sum_{j=1}^{N} \left( \phi \left( u_{h,j+\frac{1}{2}}^- \right) - \phi \left( u_{h,j-\frac{1}{2}}^+ \right) - \hat{f}_{j+\frac{1}{2}} u_{h,j+\frac{1}{2}}^- + \hat{f}_{j-\frac{1}{2}} u_{h,j-\frac{1}{2}}^+ \right) \, ds \right]
\]
\[
= C + \| u_h(\cdot, 0) \|^2 + C \int_0^t \mathbb{E} \left[ \| u_h(\cdot, s) \|^2 \right] \, ds
\]
\[
+ 2\mathbb{E} \left[ \int_0^t \sum_{j=1}^{N} \left( \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} + \Theta_{j-\frac{1}{2}} \right) \, ds \right],
\]
(3.9)

where
\[
\phi(u) = \int^u f(a) \, da,
\]
\[
\hat{F}_{j+\frac{1}{2}} = \left( \phi(u^-_{h}) - \hat{f} u^-_{h} \right)_{j+\frac{1}{2}},
\]
\[
\Theta_{j-\frac{1}{2}} = \left( \phi(u^-_{h}) - \phi(u^+_{h}) + \hat{f} u^+_{h} - \hat{f} u^-_{h} \right)_{j-\frac{1}{2}}.
\]

By periodicity, we have
\[
\sum_{j=1}^{N} \left( \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} \right) = 0.
\]

Note that
\[
\Theta = \phi(u^-_{h}) - \phi(u^+_{h}) + \hat{f} u^+_{h} - \hat{f} u^-_{h}
\]
\[
= \phi(\xi)(u^+_{h} - u^-_{h}) + \hat{f} \left( \left( \hat{f}(u^-_{h}, u^+_{h}) - \hat{f}(\xi, \xi) \right) \left( u^+_{h} - u^-_{h} \right) \right)
\]
\[
= \left( \hat{f}(u^-_{h}, u^+_{h}) - \hat{f}(\xi, \xi) \right) \left( u^+_{h} - u^-_{h} \right) \leq 0.
\]

Then by (3.9), we have
\[
\mathbb{E} \left[ \| u_h(\cdot, t) \|^2 \right] \leq C + \| u_h(\cdot, 0) \|^2 + C \int_0^t \mathbb{E} \left[ \| u_h(\cdot, s) \|^2 \right] \, ds.
\]

Using Gronwall’s inequality, we have
\[
\mathbb{E} \left[ \| u_h(\cdot, t) \|^2 \right] \leq \left( C + \| u_h(\cdot, 0) \|^2 \right) e^{Ct}.
\]
3.2 The DG method for variable-coefficient semi-linear stochastic conservation laws

The flux $f$ in Equation (1.1) depends only on the variable $u$. Now we consider a more general case where the flux depends on $u$ linearly via the variable coefficient $a(\omega, x, t)$ as follows:

$$
\begin{cases}
  du + (a(\omega, x, t)u) \, dt &= g(\omega, x, t, u) \, dW_t \quad \text{in } \Omega \times [0, 2\pi] \times (0, T), \\
  u(\omega, x, 0) &= u_0(x), \quad \omega \in \Omega, x \in [0, 2\pi],
\end{cases}
$$

(3.10)

where $a$ satisfies the following assumption,

(H4) The function $a(\omega, \cdot, t)$ is periodic and smooth for any $(\omega, t) \in \Omega \times [0, T]$ and for any positive integer $l$,

$$
\sup_{(\omega, x, t) \in \Omega \times [0, 2\pi] \times [0, T]} \left\{ |a(\omega, x, t)| + \sum_{i=1}^{l} \left| \frac{d^l a}{dx^l}(\omega, x, t) \right| \right\} < \infty.
$$

Analogous to the deterministic case, we present the DG method for variable-coefficient semi-linear stochastic conservation laws. For any $(\omega, t) \in \Omega \times [0, T]$, find $u_h(\omega, \cdot, t) \in V_h$ such that for any $v \in V_h$,

$$
\int_{I_j} (du_h(\omega, x, t)) \cdot v(x) \, dx = \left( \int_{I_j} a(\omega, x, t)u_h(\omega, x, t)v_x(x) \, dx - \hat{a}_{j+\frac{1}{2}} v_{j+\frac{1}{2}}^{-} + \hat{a}_{j-\frac{1}{2}} v_{j-\frac{1}{2}}^{+} \right) \, dt \\
+ \left( \int_{I_j} g(\omega, x, t, u_h(\omega, x, t)) v(x) \, dx \right) \, dW_t
$$

(3.11)

with

$$
\hat{a}_{j+\frac{1}{2}} := a_{+}(\omega, x_{j+\frac{1}{2}}, t)u_{h}(\omega, x_{j+\frac{1}{2}}, t) - a_{-}(\omega, x_{j+\frac{1}{2}}, t)u_{h}(\omega, x_{j+\frac{1}{2}}, t),
$$

where $a_{+}$ and $a_{-}$ are the positive and negative parts of the real number $a$, and thus $a = a_{+} - a_{-}$.

Similar to the above subsection, for $x \in I_j$, the approximating solution should have the form

$$
u_h(\omega, x, t) = \sum_{l=0}^{k} c^j_l(\omega, t) \varphi^j_l(x).
$$

We want to solve (3.11) to get \{\varphi^j_l(\omega, t) : l = 0, 1, ..., k, j = 0, 1, ..., N + 1\}. Taking $v = \varphi^j_m$, $m = 0, 1, ..., k$, we obtain

$$
\sum_{l=0}^{k} \left( \int_{I_j} \varphi^j_m(x) \varphi^j_l(x) \, dx \right) dc^j_l(\omega, t) = \left( \int_{I_j} a(\omega, x, t) \left( \sum_{l=0}^{k} c^j_l(\omega, t) \varphi^j_l(x) \right) \varphi^j_m(x) \, dx - \hat{a}_{j+\frac{1}{2}} \varphi^j_m(x_{j+\frac{1}{2}}) + \hat{a}_{j-\frac{1}{2}} \varphi^j_m(x_{j-\frac{1}{2}}) \right) \, dt
$$

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\[
\begin{align*}
+ \left( \int_{I_j} g \left( \omega, x, t, \sum_{l=0}^{k} c_j^l(\omega, t) \varphi_j^l(x) \right) \varphi_m^j(x) \, dx \right) \, dW_t.
\end{align*}
\]

Then the problem is reduced to solve a \((k + 1) \times (N + 2)\)-dimensional SDE as following

\[
dc(\omega, t) = F(\omega, t, c(\omega, t)) \, dt + G(\omega, t, c(\omega, t)) \, dW_t,
\]

where

\[
c_{l,j}(\omega, t) = c_j^l(\omega, t),
\]

\[
F_{l,j}(\omega, t, c) = \int_{I_j} a(\omega, x, t) \left( \sum_{n=0}^{k} c_{n,j} \varphi_n^j(x) \right) \sum_{m=0}^{k} A_{lm}^{j-1} \varphi_m^j(x) \, dx
\]

\[
- a_+(\omega, x_{j+\frac{1}{2}}, t) \left( \sum_{n=0}^{k} c_{n,j+1} \varphi_n^{j+1}(x_{j+\frac{1}{2}}) \right) \sum_{m=0}^{k} A_{lm}^{j-1} \varphi_m^j(x_{j+\frac{1}{2}})
\]

\[
+ a_-(\omega, x_{j+\frac{1}{2}}, t) \left( \sum_{n=0}^{k} c_{n,j-1} \varphi_n^{j-1}(x_{j+\frac{1}{2}}) \right) \sum_{m=0}^{k} A_{lm}^{j-1} \varphi_m^j(x_{j+\frac{1}{2}})
\]

\[
+ a_+(\omega, x_{j-\frac{1}{2}}, t) \left( \sum_{n=0}^{k} c_{n,j} \varphi_n^j(x_{j-\frac{1}{2}}) \right) \sum_{m=0}^{k} A_{lm}^{j-1} \varphi_m^j(x_{j-\frac{1}{2}})
\]

\[
- a_-(\omega, x_{j-\frac{1}{2}}, t) \left( \sum_{n=0}^{k} c_{n,j+1} \varphi_n^{j+1}(x_{j-\frac{1}{2}}) \right) \sum_{m=0}^{k} A_{lm}^{j-1} \varphi_m^j(x_{j-\frac{1}{2}}),
\]

and

\[
G_{l,j}(\omega, t, c) = \int_{I_j} g \left( \omega, x, t, \sum_{n=0}^{k} c_{n,j} \varphi_n^j(x) \right) \sum_{m=0}^{k} A_{lm}^{j-1} \varphi_m^j(x) \, dx.
\]

Again we use the \(L^2\)-projection coefficients of \(u_0\) as the initial value of \(c\). That is,

\[
c_j^0(0) = \sum_{m=0}^{k} A_{lm}^{j-1} \int_{I_j} u_0(x) \varphi_m^j(x) \, dx.
\]

Similar to Lemma 3.1, we have from both Hypotheses (H3) and (H4) that \(F\) and \(G\) are uniformly Lipschitz-continuous in the variable \(c\). Since \(u_0\) is a deterministic function, then \(c(0)\) is a deterministic matrix, which is \(L^p(\Omega)\)-integrable for any \(p \geq 1\). Thus according to classical results of stochastic differential equations, SDE (3.12) has a unique solution \(\{c(t)\}_{0 \leq t \leq T}\) such that for any \(p \geq 1\),

