Abstract. We propose a low order discontinuous Galerkin method for incompressible flows. Stability of the discretization of the Laplace operator is obtained by enriching the space element wise with a non-conforming quadratic bubble. This enriched space allows for a wider range of pressure spaces. We prove optimal convergence estimates and local conservation of both mass and linear momentum independent of numerical parameters.

1. Introduction

Discontinuous Galerkin (DG) methods for incompressible flow has been studied in Hansbo and Larson [9] in the framework of Nitsche’s method and inf-sup stable velocity pressure pairs. The analysis was extended by Toselli to the $hp$-framework using mixed or equal order stabilized formulations in [12]. Local discontinuous Galerkin methods with equal order velocity and pressure spaces stabilized using penalty on the interelement pressure jumps was proposed by Cockburn et al. [6]. The Navier-Stokes equations discretized using DG has recently been given a full analysis in the framework of domain decomposition on non-matching meshes using DG techniques by Girault et al. [8].

In this paper we extend our previous work on low order discontinuous Galerkin methods for scalar second order elliptic problems to the case of incompressible flow problems [2, 3]. Using piecewise affine discontinuous finite elements enriched with non-conforming bubbles we can eliminate all stabilization terms from the formulation without compromising the adjoint consistency. The upshot is that linear momentum is conserved locally and optimal convergence in the $L^2$-norm may be proven using a duality argument. This is a consequence of the fact that the vectors of the velocity gradient matrix are functions in the lowest order Raviart-Thomas space for our choice of velocity finite element space (see also [1]).

Several choices for the pressure space are possible without introducing a penalty term. Indeed we can use either globally continuous, piecewise affine functions, piecewise constant, discontinuous functions or functions from a direct sum of these two sets of functions and still satisfy the inf-sup condition uniformly with respect to the mesh size.

Depending on the choice of pressure space slightly different results may be obtained. When the pressure space consists of continuous functions we prove that the stresses are continuous. Using spaces with discontinuous functions for the pressure on the other hand leads to a method that also enjoys local mass conservation and
for which the divergence of the non-pressure stress has optimal convergence. We give a unified analysis for these three choices of pressure space and we then discuss the differences of these approaches from numerical and analytical point of view.

2. Notation

Let $\Omega$ be a convex polygon (polyhedron in three space dimensions) in $\mathbb{R}^d$, $d = 2, 3$, with outer normal $\mathbf{n}$. Let $\mathcal{K}$ be a subdivision of $\Omega \subset \mathbb{R}^d$ into non-overlapping $d$-simplices $\kappa$ and denote by $N_\mathcal{K}$ the number of simplices of the mesh. Suppose that each $\kappa \in \mathcal{K}$ is an affine image of the reference element $\tilde{\kappa}$, i.e. for each element $\kappa$ there exists an affine transformation $T_\kappa : \tilde{\kappa} \rightarrow \kappa$.

Let $\mathcal{F}_i$ denote the set of interior faces ($(d - 1)$-manifolds) of the mesh, i.e. the set of faces that are not included in the boundary $\partial \Omega$. The set $\mathcal{F}_e$ denotes the faces that are included in $\partial \Omega$ and define $\mathcal{F} = \mathcal{F}_i \cup \mathcal{F}_e$. Define by $N_\mathcal{F} = \text{card}(\mathcal{F})$ and $N_{\mathcal{F}_e} = \text{card}(\mathcal{F}_e)$ the number of faces resp. interior faces of the mesh.

Assume that $\mathcal{K}$ is shape-regular, does not contain any hanging node and covers $\overline{\Omega}$ exactly. For an element $\kappa \in \mathcal{K}$, $h_\kappa$ denotes its diameter and for a face $F \in \mathcal{F}$, $h_F$ denotes the diameter of $F$. Set $\hat{h} = \max_{\kappa \in \mathcal{K}} h_\kappa$ and let $\hat{h}$ be the function such that $\hat{h}|_\kappa = h_\kappa$ and $\hat{h}|_F = h_F$ for all $\kappa \in \mathcal{K}$ and $F \in \mathcal{F}$.

For a subset $R \subset \Omega$ or $R \subset \mathcal{F}$, $(\cdot, \cdot)_R$ denotes the $L^2(R)$-scalar product, $\| \cdot \|_R = (\cdot, \cdot)_R^{1/2}$ the corresponding norm, and $\| \cdot \|_{s,R}$ the $H^s(R)$-norm. The element-wise counterparts will be distinguished using the discrete partition as subscript, for example $(\cdot, \cdot)_\kappa = \sum_{\kappa \in \mathcal{K}} (\cdot, \cdot)_\kappa$. For $s \geq 1$, let $H^s(\mathcal{K})$ be the space of piecewise Sobolev $H^s$-functions and denote its norm by $\| \cdot \|_{s, \mathcal{K}}$. Further let us denote $H^s(\Omega) = [H^s(\Omega)]^d$ and $H^s(\mathcal{K}) = [H^s(\mathcal{K})]^d$. Moreover the following space is defined

$$L^2_0(\Omega) = \{ v \in L^2(\Omega) \mid \int_\Omega v \, dx = 0 \}.$$  

Let $\mathbf{v} = (v_1, \ldots, v_d)^T \in H^1(\mathcal{K})$, then we define $\nabla \mathbf{v}|_\kappa \in [L^2(\kappa)]^{d \times d}$ by $(\nabla \mathbf{v})_{i,j} = \partial_x v_i, 1 \leq i, j \leq d$, for each $\kappa \in \mathcal{K}$. Based on the scalar $L^2$-product we define

$$(\mathbf{v}, \mathbf{w})_\kappa = \sum_{i=1}^d (v_i, w_i)_\kappa \quad \text{and} \quad (\nabla \mathbf{v}, \nabla \mathbf{w})_\kappa = \sum_{i,j=1}^d (\partial_x v_i, \partial_x w_i)_\kappa$$

for all $\mathbf{v}, \mathbf{w} \in H^1(\mathcal{K})$.

Further let us define the jump and average operators. Fix $F \in \mathcal{F}_i$ and thus $F = \kappa_1 \cap \kappa_2$ with $\kappa_1, \kappa_2 \in \mathcal{K}$. Let $v \in H^1(\mathcal{K})$, $v \in H^1(\mathcal{K})$ and denote by $v_1$, $v_2$ resp. $\mathbf{v}_1$, $\mathbf{v}_2$ the restriction of $v$, $\mathbf{v}$ to the element $\kappa_1$, $\kappa_2$, i.e. $v_i = v|_{\kappa_i}$, resp. $v_1 = v|_{\kappa_1}$, $i = 1, 2$, and denote by $\mathbf{n}_1$, $\mathbf{n}_2$ the exterior normal of $\kappa_1$ resp. $\kappa_2$. We then define the average and jump operators by

$$\begin{align*}
\{ v \} &= \frac{1}{2}(v_1 + v_2), & [ v ] &= v_1 \mathbf{n}_1 + v_2 \mathbf{n}_2, \\
\{ \mathbf{v} \} &= \frac{1}{2}(v_1 + v_2), & [ \mathbf{v} ] &= v_1 \mathbf{n}_1 + v_2 \cdot \mathbf{n}_2,
\end{align*}$$

and

$$\begin{align*}
\{ \nabla \mathbf{v} \} &= \frac{1}{2}(\nabla v_1 + \nabla v_2), & [ \nabla \mathbf{v} ] &= \nabla v_1 \mathbf{n}_1 + \nabla v_2 \mathbf{n}_2,
\end{align*}$$

for $\mathbf{v} \in H^1(\mathcal{K})$. 

where \( \mathbf{a} \otimes \mathbf{b} \in \mathbb{R}^{d \times d} \) is defined by \((\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j, 1 \leq i, j \leq d, \) for all \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^d \).

