HIGH-ORDER MULTISCALE FINITE ELEMENT METHODS FOR ELLIPTIC PROBLEMS

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Abstract. In this paper, a new high-order multiscale finite element method is developed for elliptic problems with highly oscillating coefficients. The method is inspired by the multiscale finite element method developed in [3], but a more explicit multiscale finite element space is constructed. The approximation space is nonconforming when oversampling technique is used. We use a Petrov-Galerkin formulation suggested in [14] to simplify the implementation and to improve the accuracy. The method is natural for high-order finite element methods used with advantage to solve the coarse grained problem. We prove optimal error estimates in the case of periodically oscillating coefficients and support the findings by various numerical experiments.

1. Introduction. The development of numerical methods for problems with highly oscillating coefficients is an increasingly active field of research. To overcome the computational cost of resolving the fine scale, multiscale finite element methods (MsFEM) have been developed in [12, 13, 11, 7, 14, 10]. Accuracy is achieved by solving a fine scale problem locally. These solutions are used to build the multiscale finite element basis to capture the small scale information of the leading order differential operator. Alternatives to this approach for multiscale methods are several, for example, the multiscale variational method [15] and the heterogeneous multiscale method (HMM) [1] and additional methods are discussed in [4, 19, 6]. In this work, however, we focus on techniques based on multiscale finite element formulations.

Originally, MsFEM was proposed for linear finite elements, see [12, 13]. For many applications, e.g., elliptic problems with singular forcing or non-convex domains, and wave equations, high-order finite elements are known to be advantageous in terms of accuracy and efficiency, in particular for large problems. Allaire and Brizzi [3] generalized the original approach to enable the use of high-order elements by local harmonic coordinates. This method uses a composite rule to change the local coordinates. Inspired by this work, we propose in this work a new high-order accurate multiscale finite element method. However, in contrast to the previous work, we do not use a composite rule but approach the development in a more explicit way. Additionally, we also solve local oscillating functions by high-order finite elements, as is necessary in many cases to preserve the accuracy. This overall approach proves itself to be cheaper to implement. To further improve the accuracy and simplify the implementation, an over-sampling technique and Petrov-Galerkin formulations following [11, 14] are used. The bases are nonconforming when the over-sampling technique is used. Similar to the method proposed in [3], the analysis is restricted to the periodic case but the method is derived and applied to the general non-periodic cases. Note that the analysis in [14] assume that the local problem is solved exactly in $H^1$. In this paper, we improve these results to the more practical situation in which we assume that the local problem is solved by some high-order finite elements as in [3]. New discussion of the harmonic coordinate in the multiscale method can be found in [19] where the globally defined harmonic coordinate is solved. Other high order methods for generalized finite element methods are discussed in [17].

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The rest of the paper is organized as follows. The formulations of the model elliptic problem and the motivation for the new multiscale finite element method are discussed in Sec. 2. The new high-order multiscale finite element method is introduced in Sec 3. In Sec. 4, convergence result are proved for the periodic case. Implementations and numerical experiments are discussed in Sec. 5 and a few final remarks and outlook for continued work is discussed in Sec. 6.

2. Model problem and motivations. Let Ω be a bounded polygonal domain in \( \mathbb{R}^n \) and consider the elliptic model problem:

\[
\begin{cases}
-\nabla \cdot (A^\epsilon \nabla u^\epsilon) = f & \text{in } \Omega, \\
u^\epsilon = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(2.1)

The matrix \( A^\epsilon \in L^\infty(\Omega, \mathcal{M}_{\alpha_A, \gamma_A}) \), where \( \mathcal{M}_{\alpha_A, \gamma_A} \) is the set of uniformly positive definite matrices with uniformly positive definite inverse, that is, for any \( \xi \in \mathbb{R}^d \),

\[
\|\xi\|_{\ell^2} = 1, 0 < \alpha_A \leq \xi^T A^\epsilon \xi \leq \gamma_A^{-1}.
\]

In the periodic case we have

\[
A^\epsilon(x) = A(x/\epsilon),
\]

where \( y \mapsto A(y) \) is a \( Y \)-periodic function with \( Y = (0, 1)^n \).