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |c(t)|^p \right] < \infty.
\]

(3.13)

Similar to Theorem 3.1, we could also obtain that the scheme (3.11) is stable.
Theorem 3.2. If the assumptions (H1), (H3) and (H4) hold, then there exists a constant $C > 0$, which is independent with $h$, such that for any $t \in [0,T]$,

$$
\mathbb{E} \left[ \| u_h(\cdot, t) \|^2 \right] \leq (C + \| u_h(\cdot, 0) \|^2) e^{Ct}.
$$

Proof. For any $N \in \mathbb{N}_+$ and $(\omega, t) \in \Omega \times [0,T]$, we define a bilinear functional on piecewisely smooth function space. For any piecewise smooth functions $u, v$, define

$$
H_j(a, \omega, t; u, v) = \int_{I_j} a(\omega, x, t) u(x)v(x) \, dx \\
- \left( a_+(\omega, x_{j+\frac{1}{2}}, t) u_{j+\frac{1}{2}}^+ - a_-(\omega, x_{j+\frac{1}{2}}, t) u_{j+\frac{1}{2}}^- \right) v_{j+\frac{1}{2}}^- \\
+ \left( a_+(\omega, x_{j-\frac{1}{2}}, t) u_{j-\frac{1}{2}}^- - a_-(\omega, x_{j-\frac{1}{2}}, t) u_{j-\frac{1}{2}}^+ \right) v_{j-\frac{1}{2}}^+.
$$

Note that

$$
\int_{I_j} a(\omega, x, t) u(x)u(x) \, dx = -\frac{1}{2} \int_{I_j} a_x(\omega, x, t) |u(x)|^2 \, dx + \frac{1}{2} \left[ a_{j+\frac{1}{2}} \left| u_{j+\frac{1}{2}}^- \right|^2 - a_{j-\frac{1}{2}} \left| u_{j-\frac{1}{2}}^+ \right|^2 \right]
$$

where $a_{j+\frac{1}{2}} = a(\omega, x_{j+\frac{1}{2}}, t)$, $j = 0, 1, ..., N$. Then

$$
H_j(a, \omega, t; u, u) = -\frac{1}{2} \int_{I_j} a_x(\omega, x, t) |u(x)|^2 \, dx + \frac{1}{2} \left[ a_{j+\frac{1}{2}} \left| u_{j+\frac{1}{2}}^- \right|^2 - a_{j-\frac{1}{2}} \left| u_{j-\frac{1}{2}}^+ \right|^2 \right]
$$

$$
- a_{j+\frac{1}{2},+} \left| u_{j+\frac{1}{2}}^- \right|^2 + a_{j+\frac{1}{2},-} \left| u_{j+\frac{1}{2}}^+ \right|^2 + a_{j-\frac{1}{2},+} \left| u_{j-\frac{1}{2}}^- \right|^2 + a_{j-\frac{1}{2},-} \left| u_{j-\frac{1}{2}}^+ \right|^2
$$

$$
= -\frac{1}{2} \int_{I_j} a_x(\omega, x, t) |u(x)|^2 \, dx - \hat{F}_{j+\frac{1}{2}} + \hat{F}_{j-\frac{1}{2}}
$$

$$
+ a_{j+\frac{1}{2},-} u_{j+1}^- u_{j+\frac{1}{2}}^- - \frac{1}{2} a_{j+\frac{1}{2},-} \left| u_{j+\frac{1}{2}}^- \right|^2 - \frac{1}{2} a_{j+\frac{1}{2},+} \left| u_{j+\frac{1}{2}}^+ \right|^2
$$

$$
+ a_{j-\frac{1}{2},+} u_{j-1}^- u_{j-\frac{1}{2}}^- - \frac{1}{2} a_{j-\frac{1}{2},+} \left| u_{j-\frac{1}{2}}^- \right|^2 - \frac{1}{2} a_{j-\frac{1}{2},-} \left| u_{j-\frac{1}{2}}^+ \right|^2
$$

$$
= -\frac{1}{2} \int_{I_j} a_x(\omega, x, t) |u(x)|^2 \, dx - \hat{F}_{j+\frac{1}{2}} + \hat{F}_{j-\frac{1}{2}} - \frac{1}{2} \left\| a_x \right\|_{\infty} \int_{I_j} |u(x)|^2 \, dx
$$

$$
\leq \left\| a_x \right\|_{\infty} \int_{I_j} |u(x)|^2 \, dx - \left( \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} \right),$$

where

$$
\hat{F}_{j+\frac{1}{2}} = a_{j+\frac{1}{2},-} u_{j+\frac{1}{2},-} - \frac{1}{2} a_{j+\frac{1}{2},+} \left| u_{j+\frac{1}{2}}^+ \right|^2 - \frac{1}{2} a_{j+\frac{1}{2},-} \left| u_{j+\frac{1}{2}}^+ \right|^2.
$$
By periodicity we know that
\[ \sum_{j=1}^{N} \left( \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} \right) = 0. \]

Thus
\[ \sum_{j=1}^{N} H_j(a, \omega, t; u, u) \leq \frac{1}{2} \|a_x\|_\infty \|u\|^2. \] (3.14)

For any \( N \in \mathbb{N}_+ \) and \( (\omega, t) \in \Omega \times [0, T) \), take \( v = u_h(\omega, \cdot, t) \) in (3.11) and do the summation from \( j = 1 \) to \( j = N \),
\[ \int_0^{2\pi} (du_h(x,t)) \cdot u_h(x,t) \, dx = \sum_{j=1}^{N} H_j(a, t; u_h(\cdot, t), u_h(\cdot, t)) \, dt \\
+ \int_0^{2\pi} g(x, t, u_h(x,t)) u_h(x,t) \, dx \, dW_t, \\
\leq \frac{1}{2} \|a_x\|_\infty \|u_h(\cdot, t)\|^2 \, dt + \int_0^{2\pi} g(x, t, u_h(x,t)) u_h(x,t) \, dx \, dW_t. \]

Similar to the calculation for (3.6) in Theorem 3.1, we have
\[ \int_0^{2\pi} du_h(x,t) \cdot du_h(x,t) \, dx \leq C \int_0^{2\pi} (1 + \|u_h(x,t)\|^2) \, dx \, dt. \]

According to Itô formula we have
\[ d\|u_h(\cdot, t)\|^2 = 2u_h(x,t) \, du_h(x,t) + du_h(x,t) \cdot du_h(x,t). \]

Thus
\[ \int_0^{2\pi} (d\|u_h(x,t)\|^2) \, dx \leq \|a_x\|_\infty \|u_h(\cdot, t)\|^2 \, dt + 2 \int_0^{2\pi} g(x, t, u_h(x,t)) u_h(x,t) \, dx \, dW_t \\
+ C \int_0^{2\pi} (1 + \|u_h(x,t)\|^2) \, dx \, dt. \] (3.15)

By similar arguments in Theorem 3.1 we know that
\[ \left\{ \int_0^t \int_0^{2\pi} g(x, s, u_h(x,s)) u_h(x,s) \, dx \, dW_s, \quad 0 \leq t \leq T \right\} \]
is a martingale. Integrating from \( t = 0 \) and taking expectation on both sides of (3.15) we have
\[ \mathbb{E} \left[ \|u_h(\cdot, t)\|^2 \right] \leq C + \|u_h(\cdot, 0)\|^2 + C \int_0^t \mathbb{E} \left[ \|u_h(\cdot, s)\|^2 \right] \, ds. \]

Lastly Gronwall’s inequality tells us that
\[ \mathbb{E} \left[ \|u_h(\cdot, t)\|^2 \right] \leq \left( C + \|u_h(\cdot, 0)\|^2 \right) e^{Ct}. \]

Now we state the error estimates of the DG method (3.11).
Theorem 3.3. Suppose that assumptions (H1), (H3) and (H4) hold and equation (3.10) has a unique strong solution \( u(\cdot) \) such that

\[
(H5) \quad u(\cdot) \in L^2(\Omega \times [0, T]; H^{k+2}) \cap L^4(\Omega \times [0, 2\pi] \times [0, T]; \mathbb{R}) \cap L^\infty(0, T; L^2(\Omega; H^{k+1}));
\]

\[
(H6) \quad g(\cdot, u(\cdot)) \in L^2(\Omega \times [0, T]; H^{k+1}).
\]

Then there exists a constant \( C > 0 \), which is independent with \( h \), such that for any \( t \in [0, T] \),

\[
(\mathbb{E}[\|u(\cdot, t) - u_h(\cdot, t)\|^2])^{\frac{1}{2}} \leq C e^{Ct} h^{k+1}.
\]

Proof. Notice that the scheme (3.11) is also satisfied when the numerical solution \( u_h(\cdot) \) is replaced by the exact solution \( u(\cdot) \), for any \( v \in V_h \),

\[
\int_{I_j} (du(\omega, x, t)) \cdot v(x) dx = \left( \int_{I_j} a(\omega, x, t)u(\omega, x, t)v_x(x) dx - \widehat{a}_j^{u+}v_{j+} - \widehat{a}_j^{u-}v_{j-} \right) dt
\]

\[
+ \left( \int_{I_j} g(\omega, x, t, u(\omega, x, t)) v(x) dx \right) dW_t,
\]

with

\[
\widehat{a}_j^{u+} = a_+(\omega, x_{j+\frac{1}{2}}, t)u(\omega, x_{j+\frac{1}{2}}, t) - a_-(\omega, x_{j+\frac{1}{2}}, t)u(\omega, x_{j+\frac{1}{2}}, t).
\]

Define

\[
e(\omega, x, t) = u(\omega, x, t) - u_h(\omega, x, t) = \xi(\omega, x, t) - \eta(\omega, x, t),
\]

with

\[
\xi(\omega, x, t) = \mathcal{P}u(\omega, x, t) - u_h(\omega, x, t), \quad \eta(\omega, x, t) = \mathcal{P}u(\omega, x, t) - u(\omega, x, t),
\]

where \( \mathcal{P} \) is a projection from \( H^{k+1} \) onto \( V_h \), which will be specified later.

