A Semi-Lagrangian High-Order Method for Navier-Stokes Equations

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Abstract

We present a semi-Lagrangian method for advection-diffusion and incompressible Navier-Stokes equations. The focus is on constructing stable schemes of second-order temporal accuracy, as this is a crucial element for the successful application of semi-Lagrangian methods to turbulence simulations. We implement the method in the context of unstructured spectral/hp element discretizations, which allows for efficient search-interpolation procedures as well as for illumination of the non-monotonic behavior of the temporal (advection) error of the form: $O(\Delta t^k + \frac{\Delta x^{p+1}}{\Delta t})$. We present numerical results that validate this error estimate for the advection-diffusion equation, and we document that such estimate is also valid for the Navier-Stokes equations at high Reynolds number. Two- and three-dimensional flow simulations suggest that semi-Lagrangian schemes are at least one order of magnitude more efficient than their Eulerian counterparts.

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1 Introduction

The case against direct numerical simulation of turbulent flows (DNS) at high Reynolds number (Re) is often made based on the enormous amount of spatial scales that need to be resolved. Indeed, simple estimates based on the Kolmogorov dissipative length scale suggest that the required number of degrees of freedom scale as Re$^{9/4}$ in three-dimensions [10]. What is not factored, however, in such estimate is the computational cost associated with the time-integration of the Navier-Stokes equations, which in practice, may be the prohibitive factor. After all, in a parallel computation the spatial resolution requirements can be alleviated by domain decomposition whereas the time-stepping cost cannot be avoided.

To illustrate the current inefficiency of time-discretization, let us consider the often-used semi-implicit Eulerian scheme, where advection is treated explicitly. The maximum allowable time step is dictated by the CFL number, which is typically of order one, and thus

$$\Delta t_{CFL} \propto \frac{\mathcal{L}}{u N^\alpha}.$$ 

Here $\mathcal{L}$ is an integral length scale, $u$ is a characteristic velocity scale, e.g., the $rms$ value, and $N^\alpha$ represents the scaling of the maximum eigenvalue associated with the spectral discretization, with $N$ the number of nodes in one dimension. For example, for a Fourier discretization $\alpha = 1$; for Chebyshev discretization used often in DNS $\alpha = 2$; and for spectral/$hp$ element methods $\alpha \approx 3/2$ (see [11], ch. 6). We want to compare this time step to the Kolmogorov time scale

$$\tau = \sqrt{(\nu \mathcal{L}/u^2)}.$$ 

To this end, we recall that according to Kolmogorov theory, valid at high Reynolds numbers $Re$, to resolve the Kolmogorov spatial length scale we need to employ approximately

$$N \approx Re^{3/4}$$

nodes per each spatial direction. Using these estimates, we can obtain the ratio of the maximum allowable time step to Kolmogorov’s time scale, i.e.

$$\frac{\Delta t_{CFL}}{\tau} \propto Re^{1/2} \frac{1}{N^\alpha} \propto Re^{1/2-3\alpha/4}.$$ (1)

It is clear from equation (1) that even at a modest Reynolds number of 10,000, the maximum allowable time step can be one to four orders of magnitude (depending on $\alpha$) smaller than the temporal Kolmogorov scale. Therefore, in most current DNS of inhomogeneous turbulence there is an uneven distribution of resolution in space and time, with the smallest spatial scale approximately matched but with the temporal scale over-resolved by at least two to three
orders of magnitude\(^1\). This inefficiency of currently employed semi-implicit schemes for DNS has been recognized before, and attempts have been made to employ fully implicit schemes. However, this requires Newton iterations and non-symmetric solvers that render the overall approach inefficient.

Progress can be made by employing semi-Lagrangian time-discretization, which could increase significantly the maximum allowable time step while maintaining the efficiency of symmetric solves. The semi-Lagrangian approach has long been used in meteorology for numerical weather prediction, where the use of large time step is essential for efficiency. This approach has been introduced at the beginning of the eighties [20], and the basic idea is to discretize the Lagrangian derivative of the solution in time instead of the Eulerian derivative. It involves backward time integration of a characteristic equation to find the departure point of a fluid particle arriving at an Eulerian grid point. The solution value at the departure point is then obtained by interpolation. There is no mesh deformation as in Lagrangian methods because the “arrival points” employed coincide with the grid points. However, there may be significant interpolation cost to obtain the solution values at the “departure points”.

The accuracy of semi-Lagrangian method is particularly sensitive to the method of backward integration of the characteristic equation as well as the interpolation scheme to evaluate the solution at departure points. This has been shown by Falcone and Ferretti [3] who conducted a rigorous analysis of the stability and convergence properties of semi-Lagrangian schemes. Typically, the backward integration is performed by employing second-order schemes (i.e., mid-point rule), explicitly or implicitly. In [8] and [14], the fourth-order Runge-Kutta method was employed but their results did not show any improvement over the second-order schemes. This finding is perhaps due to low spatial resolution used in these works, which is crucial for the overall accuracy of semi-Lagrangian schemes. It has been shown that the simplest semi-Lagrangian scheme with linear interpolation is equivalent to the classical first-order upwinding scheme [17], which is excessively dissipative (see [20] and [22]). A popular choice for interpolation methods in previous works has been the cubic spline methods [13].