On outer faces \( F \in \mathcal{F}_e \) we define them by
\[
\{v\} = v, \quad \{v\} = v, \quad \{v\} = v, \quad \{v\} = v, \quad \{\nabla v\} = \nabla v, \\
[v] = vn, \quad [v] = v \cdot n, \quad [v] = v \otimes n, \quad [\nabla v] = \nabla vn,
\]
where \( n \) is the outer normal of the domain \( \Omega \). Further we introduce some additional notation of the jump and average operators. Let \( n_F \in \{n_1, n_2\} \) be arbitrarily chosen but fixed and introduce the vectorial quantities, indexed by \( v \),
\[
\{\nabla v\}_v = \{\nabla v\}_v n_F \quad \text{and} \quad [v]_v = [v] n_F.
\]
This notation is necessary to define some projection, see Lemma 3.8. Observe that using these definitions that
\[
(1) \quad \{v\} : \{\nabla w\} = \{v\}_v : \{\nabla w\}_v
\]
for all \( v \in H^1(\mathcal{K}), \ v, w \in H^1(\mathcal{K}) \) and where \( a : b = \sum_{i,j=1}^d a_{ij} b_{ij} \) for all \( a, b \in \mathbb{R}^{d \times d} \).

Additionally remark that
\[
\{v\} = \{v\}_v \otimes n_F, \quad [v] = [v] n_F \cdot n_F
\]
and thus
\[
(2) \quad \|\{v\}\|_F = \|\{v\}_v\|_F, \quad \|\{v\}\|_F \leq c \|\{v\}\|_F
\]
for some mesh independent constant \( c > 0 \).

**Lemma 2.1 (Integration by parts).** Let \( v \in H^1(\mathcal{K}) \) and \( v, w \in H^1(\mathcal{K}) \), then
\[
(v, \nabla \cdot w)_K = -(\nabla v, w)_K + (\{v\}, [w])_F + ([v], \{w\})_F, \\
(\nabla v, \nabla w)_K = -(\Delta v, w)_K + ([\nabla v], [w])_F + (\{\nabla v\}, \{w\})_F.
\]

**Proof.** Element-wise integration by parts and applying the definitions of the jump and average operators leads to the result. \( \square \)

Respecting the equalities (1) leads to the following corollary.

**Corollary 2.2 (Integration by parts).** Let \( v \in H^1(\mathcal{K}) \) and \( v, w \in H^1(\mathcal{K}) \), then
\[
(\nabla v, \nabla w)_K = -(\Delta v, w)_K + ([\nabla v], [w])_F + (\{\nabla v\}, \{w\})_F.
\]

3. **Bubble stabilized finite element space**

Let us denote by \( V^p_h \) the standard discontinuous finite element space of degree \( p \geq 0 \) defined by
\[
V^p_h = \{v_h \in L^2(\Omega) : v_h|_\kappa \in \mathbb{P}_p(\kappa), \forall \kappa \in \mathcal{K}\},
\]
where \( \mathbb{P}_p(\kappa) \) denotes the set of polynomials of maximum degree \( p \) on \( \kappa \). Consider then the enriched finite element space
\[
V_{bs} = V^1_h + \{v_h \in L^2(\Omega) : v_h(x) = ax \cdot x, \ a \in V^0_h\},
\]
where \( x = (x_1, \ldots, x_d) \) denotes the physical variables. Let us additionally define some functional spaces that consists of functions only defined on the skeleton of the mesh:
\[
W^0_h = \{v_h \in L^2(\mathcal{F}) : v_h|_F \in \mathbb{P}_0(F), \forall F \in \mathcal{F}\},
\]
Define also the vectorial versions \( V^p_h = [V^p_h]^d, V_{bs} = [V_{bs}]^d \) and \( W^0_h = [W^0_h]^d \).
Let \( \mathbf{v} \in [L^2(\mathcal{F})]^m \), \( m \in \{1, d, d^2\} \), and define by \( \bar{\mathbf{v}} \) the \( L^2 \)-projection of \( \mathbf{v} \) onto \( [W_h^0]^m \), i.e.

\[
(\bar{\mathbf{v}}, \mathbf{w}_h)_F = (\mathbf{v}, \mathbf{w}_h)_F, \quad \forall \mathbf{w}_h \in [W_h^0]^m.
\]

3.1. Properties of the bubble stabilized finite element space. Let us discuss some important properties of the space \( V_{bs} \).

**Lemma 3.1.** For \( \mathbf{v}_h \in V_{bs} \) we have that

\[
\Delta \mathbf{v}_h \in V_h^0.
\]

Moreover the application \( \Delta : V_{bs}/V_h^1 \rightarrow V_h^0 \) is bijective.

**Proof.** Observe that \( \Delta \mathbf{w}_h = 0 \) for all \( \mathbf{w}_h \in V_h^1 \) and that \( \Delta(\alpha \mathbf{x} \cdot \mathbf{x}) = 2d\alpha \in V_h^0 \) where \( d \) is the dimension of \( \Omega \).

Let us denote by \( RT_0(\kappa) \) the local lowest order Raviart-Thomas space on \( \kappa \). The following Lemma holds.

**Lemma 3.2.** For all \( \mathbf{v}_h \in V_{bs} \) with \( \mathbf{v}_h = (v_{h,1}, \ldots, v_{h,d})^\top \) there holds

\[
\nabla v_{h,i}|_\kappa \in RT_0(\kappa), \quad \forall \kappa \in \mathcal{K},
\]

and for all \( \kappa \in \mathcal{K} \) and \( \mathbf{r}_h = (r_{h,1}, \ldots, r_{h,d}) \) with \( r_{h,i} \in RT_0(\kappa) \), there exists \( \mathbf{v}_h \in V_{bs} \) such that \( \nabla v_{h,i}|_\kappa = r_{h,i} \) for all \( 1 \leq i \leq d \).

**Proof.** For the scalar case we refer to the proof of Lemma 3.4 in [2] and the vectorial counterpart follows from the componentwise construction.

**Corollary 3.3.** For \( \mathbf{v}_h \in V_{bs} \) we have that

\[
\|\nabla \mathbf{v}_h\|_v \in W_h^0, \quad [\nabla \mathbf{v}_h] \in W_h^0.
\]

Moreover the applications \( \|\nabla \cdot \|_v : V_{bs} \rightarrow W_h^0 \) and \( \|\nabla \cdot \| : V_{bs} \rightarrow W_h^0 \) are surjective.

**Lemma 3.4.** There is a constant \( c > 0 \) independent of \( h \) such that for all \( \mathbf{v}_h \in V_{bs} \) there holds

\[
\|
\nabla \mathbf{v}_h\|_F + \frac{\sqrt{h}}{2} \|
\nabla \mathbf{v}_h\|_K \leq c \left( \|
\nabla \mathbf{v}_h\|_F + \|
\nabla \mathbf{v}_h\|_K \right).
\]

**Proof.** We refer to the proof of Lemma 4.1 in [2] for the scalar case. Its vectorial counterpart follows from the componentwise construction.

**Lemma 3.5** (Poincaré inequality). There is a constant \( c > 0 \) independent of \( h \) such that for all \( \mathbf{v}_h \in V_{bs} \) there holds

\[
\|
\nabla \mathbf{v}_h\|_K \leq c_F \left( \|
\nabla \mathbf{v}_h\|_F + \|
\nabla \mathbf{v}_h\|_K \right).
\]

**Proof.** We refer to the proof of Corollary 4.2 in [2] for the scalar case. Its vectorial counterpart follows from the componentwise construction.