By (3.11) and (3.16), we have the error equation

\[
\int_{I_j} (de(\omega, x, t)) \cdot v(x) dx
\]

\[
= \left( \int_{I_j} a(\omega, x, t)e(\omega, x, t)v_x(x) dx - \widehat{a}_j^{e+}v_{j+} - \widehat{a}_j^{e-}v_{j-} \right) dt
\]

\[
+ \int_{I_j} \{ g(\omega, x, t, u(\omega, x, t)) - g(\omega, x, t, u_h(\omega, x, t)) \} v(x) dx dW_t,
\]

with

\[
\widehat{a}_j^{e+} = a_+(\omega, x_{j+\frac{1}{2}}, t)e(\omega, x_{j+\frac{1}{2}}, t) - a_-(\omega, x_{j+\frac{1}{2}}, t)e(\omega, x_{j+\frac{1}{2}}, t).
\]

It turns out that

\[
\int_{I_j} (d\xi(\omega, t)) \cdot v(x) dx
\]

\[
= \left( \int_{I_j} (d\eta(\omega, t)) \cdot v(x) dx + H_j(a, \omega, t; \xi(\cdot, t), v) - H_j(a, \omega, t; \eta(\cdot, t), v) \right) dt
\]
\[ + \int_{I_j} \{ g(x, t, u(x, t)) - g(x, t, u_h(x, t)) \} v(x) \, dx \, dW_t. \]

Taking \( v = \xi(\cdot, t) \) and doing the summation from \( j = 1 \) to \( j = N \) we have

\[
\int_0^{2\pi} \xi(x, t) d\xi(x, t) \, dx \\
= \int_0^{2\pi} \xi(x, t) d\eta(x, t) \, dx + \sum_{j=1}^N \left( H_j(a, \omega, \xi(\cdot, t), \xi(\cdot, t)) - H_j(a, \omega, \eta(\cdot, t), \xi(\cdot, t)) \right) \, dt \\
+ \int_0^{2\pi} \{ g(x, t, u(x, t)) - g(x, t, u_h(x, t)) \} \xi(x, t) \, dx \, dW_t.
\]

According to Itô formula we get

\[
d |\xi(x, t)|^2 = 2\xi(x, t) \, d\xi(x, t) + d\xi(x, t) \cdot d\xi(x, t).
\]

Then we have

\[
E \left[ \|\xi(\cdot, t)\|^2 \right] = \|\xi(\cdot, 0)\|^2 + T_1(t) + T_2(t) + T_3(t) + T_4(t) + T_5(t),
\]

where

\[
T_1(t) = -2E \left[ \int_0^t \sum_{j=1}^N H_j(a, \omega, \xi(\cdot, s), \xi(\cdot, s)) \, ds \right],
\]

\[
T_2(t) = E \left[ \int_0^{2\pi} \int_0^t d\xi(x, s) \cdot d\xi(x, s) \, dx \right],
\]

\[
T_3(t) = 2E \left[ \int_0^{2\pi} \int_0^t \xi(x, s) \, d\eta(x, s) \, dx \right],
\]

\[
T_4(t) = 2E \left[ \int_0^t \sum_{j=1}^N H_j(a, \omega, \xi(\cdot, s), \xi(\cdot, s)) \, ds \right],
\]

\[
T_5(t) = 2E \left[ \int_0^t \int_0^{2\pi} \{ g(x, s, u(x, s)) - g(x, s, u_h(x, s)) \} \xi(x, s) \, dx \, dW_s \right]
\]

will be estimated separately later.

- \( T_1(t) \) term.

We define the projection \( P \) on piecewise smooth function space as the following way. For any fixed \((\omega, t) \in \Omega \times [0, T], j \in \{1, 2, ..., N\}\) and piecewise smooth function \( u \),

Case 1: If \( a(\omega, x_{j-\frac{1}{2}}, t) < 0 \) and \( a(\omega, x_{j+\frac{1}{2}}, t) > 0 \),

\[
\begin{align*}
\int_{I_j} (P u - u)^-(x) \cdot v(x) \, dx &= 0, \quad \forall v \in P^{k-2}(I_j) \\
(P u - u)^-_{j+\frac{1}{2}} &= 0, \\
(P u - u)^+_{j-\frac{1}{2}} &= 0.
\end{align*}
\]
Case 2: If \( a(\omega, x_{j-\frac{1}{2}}, t) > 0 \) and \( a(\omega, x_{j+\frac{1}{2}}, t) > 0 \),
\[
\begin{cases}
\int_{I_j} \langle Pu - u \rangle (x) \cdot v(x) \, dx = 0, & \forall v \in P^{k-1}(I_j) \\
(Pu - u)_{j+\frac{1}{2}}^+ = 0.
\end{cases}
\]

Case 3: If \( a(\omega, x_{j-\frac{1}{2}}, t) < 0 \) and \( a(\omega, x_{j+\frac{1}{2}}, t) < 0 \),
\[
\begin{cases}
\int_{I_j} \langle Pu - u \rangle (x) \cdot v(x) \, dx = 0, & \forall v \in P^{k-1}(I_j) \\
(Pu - u)_{j-\frac{1}{2}}^- = 0.
\end{cases}
\]

Case 4: If \( a(\omega, x_{j-\frac{1}{2}}, t) > 0 \) and \( a(\omega, x_{j+\frac{1}{2}}, t) < 0 \),
\[
\int_{I_j} \langle Pu - u \rangle (x) \cdot v(x) \, dx = 0, & \forall v \in P^k(I_j).
\]

Then according to the classical projection theory and (H4), we know that there is a constant \( C > 0 \) that is independent with \( \omega, t, u \) and \( h \), such that
\[
\|u - Pu\| \leq C \|u\|_{H^{k+1}} h^{k+1}. \tag{3.17}
\]

Note that
\[
H_j(a, \omega, t; \eta(\cdot, s), \xi(\cdot, s)) = \int_{I_j} a(\omega, x, t)\eta(\omega, x, s)\xi_x(\omega, x, s) \, dx
- \left( a_+(\omega, x_{j+\frac{1}{2}}, t)\eta_{j+\frac{1}{2}}^- - a_-(\omega, x_{j+\frac{1}{2}}, t)\eta_{j+\frac{1}{2}}^+ \right) \xi_{j+\frac{1}{2}}^{-}
+ \left( a_+(\omega, x_{j-\frac{1}{2}}, t)\eta_{j-\frac{1}{2}}^- - a_-(\omega, x_{j-\frac{1}{2}}, t)\eta_{j-\frac{1}{2}}^+ \right) \xi_{j-\frac{1}{2}}^+.
\]

By the properties of the projection \( P \), we can verify that for all \( j \in \{1, 2, ..., N\} \)
\[
- \left( a_+(\omega, x_{j+\frac{1}{2}}, t)\eta_{j+\frac{1}{2}}^- - a_-(\omega, x_{j+\frac{1}{2}}, t)\eta_{j+\frac{1}{2}}^+ \right) \xi_{j+\frac{1}{2}}^{-}
+ \left( a_+(\omega, x_{j-\frac{1}{2}}, t)\eta_{j-\frac{1}{2}}^- - a_-(\omega, x_{j-\frac{1}{2}}, t)\eta_{j-\frac{1}{2}}^+ \right) \xi_{j-\frac{1}{2}}^+ = 0.
\]

For the term \( \int_{I_j} a(\omega, x, t)\eta(\omega, x, s)\xi_x(\omega, x, s) \, dx \), we study it case by case.