Let \( \chi_i, i = 1, \cdots, n \) be the solution to the cell problem

\[
\begin{cases}
-\text{div}_y (A(y) \nabla_y \chi_i) = \text{div}_y (A(y)e_i) & \text{in } Y, \\
y \mapsto x_i(y) & \text{\( Y \)-periodic}.
\end{cases}
\]

(2.2)

The notation \( \langle \cdot \rangle_Y \) is used to denote the mean of a function in domain \( Y \):

\[
\langle f \rangle_Y = \frac{1}{Y} \int_Y f(x) dx.
\]

From classical homogenization theory, see e.g., \([5, 9, 2]\), we have the following approximation results

\[
u^\epsilon(x) \approx u^*(x) + \epsilon \sum_{i=1}^n \chi_i \left( \frac{x}{\epsilon} \right) \frac{\partial u^*}{\partial x_i}(x),
\]

(2.3)

and

\[
\nabla u^\epsilon(x) \approx \nabla u^*(x) + \epsilon \sum_{i=1}^n (\nabla_y \chi_i) \left( \frac{x}{\epsilon} \right) \frac{\partial u^*}{\partial x_i}(x).
\]

(2.4)

Here \( u^* \) is the solution of the homogenization problem:

\[
\begin{cases}
-\nabla \cdot (A^* \nabla u^*) = f & \text{in } \Omega, \\
u^* = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(2.5)

where \( A^* \) is a constant homogenized matrix given by the explicit formula

\[
A^*e_i = \int_Y A(y)(e_i + \nabla_y \chi_i) dy \quad \text{with } e_i \text{ be the } i\text{-th canonical basis of } \mathbb{R}^n.
\]

Formula (2.3) looks like a first-order Taylor expansion of the map,
Based on this observation, [3] introduces a multiscale finite element basis using the composition rule,

$$
\Phi^{\epsilon,h}(x) = \Phi^h \left( \frac{\hat{w}^{\epsilon,h}(x)}{\epsilon} \right),
$$

where $\Phi^h$ is a standard finite element basis function on a coarse mesh $h > \epsilon$ and $\hat{w}^{\epsilon,h}$ is a numerical solution of a local multiscale problem on an element $K$ or an oversampled domain $S$ containing $K$ with finite element spaces defined on local meshes on $K$ or $S$. For the periodic case, $w^\epsilon$ is a numerical approximation of $\epsilon \chi \left( \frac{x}{\epsilon} \right) - x$.

Taking a second look at (2.3), ignoring the apparent connection to a Taylor expansion or composite rule, we propose the following multiscale finite element basis functions:

$$
\Phi^{\epsilon,h}(x) = \Phi^h + \left( \frac{\hat{w}^{\epsilon,h}(x)}{\epsilon} - x \right) \cdot \nabla \Phi^h.
$$

This new multiscale finite element basis will clearly lead to a non-conforming element method when oversampling is used. However, as we shall discuss, this will not be an issue either for the analysis or the implementation. When both $\Phi^h$ and $\hat{w}^{\epsilon,h}$ are of high order in order to maintain the accuracy, the new formulation is cheaper to implement in the sense that its polynomial degree is lower.

### 3. Multiscale finite element methods.

In this section, we introduce and derive the details of the multiscale finite element method for the general non-periodic case.

For simplicity of the presentation, we consider only triangular elements. Let $T_h = \{K\}$ be a finite element partition of the domain $\Omega$. Assume that the triangulation $T_h$ is regular and quasi-uniform with size $h$. Let $P_k(K)$ be the space of polynomials of degree $k$ on element $K$.

Without confusion, we will use $v = (v_1, \ldots, v_n)$ to denote an $n$-dimensional vector. Then $\nabla v$ is a matrix, and we denote it by $[\nabla v]$ to emphasize it is a matrix.

#### 3.1. Local element problems.

Let us introduce the following local problem:

$$
\begin{cases}
- \nabla \cdot (A^\epsilon \nabla \hat{w}^{\epsilon,K}_i) = - \nabla \cdot (A^*_K \nabla x_i) & \text{in } K, \\
\hat{w}^{\epsilon,K}_i = x_i & \text{on } \partial K.
\end{cases}
$$

(3.1)

If the homogenized matrix $A^*$ is a constant matrix, or $A^*$ is piecewise smooth, and the interface of discontinuity is aligned with the mesh, when the size of $K$ is small enough (when it is piecewise constant, $K$ needs not be too small), we can take $A^*_K$ as a local approximation of $A^*$ to be a constant matrix in $K$. In this case, $\nabla \cdot (A^*_K \nabla x_i) = 0$ in $K$.