Observe that Lemma 3.4 and 3.5 are only valid on discrete spaces and thus does not hold for functions in \( H^1(\mathcal{K}) \). Thus we define a norm for \( H^1(\mathcal{K}) \) by

\[
\|
\mathbf{v}\|_K^2 = \|
\nabla \mathbf{v}\|_K^2 + \|
\frac{\sqrt{h}}{2} \nabla \mathbf{v}\|_F^2
\]

for all \( \mathbf{v} \in H^1(\mathcal{K}) \). Nevertheless observe that there exists a constant \( c_d > 0 \) such that

\[
c_d \|
\nabla \mathbf{v}_h\|_K^2 + \|
\frac{\sqrt{h}}{2} \nabla \mathbf{v}_h\|_F^2 \leq \|
\mathbf{v}_h\|_K^2
\]

for all \( \mathbf{v}_h \in V_{bs} \).
3.2. Technical lemmas. Let us cite some well known results. For the proofs we refer to [4].

**Lemma 3.6** (Inverse inequality). Let \( v_h \in [V_{bs}]^m \), \( m \in \{1, d\} \), then there exists a constant \( c_I > 0 \) independent of \( h \) such that

\[
c_I^{-1} \| \tilde{h}^2 \Delta v_h \|_K \leq \| \tilde{h} \nabla v_h \|_K \leq c_I \| v_h \|_K.
\]

**Lemma 3.7** (Trace inequality). Let \( v \in [H^1(K)]^{m_1} \) and \( w_h \in [V_{bs}]^{m_2} \), \( m_1, m_2 \in \{1, d\} \) resp. \( w \in [H^1(K)]^{m_2} \) and \( w_h \in [V_{bs}]^{m_2} \), then there holds

\[
\| \{ \{ v \} \} \|_X + \| \{ w \} \|_X \leq c_T \left( \| \tilde{h}^{-\frac{1}{2}} v \|_K + \| \tilde{h}^{\frac{1}{2}} \nabla v \|_K \right),
\]

\[
\| \{ v_h \} \|_X + \| \{ w_h \} \|_X \leq c_T \| \tilde{h}^{-\frac{1}{2}} v_h \|_K
\]

and

\[
\| \{ \{ w \} \} \|_X + \| \{ w \} \|_X \leq c_T \left( \| \tilde{h}^{-\frac{1}{2}} w \|_K + \| \tilde{h}^{\frac{1}{2}} \nabla w \|_K \right),
\]

\[
\| \{ w_h \} \|_X + \| \{ w_h \} \|_X \leq c_T \| \tilde{h}^{-\frac{1}{2}} w_h \|_K
\]

where \( c_T > 0 \) is a constant independent of \( h \).

3.3. Projections. Moreover denote by \( \pi_p : L^2(\Omega) \rightarrow V^p_h \) the \( L^2 \)-projection onto \( V^p_h \) defined by

\[
\int_{\Omega} \pi_p(v_h) w_h \, dx = \int_{\Omega} v_h w_h \, dx \quad \forall w_h \in V^p_h
\]

and by \( \pi_p : L^2(\Omega) \rightarrow V^p_h \) its vectorial counterpart defined by \( (\pi_p v)_i = \pi_p v_i \), \( 1 \leq i \leq d \). Then \( \pi_p \) and \( \pi_p \) satisfy the following approximation result: Let \( v \in H^{p+1}(K) \) resp. \( v \in H^{p+1}(K) \), then

\[
\| v - \pi_p v \|_K + \| \tilde{h} \nabla (v - \pi_p v) \|_K \leq ch^{p+1} \| v \|_{p+1,K},
\]

\[
\| v - \pi_p v \|_K + \| \tilde{h} \nabla (v - \pi_p v) \|_K \leq ch^{p+1} \| v \|_{p+1,K}.
\]

Denote the Clément-Raviart space by \( \text{CR} \) and its vectorial counterpart by \( \text{CR} \). Additionally let us denote by \( i : H^1(\Omega) \rightarrow \text{CR} \) the vectorial Clément-Raviart interpolant satisfying the following approximation result: if \( v \in H^2(\Omega) \), then

\[
\| v - i, v \|_K + \| \tilde{h} \nabla (v - i, v) \|_K \leq ch^2 \| v \|_{2,K}.
\]

Further denote by \( C_h : L^2(\Omega) \rightarrow V^1_{h,c} \) the Clément interpolant [5] where \( V^1_{h,c} = \{ v_h \in C^0(\Omega) \mid v_h|_K \in P_p(K), \forall K \in \mathcal{T} \} \) is the continuous finite element space of degree 1. Remind that the Clément interpolant satisfies

\[
\| v - C_h v \|_K + \| \tilde{h} \nabla (v - C_h v) \|_K \leq ch^{\gamma+1} \| v \|_{\gamma+1,K}
\]

for all \( v \in H^{\gamma+1}(K) \), \( \gamma \in \{0, 1\} \). Note that one can prove that the Clément interpolant conserves the mean of a function over \( \Omega \), i.e., \( \int_{\Omega} C_h v(x) \, dx = \int_{\Omega} v(x) \, dx \), and thus \( C_h(L^2_0(\Omega)) \subset L^2_0(\Omega) \). Note also that \( C_h \) is \( H^1 \)-stable, i.e. \( \| \nabla (C_h v) \|_K \leq c \| \nabla v \|_K \). We denote by \( C_h \) the vectorial version of \( C_h \) sharing all properties.

Furthermore, we present the following projection which will be used in the analysis.
Lemma 3.8. Let \( a_h \in V_h^0 \) and \( b_h, c_h \in W_h^0 \) be fixed. Then, there exists a unique function \( \phi_h \in V_h \) such that

\[
\begin{align*}
\pi_0 \phi_h &= a_h, \\
\|\nabla \phi_h\|_{\mathcal{F}} &= b_h \quad \forall F \in \mathcal{F}, \\
\|\phi_h\|_{\mathcal{F}} &= c_h \quad \forall F \in \mathcal{F}.
\end{align*}
\]

Moreover \( \phi_h \) satisfies the following stability result

\[
\|\tilde{h}^{-1/2} \phi_h\|_K^2 \leq c \left( \|\tilde{h}^{-1} a_h\|_K^2 + \|\tilde{h}^{1/2} b_h\|_F^2 + \|\tilde{h}^{-1/2} c_h\|_{\mathcal{F}, k}^2 \right).
\]

Proof. Let us first establish the a priori estimate. Firstly by the trace inequality observe that

\[
\|\tilde{h}^{-1} a_h\|_K^2 \leq \|\tilde{h}^{-1} \pi_0 \phi_h\|_K^2 + \|\tilde{h}^{-1} (\phi_h - \pi_0 \phi_h)\|_K^2 \leq \|\tilde{h}^{-1} a_h\|_K^2 + c_a \|\nabla \phi_h\|_K^2
\]

for some constant \( c_a > 0 \). Secondly integrate by parts, use Lemma 3.1 and Corollary 3.3

\[
\begin{align*}
\|\nabla \phi_h\|_K^2 &= -(\Delta \phi_h, \pi_0 \phi_h)_K + (\|\nabla \phi_h\|_{\mathcal{F}}, [\|\phi_h\|_{\mathcal{F}}])_F + ([\|\phi_h\|_{\mathcal{F}}], [\|\phi_h\|_{\mathcal{F}}])_F \\
&= -(\Delta \phi_h, a_h)_K + (\|\phi_h\|_{\mathcal{F}}, b_h)_F + ([\|\phi_h\|_{\mathcal{F}}], c_h)_F.
\end{align*}
\]