Case 1 & Case 4: Since \( a(\omega, x_{j-\frac{1}{2}}, t) \cdot a(\omega, x_{j+\frac{1}{2}}, t) < 0 \), there must exist \( y_j \in I_j \) such that \( a(\omega, y_j, t) = 0 \). Then according to (H4) we have
\[
|a(\omega, x, t)| = |a(\omega, y_j, t) + a_x(\omega, \xi, t)(x - y_j)| \leq Ch.
\]

By inverse inequality (2.1), we have
\[
\left| \int_{I_j} a(\omega, x, t)\eta(\omega, x, s)\xi_x(\omega, x, s) \, dx \right|
\]
\[ \leq C h \int_{I_j} |\eta(\omega, x, t)\xi_x(\omega, x, t)| \, dx \]
\[ \leq C h \|\eta(\cdot, t)\|_{I_j} \|\xi_x(\cdot, t)\|_{I_j} \leq C h \|\eta(\cdot, t)\|_{I_j}^{-1} \|\xi_x(\cdot, t)\|_{I_j} \]
\[ \leq C \|\eta(\cdot, t)\|_{I_j}^2 + C \|\xi(\cdot, t)\|_{I_j}^2. \]

Case 2 & Case 3: Note that
\[ a(\omega, x, t) = a(\omega, x_j, t) + a_x(\omega, \xi_2, t)(x - x_j). \]

It follows that
\[ \left| \int_{I_j} a(\omega, x, t)\eta(\omega, x, s)\xi_x(\omega, x, s) \, dx \right| \]
\[ \leq \left| a(\omega, x_j, t) \int_{I_j} \eta(\omega, x, s)\xi_x(\omega, x, s) \, dx \right| + C h \int_{I_j} \eta(\omega, x, t)\xi_x(\omega, x, t) \, dx \]
\[ \leq C h \|\eta(\cdot, t)\|_{I_j} \|\xi_x(\cdot, t)\|_{I_j} \leq C h \|\eta(\cdot, t)\|_{I_j}^{-1} \|\xi(\cdot, t)\|_{I_j} \]
\[ \leq C \|\eta(\cdot, t)\|_{I_j}^2 + C \|\xi(\cdot, t)\|_{I_j}^2. \]

Then we have from (H5) and (3.17),
\[ T_1(t) = -2 \mathbb{E} \left[ \int_0^t \sum_{j=1}^N H_j(a, \omega, s; \eta(\cdot, s), \xi(\cdot, s)) \, ds \right] \]
\[ \leq C \mathbb{E} \left[ \int_0^t \|\eta(\cdot, s)\|^2 \, ds \right] + C \mathbb{E} \left[ \int_0^t \|\xi(\cdot, s)\|^2 \, ds \right] \]
\[ \leq C \mathbb{E} \left[ \int_0^t \|u(\cdot, s)\|_{H^{k+1}}^2 \, ds \right] h^{2k+2} + C \mathbb{E} \left[ \int_0^t \|\xi(\cdot, s)\|^2 \, ds \right] \]
\[ \leq C h^{2k+2} + C \int_0^t \mathbb{E} \left[ \|\xi(\cdot, s)\|^2 \right] \, ds. \]

• \( T_2(t) \) term.

Since
\[ d_t(Pu)(\omega, x, t) = P(d_t u)(\omega, x, t) \]
\[ = -P((au)_x)(\omega, x, t) \, dt + P\left( g(\omega, \cdot, t, u(\omega, \cdot, t)) \right) (x) \, dW_t, \]
we have
\[ \int_{I_j} (dPu(\omega, x, t)) \cdot v(x) \, dx \]
\[ = \left( \int_{I_j} -P((au)_x)(\omega, x, t) \cdot v(x) \, dx \right) \, dt \]
From (3.11) and (3.19), we have

$$l \int l \rho \cdot v(x) dx = 0$$

Since

$$l \int l \rho \cdot v(x) dx$$

Using the scaled Legendre basis, similar to the arguments in Theorem 3.1, we have for

$$l \int l \rho \cdot v(x) dx$$

Then we have

$$l \int l \rho \cdot v(x) dx = 0$$

Since $$\xi(\omega, \cdot, t) \in V_h$$ for any $$(\omega, t) \in \Omega \times [0, T]$$, then $$\xi$$ should have the form

$$\xi(\omega, x, t) = \sum_{l=0}^{k} \xi^j_l(\omega, t) \varphi^j_l(x), \quad x \in I_j.$$  

Using the scaled Legendre basis, similar to the arguments in Theorem 3.1, we have for

$$l = 0, 1, \ldots, k,$$

$$d \xi^j_l(t) \cdot d \xi^j_l(t) = \left\{ A_{ij}^{l-1} \int I_j \left( P(g(\cdot, t, u(\cdot, t))) - g(\cdot, t, u_h(\cdot, t)) \right) (x) \varphi^j_l(x) dx \right\}^2 dt$$

$$\leq 2 \left( A_{ij}^{l-1} \right)^2 \int I_j \left| P(g(\cdot, t, u(\cdot, t))) - g(\cdot, t, u(\cdot, t)) \right|^2 (x) dx \int I_j \left| \varphi^j_l(x) \right|^2 dx dt$$

$$+ 2 \left( A_{ij}^{l-1} \right)^2 \int I_j \left| g(x, t, u(x, t)) - g(x, t, u_h(x, t)) \right|^2 dx \int I_j \left| \varphi^j_l(x) \right|^2 dx dt$$

$$\leq 2A_{ii}^{l-1} \int I_j \left| P(g(\cdot, t, u(\cdot, t))) - g(\cdot, t, u(\cdot, t)) \right|^2 (x) dx dt$$

$$+ 2L_g^2 A_{ii}^{l-1} \int I_j \left| u(x, t) - u_h(x, t) \right|^2 dx dt$$

$$\leq CA_{ii}^{l-1} \int I_j \left| P(g(\cdot, t, u(\cdot, t))) - g(\cdot, t, u(\cdot, t)) \right|^2 (x) dx dt$$

$$+ CA_{ii}^{l-1} \int I_j \left| \eta(x, t) \right|^2 dx dt + CA_{ii}^{l-1} \int I_j \left| \xi(x, t) \right|^2 dx dt.$$  

Then we have

$$l \int I_j d \xi(\omega, x, t) \cdot d \xi(\omega, x, t) dx = \int I_j \left( \sum_{m=0}^{k} \varphi^j_m(x) d \xi^j_m(t) \right)^2 dx = \sum_{l=0}^{k} A_{ii}^{l} d \xi^j_l(t) \cdot d \xi^j_l(t)$$

$$\leq C \int I_j \left| P(g(\cdot, t, u(\cdot, t))) - g(\cdot, t, u(\cdot, t)) \right|^2 (x) dx dt$$
+ C \int_{I_j} |\eta(x, t)|^2 \, dx \, dt + C \int_{I_j} |\xi(x, t)|^2 \, dx \, dt.

It follows that
\begin{align*}
\int_0^{2\pi} d\xi(x, t) \cdot d\xi(x, t) \\
\leq C \|P g(\cdot, t, u(\cdot, t)) - g(\cdot, t, u(\cdot, t))\|^2 dt + C \|\mathcal{P} u(\cdot, t) - u(\cdot, t)\|^2 dt + C \|\xi(\cdot, t)\|^2 dt \\
\leq C \left(\|g(\cdot, t, u(\cdot, t))\|^2_{H^{k+1}} + \|u(\cdot, t)\|^2_{H^{k+1}}\right) h^{2k+2} dt + C \|\xi(\cdot, t)\|^2 dt.
\end{align*}

Then by (H5) and (H6) we get
\[ T_2(t) = \mathbb{E} \left[ \int_0^{2\pi} \int_0^t d\xi(x, s) \cdot d\xi(x, s) \, dx \right] \leq Ch^{2k+2} + C \int_0^t \mathbb{E} \left[ \|\xi(\cdot, s)\|^2 \right] \, ds.
\]

- **T_3(t) term.**

By (3.10) and (3.18), we have
\[ d\eta(\omega, x, t) = \{ (au)_x(\omega, x, t) - \mathcal{P}( (au)_x)(\omega, x, t) \} dt \\
+ \{ \mathcal{P}(g(\cdot, t, u(\cdot, t)))(x) - g(\omega, x, t, u(\omega, x, t)) \} dW_t.
\]

Thus
\begin{align*}
\int_0^{2\pi} \int_0^t \xi(x, s)d\eta(x, s) \, dx \\
= \int_0^{2\pi} \int_0^t \left\{ (au)_x(x, s) - \mathcal{P}( (au)_x)(x, s) \right\} \xi(x, s) \, ds \, dx \\
+ \int_0^t \int_0^{2\pi} \{ \mathcal{P}(g(\cdot, t, u(\cdot, t)))(x) - g(x, s, u(x, s)) \} \xi(x, s) \, dx \, dW_s.
\end{align*}

According to (3.13) and \( u \in L^4(\Omega \times [0, 2\pi] \times [0, T]; \mathbb{R}) \), we know that the process
\[ \left\{ \int_0^t \int_0^{2\pi} \{ \mathcal{P}(g(\cdot, t, u(\cdot, t)))(x) - g(x, s, u(x, s)) \} \xi(x, s) \, dx \, dW_s, \quad 0 \leq t \leq T \right\}
\]
is a martingale. Then
\begin{align*}
T_3(t) &= 2\mathbb{E} \left[ \int_0^{2\pi} \int_0^t \xi(x, s)d\eta(x, s) \, dx \right] \\
= 2\mathbb{E} \left[ \int_0^{2\pi} \int_0^t \left\{ (au)_x(x, s) - \mathcal{P}( (au)_x)(x, s) \right\} \xi(x, s) \, ds \, dx \right] \\
\leq \mathbb{E} \left[ \int_0^t \int_0^{2\pi} \left| (au)_x(x, s) - \mathcal{P}( (au)_x)(x, s) \right|^2 \, dx \, ds \right] + \int_0^t \mathbb{E} \left[ \|\xi(\cdot, s)\|^2 \right] \, ds \\
\leq C\mathbb{E} \left[ \int_0^t \| (au)_x(\cdot, s) \|^2_{H^{k+1}} \, ds \right] h^{2k+2} + \int_0^t \mathbb{E} \left[ \|\xi(\cdot, s)\|^2 \right] \, ds.
\]
Since
\[ \|(au)_x(t, s)\|_{H^{k+1}} \leq C \|(au)(\cdot, s)\|_{H^{k+2}} \leq C \|u(\cdot, s)\|_{H^{k+2}} \]
and \( u(\cdot) \in L^2(\Omega \times [0, T]; H^{k+2}) \), we get
\[ T_3(t) \leq Ch^{2k+2} + C \int_0^t \mathbb{E} \left[ \|\xi(\cdot, s)\|^2 \right] \, ds. \]