An intriguing finding is that the error of semi-Lagrangian schemes in solving advection-diffusion equations decreases as the time step increases in a certain range of parameters, and this has initially led to some erroneous justifications ([16], [15]). The error analysis in [3] showed that the overall error of semi-Lagrangian method is indeed not monotonic with respect to time step \(\Delta t\), and, in particular, it has the form

\[ O(\Delta t^k + \frac{\Delta x^{p+1}}{\Delta t}), \]

where \(k\) refers to the order of backward time integration and \(p\) to the interpolation order. Another interesting result

\(^1\)One could argue that discretization in space is typically high-order but in time is low-order, however even with this factor taken into account the uneven distribution argument is still valid.
was obtained by Giraldo [6] who combined the semi-Lagrangian approach with spectral element discretization for the advection-diffusion equation. He found that for polynomial order $p \geq 4$ the combined scheme exhibits neither dissipation nor dispersion errors.

The extension of semi-Lagrangian method to the solution of Navier-Stokes equations was presented in the pioneering work of Pironneau (1982) [18]. He demonstrated the nonlinear stability of the method even as the viscosity approaches to zero. He also obtained suboptimal error estimates, which were improved later by Sâli (1988) [23]. Most of the previous analysis and numerical implementations have employed the Taylor-Hood finite element and are first-order in time ([2], [8] and [19]). In a recent paper [1], an error analysis was conducted for the fractional-step method for incompressible Navier-Stokes equations. In particular, the pressure-correction version of the fractional scheme with first-order time stepping was analyzed and an extension to a second-order was proposed but not analyzed. An attempt at a second-order scheme was made in [2] but no convergence rates were documented in that work. Moreover, results presented for a standard benchmark problem are markedly different than accepted results in the literature, possibly due to an erroneous treatment of the pressure term.

In this paper, we present a semi-Lagrangian spectral/$hp$ element (SLSE) method for the two- and three-dimensional Navier-Stokes equations, inspired by the results of Giraldo [6] for the advection-diffusion equation. In particular, we apply the newer modal version of the spectral element method [11] to the advection diffusion. We study in detail the dependence of the overall accuracy upon the time step, demonstrating the non-monotonic trends suggested by the theory of Falcone & Ferretti [3]. We then extend the semi-Lagrangian spectral/$hp$ element method to incompressible Navier-Stokes equations targeting a high-order temporal discretization. To this end, we propose a discretization based on a second-order stiffly-stable scheme, to circumvent instabilities or inaccuracies associated with the treatment of the pressure term in previous efforts. We demonstrate that the new SLSE method achieves second-order accuracy in time and retains spectral accuracy in space. We also demonstrate the non-monotonic dependence of error upon time step for a certain range of parameters, similar to that of advection-diffusion equation. The new SLSE method is applied to two- and three-dimensional incompressible flow problems with typical time steps more than twenty times the time steps of the corresponding Eulerian discretization. As regards efficiency, a new search-interpolation procedure is designed which makes the SLSE method at most twice expensive per time step compared to its Eulerian counterpart for advection-diffusion equation. For high spectral order ($P \geq 10$) the SLSE method is less than 25% more expensive per time step than the Eulerian approach, resulting in an overall gain one- to two-orders of magnitude in favor of the SLSE. This makes the SLSE method a great candidate for turbulence simulations, especially at high Reynolds numbers.
The paper is organized as follows: In the next section we present the main search, interpolation and backward integration algorithms, and validate the method for the advection-diffusion equation. We then extend the method to Navier-Stokes equations examining different splitting schemes and present two- and three-dimensional flow simulations for validation of the method.

2 Advection-Diffusion Equation

We first consider the advection-diffusion equation

$$\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi = \nu \nabla^2 \phi, \quad (2)$$

and we use a semi-implicit scheme for time discretization, i.e., we employ a second-order Adams-Bashforth scheme for the advection term and a Crank-Nicolson scheme for the diffusion term

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} + u \cdot \nabla \left( \frac{3\phi^n - \phi^{n-1}}{2} \right) = \nu \nabla^2 \left( \frac{\phi^{n+1} + \phi^n}{2} \right). \quad (3)$$

The semi-implicit scheme avoids the stability constraint from the diffusion term but it is subject to the CFL condition due to the explicit treatment of the advection term. The CFL restriction is more severe for the spectral/hp element method as the permissible time step $\Delta t$ scales approximately as the square of the spectral order [11].