Applying (11) and the Cauchy-Schwarz, the inverse or the trace and Young's inequality for each term respectively

\[
\begin{align*}
I &\leq c_T \|\nabla \phi_h\|_K \|\tilde{h}^{-1} a_h\|_K^2 \leq \frac{1}{4} \|\nabla \phi_h\|_K^2 + c_T^2 \|\tilde{h}^{-1} a_h\|_K^2 \\
II &\leq c_t \|\tilde{h}^{-1/2} \phi_h\|_{\mathcal{F}} \|\tilde{h}^{1/2} b_h\|_{\mathcal{F}} \leq \frac{1}{4} \|\nabla \phi_h\|_K^2 + \frac{1}{4c_a} \|\tilde{h}^{-1} a_h\|_K^2 + c_c c_T \|\tilde{h}^{1/2} b_h\|_{\mathcal{F}}^2, \\
III &\leq c_T \|\nabla \phi_h\|_K \|\tilde{h}^{-1/2} c_h\|_{\mathcal{F}} \leq \frac{1}{4} \|\nabla \phi_h\|_K^2 + c_T^2 \|\tilde{h}^{-1/2} c_h\|_{\mathcal{F}}^2,
\end{align*}
\]

and thus

\[
\|\nabla \phi_h\|_K^2 \leq c \left( \|\tilde{h}^{-1} a_h\|_K^2 + \|\tilde{h}^{1/2} b_h\|_{\mathcal{F}}^2 + \|\tilde{h}^{-1/2} c_h\|_{\mathcal{F}}^2 \right),
\]

which, combined with (3), (10) and (11), completes the a priori estimate. To conclude the proof, it now suffices to observe that (8) is a square linear system of size \( d (N_K + N_F + N_{F'}) \). Hence, existence and uniqueness of a solution of the linear system are equivalent. Let us denote by \( Aw = b \) the square linear system and assume that there is a vector \( w_1 \) and \( w_2 \) such that \( Aw_i = b_i, i = 1, 2 \). Further let us denote the difference between them by \( e = w_1 - w_2 \) and therefore \( Ae = 0 \). The a priori estimate (9) implies that \( e = 0 \) and thus the solution is unique and hence the matrix is regular. \( \square \)

4. BUBBLE STABILIZED DISCONTINUOUS GALERKIN METHOD FOR STOKES’ PROBLEM

Consider the steady Stokes problem: find \( u \in H^1(\Omega) \) and \( p \in L_0^2(\Omega) \) such that

\[
\begin{align*}
\begin{cases}
-\Delta u + \nabla p &= f \quad \text{in} \ \Omega, \\
\nabla \cdot u &= 0 \quad \text{in} \ \Omega, \\
u &= g \quad \text{on} \ \partial \Omega,
\end{cases}
\end{align*}
\]

where \( f \in H^{-1}(\Omega) \) and \( g \in H^{1/2}(\partial \Omega) \) such that \( \int_{\partial \Omega} g \cdot n \, ds = 0 \). This setting ensures a unique solution to the model problem (12), see [7].
4.1. Bubble stabilized discontinuous Galerkin method. Let \( Q_h \subset L^2_0(\Omega) \) be some scalar finite element space that will be precised later. We introduce the bubble stabilized Galerkin method by: find \((u_h, p_h) \in V_{bs} \times Q_h\) such that

\[
a(u_h, v_h) + b(p_h, v_h) - b(q_h, u_h) = F(v_h, q_h) \quad \forall (v_h, q_h) \in V_{bs} \times Q_h
\]

where the linear form \( F(\cdot, \cdot) \) and the bilinear forms \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \) are defined by

\[
a(u_h, v_h) = (\nabla u_h, \nabla v_h)_K - (\|\nabla u_h\|_K, [v_h])_F - ([u_h], \|\nabla v_h\|_K)_F,
\]

\[
b(p_h, v_h) = -(p_h, \nabla \cdot v_h)_K + ([p_h], [v_h])_F,
\]

\[
F(v_h, q_h) = (f, v_h)_K - (g \cdot n, q_h)_F - (g, \nabla v_h n)_F.
\]

Remark 4.1. The discrete solution \( u_h \) and \( p_h \) of (13) satisfies the following local linear momentum conservation property

\[
- \int_{\partial \kappa} \|\nabla u_h\| n_\kappa \; ds + \int_{\partial \kappa} \{p_h\} n_\kappa \; ds = \int_{\kappa} f \; dx
\]

for all \( \kappa \in \mathcal{K} \) and where \( n_\kappa \) denotes the outer normal of \( \kappa \).

Lemma 4.2 (Consistency). Let the couple \((u, p) \in H^2(\Omega) \times H^1(\Omega)\) be the exact solution of (12) and let \((u_h, p_h) \in V_{bs} \times Q_h\) be the approximation defined by (13). Then, there holds that

\[
a(u - u_h, v_h) + b(p - p_h, v_h) - b(q_h, u - u_h) = 0
\]

for all \((v_h, q_h) \in V_{bs} \times Q_h\).

Proof. By integration by parts, Lemma 2.1. Details are left to the reader. \(\square\)

5. Convergence analysis

Assume for simplicity homogeneous boundary conditions, i.e. \( g = 0 \), in this section.

We will specify the choice for the pressure space \( Q_h \) to get inf-sup stable approximations. Let us first introduce three possible choices for \( Q_h \). The first alternative to define the pressure approximation space is as the continuous piecewise linear finite element space defined by

\[
Q_{h,c}^1 = \{ v_h \in C^0(\Omega) \cap L^2_0(\Omega) \mid v_h|_\kappa \in \mathbb{P}_1(\kappa), \forall \kappa \in \mathcal{K} \}
\]

and the second one is the space of piecewise constant functions

\[
Q_{h,d}^0 = \{ v_h \in L^2_0(\Omega) \mid v_h|_\kappa \in \mathbb{P}_0(\kappa), \forall \kappa \in \mathcal{K} \}.
\]

Further let us also consider the direct sum of the above defined spaces \( Q_{h,c}^1 \oplus Q_{h,d}^0 \). Remark that \( \|\tilde{h} \nabla q_h\|_K \) and \( \|\tilde{h}^2 [q_h]\|_F \) is a norm for \( q_h \in Q_{h,c}^1 \) resp. \( q_h \in Q_{h,d}^0 \), and thus

\[
\|\tilde{h} \nabla q_h\|_K^2 = \|\tilde{h} \nabla q_h\|_K^2 + \|\tilde{h}^2 [q_h]\|_F^2
\]

is a norm for \( q_h \in Q_{h,c}^1 \oplus Q_{h,d}^0 \). For the following we chose \( Q_h = Q_{h,c}^1 \oplus Q_{h,d}^0 \) and observe that the cases \( Q_h = Q_{h,c}^1 \) and \( Q_h = Q_{h,d}^0 \) are covered by the analysis as well.

Thus let us define the following triple norm

\[
\|v_h, q_h\|^2 = \|v_h\|^2 + \|q_h\|^2
\]

for all \( v_h, q_h \in V_{bs} \times Q_h \).
Proposition 5.1 (Inf-sup condition). There exists a constant $c > 0$ independent of $h$ such that there holds $\forall (u_h, p_h) \in V_{bs} \times Q_h$

$$c \| u_h, p_h \|^2 \leq \sup_{0 \neq (v_h, q_h) \in V_{bs} \times Q_h} \frac{a(u_h, v_h) + b(p_h, v_h) - b(q_h, u_h)}{\| v_h, q_h \|}.$$ 

Proof. For the proof of Proposition 5.1 we introduce the following two lemmas.

Lemma 5.2. There exists a constant $c > 0$ independent of $h$ such that there exists for each fixed couple $(u_h, p_h) \in V_{bs} \times Q_h$ a couple $(v_h, q_h) \in V_{bs} \times Q_h$ with

$$c \| u_h, p_h \|^2 \leq a(u_h, v_h) + b(p_h, v_h) - b(q_h, u_h).$$

Lemma 5.3. There exists a constant $c > 0$ independent of $h$ such that for each $(u_h, p_h)$ and $(v_h, q_h)$ defined by Lemma 5.2 there holds

$$\| v_h, q_h \| \leq c \| u_h, p_h \|.$$ 

Indeed combining Lemma 5.2 and 5.3 yields

$$a(u_h, v_h) + b(p_h, v_h) - b(q_h, u_h) \geq c \| u_h, p_h \|^2 \geq c \| u_h, p_h \| \| v_h, q_h \|.$$ 

\[\square\]

Proof of Lemma 5.2. Firstly fix $u_h \in V_{bs}$ and $p_h \in Q_h$. Observe that choosing $v_h = u_h$ and $q_h = p_h$ in (13) yields

$$a(u_h, u_h) = \| \nabla u_h \|_{X}^2 - 2(\{ \nabla u_h \}, \{ u_h \})_{X} \geq \| \nabla u_h \|_{X}^2 - 2\| \hat{h}^{-\frac{1}{2}} \{ \nabla u_h \} \|_{X} \| \hat{h}^{-\frac{1}{2}} u_h \|_{X}$$

using Corollary 3.3, the Cauchy-Schwarz, the trace and Young’s inequality.