- **\( T_4(t) \) term.**
  According to (3.14), we get
  \[ T_4(t) = 2 \mathbb{E} \left[ \int_0^t \sum_{j=1}^N H_j(a, \omega, s; \xi(\cdot, s), \xi(\cdot, s)) \right] \, ds \]
  \[ \leq \|a_x\|_{\infty} \int_0^t \mathbb{E} \left[ \|\xi(\cdot, s)\|^2 \right] \, ds. \]

- **\( T_5(t) \) term.**
  By virtue of (H5) and (3.13), we know that the process
  \[ \left\{ \int_0^t \int_0^{2\pi} \left( g(x, s, u(x, s)) - g(x, s, u_h(x, s)) \right) \xi(x, s) \, dx \, dW_s, \quad 0 \leq t \leq T \right\} \]
  is a martingale. Then
  \[ T_5(t) = 2 \mathbb{E} \left[ \int_0^t \int_0^{2\pi} \left( g(x, s, u(x, s)) - g(x, s, u_h(x, s)) \right) \xi(x, s) \, dx \, dW_s \right] = 0. \]

Concluding the above arguments, we have
\[ \mathbb{E} \left[ \|\xi(\cdot, t)\|^2 \right] \leq Ch^{2k+2} + C \int_0^t \mathbb{E} \left[ \|\xi(\cdot, s)\|^2 \right] \, ds. \]
Using Gronwall’s inequality, we have
\[ \left( \mathbb{E} \left[ \|\xi(\cdot, t)\|^2 \right] \right)^\frac{1}{2} \leq Ch^{k+1} e^{Ct}. \]
According to (3.17) and \( u \in L^\infty(0, T; L^2(\Omega; H^{k+1})) \), we have
\[ \left( \mathbb{E} \left[ \|\eta(\cdot, t)\|^2 \right] \right)^\frac{1}{2} \leq C \left( \mathbb{E} \left[ \|u(\cdot, t)\|_{H^{k+1}}^2 \right] \right)^\frac{1}{2} h^{k+1} \leq Ch^{k+1}. \]
It turns out that
\[ \left( \mathbb{E} \left[ \|u(\cdot, t) - u_h(\cdot, t)\|^2 \right] \right)^\frac{1}{2} \leq \left( \mathbb{E} \left[ \|\xi(\cdot, t)\|^2 \right] \right)^\frac{1}{2} + \left( \mathbb{E} \left[ \|\eta(\cdot, t)\|^2 \right] \right)^\frac{1}{2} \leq C e^{Ct} h^{k+1}. \]

**Remark 3.1.** The solution of stochastic conservation law usually does not have a uniform bound with respect to the variable \( \omega \in \Omega \). Thus we could not generalize the method in Zhang and Shu [26] to get the error estimates for fully nonlinear stochastic conservation law, in which they made use of the uniform boundedness of the approximate solutions. But interestingly, numerical examples in Section 5.3 verify the optimal order \( \mathcal{O}(h^{k+1}) \) for nonlinear stochastic equations.
4 Time discretization

The DG method only involves the spatial discretization, and transfers the primal stochastic partial differential equation into a SDE. Thus we need to derive an implementable high-order time discretization. For notational simplicity, we shall mainly state the schemes for the autonomous case. Consider the following matrix-valued SDE:

\[
\begin{cases}
  dX_{t}^{i,j} = a^{i,j}(X_{t}) \, dt + b^{i,j}(X_{t}) \, dW_{t}, & t > 0 \\
  X_{0}^{i,j} = x_{0}^{i,j},
\end{cases}
\]

where \( i = 0, 1, ..., k \) and \( j = 0, 1, ..., N + 1 \). We aim to use \( Y_{n}^{i,j} \) to approximate \( X_{t_{n}}^{i,j} \). Define \( Y_{0}^{i,j} := x_{0}^{i,j} \). Suppose we already have \( \{ Y_{n}^{i,j} : i = 0, 1, ..., k \text{ and } j = 0, 1, ..., N + 1 \} \).

4.1 Taylor order 2.0 strong scheme

Define the following operators

\[
\mathcal{L}^{0} f := \sum_{j=0}^{N+1} \sum_{i=0}^{k} a^{i,j} \frac{\partial f}{\partial x_{ij}} + \frac{1}{2} \sum_{l,j=0}^{N+1} \sum_{m,i=0}^{k} b^{i,j} b^{m,l} \frac{\partial^{2} f}{\partial x_{ij} \partial x_{ml}},
\]

and

\[
\mathcal{L}^{1} f := \sum_{j=0}^{N+1} \sum_{i=0}^{k} b^{i,j} \frac{\partial f}{\partial x_{ij}},
\]

where \( f : \mathbb{R}^{(k+1) \times (N+2)} \rightarrow \mathbb{R} \) is twice differentiable.

According to [18, Theorem 11.5.1, page 391], the order 2.0 strong Taylor scheme is

\[
Y_{n+1}^{i,j} = Y_{n}^{i,j} + a^{i,j}(Y_{n})(t_{n+1} - t_{n}) + b^{i,j}(Y_{n})(W_{t_{n+1}} - W_{t_{n}})
\]

(order 0.5)

\[
+ \mathcal{L}^{1} b^{i,j}(Y_{n}) \int_{t_{n}}^{t_{n+1}} \int_{t_{n}}^{s} dW_{r} dW_{s}
\]

(order 1.0)

\[
+ \frac{1}{2} \mathcal{L}^{0} a^{i,j}(Y_{n}) (t_{n+1} - t_{n})^{2} + \mathcal{L}^{0} b^{i,j}(Y_{n}) \int_{t_{n}}^{t_{n+1}} \int_{t_{n}}^{s} dr dW_{s}
\]

\[
+ \mathcal{L}^{1} a^{i,j}(Y_{n}) \int_{t_{n}}^{t_{n+1}} \int_{t_{n}}^{s} dW_{r} ds + \mathcal{L}^{1} \mathcal{L}^{1} b^{i,j}(Y_{n}) \int_{t_{n}}^{t_{n+1}} \int_{t_{n}}^{s} \int_{t_{n}}^{r} dW_{u} dW_{r} dW_{s}
\]

(order 1.5)

\[
+ \mathcal{L}^{1} \mathcal{L}^{0} b^{i,j}(Y_{n}) \int_{t_{n}}^{t_{n+1}} \int_{t_{n}}^{s} \int_{t_{n}}^{r} dW_{u} dr dW_{s}
\]

\[
+ \mathcal{L}^{1} \mathcal{L}^{1} a^{i,j}(Y_{n}) \int_{t_{n}}^{t_{n+1}} \int_{t_{n}}^{s} \int_{t_{n}}^{r} dW_{u} dW_{r} ds
\]
\[ + \mathcal{L}^0 \mathcal{L}^1 b^{i,j}(Y_n) \int_{t_n}^{t_{n+1}} \int_{t_n}^s \int_{t_n}^r dudW_r dW_s \]
\[ + \mathcal{L}^1 \mathcal{L}^0 b^{i,j}(Y_n) \int_{t_n}^{t_{n+1}} \int_{t_n}^s \int_{t_n}^r \int_{t_n}^u dW_r dW_u dW_r dW_u. \]
(order 2.0) \hspace{1cm} (4.1)

Define
\[ \Delta_n = t_{n+1} - t_n, \quad \Delta W_n = W_{t_{n+1}} - W_{t_n}, \]
and
\[ \Delta Z_n = \int_{t_n}^{t_{n+1}} (W_s - W_{t_n}) \, ds, \quad \Delta U_n = \int_{t_n}^{t_{n+1}} (W_s - W_{t_n})^2 \, ds. \]

By Itô formula we have
\[ \int_{t_n}^{t_{n+1}} \int_{t_n}^s dr dW_s = \Delta W_n \Delta_n - \Delta Z_n, \]
\[ \int_{t_n}^{t_{n+1}} \int_{t_n}^s \int_{t_n}^r dW_r dW_s = \frac{1}{6} (\Delta W_n^2 - 3 \Delta_n) \Delta_n, \]
\[ \int_{t_n}^{t_{n+1}} \int_{t_n}^s \int_{t_n}^r dW_r dr dW_s = -\Delta U_n + \Delta W_n \Delta Z_n, \]
\[ \int_{t_n}^{t_{n+1}} \int_{t_n}^s \int_{t_n}^r dW_u dr dW_s = \frac{1}{2} \Delta U_n - \frac{1}{4} \Delta_n^2, \]
\[ \int_{t_n}^{t_{n+1}} \int_{t_n}^s \int_{t_n}^r dudW_r dW_s = \frac{1}{2} \Delta U_n - \Delta W_n \Delta Z_n + \frac{1}{2} (\Delta W_n^2) \Delta_n - \frac{1}{4} \Delta_n^2, \]
\[ \int_{t_n}^{t_{n+1}} \int_{t_n}^s \int_{t_n}^r \int_{t_n}^u dW_u dW_u dW_r dW_s = \frac{1}{24} (\Delta W_n^4 - 6 (\Delta W_n^2) \Delta_n + 3 \Delta_n^2). \]