The Lagrangian form of (2) is

$$\frac{d\phi}{dt} = \nu \nabla^2 \phi, \quad (4)$$

$$\frac{dx}{dt} = u(x, t). \quad (5)$$

The idea of a pure Lagrangian approach is to solve equation (4) along the characteristic lines (5). This leads to an effective decoupling of the advection and diffusion terms and an unconditionally stable scheme. However, as the fluid particles move along, they may result in heavily distorted and irregular mesh. Hence, expensive remeshing is required between time steps. In the semi-Lagrangian approach, the computational mesh is fixed. At each time step, a discrete set of particles arriving at the grid points is tracked backwards over a single time step along its characteristic line up to its departure points. The solution value at the departure points is then obtained by interpolation. The second-order Crank-Nicolson semi-Lagrangian scheme is of the form

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = \nu \nabla^2 \left( \frac{\phi^{n+1} + \phi^n}{2} \right), \quad (6)$$

$$\frac{dx}{dt} = u(x, t), \quad x^{n+1} = x(t^{n+1}) = x_a. \quad (7)$$
Here \( \phi^n_d \) denotes the value of \( \phi \) at the departure points \( x_d \) at time level \( n \), and \( x_a \) is the position of the arrival points which are the grid points. The characteristic equation (7) is solved backward, i.e., we solve for the departure point at time level \( n \), \( x^n_d \), with the initial condition \( x^{n+1} = x_a \).

### 2.1 Details of the Algorithm

The departure points do not coincide with the grid points, and thus a search-interpolation procedure is needed. Also, the overall accuracy of semi-Lagrangian depends critically on both the accuracy of the backward integration of (7) as well as the accuracy of the interpolation method. In the following we provide details on how we implement both algorithms.

#### 2.1.1 Backward Integration

We solve equation (7) for one single time step in order to obtain \( x_d = x(t^n) \) by the explicit second-order mid-point rule

\[
\dot{x} = x_a - \frac{\Delta t}{2} u(x_a, t^n),
\]

\[ x_d = x_a - \Delta t u(\dot{x}, t^n + \frac{\Delta t}{2}). \tag{9} \]

By defining

\[ \alpha = x_a - x_d, \]

we can re-write the explicit mid-point rule as

\[ \alpha = \Delta t u \left( x_a - \frac{\Delta t}{2} u(x_a, t^n), t^n + \frac{\Delta t}{2} \right). \tag{10} \]

Similarly, we employ implicit integration for equation (8) setting

\[ \dot{x} = x_a - \frac{\Delta t}{2} u(\dot{x}, t^n + \frac{\Delta t}{2}), \]

to arrive at the implicit mid-point rule

\[ \alpha = \Delta t u \left( x_a - \frac{\alpha}{2}, t^n + \frac{\Delta t}{2} \right). \tag{11} \]

This is the backward integration algorithm used in most of previous semi-Lagrangian schemes. Although the explicit and implicit schemes are formally of second-order, a small accuracy improvement has been reported for the implicit method. Equation (11) has to be solved iteratively, but numerical experiments show that only a few iterations are needed for convergence (typically around five). For an advection-diffusion equation with the velocity field known
analytically, this additional cost is negligible. However, for a velocity field known only in numerical form, the iteration is costly because each substep requires a search-interpolation procedure. Our numerical results show that the two methods give almost identical results and for more general problems, especially for Navier-Stokes equations, the explicit method (equation (10)) is preferred. To enhance the accuracy further, higher-order methods can be used; in the following we will be using the fourth-order Runge-Kutta method.

2.1.2 Search-Interpolation Procedure

In general, the departure points do not coincide with the grid points. To evaluate the solution at these points, we have developed the following algorithms:

Searching Algorithm

We first determine in which element a particular departure point lies. Although for structured grids this task can be as simple as index checking, it is nontrivial for unstructured grids. For general unstructured grids, the QuadTree algorithm has been employed by Giraldo [6] for a quadrilateral grid. In this paper, we present a new fast searching algorithm especially designed for the semi-Lagrangian method.

![Figure 1: Sketch to illustrate the searching algorithm.](image)

The searching starts with the 'parent' element, which could be the element where the point physically lies initially or where it is last found (see figure 1). To decide if the point is in the element or not, the scalar product between the vector starting from the the vertex of the element to the point and the inward normal vector of the corresponding edge is taken. The point is inside the element if all the scalar products of each edge are positive. If any of them is negative, the point falls outside the element in that direction. The searching is then moved to the neighbouring
element in that direction by using the connectivity information of the unstructured grid until the parent element is found. In the worst case scenario, the number of elements checked is the same as the number of elements the particle travels. Thus, the method is very fast, as in the spectral/hp element method a substantially smaller number of elements is employed compared to low-order finite elements. Our numerical tests show that the cost of searching is negligible compared to the other costs.

Interpolation Algorithm

A clear advantage of the spectral/hp element method is that it employs a high-order polynomial basis and thus there is no need for constructing explicitly special interpolation functions (e.g., cubic splines). A Lagrangian interpolation is then performed in the parent element of the departure points using the spectral basis, which is a hierarchical set of Jacobi polynomials [11]. The order of the polynomial basis used throughout this paper ranges from $P = 1$ to $P = 14$ with a typical value $P = 8$.

2.2 Numerical Results

In this section we present the results and error analysis of the SLSE method applied to advection-diffusion equation (2).