Further let $w_h \in V_{bs}$ be the projection defined by Lemma 3.8 with $a_h = 0$, $b_h = \hat{h}^{-1} [w_h]$, and $c_h = 0$. By its properties, integration by parts, Lemma 3.1, Corollary 3.3 and (2) we get

$$a(u_h, w_h) = -\langle \Delta u_h, w_h \rangle_{X} + (\{ \nabla u_h \}, \{ w_h \})_{X} - (\{ u_h \}, \{ \nabla w_h \})_{X}$$

$$= -\langle \Delta u_h, \pi_0 w_h \rangle_{X} + (\{ \nabla u_h \}, \{ w_h \})_{X} - (\{ u_h \}, \{ \nabla w_h \})_{X}$$

and again by integration by parts

$$b(p_h, w_h) = (\nabla p_h, \pi_0 w_h)_{X} - (\{ p_h \}, \{ w_h \})_{X} = 0$$

since $\nabla p_h \in V_{h}^{0}$ and by the property of the projection $w_h$. Further since $p_h \in Q_h$ we write $p_h = p_{h,c} + p_{h,d}$ with $p_{h,c} \in Q_{h,c}^{0}$ and $p_{h,d} \in Q_{h,d}^{0}$ and thus

$$b(p_h, w_h) = -\langle [p_{h,d}], \{ w_h \} \rangle_{X} = 0$$

since $[p_{h,c}] = 0$ and by the property of the projection $w_h$.

Let $z_h$ be the projection defined by Lemma 3.8 with $a_h = \hat{h}^2 \nabla p_h$, $b_h = 0$ and $c_h = -\hat{h}[p_{h,d}]$ and observe that

$$b(p_h, z_h) = (\nabla p_h, \pi_0 z_h)_{X} - (\{ p_{h,d} \}, \{ z_h \})_{X} = \| \hat{h} \nabla p_h \|_{X}^2 + \| \hat{h}^2 [p_{h,d}] \|_{X}^2 = \| p_h \|_{q}^2.$$

\[\square\]
Also, applying integration by parts combined with the Cauchy-Schwarz and Young’s inequality, yields

\[ a(u_h, z_h) = -\langle \Delta u_h, \pi_0 z_h \rangle_K + \langle \| \nabla u_h \|, \| z_h \| \rangle_F - \langle \langle u_h \rangle_v, \| \nabla z_h \| \rangle_F = -\langle \Delta u_h, \tilde{h}^2 \nabla p_h \rangle_K - \langle \| \nabla u_h \|, \tilde{h} \rangle_F, \]

\[ \geq -c \| \nabla u_h \|_K (\| \tilde{h} \nabla p_h \|_K + \| \tilde{h}^2 \|_F \| p_h \|_2) \]

\[ \geq -c \| \nabla u_h \|_K^2 - \frac{1}{2} \| p_h \|_q^2 \]

which implies that

\[ a(u_h, z_h) + b(p_h, z_h) \geq \frac{1}{2} \| p_h \|_q^2 - c_2 \| \nabla u_h \|_K^2. \]

Defining \( v_h = u_h + (c_1 + \frac{1}{2})w_h + \frac{1}{4c_2}z_h, \) \( q_h = p_h \) and respecting (14)-(16) yields

\[ a(u_h, v_h) + b(p_h, v_h) - b(q_h, u_h) \geq c \| u_h, p_h \|^2 \]

using (3).

\[ \square \]

Proof of Lemma 5.3. Observe that by (9) we get

\[ \| w_h \|^2 \leq c \| u_h \|^2 \quad \text{and} \quad \| z_h \|^2 \leq c \| p_h \|^2, \]

and therefore using the definition of \( v_h \) implies

\[ \| v_h, q_h \|^2 = \| v_h \|^2 + \| p_h \|^2 \leq 4 \| u_h \|^2 + 4 \| w_h \|^2 + 4 \| z_h \|^2 + \| p_h \|^2_q \]

\[ \leq c (\| u_h \|^2 + \| p_h \|^2_q) = c \| u_h, p_h \|^2. \]

\[ \square \]

Thus the numerical scheme is inf-sup stable for all three choices \( Q_h = Q^1_{h,c}, \)
\( Q_h = Q^0_{h,d} \) and \( Q_h = Q^1_{h,c} \oplus Q^0_{h,d} \). Let us define the following auxilliary norm

\[ \| v, q \|_n^2 = \| \tilde{h}^{-1} v \|_K^2 + \| \nabla v \|_K^2 + \| \tilde{h}^{-\frac{1}{2}} \{ v \} \|_F^2 + \| \tilde{h}^{\frac{1}{2}} \{ \nabla v \} \|_F^2 + \| q \|_K^2 + \| \tilde{h}^{\frac{1}{2}} \{ q \} \|_F^2 \]

for all \( v \in H^2(K) \) and \( q \in L^2(\Omega) \) which will be used for the continuity result. We denote further by \( A + B \) the (in general not direct) sum of the functional spaces \( A \) and \( B \). Then we prove the following continuity result.

Proposition 5.4 (Continuity). Let \( v \in H^2(K) + CR, \) \( v_h \in V_{bs}, q \in H^1(K) \) and \( q_h \in Q_h \). There exists a constant \( c > 0 \) independent of \( h \) such that

\[ a(v, v_h) + b(p, v_h) - b(q_h, v) \leq c \| v, q \|_n \| v_h, q_h \|. \]
Proof. Applying the Cauchy-Schwarz inequality, Lemma 3.4 and (2) yields
\[ a(v, v_h) = \langle \nabla v, \nabla v_h \rangle_K - \langle \frac{1}{2} \nabla v_h, [v_h] \rangle_F \]
\[ \leq \left( \| \nabla v \|_K^2 + \| \nabla v \|_F^2 \right)^{\frac{1}{2}} \left( \| \nabla v_h \|_K^2 + \frac{1}{2} \| [v_h] \|_F^2 \right)^{\frac{1}{2}} \]
\[ \leq \| v, q \|_a \| v_h, q_h \|, \]
\[ b(q, v_h) = -\langle q, \nabla \cdot v_h \rangle_K + \{ q \}, [v_h] \rangle_F \]
\[ \leq \left( \| q \|_K^2 + \| \nabla \cdot q \|_F^2 \right)^{\frac{1}{2}} \left( \| \nabla v_h \|_K^2 + \frac{1}{2} \| [v_h] \|_F^2 \right)^{\frac{1}{2}} \]
\[ \leq \| v, q \|_a \| v_h, q_h \|, \]
\[-b(q_h, v) = \langle \nabla q_h, v \rangle_K -(q_h, \{ v \} \rangle_F \]
\[ \leq \left( \| \nabla q_h \|_K^2 + \| \nabla \cdot q_h \|_F^2 \right)^{\frac{1}{2}} \left( \| \nabla^{-1} v \|_K^2 + \frac{1}{2} \| \{ v \} \|_F^2 \right)^{\frac{1}{2}} \]
\[ \leq c \| v, q \|_a \| v_h, q_h \|. \]

Assume that \( Q_h = \alpha Q^{(1)}_{h,c} + \beta Q^{(0)}_{h,d} \) for \((\alpha, \beta) \in \{(1,0), (0,1), (1,1)\}\) and define in a standard manner the following quantities
\[ \eta_u = u - i_u, u, \quad \xi_u = u - i_u, u, \quad \eta_p = p - P_a p, \quad \xi_p = p_h - P_a p, \]
where the projection \( P_a : L^2(\Omega) \rightarrow Q_h \) is defined by \( P_a = (1 - \alpha) \pi_0 + \alpha C_h \). Observe that \( \xi_u \in V_{bs} \) and \( \xi_p \in Q_h \) since the projection \( P_a \) preserves the property of zero mean. Further \( P_a \) satisfies the error estimate
\[ \| v - P_a v \|_K + \| \nabla (v - P_a v) \|_K \leq c h^{\alpha + 1} |v|_{1+\alpha, K} \]
for all \( v \in H^{\gamma+1}(\Omega), \gamma \in \{0,1\} \).