Thus we could rewrite the Taylor scheme (4.1) as follows,
\[ Y_{n+1}^{i,j} = Y_{n}^{i,j} + a^{i,j}(Y_n) \Delta_n + b^{i,j}(Y_n) \Delta W_n \]
(order 0.5)
\[ + \frac{1}{2} \mathcal{L}^1 b^{i,j}(Y_n) \{ (\Delta W_n)^2 - \Delta_n \} \]
(order 1.0)
\[ + \frac{1}{2} \mathcal{L}^0 a^{i,j}(Y_n) \Delta_n^2 + \mathcal{L}^0 b^{i,j}(Y_n) \{ \Delta W_n \Delta_n - \Delta Z_n \} \]
\[ + \mathcal{L}^1 a^{i,j}(Y_n) \Delta Z_n + \frac{1}{6} \mathcal{L}^1 b^{i,j}(Y_n) \{ (\Delta W_n)^2 - 3 \Delta_n \} \Delta W_n \]
(order 1.5)

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\[ + \mathcal{L}^1 \mathcal{L}^0 a^{i,j} (Y_n) \{ a^{i,j} (Y_n) - 2a^{i,j} (Y_n) + a^{i,j} (Y_n) \} + \mathcal{O}(\Delta_n), \]

where the method of modeling the stochastic variables \( \Delta W_n, \Delta Z_n \) and \( \Delta U_n \) will be specified later.

### 4.2 Explicit order 2.0 strong scheme

A disadvantage of the strong Taylor approximations is that the derivatives of various orders of the drift and diffusion coefficients must be evaluated at each step, in addition to the coefficients themselves. This can make implementation of such schemes a complicated undertaking. In this subsection we will propose a strong scheme which avoids the usage of derivatives in much the same way that Runge-Kutta schemes do in the deterministic setting.

#### 4.2.1 Derivative-free scheme

Following the idea of [18], we could derive a derivative-free scheme of order 2.0 by replacing the derivatives in the order 2.0 strong Taylor scheme (4.2) by corresponding finite differences.

We set

\[ \begin{align*}
\gamma_{m,l}^{m,l} &= Y_{n}^{m,l} + a^{m,l}(Y_n) \Delta_n \pm b^{m,l}(Y_n) \sqrt{\Delta_n}, \\
\eta_{m,l}^{m,l} &= Y_{n}^{m,l} \pm b^{m,l}(Y_n) \Delta_n; \\
\phi_{+,+}^{m,l} &= \gamma_{+}^{m,l} + a^{m,l}(\gamma_{+}) \Delta_n \pm b^{m,l}(\gamma_{+}) \sqrt{\Delta_n}, \\
\phi_{+,-}^{m,l} &= \gamma_{-}^{m,l} + a^{m,l}(\gamma_{-}) \Delta_n \pm b^{m,l}(\gamma_{-}) \sqrt{\Delta_n}; \\
\beta_{+,+}^{m,l} &= \phi_{+,+}^{m,l} \pm b^{m,l}(\phi_{+,+}) \sqrt{\Delta_n}, \\
\beta_{+,-}^{m,l} &= \phi_{+,-}^{m,l} \pm b^{m,l}(\phi_{+,-}) \sqrt{\Delta_n}. \\
\end{align*} \]

One could easily verify that

\[ \mathcal{L}^1 b^{i,j} (Y_n) = \frac{1}{2\Delta_n} \{ b^{i,j} (\eta_{+}) \pm b^{i,j} (\eta_{-}) \} + \mathcal{O}(\Delta^2_n), \]

\[ \mathcal{L}^0 a^{i,j} (Y_n) = \frac{1}{2\Delta_n} \{ a^{i,j} (\gamma_{+}) \pm 2a^{i,j} (Y_n) + a^{i,j} (\gamma_{-}) \} + \mathcal{O}(\Delta_n), \]
\[
\mathcal{L}^0 b^{i,j}(Y_n) = \frac{1}{2\Delta_n} \left\{ b^{i,j}(\gamma_+ - 2b^{i,j}(Y_n) + b^{i,j}(\gamma_-) \right\} + \mathcal{O}(\Delta_n),
\]

\[
\mathcal{L}^1 a^{i,j}(Y_n) = \frac{1}{2\sqrt{\Delta_n}} \left\{ a^{i,j}(\gamma_+) - a^{i,j}(\gamma_-) \right\} + \mathcal{O}(\Delta_n),
\]

\[
\mathcal{L}^1 \mathcal{L}^0 b^{i,j}(Y_n) = \frac{1}{4\Delta_n} \left\{ b^{i,j}(\phi_{+,+}) - b^{i,j}(\phi_{++,}) - b^{i,j}(\phi_{-,-}) + b^{i,j}(\phi_{-,+}) \right\} + \mathcal{O}(\Delta_n),
\]

\[
\mathcal{L}^1 \mathcal{L}^0 b^{i,j}(Y_n) = \frac{1}{2\Delta_n} \left\{ b^{i,j}(\phi_{+,+}) + b^{i,j}(\phi_{-,-}) - 3b^{i,j}(\gamma_+) + b^{i,j}(\gamma_-) + 2b^{i,j}(Y_n) \right\} + \mathcal{O}(\sqrt{\Delta_n}),
\]

\[
\mathcal{L}^1 \mathcal{L}^1 a^{i,j}(Y_n) = \frac{1}{2\Delta_n} \left\{ a^{i,j}(\phi_{+,+}) - a^{i,j}(\phi_{-,-}) - a^{i,j}(\gamma_+) + a^{i,j}(\gamma_-) \right\} + \mathcal{O}(\sqrt{\Delta_n}),
\]

\[
\mathcal{L}^0 \mathcal{L}^1 b^{i,j}(Y_n) = \frac{1}{4\Delta_n^2} \left\{ b^{i,j}(\phi_{+,+}) - b^{i,j}(\phi_{-,-}) + b^{i,j}(\beta_{+,+}) - b^{i,j}(\beta_{-,+}) - 2b^{i,j}(\gamma_+) + 2b^{i,j}(\gamma_-) \right\} + \mathcal{O}(\sqrt{\Delta_n}),
\]

\[
\mathcal{L}^1 \mathcal{L}^1 b^{i,j}(Y_n) = \frac{1}{4\Delta_n^2} \left\{ b^{i,j}(\beta_{+,+}) - b^{i,j}(\beta_{-,-}) - b^{i,j}(\beta_{+,+}) + b^{i,j}(\beta_{-,+}) - b^{i,j}(\phi_{+,+}) + b^{i,j}(\phi_{-,-}) - b^{i,j}(\phi_{-,+}) \right\} + \mathcal{O}(\sqrt{\Delta_n}).
\]

Then we could rewrite scheme (4.2) as the following scheme

\[
Y_{n+1}^{i,j} = Y_n^{i,j} + a^{i,j}(Y_n)\Delta_n + b^{i,j}(Y_n)\Delta W_n
\]

(order 0.5)

\[
+ \frac{1}{4\Delta_n} \left\{ b^{i,j}(\eta_+) - b^{i,j}(\eta_-) \right\} \left( (\Delta W_n)^2 - \Delta_n \right)
\]

(order 1.0)

\[
+ \frac{1}{4} \left\{ a^{i,j}(\gamma_+) - 2a^{i,j}(Y_n) + a^{i,j}(\gamma_-) \right\} \Delta_n
\]

\[
+ \frac{1}{2\sqrt{\Delta_n}} \left\{ a^{i,j}(\gamma_+) - a^{i,j}(\gamma_-) \right\} \Delta Z_n
\]

\[
+ \frac{1}{2\Delta_n} \left\{ b^{i,j}(\gamma_+) - 2b^{i,j}(Y_n) + b^{i,j}(\gamma_-) \right\} \left\{ \Delta W_n \Delta_n - \Delta Z_n \right\}
\]

25
random variables ∆

We have proposed a derivative-free scheme (4.4). Now it remains to model at each step three random variables approximately. The method of modeling can be found in [23]. For convenience of the reader, we give a full detailed description of the modeling algorithm here.

4.2.2 Modeling of Itô integrals

It is obvious that \( v(s) = \frac{W_{t_n+\Delta_n s} - W_{t_n}}{\sqrt{\Delta_n}} \), \( 0 \leq s \leq 1 \). It is clear that \{v(s), 0 \leq s \leq 1\} is a standard Wiener process. We have

\[
\Delta W_n = \Delta \frac{1}{3} v(1), \quad \Delta Z_n = \Delta \frac{3}{2} \int_0^1 v(s) \, ds, \quad \Delta U_n = \Delta \frac{2}{3} \int_0^1 v^2(s) \, ds.
\]

Then the problem of modeling the random variables \( \Delta W_n, \Delta Z_n \) and \( \Delta U_n \) could be reduced to that of modeling the variables \( v(1), \int_0^1 v(s) \, ds \) and \( \int_0^1 v^2(s) \, ds \). These variables are the solution of the system of equations

\[
\begin{align*}
dx &= dv(s), & x(0) &= 0, \\
dy &= x \, ds, & y(0) &= 0, \\
dz &= x^2 \, ds, & z(0) &= 0,
\end{align*}
\]

at the moment \( s = 1 \).