2.2.1 Benchmark Problem

We use the same Gaussian-cone problem as Giraldo [6] to test the SLSE method. The transport velocity field is

\[ u = +y, \quad v = -x. \]

The initial condition is

\[ \phi(x, y, 0) = e^{-[(x-x_0)^2+(y-y_0)^2]/2\lambda^2}, \]

and the exact solution is

\[ \phi(x, y, t) = \frac{\lambda^2}{\lambda^2 + 2vt} e^{-[(x-x_0)^2+y^2]/2(\lambda^2+2vt)}, \]

where

\[ \tilde{x} = x - x_0 \cos t - y_0 \sin t, \quad \tilde{y} = y + x_0 \sin t - y_0 \cos t. \]

The constants are fixed at $\lambda = \frac{1}{2}$, and $(x_0, y_0) = (-\frac{1}{2}, 0)$. A mesh consisting of $10 \times 10$ quadrilateral elements is used, and the time-integration is performed for one revolution corresponding to $t = 2\pi$.

The $L^2$ error norm is used to examine the accuracy, i.e.,

\[ e_{L^2} = \frac{\int (\phi_{\text{exact}} - \phi)^2 d\Omega}{\int \phi_{\text{exact}}^2 d\Omega}. \] (12)
The Courant number (CFL) and the non-dimensional diffusion coefficient are defined as

$$\sigma = \max \left( \frac{U \Delta t}{\Delta x} \right) \quad \text{and} \quad \mu = \max \left( \frac{\nu \Delta t}{\Delta x^2} \right),$$

where $U = \sqrt{u^2 + v^2}$ and $\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$; these definitions are the same as in [6].

Figure 2: Spatial convergence of Eulerian and semi-Lagrangian methods with a spectral/hp element discretization at small time step: Gaussian-cone problem.

In figure 2 we plot the results from an Eulerian method (Adams-Bashforth/Crank-Nicolson, ABCN) and the SLSE method with fixed $\sigma = 0.5$ and $\mu = 0.01$. The backward integration is the explicit mid-point rule (equation (10)), denoted here as RK2 method. The spectral order varies from $P = 2$ to $8$. We observe on this semi-log plot that spectral convergence is achieved for both methods. The SLSE method gives relatively larger error at lower polynomial order $P$, but it quickly reaches the $O(\Delta t^2)$ temporal error limit at $P = 8$. In figure 3 we plot the results with larger time step. The base time step $\Delta t$ corresponds to CFL number $\sigma = 0.5$. Results with $10\Delta t$ and $20\Delta t$, which correspond to CFL number 5 and 10, are also plotted. We observe that as the time step increases, the error is reduced, matching the error of the Eulerian scheme but at time step size twenty times larger. Also, a further improvement with the fourth-order Runge-Kutta method (RK4) is obtained at $20\Delta t$ with polynomial order $P = 6$. 
2.2.2 Error Analysis

The error of semi-Lagrangian method consists of two parts: the error of the backward integration $O(\Delta t^{k+1})$ and the error from interpolation $E(\Delta x) = O(\Delta x^{P+1})$, where $k$ is the order of integration method and $P$ is the order of the polynomial basis. Therefore, the overall accuracy of semi-Lagrangian method is

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{du}{dt} + O(\Delta t^k + \frac{O(\Delta x^{P+1})}{\Delta t}).$$  \hspace{1cm} (13)

A rigorous derivation of the above expression can be found in [3]. Equation (13) shows that the error is not monotonic with respect to $\Delta t$. When the polynomial order $P$ is small, the interpolation error dominates. As $\Delta t$ increases, the overall error decreases. It can also be appreciated that as the first term $O(\Delta t^k)$ is subdominant, further increasing $k$ will not improve the overall accuracy. This explains the reason that [8] and [14] did not see improvement of fourth-order Runge-Kutta method over the second-order methods. On the other hand, when the spatial error is subdominant at high $P$, increasing $\Delta t$ increases the first error term in (13) and thus the overall error is larger. In this case, a higher-order backward integration method (higher $k$, e.g. Runge-Kutta of fourth-order) reduces the dominant first term and improves the solution.

To further study the structure of the error, we test the SLSE method at different time steps and different spectral orders. We set the viscosity to a small value, $\nu = 4.6 \times 10^{-6}$, in order to emphasize the effect of the advection. The
range for $\Delta t$ is 0.01 to 0.05, which corresponds to CFL number 5 to 25, for $P = 10$. In figure 4 we plot results obtained with second-order backward integration ($k = 2$) in log-log axes. We make the following observations:

- $P = 4$: The interpolation error is relatively large, and thus the second error term in (13) dominates. As $\Delta t$ increases, the overall accuracy improves almost monotonically up to a large $\Delta t$ when the first error term becomes significant.

- $P = 6$: The interpolation error is smaller and the first error term in (13), $O(\Delta t^2)$, is comparable with the second term. As $\Delta t$ increases, the error starts to decrease first. The $O(\Delta t^2)$ term then becomes dominant and the overall error starts to increase. At this intermediate spatial resolution, there is clearly a competition between the two error terms resulting in the minimum error around $\Delta t \approx 0.024$.

- $P = 8$: The interpolation error is sufficiently small and thus the $O(\Delta t^2)$ dominates. The overall error then grows at an algebraic second-order rate.

- $P = 10$: The result is identical to that of $P = 8$, because the dominant error is the $O(\Delta t^2)$ term, which does not depend on $P$.