Thus, we introduce the simplified local problem: For each $K \in T_h$, define $\hat{w}^{\epsilon,K}_i$, $i = 1, \ldots, n$, as the solution of

$$
\begin{cases}
- \nabla \cdot (A^\epsilon \nabla \hat{w}^{\epsilon,K}_i) = 0 & \text{in } K, \\
\hat{w}^{\epsilon,K}_i = x_i & \text{on } \partial K.
\end{cases}
$$

(3.2)
We will use (3.1) if a better non-constant approximation of $A^*$ is known. The vector $\hat{w}^\varepsilon$ is defined as $\hat{w}^\varepsilon = (\hat{w}_{i1}, \ldots, \hat{w}_{in}) \in H^1(\Omega)^n$, where $\hat{w}_i^\varepsilon \in H^1(\Omega)$ with $\hat{w}_i^\varepsilon = \hat{w}_i^{\varepsilon,K}$ for each $K \in T_h$.

For each $K \in T$, a quasi-uniform fine mesh $T_{h'}(K)$ with element size $h'$ is introduced. Define

$$W_{h'}(K) = \{ w \in C^0(K) : w|_T \in P_{h'}(T), \forall T \in T_{h'}(K) \},$$

and let $w^{\varepsilon,K}_i$ be the $P_{h'}$ finite element approximation of $\hat{w}_i^{\varepsilon,K}$ in (3.2) using mesh $T_{h'}^K$. Find $w^{\varepsilon,K}_i \in W_{h'}(K)$, $w^{\varepsilon,K}_i = x_i$ on $\partial K$, such that

$$\begin{align*}
(A^* \nabla \hat{w}_i^{\varepsilon,K}, \nabla v) &= 0, \quad v \in W_{h'}(K) \cap H^1_0(K).
\end{align*}$$

Define $w^{\varepsilon,h} = (w_{i1}^{\varepsilon,h}, \ldots, w_{in}^{\varepsilon,h}) \in H^1(\Omega)^n$, where $w_{i1}^{\varepsilon,h} \in H^1(\Omega)$ with $w_{i1}^{\varepsilon,h} = w_i^{\varepsilon,K}$ for each $K \in T_h$.

### 3.1.1. Oversampled local problem

In order to remove the resonance effect [10] associated with the lack of correct boundary conditions on the local problem, we shall use an oversampling method on a domain $S \supset K$. Define $\hat{w}_i^{\varepsilon,S}$, $i = 1, \ldots, n$, as the solution of

$$\begin{cases}
-\nabla \cdot (A^* \nabla \hat{w}_i^{\varepsilon,S}) &= -\nabla \cdot (A^*_S \nabla x_i) \quad \text{in } S, \\
\hat{w}_i^{\varepsilon,S} &= x_i \quad \text{on } \partial S.
\end{cases} \tag{3.4}$$

When the size of $S$ is small enough, and there is no discontinuity of $A^*$ inside $S$, we take $A^*_S$, a local approximation of $A^*$, to be a constant matrix in $S$. In this case, the right-hand side of (3.4) is 0:

$$\begin{cases}
-\nabla \cdot (A^* \nabla \hat{w}_i^{\varepsilon,S}) &= 0 \quad \text{in } S, \\
\hat{w}_i^{\varepsilon,S} &= x_i \quad \text{on } \partial S.
\end{cases} \tag{3.5}$$

We define $\hat{w}_i^{\varepsilon,K} = \hat{w}_i^{\varepsilon,S}|_K$, $i = 1, \ldots, n$, and define $\hat{w}^\varepsilon$ accordingly. In general, $\hat{w}^\varepsilon$ obtained from the oversampling method is not in $H^1(\Omega)^n$.