(4.5)
Let \( x_k = \bar{x}(s_k), y_k = \bar{y}(s_k), z_k = \bar{z}(s_k), 0 = s_0 < s_1 < \cdots < s_{N_n} = 1, s_{k+1} - s_k = \delta = \frac{1}{N_n} \), be an approximate solution of (4.5), where \( N_n \) is to be determined. We will now use a method of order 1.5 to integrate (4.5).

\[
\begin{align*}
x_{k+1} &= x_k + (v(s_{k+1}) - v(s_k)), \\
y_{k+1} &= y_k + x_k\delta + \int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k)) \, d\theta, \\
z_{k+1} &= z_k + x_k^2\delta + 2x_k \int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k)) \, d\theta + \frac{\delta^2}{2}.
\end{align*}
\]

(4.6)

Here the additional random variable \( \int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k)) \, d\theta \) is normally distributed with mean, variance and correlation

\[
\begin{align*}
\mathbb{E} \left[ \int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k)) \, d\theta \right] &= 0, \\
\mathbb{E} \left[ \left( \int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k)) \, d\theta \right)^2 \right] &= \frac{1}{3} \delta^3, \\
\mathbb{E} \left[ \left( v(s_{k+1}) - v(s_k) \right) \cdot \left( \int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k)) \, d\theta \right) \right] &= \frac{1}{2} \delta^2,
\end{align*}
\]

respectively. We note that there is no difficulty in generating the pair of correlated normally distributed random variables \( v(s_{k+1}) - v(s_k) \) and \( \int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k)) \, d\theta \) using the transformation

\[
v(s_{k+1}) - v(s_k) = \zeta_{k,1}\delta^\frac{1}{2}, \quad \int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k)) \, d\theta = \frac{1}{2} \left( \zeta_{k,1} + \frac{1}{\sqrt{3}} \zeta_{k,2} \right)\delta^\frac{3}{2},
\]

(4.7)

where \( \zeta_{k,1} \) and \( \zeta_{k,2} \) are independent normally \( N(0; 1) \) distributed random variables.

The method (4.6) has the following properties. Firstly \( x_k \) and \( y_k \) are equal to \( v(s_k) \) and \( \int_0^{s_k} v(\theta) \, d\theta \) exactly. Secondly we have

\[
\left( \mathbb{E} \left[ z_{N_n} - \int_0^1 v^2(s) \, ds \right] ^2 \right) ^\frac{1}{2} = O(\delta^\frac{3}{2}).
\]

We choose \( \delta \) such that \( \delta = O(\Delta_n^{\frac{1}{2}}) \) i.e.

\[
N_n = \left\lceil \Delta_n^{-\frac{3}{4}} \right\rceil,
\]

(4.8)

with \( \lceil \cdot \rceil \) standing for the ceiling function.

Then we have \( \Delta_n^{\frac{1}{2}} x_{N_n} = \Delta W_n, \Delta_n^{\frac{3}{2}} y_{N_n} = \Delta Z_n \) and

\[
\left( \mathbb{E} \left[ |\Delta_n^{\frac{1}{2}} z_{N_n} - \Delta U_n| ^2 \right] \right) ^\frac{1}{2} = O(\Delta_n^{\frac{3}{2}}).
\]

Thus according to [23, Theorem 4.2, page 50], in a method of second order of accuracy with time step \( \Delta_n \) such as scheme (4.4), we could replace \( \Delta W_n, \Delta Z_n \) and \( \Delta U_n \) by \( \Delta_n^{\frac{1}{2}} x_{N_n}, \Delta_n^{3/2} y_{N_n} \), and \( \Delta_n^{3/2} z_{N_n} \) respectively.
we set $\Delta t$ a positive real number such that the scheme in time is effectively third-order. In all experiments of DG scheme with $k = 2$, we have adjusted the time step to $\Delta t \approx (\Delta x)^{\frac{3}{2}}$ so that the scheme in time is effectively third-order.

5 Numerical experiments

In this section we consider the application of the numerical methods, which we defined in section 3 and section 4, on some model problems. Here, $M$ is the number of realizations of the stochastic approximate solutions. We use the average of $M$ realizations to approximate the mathematical expectation. The degree of the piecewise-polynomial space $V_h$ is $k$. The positive real number $T$ is the terminal time. In all experiments of DG scheme with $k = 1$, we set $\Delta t = \frac{\Delta x}{2k+1}$ so that scheme (4.9) is efficiently second-order and the CFL condition is satisfied. In all experiments of DG scheme with $k = 2$, we have adjusted the time step to $\Delta t \sim (\Delta x)^{\frac{3}{2}}$ so that the scheme in time is effectively third-order.
5.1 Constant-Coefficient linear stochastic equation

We consider the following linear equation

\[
\begin{align*}
\begin{cases}
du + u_x \, dt &= bu \, dW_t & \text{in } \Omega \times [0, 2\pi] \times (0, T), \\
u(\omega, x, 0) &= u_0(x), & \omega \in \Omega, \ x \in [0, 2\pi].
\end{cases}
\end{align*}
\tag{5.1}
\]

The exact solution of (5.1) is

\[u(\omega, x, t) = u_0(x-t) e^{bW_t(\omega) - \frac{1}{2}b^2t}.
\]

The numerical flux is taken as the simple upwind flux \( \hat{f}(u^-, u^+) = u^- \). In Table 1 and Table 2, we show the errors of DG scheme (3.1) with 10000 realizations and \( u_0(x) = \sin(x) \).

<table>
<thead>
<tr>
<th>N</th>
<th>( b = 0.1 )</th>
<th>( b = 0.5 )</th>
<th>( b = 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( L^2 ) Error</td>
<td>order</td>
<td>( L^2 ) Error</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>10</td>
<td>4.12E-02</td>
<td>-</td>
<td>4.38E-02</td>
</tr>
<tr>
<td>20</td>
<td>1.06E-02</td>
<td>1.96</td>
<td>1.12E-02</td>
</tr>
<tr>
<td>40</td>
<td>2.66E-03</td>
<td>1.99</td>
<td>2.82E-03</td>
</tr>
<tr>
<td>80</td>
<td>6.67E-04</td>
<td>2.00</td>
<td>7.08E-04</td>
</tr>
<tr>
<td>160</td>
<td>1.67E-04</td>
<td>2.00</td>
<td>1.78E-04</td>
</tr>
<tr>
<td>320</td>
<td>4.17E-05</td>
<td>2.00</td>
<td>4.42E-05</td>
</tr>
</tbody>
</table>

From Table 1 and Table 2, we see that the order of accuracy of the DG scheme (3.1) is \( k + 1 \), which is consistent with the result in Theorem 3.3. We could also observe that the DG scheme with \( k = 2 \) is more efficient than the one with \( k = 1 \), and the error increases when \( T \) and \( b \) become larger.

In practice, we may need to simulate the solution with one certain stochastic path, i.e., \( M = 1 \). We show the results with \( M = 1 \) in Table 3 and Table 4 at \( T = 2\pi \).
Table 2: Accuracy on (5.1) with $k = 2$, $M = 10000$, $u_0(x) = \sin(x)$

<table>
<thead>
<tr>
<th>$T$</th>
<th>$b = 0.1$</th>
<th>$b = 0.5$</th>
<th>$b = 1.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N$</td>
<td>$L^2$ Error</td>
<td>order</td>
</tr>
<tr>
<td>0.5</td>
<td>10</td>
<td>2.12E-03</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.70E-04</td>
<td>2.97</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>3.37E-05</td>
<td>3.00</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>4.22E-06</td>
<td>3.00</td>
</tr>
<tr>
<td></td>
<td>160</td>
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<td>3.00</td>
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<td></td>
<td>320</td>
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<td>3.00</td>
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<tr>
<td>1.0</td>
<td>10</td>
<td>2.58E-03</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>3.17E-04</td>
<td>3.03</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>3.93E-05</td>
<td>3.01</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>4.92E-06</td>
<td>3.00</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>6.14E-07</td>
<td>3.00</td>
</tr>
<tr>
<td></td>
<td>320</td>
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</tr>
<tr>
<td>2.0</td>
<td>10</td>
<td>1.15E-02</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.32E-03</td>
<td>3.12</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>1.61E-04</td>
<td>3.04</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>2.00E-05</td>
<td>3.01</td>
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<td>160</td>
<td>2.50E-06</td>
<td>3.00</td>
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<tr>
<td></td>
<td>320</td>
<td>3.12E-07</td>
<td>3.01</td>
</tr>
</tbody>
</table>

According to Table 3 and Table 4, we observe that if the stochastic noise of the concerning problem is small enough, such as the case $b = 0.001$, the result with single path is as good as the classical deterministic case.

We also consider the case that initial condition is discontinuous

$$u_0(x) = \begin{cases} 
1, & \text{if } \frac{\pi}{2} \leq x \leq \frac{3\pi}{2}, \\
0, & \text{if } 0 \leq x < \frac{\pi}{2} \text{ or } \frac{3\pi}{2} < x \leq 2\pi. 
\end{cases}$$

We plot the approximate solution and the true solution at $T = 2\pi$ with only one realization $M = 1$ to get figure 1.

In view of figure 1, we could see that the DG scheme works well for the discontinuous case and the numerical solution approximates the true solution more accurately when $k$ and $N$ become larger. Similar to the deterministic cases, there are oscillations arising near discontinuities of the solution.