A similar but less obvious behavior can be observed in figure 5 with fourth-order Runge-Kutta backward integration ($k = 4$). It is worth noticing that with high-order interpolation polynomials, $P = 8$ and $P = 10$, the
interpolation error is very small and so is the backward integration error ($O(\Delta t^4)$). The competition between the two error terms results in a plateau, and the overall accuracy is preserved over a large range of time step. A slight anomaly is observed for $P = 6$ where a second decaying trend appears towards the larger step.

The above results show that when the solution is well-resolved in space with high-order polynomial interpolation, the overall accuracy is solely controlled by the method of backward integration. In figure 6 we plots results obtained with polynomial order $P = 10$; second-, third- and fourth-order Runge-Kutta backward integrations are employed and referred to in the legend as RK2, RK3 and RK4, respectively. The second-order scheme, RK2, shows an algebraic growth rate of order two, as the $O(\Delta t^2)$ error term always dominates. For third- and fourth-order schemes, RK3 and RK4, the trend is less obvious. When $\Delta t$ is not too large, the two error terms in the estimate (13) are comparable and the competition between them is nearly in balance resulting in a very slow growth in overall error. It is only at relatively large $\Delta t$ that the two methods deviate and show different growth rates (see right plot).

2.2.3 Solution of Finite Smoothness

To further examine the accuracy and robustness of the SLE method, we solved the advection equation for a solution with finite regularity. The initial condition is a parabolic cone, which has a discontinuity in the derivatives, unlike
Figure 6: Error dependence upon $\Delta t$ with $P = 10$: Left: RK2, RK3, and RK4; Right: close-up view of RK3 and RK4.

Figure 7: Pointwise error for the parabolic cone advection problem. Left: Eulerian SE method, and Right: SLSE method.
the previously tested smooth Gaussian cone:

$$\phi(x, y, t = 0) = \begin{cases} 
-16 \left( \frac{r_0^2}{r_0^2 - \frac{1}{4}} \right) & \text{if } r_0 < \frac{1}{4} \\
0 & \text{elsewhere}
\end{cases},$$

where $r_0^2 = (x - x_0)^2 + (y - y_0)^2$ and $(x_0, y_0)$ is the initial center position of the cone. The transport velocity field is the same as used in the previous example. This problem is solved with both Eulerian and semi-Lagrangian spectral $hp$ element methods. A mesh consisting of $10 \times 10$ quadrilateral elements with polynomial order $P = 8$ was used. The point-wise error of the solution after one revolution is plotted in figure 7. We see that the errors for the SLSE method are localized near the discontinuity, in agreement with previously reported results [3]. However, for the Eulerian scheme error oscillations are spread everywhere in the domain.

### 2.2.4 Computational Cost

The SLSE method is generally more expensive per time step than its Eulerian counterpart. However, by using the fast searching algorithm presented above, the computational cost of SLSE can be of the same order as the Eulerian spectral $hp$ element method. In table 1 we compare the computational cost of the two approaches for

<table>
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<th>Polynomial order $P$</th>
<th>SLSE - RK4 (sec/step)</th>
<th>ABCN Eulerian (sec/step)</th>
<th>Ratio (SLSE/ABCN)</th>
<th>Overall speed-up (SLSE/ABCN)</th>
</tr>
</thead>
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</tr>
<tr>
<td>10</td>
<td>1.49</td>
<td>1.19</td>
<td>1.25</td>
<td>16.0</td>
</tr>
</tbody>
</table>

Table 1: Comparison of computational cost of Eulerian SE and SLSE methods.

the aforementioned Gaussian-cone problem. We see that the SLSE method is less than twice more expensive than the Eulerian method, and as the spectral order increases the two costs are comparable. However, with much larger allowable CFL numbers, the total CPU time required for the SLSE method to reach a certain time level is significantly less than that of Eulerian method. The overall speed-up list in the table is obtained at CFL number of 20. Therefore, at least one order of magnitude in speed-up is achieved. The observed drop in the cost ratio as $P$ increases can be explained as follows: In the current implementation of the SLSE method, the common edges of interior elements are calculated twice. As the number of degrees-of-freedom of these common edges counts for a smaller portion as $P$ increases, the cost ratio decreases. No full optimization has been performed yet to further reduce the search-interpolation CPU time.
3 Incompressible Navier-Stokes Equations

In this section we extend the SLSE method to incompressible Navier-Stokes equations. Our goal is to design a robust scheme with high-order accuracy in time. We consider the Navier-Stokes equations in Lagrangian form

\[
\frac{du}{dt} = -\nabla p + \nu \nabla^2 u, \tag{14}
\]

\[
\nabla \cdot u = 0, \tag{15}
\]

and present three different time-discretization schemes.

3.1 The First-Order Scheme

The first-order semi-Lagrangian scheme can be obtained readily by treating the right-hand-side of (14) implicitly, i.e.,

\[
\frac{u^{n+1} - u^n}{\Delta t} = -\nabla p^{n+1} + \nu \nabla^2 u^{n+1}, \tag{16}
\]

\[
\nabla \cdot u^{n+1} = 0. \tag{17}
\]

This approach leads to unconditional stability and has been used in [8], [18] and [23]. However, it is only first-order in time and thus accuracy considerations limit its effectiveness.