**Proposition 5.5** (Approximability). Let \( \eta_u \in H^2(\Omega) + CR \) and \( \eta_p \in H^{\gamma+1}(\Omega) + Q_h \), with \( \gamma \in \{0,1\} \) and \( Q_h = \alpha Q^{(1)}_{h,c} + \beta Q^{(0)}_{h,d} \) for \((\alpha, \beta) \in \{(1,0), (0,1), (1,1)\}\), be defined by (17). Then, there exists a constant \( c > 0 \) independent of \( h \) such that
\[ \| \eta_u, \eta_p \| + \| \eta_u, \eta_p \|_a \leq ch |u|_{2,K} + ch^{1+\alpha} |p|_{1+\alpha, K}. \]

**Proof.** This is a direct consequence of the trace inequality and the error estimates (6) and (18).

**Theorem 5.6** (Convergence in Energy norm). Let \( u \in H^2(\Omega), p \in H^{\gamma+1}(\Omega), \gamma \in \{0,1\}, \) be the exact solution of problem (12) and let \( u_h \in V_{bs}, p_h \in Q_h \), with \( Q_h = \alpha Q^{(1)}_{h,c} + \beta Q^{(0)}_{h,d} \) for \((\alpha, \beta) \in \{(1,0), (0,1), (1,1)\}\), be the approximation defined by (13). Then, there exists a constant \( c > 0 \) independent of \( h \) such that
\[ \| u - u_h \| + \| p - p_h \|_K + \| p - p_h \|_q \leq ch |u|_{2,K} + ch^{1+\alpha} |p|_{1+\alpha, K}. \]

**Proof.** Let us first establish the a priori estimate for the triple norm \( \| \cdot \| \). Split the error in a standard manner in two parts
\[ \| u - u_h, p - p_h \| \leq \| \eta_u, \eta_p \| + \| \xi_u, \xi_p \|. \]
By Proposition 5.5 it follows that
\[ \| \eta_u, \eta_p \| \leq ch |u|_{2,K} + ch^{1+\alpha} |p|_{1+\alpha, K}. \]
For the second term of the right hand side of (19) observe that applying the inf-sup condition of Proposition 5.1, the consistency of Lemma 4.2, the continuity of Proposition 5.4 and the approximability of Proposition 5.5 yields

\[
\|\xi u, \xi p\| \leq \sup_{0 \neq (v_h, q_h) \in V_h \times Q_h} \frac{a(\xi u, v_h) + b(\xi p, v_h) - b(q_h, \xi u)}{\|v_h, q_h\|} = \sup_{0 \neq (v_h, q_h) \in V_h \times Q_h} \frac{a(\eta u, v_h) + b(\eta p, v_h) - b(q_h, \eta u)}{\|v_h, q_h\|} \leq c\|\eta u, \eta p\| \leq c h |u|_{2, \Omega} + c h^{1+\alpha\gamma} |p|_{1+\alpha\gamma, \Omega}.
\]

Therefore we conclude that

\[
\|u - u_h\| + \|p - p_h\| \leq c h |u|_{2, \Omega} + c h^{1+\alpha\gamma} |p|_{1+\alpha\gamma, \Omega}.
\]

In order to prove the error estimate for the quantity \(\|p - p_h\|_{\Omega}\) we follow the standard technique, see [7, 11]. Let \(H^1_0(\Omega) = \{v \in H^1(\Omega) \mid v|_{\partial \Omega} = 0\}\). Since \(p - p_h \in L^2(\Omega)\) there exists \(v_p \in H^1_0(\Omega)\) such that \(\nabla \cdot v_p = p - p_h\) and \(\|\nabla v_p\|_{\Omega} \leq c |p - p_h|_{\Omega}\).

Applying integration by parts yields

\[
\|p - p_h\|_{\Omega}^2 = (p - p_h, \nabla \cdot v_p)_{\Omega} = -(\nabla (p - p_h), v_p)_{\Omega} + ([p - p_h], (v_p))_{\Omega}.
\]

Further split this equation

\[
(21) \quad \|p - p_h\|_{\Omega}^2 = I + II
\]

with

\[
I = -(\nabla (p - p_h), v_p - C_h v_p)_{\Omega} + ([p - p_h], (v_p - C_h v_p))_{\Omega},
\]

\[
II = -(\nabla (p - p_h), C_h v_p)_{\Omega} + ([p - p_h], (C_h v_p))_{\Omega},
\]

and where \(C_h\) denotes the vectorial Clément interpolation operator. Firstly observe using the Cauchy-Schwarz inequality, the trace inequality and the approximation properties of \(C_h\) that

\[
(22) \quad I \leq c (\|\tilde{h} \nabla (p - p_h)\|_{\Omega} + \|\tilde{h}^{1/2} |p - p_h||_{\Omega})\|\nabla v_p\|_{\Omega} \leq c ||p - p_h||_{\Omega} \|p - p_h\|_{\Omega}.
\]

Secondly using the consistency of Lemma 4.2 implies

\[
II = -b(p - p_h, C_h v_p) = a(u - u_h, C_h v_p)
\]

\[
= (\nabla (u - u_h), \nabla C_h v_p)_{\Omega} - ([u - u_h], \|\nabla C_h v_p\|_{\Omega})_{\Omega}
\]

\[
\leq (\|\nabla (u - u_h)\|_{\Omega}^2 + \|\tilde{h}^{1/2} |u - u_h||_{\Omega}^2)\|\nabla C_h v_p\|_{\Omega}^2 + \|\tilde{h}^{1/2} \|\nabla C_h v_p\|_{\Omega}^2)^{1/2}
\]

\[
\leq c \left( \|\nabla (u - u_h)\|_{\Omega} + \|\tilde{h}^{1/2} |u - u_h||_{\Omega} \right) \|\nabla C_h v_p\|_{\Omega}
\]

\[
(23) \leq c \left( \|\nabla (u - u_h)\|_{\Omega} + \|\tilde{h}^{1/2} |u - u_h||_{\Omega} \right) |p - p_h|_{\Omega}
\]

using that \(C_h v_p \in H^1_0(\Omega)\), the Cauchy-Schwarz inequality, the trace inequality, the \(H^1(\Omega)\)-stability of \(C_h\) and the stability of \(v_p\). Combining (20)-(23) leads to the result.