**Remark 5.1.** For the discontinuous cases, the $L^2$-stability, although helpful, is not enough to control spurious numerical oscillations near discontinuities. In practice, especially for problems containing strong discontinuities, it is worth trying to apply nonlinear limiters to control these oscillations, which is an interesting future work to accomplish.
Table 3: Accuracy on (5.1) with \(k = 1, M = 1, T = 2\pi, u_0(x) = \sin(x)\)

<table>
<thead>
<tr>
<th>(N)</th>
<th>(L^2) Error order</th>
<th>(L^2) Error order</th>
<th>(L^2) Error order</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>9.76E-02 -</td>
<td>9.69E-02 -</td>
<td>9.44E-02 -</td>
</tr>
<tr>
<td>20</td>
<td>2.31E-02 2.08</td>
<td>2.34E-02 2.05</td>
<td>2.50E-02 1.91</td>
</tr>
<tr>
<td>40</td>
<td>5.56E-02 2.05</td>
<td>5.80E-02 2.02</td>
<td>6.95E-02 1.85</td>
</tr>
<tr>
<td>80</td>
<td>1.35E-02 2.04</td>
<td>1.32E-02 2.13</td>
<td>1.23E-02 2.50</td>
</tr>
<tr>
<td>160</td>
<td>3.35E-04 2.01</td>
<td>3.27E-04 2.02</td>
<td>2.95E-04 2.05</td>
</tr>
<tr>
<td>320</td>
<td>8.38E-05 2.00</td>
<td>8.60E-05 1.93</td>
<td>9.64E-05 1.61</td>
</tr>
<tr>
<td>640</td>
<td>2.09E-05 2.00</td>
<td>2.12E-05 2.02</td>
<td>2.23E-05 2.11</td>
</tr>
</tbody>
</table>

Table 4: Accuracy on (5.1) with \(k = 2, M = 1, T = 2\pi, u_0(x) = \sin(x)\)

<table>
<thead>
<tr>
<th>(N)</th>
<th>(L^2) Error order</th>
<th>(L^2) Error order</th>
<th>(L^2) Error order</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>8.59E-03 -</td>
<td>8.72E-03 -</td>
<td>9.31E-03 -</td>
</tr>
<tr>
<td>20</td>
<td>1.05E-03 3.03</td>
<td>1.05E-03 3.06</td>
<td>1.00E-03 3.22</td>
</tr>
<tr>
<td>40</td>
<td>1.32E-04 3.00</td>
<td>1.31E-04 3.00</td>
<td>1.25E-04 3.00</td>
</tr>
<tr>
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<td>1.65E-05 3.00</td>
<td>1.68E-05 2.96</td>
<td>1.82E-05 2.78</td>
</tr>
<tr>
<td>160</td>
<td>2.06E-06 3.00</td>
<td>2.06E-06 3.03</td>
<td>2.05E-06 3.15</td>
</tr>
<tr>
<td>320</td>
<td>2.56E-07 3.00</td>
<td>2.52E-07 3.03</td>
<td>2.33E-07 3.14</td>
</tr>
<tr>
<td>640</td>
<td>3.22E-08 2.99</td>
<td>3.28E-08 2.94</td>
<td>3.55E-08 2.72</td>
</tr>
</tbody>
</table>

5.2 Linear stochastic equation with a variable coefficient

In the following we test the accuracy of the DG scheme (3.11) for the linear equation with a variable coefficient

\[
\begin{cases}
    \frac{du}{dt} + \frac{\partial}{\partial x} (\sin(x) \cdot u) dt = bu dW_t & \text{in } \Omega \times [0, 2\pi] \times (0, T), \\
    u(\omega, x, 0) = \sin(x), & \omega \in \Omega, x \in [0, 2\pi].
\end{cases}
\]

The exact solution of (5.2) is

\[u(\omega, x, t) = v(x, t)e^{bW_t(\omega) - \frac{1}{2} b^2 t},\]

where \(v\) is the unique solution of the following deterministic equation

\[
\begin{cases}
    \frac{dv}{dt} + \frac{\partial}{\partial x} (\sin(x) \cdot v) = 0 & \text{in } [0, 2\pi] \times (0, T), \\
    v(x, 0) = \sin(x), & x \in [0, 2\pi].
\end{cases}
\]

In Table 5 and Table 6, we show the error of the DG scheme (3.11) with 10000 realizations for \(k = 1\) and 1000 realizations for \(k = 2\), respectively.
By Table 5 and Table 6 we observe that the order of accuracy converges to $k + 1$ when $N$ increases. Similar to the constant-coefficient linear case, the error increases when $T$ and $b$ increase. The scheme with $k = 2$ is more efficient than the one with $k = 1$. All of the results are consistent with Theorem 3.3.

### 5.3 Stochastic Burgers equation

Although we cannot give the error estimates for the fully nonlinear problems with locally Lipschitz-continuous physical flux, it is worth trying to apply the DG scheme (3.1) to some nonlinear equation. So the next example is stochastic Burgers equation

\[
\begin{aligned}
&du + \frac{\partial}{\partial x}\left( \frac{1}{2} u^2 \right) \, dt = b \, dW_t, \quad \text{in } \Omega \times [0, 2\pi] \times (0, T), \\
&u(\omega, x, 0) = \sin(x), \quad \omega \in \Omega, \ x \in [0, 2\pi].
\end{aligned}
\] (5.4)
The exact solution of (5.4) is

\[ u(\omega, x, t) = v \left( x - b \int_0^t W_s ds, t \right) + bW_t(\omega), \]

where \( v \) is the solution of the following deterministic equation

\[
\begin{aligned}
\frac{v_t}{\pi} + \frac{\partial}{\partial x} \left( \frac{1}{2} v^2 \right) &= 0 \quad \text{in } [0, 2\pi] \times (0, T), \\
v(x, 0) &= \sin(x), \quad x \in [0, 2\pi],
\end{aligned}
\]

(5.5)

and the random variable \( \int_0^t W_s ds \) could be computed exactly by (4.7).

We use the simple Lax-Friedrichs flux

\[
\tilde{f}(u^-, u^+) = \frac{1}{4} \left\{ (u^-)^2 + (u^+)^2 \right\} - \frac{1}{2} \alpha (u^+ - u^-),
\]

where

\[
\alpha = \max_j \left\{ \left| u^-_{j+\frac{1}{2}} \right|, \left| u^+_{j+\frac{1}{2}} \right| \right\}.
\]

In Table 7 and Table 8, we show the errors of the DG scheme (3.1) with 100 realizations.
Notice that the solution of (5.5) has an infinite slope - the wave "breaks" and a shock forms at
\[ T_b = \frac{-1}{\min v_0'(x)} = 1, \]
see [21].

From Table 7 and Table 8, we observe that the order of accuracy converges to \( k + 1 \) when \( N \) increases for the case that \( T < T_b \). The scheme with \( k = 2 \) is more efficient than the one with \( k = 1 \).

Unlike the diffusion effect of the stochastic terms on the solutions of (5.1) and (5.2), here the stochastic term only has the drift effect on the solution of (5.4) since the stochastic perturbation in (5.4) is additive. Thus the value of \( b \) has little influence on the error and \( M = 100 \) is good enough to approximate the mathematical expectation.

When \( T \) increases, the scheme converges slowly and becomes more inefficient. When \( T > T_b \), the DG scheme lose its order of accuracy. To see the behaviour of the approximate solution with \( T > T_b \), we plot the approximate solution and the true solution at \( T = 1.5 \) with only one realization \( M = 1 \) to get figure 2.

From figure 2, we see that the DG schemes works well and the numerical solution approximates the true solution more accurately when \( k \) and \( N \) increase.

---

### Table 6: Accuracy on (5.2) with \( k = 2, M = 1000 \)

<table>
<thead>
<tr>
<th></th>
<th>( b = 0.1 )</th>
<th>( b = 0.5 )</th>
<th>( b = 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( N )</td>
<td>( L^2 ) Error</td>
<td>order</td>
</tr>
<tr>
<td>( T = 0.3 )</td>
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<td>-</td>
</tr>
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<td>1.54E-06</td>
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</tr>
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<td>2.02E-07</td>
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</tr>
<tr>
<td>( T = 0.6 )</td>
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<tr>
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### Table 7: Accuracy on (5.4) with $k = 1$, $M = 100$

<table>
<thead>
<tr>
<th>$T$</th>
<th>$b = 0.1$</th>
<th>$b = 1.0$</th>
<th>$b = 2.0$</th>
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### 6 Concluding remarks

In this article, we present semi-discrete DG schemes for fully nonlinear stochastic equations and semilinear variable-coefficient stochastic equations. We obtain the $L^2$ stability results of the schemes. We prove the optimal error estimates of order $O(h^{k+1})$ for semilinear stochastic conservation laws with variable coefficients. We also establish an explicit derivative-free second order time discretization scheme and perform several numerical experiments on some model problems to confirm the analytical results. It is more challenging to investigate error estimates for fully nonlinear stochastic equations and apply the DG type schemes to SPDEs with high-order spatial derivatives, which will be carried out in the future.
Table 8: Accuracy on (5.4) with $k = 2$, $M = 100$

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References


Figure 2: Figures on (5.4) with $M = 1, b = 2.0$.