3.2 The Second-Order Schemes

We propose two different approaches, the Crank-Nicolson scheme and the stiffly-stable scheme. We will show that only the latter is suitable for practical applications.

3.2.1 Crank-Nicolson Scheme

A direct extension of the Crank-Nicolson semi-Lagrangian method from the previous section yields

\[
\frac{u^{n+1} - u^n}{\Delta t} = -\nabla p^{n+1} + \nu \nabla^2 \left( \frac{u^{n+1} + u^n}{2} \right), \tag{18}
\]

\[
\nabla \cdot u^{n+1} = 0, \tag{19}
\]

where \(u^n\) denotes the velocity \(u\) at the departure point \(x, a\) and time level \(t^n\). The characteristic equation is

\[
\frac{dx}{dt} = u^{n+\frac{1}{2}}(x, t), \quad x(t^{n+1}) = x,a, \tag{20}
\]

where \(x,a\) is the position vector of the arrival points, which coincide with the grid points. The velocity at \(t^{n+1/2}\) is approximated by the second-order extrapolation

\[
u^{n+\frac{1}{2}} = \frac{3}{2}u^n - \frac{1}{2}u^{n-1}. \tag{21}\]
Equations (18) and (19) can be solved by the fractional-step method, where the discrete continuity equation results in a consistent Poisson equation for the pressure. This method seems to be second-order in time and it has been used recently in [2]. However, our numerical experiments show that it is in fact still first-order in time. It can also be justified as follows: Considering the fact that the idea of semi-Lagrangian scheme is to integrate the Navier-Stokes equations (14) along characteristic lines, i.e.,
\[
\int_{x_d}^{x_{n-1}} \frac{d\mathbf{u}}{dt} dt = \int_{x_d}^{x_{n-1}} (-\nabla p + \nu \nabla^2 \mathbf{u}) dx,
\]
a true second-order approximation, according to the mid-point rule, should take the form
\[
\frac{\mathbf{u}^{n+1} - \mathbf{u}_d^n}{\Delta t} = \frac{1}{2} \left[ \left( -\nabla p + \nu \nabla^2 \mathbf{u} \right)_d^n + \left( -\nabla p + \nu \nabla^2 \mathbf{u} \right)_d^{n+1} \right] \tag{23}
\]
Second-order accuracy can be verified by performing a Taylor expansion analysis. Comparing the above equation with equation (18), it is clear that method (18) can be at most first order in time.

The method described in equation (23) can be implemented using a high-order splitting scheme, which results in a three-step procedure [9]. However, our numerical experiments show that this method develops a long-term instability in the computation. This is due to the explicit part of pressure term in (23), and it is similar to the long-term instability observed in the Eulerian pressure correction formulation ([7] and [11]); it could be treated by additional projections.

3.2.2 Stiffly-Stable Scheme

To circumvent the long-term instability resulted from the explicit part of pressure term (see results below), we employ a variation of the backward multi-step scheme [4] to discretize the time derivative. A second-order time-discretization is
\[
\frac{3\mathbf{u}^{n+1} - 2\mathbf{u}_d^n - \frac{1}{2}\mathbf{u}_{d}^{n-1}}{\Delta t} = (-\nabla p + \nu \nabla^2 \mathbf{u})^{n+1},
\]
where \(\mathbf{u}_d^n\) is the velocity \(\mathbf{u}\) at the departure point \(x_d^n\) at time level \(t^n\) and \(\mathbf{u}_{d}^{n-1}\) is the velocity at the departure point \(x_d^{n-1}\) at time level \(t^{n-1}\). The departure point \(x_d^n\) is obtained by solving (20) with (21) over one single time level \(\Delta t\); while point \(x_d^{n-1}\) is obtained by solving
\[
\frac{dx}{dt} = \mathbf{u}(x, t), \quad x(t^{n+1}) = x_a
\]
over two time levels \(2\Delta t\). By using the above characteristic equations, the resulting scheme (24) is second-order accurate in time and can be verified by Taylor expansion analysis.
A three-step splitting scheme can be used to solve (24), i.e.,
\[ \frac{\hat{u} - 2u^n_d + \frac{1}{2}u^{n-1}_d}{\Delta t} = 0, \]  
(26)
\[ \frac{\hat{u} - \hat{u}}{\Delta t} = -\nabla p^{n+1}, \]  
(27)
\[ \frac{3u^{n+1} - \hat{u}}{\Delta t} = \nu \nabla^2 u^{n+1}. \]  
(28)

The discrete divergence-free condition \( \nabla \cdot u^{n+1} = 0 \) results in a consistent Poisson equation for the pressure
\[ \nabla^2 p^{n+1} = \frac{1}{\Delta t} \nabla \cdot \hat{u}, \]
with accurate pressure boundary conditions of the form \([9]\)
\[ \frac{\partial p}{\partial n} = -\nu \cdot n[\hat{u} + \nabla \times \omega^{n+1}], \]
where \(n\) is the unit normal, and \(\omega\) is the vorticity.