Remark 5.7 (Optimal \(L^2\)-convergence). Optimal convergence in the \(L^2\)-norm can be shown using Nitsche’s trick. The details are left to the reader.
Proposition 5.8. Let \((u_h, p_h) \in V_h \times Q_h\) be the solution of (13) with \(Q_h = \alpha Q^1_{h,c} \oplus \beta Q^2_{h,d}\) for \((\alpha, \beta) \in \{(1,0), (0,1), (1,1)\}\). Then, there holds

\[
(24) \quad h \| f + \Delta u_h - \nabla p_h \|_K + (1 - \beta) \| \hat{h}^{-\frac{1}{2}} \nabla u_h \|_X + \| \hat{h}^{-\frac{1}{2}} u_h \|_X \leq c h \| f - \pi_0 f \|_K.
\]

Additionally, if \(f \in H^1(K)\) there holds

\[
(25) \quad h \| f + \Delta u_h - \nabla p_h \|_K + (1 - \beta) \| \hat{h}^{-\frac{1}{2}} \nabla u_h \|_X + \| \hat{h}^{-\frac{1}{2}} u_h \|_X \leq c h^2 \| \nabla f \|_K.
\]

Proof. Let \(w_h\) be the function defined by Lemma 3.8 with \(a_h = h^2 (\nabla p_h - \pi_0 f - \Delta u_h), \ b_h = -\hat{h}^{-1} \nabla u_h\), and \(c_h = (1 - \beta) \hat{h} \nabla u_h\). By its properties, integration by parts, Lemma 3.1, Corollary 3.3 and (2) we get

\[
a(u_h, w_h) = - (\Delta u_h, w_h)_K + (\nabla u_h, \nabla w_h)_X - (u_h, \nabla w_h)_X = (\Delta u_h, \pi_0 w_h)_K + (\nabla u_h, \nabla w_h)_X - (u_h, \nabla w_h)_X = (\Delta u_h, a_h)_K + (1 - \beta) \| \hat{h}^{-\frac{1}{2}} \nabla u_h \|_X^2 + \| \hat{h}^{-\frac{1}{2}} u_h \|_X^2.
\]

On the other hand since \(p_h = \alpha p_{h,c} + \beta p_{h,d}\), using again integration by parts and the properties of the projections we get

\[
b(p_h, w_h) = (\nabla p_h, w_h)_K - (p_h, \nabla w_h)_X = (\nabla p_h, \pi_0 w_h)_K - \beta (\nabla p_{h,c} \pi_0 w_h)_K
\]

so that \((1 - \beta) = 0\) for the considered set of values of \(\beta\). Moreover for this set we have that \((1 - \beta) = 1 - \beta^2\) and since \(a(u_h, w_h) + b(p_h, w_h) = (f, w_h)_K\) we get

\[
h^2 \| \pi_0 f + \Delta u_h - \nabla p_h \|_K^2 + (1 - \beta)^2 \| \hat{h}^{-\frac{1}{2}} \nabla u_h \|_X^2 + \| \hat{h}^{-\frac{1}{2}} u_h \|_X^2
\]

and therefore we get

\[
h \| \pi_0 f + \Delta u_h - \nabla p_h \|_K + (1 - \beta) \| \hat{h}^{-\frac{1}{2}} \nabla u_h \|_X + \| \hat{h}^{-\frac{1}{2}} u_h \|_X \leq c h \| f - \pi_0 f \|_K.
\]

Finally we observe that

\[
\| f + \Delta u_h - \nabla p_h \|_K \leq \| f - \pi_0 f \|_K + \| \pi_0 f + \Delta u_h - \nabla p_h \|_K
\]

which yields (24) and additionally noting that for \(f \in H^1(K)\) there holds \(\| f - \pi_0 f \|_K \leq c h^2 \| \nabla f \|_K\) proves (25).

\[
\square
\]

Corollary 5.9. If \(f\) is piecewise constant, i.e. \(f \in V^0_h\), then

\[
\| u_h \|_X = 0, \quad \| \nabla u_h \|_X = 0, \quad \text{if } \alpha = 1, \beta = 0,
\]

\[
\| u_h \|_X = 0, \quad \| \Delta u - \Delta u_h \|_K = 0, \quad \text{if } \alpha = 0, \beta = 1,
\]

\[
\| u_h \|_X = 0, \quad \text{if } \alpha = 1, \beta = 1.
\]

Corollary 5.10. If \(f\) is piecewise constant, i.e. \(f \in V^0_h\), and \(Q_h = Q^2_{h,c}\) then, the discrete solution \(u_h\) and \(p_h\) of (13) satisfies the following local linear momentum conservation property

\[
- \int_{\partial \kappa} \nabla u_h n_\kappa \, ds + \int_{\partial \kappa} p_h n_\kappa \, ds = \int_{\kappa} f \, ds
\]

for all \(\kappa \in K\) and where \(n_\kappa\) denotes the outer normal of \(\kappa\).
Let us introduce the numerical examples tested in this section. We consider two numerical tests proposed in [11].

i) Problem with smooth solution
Consider problem (12) with $\Omega = (0,1)^2$ and $f(x)$ imposed such that the exact solution is given by

$$u(x) = \left( \frac{x_1^4 - 2x_1^3 + x_1^2}{(x_1^3 - 6x_1^2 + 2x_1)} \right), \quad p(x) = x_1 + x_2 - 1.$$ 

Observe that $(u, p)$ satisfies the regularity assumption and thus Theorem 5.6 and Proposition 5.8 are valid. A sequence of structured meshes is considered.

---

**Table 1.** Smooth problem: Different error quantities of the numerical solution for all three choices of the pressure approximation space with respect to the mesh size $h$. The quantities in the brackets correspond to the convergence rates.

<table>
<thead>
<tr>
<th>$Q_h$</th>
<th>$h$</th>
<th>$|u - u_h|_K$</th>
<th>$|u - u_h|_F$</th>
<th>$|p - p_h|_K$</th>
<th>$|\nabla \cdot (u - u_h)|_K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{h,c}^1$</td>
<td>0.1</td>
<td>4.47E-04</td>
<td>1.72E-02</td>
<td>1.95E-03</td>
<td>8.00E-03</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>1.16E-04 (1.95)</td>
<td>8.82E-03 (0.96)</td>
<td>7.21E-04 (1.44)</td>
<td>4.18E-03 (0.94)</td>
</tr>
<tr>
<td></td>
<td>0.025</td>
<td>2.93E-05 (1.98)</td>
<td>4.46E-03 (0.98)</td>
<td>2.54E-04 (1.51)</td>
<td>2.14E-03 (0.97)</td>
</tr>
<tr>
<td></td>
<td>0.0125</td>
<td>7.36E-06 (1.99)</td>
<td>2.24E-03 (0.98)</td>
<td>8.87E-05 (1.52)</td>
<td>4.18E-03 (0.97)</td>
</tr>
<tr>
<td>$Q_{h,d}^0$</td>
<td>0.1</td>
<td>2.88E-03</td>
<td>7.24E-02</td>
<td>4.50E-02</td>
<td>2.12E-02</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>7.55E-04 (1.93)</td>
<td>3.76E-02 (0.95)</td>
<td>2.13E-02 (1.08)</td>
<td>1.06E-02 (1.00)</td>
</tr>
<tr>
<td></td>
<td>0.025</td>
<td>1.92E-04 (1.98)</td>
<td>1.90E-02 (0.98)</td>
<td>1.04E-02 (1.04)</td>
<td>5.31E-03 (1.00)</td>
</tr>
<tr>
<td></td>
<td>0.0125</td>
<td>4.81E-05 (1.99)</td>
<td>9.55E-03 (0.99)</td>
<td>5.15E-03 (1.01)</td>
<td>2.65E-03 (1.00)</td>
</tr>
<tr>
<td>$Q_{h,c}^1 \oplus Q_{h,d}^0$</td>
<td>0.1</td>
<td>3.68E-04</td>
<td>1.50E-02</td>
<td>5.27E-03</td>
<td>6.14E-03</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>9.32E-05 (1.98)</td>
<td>7.59E-03 (0.98)</td>
<td>2.90E-03 (0.86)</td>
<td>3.12E-03 (0.98)</td>
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<tr>
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<td>0.025</td>
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<td>3.82E-03 (0.99)</td>
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<td>1.57E-03 (0.99)</td>
</tr>
<tr>
<td></td>
<td>0.0125</td>
<td>5.85E-06 (2.00)</td>
<td>1.91E-03 (1.00)</td>
<td>7.82E-04 (0.96)</td>
<td>7.86E-04 (1.00)</td>
</tr>
</tbody>
</table>

**Table 2.** Smooth problem: Different error quantities of the numerical solution for all three choices of the pressure approximation space with respect to the mesh size $h$. The quantities in the brackets correspond to the convergence rates.