For a polygonal domain $S \supset K$, define a quasi-uniform fine mesh $T_{h'}(S)$ with element size $h'$. The edges of the mesh are aligned with the sides of $K$. Define

$$W_{h'}(S) = \{ w \in C^0(S) : w|_T \in P_{h'}(T), \forall T \in T_{h'}(S) \},$$

and let $w_i^{\varepsilon,S}$ be the $P_{h'}$ finite element approximation of $\hat{w}_i^{\varepsilon,S}$ of (3.5) using mesh $T_{h'}(S)$. Find $w_i^{\varepsilon,S} \in W_{h'}(S)$, $w_i^{\varepsilon,S} = x_i$ on $\partial S$, such that

$$\begin{align*}
(A^* \nabla \hat{w}_i^{\varepsilon,S}, \nabla v) &= 0, \quad \forall v \in W_{h'}(S) \cap H^1_0(S).
\end{align*} \tag{3.6}$$

With $w_i^{\varepsilon,K} = w_i^{\varepsilon,S}|_K$, we can define $w_i^{\varepsilon,h}$ accordingly. Like $\hat{w}^\varepsilon$, in general, $w_i^{\varepsilon,h}$ is not an $H^1$ vector function.

As is generally practice, in the analysis of this paper and in our numerical tests, we will use the zero right-hand side versions of the local problems. We also assume that the local problem is solved by a large enough oversampling domain $S$. 
3.1.2. Homogenization results for the local problem. It is known [5] that
the solution to (3.5) has the following structure:
\[
\hat{w}_i^{\epsilon,S} = \hat{w}_i^{0,S} + \epsilon \chi(x/\epsilon) \cdot \nabla \hat{w}_i^{0,S} + \epsilon \theta_i^{\epsilon,S}.
\]
where \(\hat{w}_i^{0,S} \in H^2(S)\) is the solution of the homogenized equation
\[
-\nabla \cdot (A^* \nabla \hat{w}_i^{0,S}) = 0 \quad \text{in} \quad S,
\]
with boundary condition \(\hat{w}_i^{0,S} = x_i\) on \(\partial S\). One easily shows that \(\hat{w}_i^{0,S} = x_i\) on the whole element \(S\). Thus
\[
\hat{w}_i^{\epsilon,S} = x_i + \epsilon \chi(x/\epsilon)_i + \epsilon \theta_i^{\epsilon,S}.
\]
For a non-oversampled method, \(K = S\), while for the oversampling method, we assume that for \(\theta_i^{\epsilon,K} = (\theta_i^{\epsilon,S})|_K\):

**Assumption 3.1.** (Assumption 2.1 of [14]) The oversampling domain \(S\) is chosen such that for any element \(K\) in \(S\),
\[
\|\nabla \theta_i^{\epsilon,K}\|_{L^\infty(K)} \leq C \tag{3.7}
\]
where \(C\) is a constant independent of \(\epsilon\) and \(h\).

For the periodic problem, define
\[
\tilde{w}_i^\epsilon(x) = x_i + \epsilon \chi_i(x/\epsilon), i = 1, \cdots, n,
\]
and denote \(\tilde{w} = (\tilde{w}_1, \cdots, \tilde{w}_n)\).

Let \(\tilde{w}^{\epsilon,h}\) and \(w^{\epsilon,h}\) be the solutions of (3.2) and (3.6) respectively. We have the following estimates, see Theorem 4.1 of [3]:
\[
\|\nabla (\tilde{w}^\epsilon - \tilde{w}^{\epsilon,h})\|_{0,\Omega} \leq C \sqrt{\frac{\epsilon}{h}} \quad \text{and} \quad \|\nabla (\tilde{w}^{\epsilon,h} - w^{\epsilon,h})\|_{0,\Omega} \leq C \left(\frac{h'}{\epsilon}\right)^{k'}. \tag{3.8}
\]
If oversampling techniques are used, and the oversampled domain \(S\) satisfies Assumption 3.1, we have a the following result ([11, 14]):
\[
\|\nabla (\tilde{w}^\epsilon - \tilde{w}^{\epsilon,h})\|_{h,\Omega} \leq C_1 \sqrt{\epsilon} + C_2 \epsilon. \tag{3.9}
\]
where we recall that \(h'\) is the local cell size and \(k'\) is the local order of approximation.