### 3.3 Numerical Results

In this section we present numerical results by applying the above discretizations to different benchmark problems. In all the test employing the SLE method, the second-order Runge-Kutta method is employed for backward integration.

#### 3.3.1 The Taylor Vortex

We use the Taylor vortex problem, an exact solution to the unsteady Navier-Stokes equations, in order to quantify the error in the SLE method. It has the form
\[ u = -\cos x \sin y e^{-2t/Re}, \]  
(29)
\[ v = \sin x \cos y e^{-2t/Re}, \]  
(30)
\[ p = -\frac{1}{4} (\cos 2x + \cos 2y) e^{-4t/Re}. \]  
(31)

The computational domain is a square defined by the coordinates \([-\pi, \pi]\) in each direction. A mesh consisting of \(2 \times 2\) quadrilateral elements is used, and the spectral order ranges as \(P = 8, 10, 12, 14\). The Reynolds number is fixed at \(10^6\), and the \(L^\infty\) norm is used to examine the error.

In figure 8 we show the evolution of the \(L^\infty\) error of the solution over long-time integration, starting from exact initial conditions. The Crank-Nicolson SLE method develops a long-term instability and it leads to inaccurate results. It is seemingly stable for this application because of the decaying of the exact solution with time; in the
Figure 8: Time history of error over long-term integration.

Figure 9: Spectral convergence for Eulerian SE and SLSE methods.
In figure 9 we plot the dependence of the error upon the spectral polynomial order. The $L_\infty$ error of the velocity is measured at $t = 2\pi$, i.e., after one revolution of the flow. We see in this semi-log plot that spectral convergence is achieved by both Eulerian and semi-Lagrangian spectral/hp element methods. The time step here is chosen by fixing the CFL number at $\sigma = 0.6$. In figure 10 we plot the dependence of the error upon the size of time step for the stiffly-stable method. The results are similar to the behavior reported earlier for the advection-diffusion equation. Note here the accuracy of SLSE method is dictated by the term $O\left(\Delta t^2 + \frac{\Delta x_i^{P+1}}{\Delta t}\right)$. At low $P$ the interpolation error dominates, and increasing $\Delta t$ decreases the overall error, as shown for $P = 6$. When the interpolation error is small at $P = 12$, the $\Delta t^2$ term dominates and further increase in $\Delta t$ increases the overall error. The $P = 8$ curve shows the competition between these two terms. In this plot, the largest time step 0.03 corresponds to the CFL number about 4. We emphasize that it is the size of $\Delta t$ and not the CFL number that restricts the use of semi-Lagrangian method. In other words, for the SLSE method, the restriction on the size of time step is solely due to accuracy considerations but not due to stability.

In the Navier-Stokes implementation, another contribution to the error that needs to be examined is associated with the splitting procedure (equation (26)). For the second-order SLSE method proposed here, the splitting error scales as $O(\nu \Delta t^2)$. This error is only visible for relatively viscous flows. We use two extreme cases to examine this error: One is at $\nu = 1$ ($Re = 1$) where the splitting error is of the same order as temporal error; the other is at $\nu = 10^{-6}$ ($Re = 10^6$) where the splitting error is negligible. In figure 11 we show the results for these two cases, and comparison with the second-order Eulerian method for exactly the same spatial discretization. We see that at
Figure 11: Error dependence on $\Delta t$. Left: $Re = 1$, and Right: $Re = 10^6$.

$Re = 1$, the $O(\Delta t^2)$ error dominates at very small time-step ($\Delta t \sim 10^{-3}$). However, at $Re = 10^6$ the $O(\Delta t^2)$ error starts to dominate only at $\Delta t \sim 10^{-2}$. Comparison of the two responses suggests that the $O(\Delta t^2)$ error observed at $Re = 1$ is the splitting error, and it is comparable in that case to the Runge-Kutta second-order error. The second-order Eulerian method generates the exact same error in this low $Re$ number case and reinforces the above statement. It should be noted that the error from the semi-Lagrangian part, i.e., $O\left(\frac{\Delta x^{p+1}}{\Delta t}\right)$, contains another $O(\Delta t^2)$ term. In advection-dominated flows the Reynolds number is greater than one, and thus the splitting error is always subdominant.

3.3.2 Two-Dimensional Driven Cavity Flow

Figure 12: 2-D driven cavity flow at $Re=400$. Left: $u$-velocity profile along vertical centerline, Right: $v$-velocity profile along horizontal centerline.