<table>
<thead>
<tr>
<th>$Q_h$</th>
<th>$h$</th>
<th>$|h^{-1/2} u_1|_F$</th>
<th>$|h^{-1/2} u_2|_F$</th>
<th>$|h^{-1/2} \nabla u_h|_F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{h,c}^1$</td>
<td>0.1</td>
<td>8.84E-03</td>
<td>1.34E-03</td>
<td>1.94E-03</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>4.55E-03 (0.96)</td>
<td>3.64E-04 (1.88)</td>
<td>2.86E-04 (2.76)</td>
</tr>
<tr>
<td></td>
<td>0.025</td>
<td>2.30E-03 (0.99)</td>
<td>9.38E-05 (1.96)</td>
<td>3.88E-05 (2.88)</td>
</tr>
<tr>
<td></td>
<td>0.0125</td>
<td>1.12E-02 (0.97)</td>
<td>9.38E-05 (1.96)</td>
<td>4.15E-02 (0.97)</td>
</tr>
<tr>
<td>$Q_{h,d}^0$</td>
<td>0.1</td>
<td>4.14E-02</td>
<td>1.34E-03</td>
<td>1.54E-03</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>2.20E-02 (0.91)</td>
<td>3.64E-04 (1.88)</td>
<td>8.12E-02 (0.92)</td>
</tr>
<tr>
<td></td>
<td>0.025</td>
<td>1.12E-02 (0.97)</td>
<td>9.38E-05 (1.96)</td>
<td>4.15E-02 (0.97)</td>
</tr>
<tr>
<td></td>
<td>0.0125</td>
<td>5.64E-03 (0.99)</td>
<td>9.38E-05 (1.96)</td>
<td>2.09E-02 (0.99)</td>
</tr>
<tr>
<td>$Q_{h,c}^1 \oplus Q_{h,d}^0$</td>
<td>0.1</td>
<td>7.87E-03</td>
<td>1.34E-03</td>
<td>1.59E-02</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>4.00E-03 (0.98)</td>
<td>3.64E-04 (1.88)</td>
<td>8.33E-03 (0.93)</td>
</tr>
<tr>
<td></td>
<td>0.025</td>
<td>2.01E-03 (0.99)</td>
<td>9.38E-05 (1.96)</td>
<td>4.26E-03 (0.97)</td>
</tr>
<tr>
<td></td>
<td>0.0125</td>
<td>1.01E-03 (1.00)</td>
<td>2.37E-05 (1.98)</td>
<td>2.16E-03 (0.98)</td>
</tr>
</tbody>
</table>
Consider problem (12) with \( f = 0 \) and
\[
g(x) = \begin{cases} 
(8(1-x_2)(x_2-0.5), 0)^T & \text{if } x_1 = 0, \\
((1-x_2)x_2, 0)^T & \text{if } x_1 = 12, \\
0 & \text{otherwise},
\end{cases}
\]
on the domain defined by Figure 1. The domain is not convex and thus the solution does not lie in \( H^2(\Omega) \). In consequence Theorem 5.6 is no longer valid. But observe that Proposition 5.8 is independent of the geometry resp. regularity assumption and therefore Proposition 5.8 still holds. A sequence of unstructured and globally uniform meshes is considered.

We consider the approximations defined by (13) using a pressure approximation space \( Q_h = \alpha Q_{h,c}^1 \oplus \beta Q_{h,d}^0 \) for \((\alpha, \beta) \in \{(1, 0), (0, 1), (1, 1)\}\). For the computations we use FreeFem++ [10].

6.1. Smooth problem. The convergence results for test problem i) with all three choices of the pressure approximation space are illustrated in Table 1 and 2. We observe the optimal convergence as predicted by Theorem 5.6 and the superconvergence of Proposition 5.8.

6.2. Backstep channel problem. The convergence results for test problem ii) with all three choices of the pressure approximation space is illustrated in Table 3. Since the exact solution is not known only the ”non-conforming” error-quantities are given. Observe that Proposition 5.8 is still valid, in contrast to Theorem 5.6 since \( \Omega \) is non-convex, and since \( f = 0 \) the quantity \( \| \tilde{\eta}^{-\frac{1}{2}}[u_h]_\Omega \|_F \) is for any choice of pressure space equal to zero (machine precision).

Acknowledgements

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References


Table 3. Backstep channel problem: Different error quantities of the numerical solution for all three choices of the pressure approximation space with respect to the mesh size $h$. The quantities in the brackets correspond to the convergence rates and $0^*$ corresponds to zero in machine precision.

<table>
<thead>
<tr>
<th>$Q_h$</th>
<th>$h$</th>
<th>$|\nabla \cdot (u - u_h)|_\infty$</th>
<th>$|\tilde{h} - \tilde{h}^1_u|_\infty$</th>
<th>$|\tilde{h} - \tilde{h}^1_u|_\sigma$</th>
<th>$|\tilde{h} - \tilde{h}^1_u|_F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{h,c}$</td>
<td>0.25</td>
<td>6.63E-01</td>
<td>7.95E-01</td>
<td>1.03E+00</td>
<td>0*</td>
</tr>
<tr>
<td></td>
<td>0.125</td>
<td>3.99E-01 (0.73)</td>
<td>4.39E-01 (0.86)</td>
<td>6.00E-01 (0.78)</td>
<td>0*</td>
</tr>
<tr>
<td></td>
<td>0.0625</td>
<td>2.05E-01 (0.96)</td>
<td>2.54E-01 (0.79)</td>
<td>3.27E-01 (0.88)</td>
<td>0*</td>
</tr>
<tr>
<td></td>
<td>0.03125</td>
<td>1.19E-01 (0.78)</td>
<td>1.27E-01 (1.01)</td>
<td>1.73E-01 (0.91)</td>
<td>0*</td>
</tr>
<tr>
<td>$Q_{h,d}$</td>
<td>0.25</td>
<td>0*</td>
<td>9.78E-01</td>
<td>1.21E+00</td>
<td>2.61E+00</td>
</tr>
<tr>
<td></td>
<td>0.125</td>
<td>0*</td>
<td>6.70E-01 (0.54)</td>
<td>8.32E-01 (0.54)</td>
<td>1.86E+00 (0.49)</td>
</tr>
<tr>
<td></td>
<td>0.0625</td>
<td>0*</td>
<td>3.98E-01 (0.75)</td>
<td>4.84E-01 (0.78)</td>
<td>1.03E+00 (0.85)</td>
</tr>
<tr>
<td></td>
<td>0.03125</td>
<td>0*</td>
<td>2.06E-01 (0.93)</td>
<td>2.54E-01 (0.93)</td>
<td>5.61E-01 (0.87)</td>
</tr>
<tr>
<td>$Q_{h,c} \oplus Q_{h,d}$</td>
<td>0.25</td>
<td>6.76E-01</td>
<td>5.91E-01</td>
<td>7.36E-01</td>
<td>1.82E+00</td>
</tr>
<tr>
<td></td>
<td>0.125</td>
<td>3.57E-01 (0.90)</td>
<td>2.88E-01 (1.04)</td>
<td>3.83E-01 (0.94)</td>
<td>1.09E+00 (0.73)</td>
</tr>
<tr>
<td></td>
<td>0.0625</td>
<td>1.87E-01 (0.93)</td>
<td>1.79E-01 (0.68)</td>
<td>2.19E-01 (0.81)</td>
<td>5.57E-01 (0.97)</td>
</tr>
<tr>
<td></td>
<td>0.03125</td>
<td>9.28E-02 (1.01)</td>
<td>8.82E-02 (1.02)</td>
<td>1.17E-01 (0.90)</td>
<td>3.07E-01 (0.86)</td>
</tr>
</tbody>
</table>