We consider the benchmark problem of two-dimensional driven cavity flow. A mesh consisting of $10 \times 10$ quadri-
Figure 12: History of flow quantities at center of the cavity at $Re = 10^3$, 0. Left: v-velocity, Right: pressure.
lateral elements is used for most of the tests and the spectral order is set to $P = 10$. We compare the horizontal velocity profile along the vertical center line and the vertical velocity profile along the horizontal center line in figures 12, 13 and 14 corresponding to Reynolds number $Re = 400$, $Re = 3,200$ and $Re = 10,000$, respectively. The time step of Eulerian spectral/$hp$ element method is chosen by fixing CFL number at $\sigma = 0.6$, while for SLSE the time step is more than 30 times larger with CFL number at 20. We observe that the semi-Lagrangian and Eulerian methods give essentially identical results, which they agree with the accepted results of [5]. At $Re = 400$ we included results from a first-order SLSE that show visible differences with the high-order accurate results. At $Re = 10,000$ the flow has bifurcated to an unsteady state. In figure 15 we plot the time history of velocity and pressure at the center of the cavity. The periodic oscillation is consistent with the results of bifurcation studies for the driven cavity flow (see, for example, [21]). In figure 14 the velocity profiles at $Re = 10,000$ correspond to the time-averaged values; they agree with the steady state solution of [5] as the onset of bifurcation is just below $Re = 10,000$.

![Graph](image)

Figure 16: Time history of $u$-velocity at the center of the cavity with various methods; $Re = 400$.

We now return to $Re = 400$ to investigate the stability of the Crank-Nicolson semi-Lagrangian scheme. In figure 16 we plot the time history of the horizontal-velocity at the center point of the cavity after long-time integration. The flow reaches steady state after $t \sim 30$ (non-dimensional convective time units). We can see that the second-order stiffly-stable SLSE method gives almost identical result as the second-order Eulerian method. The first-order SLSE method also reaches steady state but gives less accurate results. However, the second-order Crank-Nicolson SLSE
We study the effect of the

Figure 17: 2-D driven cavity flow at $Re = 400$. Left: horizontal and gradient (unilateral) derivatives. Left: vertical profile.

Method performs quite accurately in contrast with the first-order (line) semi-Lagrange method.

We verify the results in the limit of low polynomial order the second-order semi-Lagrange
and high polynomial order semi-Lagrange elements. We select a mesh consisting of $32 \times 64$
for first-order and second-order (line). The results shown in Figure 17 were obtained on a mesh consisting of $32 \times 64$.

We finally compare results for the semi-Lagrange method using linear and quadratic elements.
three-dimensionality in the driven cavity flow as a function of the aspect ratio. The Reynolds number is set at 400
and the aspect ratio was set to 1, 3 and 5. We plot velocity profiles at the center symmetric plane in figure 18.
Three-dimensionality effects are more pronounced, as expected, for the expansion ratio 1 as the results deviate the
most from the corresponding two-dimensional profile, while for the aspect ratio 5, a trend towards two-dimensionality
is observed. The results shown in the figure are obtained by SLSE method with CFL number 20. Results of the
Eulerian spectral/hp element method are also computed with CFL number at 0.6 but are not shown in the plot
because they are essentially identical as the SLSE results. In [2], the semi-Lagrangian (quadratic) finite element
method is used to solve the 2D driven cavity flow. The reported results show significant difference with results of
[5], and the authors claim favorable comparison with three-dimensional experimental results of [12]. The present
simulations of the three-dimensional driven cavity flow suggest that such an agreement may have been fortuitous.

4 Summary

A semi-Lagrangian spectral/hp element (SLSE) method is presented and applied to advection-diffusion and incompressible Navier-Stokes equations. The method is free from the CFL-restriction and thus very large time steps can be used, dictated only by accuracy considerations. This should not be interpreted as violation of the CFL condition, as the numerical domain of dependence of the solution still contains its domain of dependence [3]. It can be understood by considering the definition of the CFL condition on a fixed number of nodes as in the finite difference approximations, and contrasting it with the effectively variable stencil involved in the semi-Lagrangian approach.

An interesting aspect of the method is the structure of the temporal (advection) error, which reveals non-monotonic trend with the time step. In particular, in a certain range of parameters typically encountered in practical computations, the overall error decreases as the time step increases. This behavior, however, depends on the interpolation procedure involved and inaccurate representations may mask this trend. From the practical point of view, it is the capability of employing significantly larger time steps compared with the Eulerian schemes that make the SLSE method particularly attractive, especially for high Reynolds number simulations. This comes at the expense of extra interpolations, which are “intrinsic” in the spectral/hp element method, but it avoids prohibitive costs associated with non-symmetric solves as in fully-implicit schemes or remeshing as in purely Lagrangian approaches. The high-order spatial discretization of the spectral/hp element method seems to significantly enhance the possibility of using very large time steps. In our numerical tests, the CFL number can be as high as 20 ~ 30 and still retain good accuracy, while the semi-Lagrangian method with finite elements is about 6 ~ 7 [22]. The total cost of SLSE method can be at least an order of magnitude less than the Eulerian method, depending on the specific problem.
The SLSE method was also extended to incompressible Navier-Stokes equations with focus on second-order temporal accuracy. Two different types of second-order SLSE methods were presented, and it was shown that only the stiffly-stable implementation is suitable for practical applications. Numerical results with the Taylor vortex problem and two- and three-dimensional cavity flows demonstrated a similar error structure as in the advection-diffusion equation. The true advantages of semi-Lagrangian methods can be demonstrated in direct numerical simulation of turbulent flows, where the time step currently used is two to three orders of magnitude smaller than the Kolmogorov temporal scale. We will report on such systematic studies in a future publication.

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