**Remark 3.2.** By comparing these results with those in [11] and [14], an extra \(h\) term is found in those papers. This term originates in the error between two homogenized solutions. In our case, there is no such error in the homogenized solution since it is exactly \(x_i\), causing this term to vanish.

3.2. Multiscale finite element method. Denote the \(P_k\) conforming finite element space associated with the triangulation \(T\) by
\[
V_h = \{ v \in H^1_0(\Omega) : v|_K \in P_k : \forall K \in T_h \}.
\]
For a function \(v \in H^1(\Omega)\), introduce a map \(J^{\epsilon,h}\) such that
\[
J^{\epsilon,h} v|_K = (v + (w^{\epsilon,h} - x) \cdot \nabla v)|_K \quad \text{on each} \quad K \in T, \tag{3.10}
\]
where \( w^{\epsilon,h} \) is obtained by accurately solving the oversampled local problem (3.6) with large enough \( S \).

Let \( (\Phi^h)_{\ell=1,\cdots,N_h} \) denote a finite element basis of \( V_h \), where \( N_h = \dim V_h \) and define

\[
\Phi^{\epsilon,h}_{\ell} = J^{\epsilon,h} \Phi^h, \quad \ell = 1, \cdots, N_h.
\]

Then the new multiscale finite element space is defined as

\[
V^{\epsilon}_h = \text{span}\{\Phi^{\epsilon,h}_{\ell}\}_{\ell=1,\cdots,N_h}.
\]

Note that \( V^{\epsilon}_h \not\subset H^1_0(\Omega) \).

Now we can define a Petrov-Galerkin multiscale finite element method: Find \( u^{\epsilon}_h \in V^{\epsilon}_h \), such that

\[
a^{\epsilon}_h(u^{\epsilon}_h, v_h) = (f, v_h) \quad \forall v_h \in V_h,
\]

(3.11)

where

\[
a^{\epsilon}_h(u, v) = \sum_{K \in T} (A^\epsilon \nabla u, \nabla v)_K \quad \forall u \in V^{\epsilon}_h, v \in V_h
\]

Or, equivalently: find \( u_h \in V_h \), such that

\[
a_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h,
\]

(3.12)

where

\[
a_h(u, v) = \sum_{K \in T} (A^\epsilon (\nabla J^{\epsilon,h} u), \nabla v)_K, \quad \forall u, v \in V_h.
\]

The function \( u_h \) is an approximation to the homogenized solution \( u^* \) and \( u^{\epsilon}_h = J^{\epsilon,h} u_h \in V^{\epsilon}_h \) is the multiscale finite element approximation to \( u^\epsilon \).

On an element \( K \in T_h \), we have

\[
\nabla(J^{\epsilon,h} u_h) = [\nabla w^{\epsilon,h}] \nabla u_h + [\nabla \nabla u_h](w^{\epsilon,h} - x).
\]

(3.13)

Now define

\[
b_h(u_h, v_h) = \sum_{K \in T_h} (A^{\epsilon}_w \nabla u_h, \nabla v_h)_K, \quad \forall u_h, v_h \in V_h,
\]

(3.14)

with

\[
(A^{\epsilon}_w)_{ij} = \sum_{k=1}^n A^{\epsilon}_w \frac{\partial w^{\epsilon,h}}{\partial x_k} \quad \text{or} \quad A^{\epsilon}_w = A^{\epsilon}[\nabla w^{\epsilon,h}],
\]

and

\[
c_h(u_h, v_h) = \sum_{K \in T_h} (A^\epsilon[\nabla(\nabla u_h)](w^{\epsilon,h} - x), \nabla v_h)_K, \quad \forall u_h, v_h \in V_h.
\]

(3.15)

We then have

\[
a_h(u_h, v_h) = b_h(u_h, v_h) + c_h(u_h, v_h) \quad \forall u_h, v_h \in V_h.
\]

(3.16)

When \( k = 1 \), \( c_h(u_h, v_h) = 0 \).

Since some functions to be analyzed are not in \( H^1_0(\Omega) \), but only \( H^1(K) \) for \( K \in T_h \), let us define an equivalent broken \( H^1 \)-norm as

\[
\|v\|_{h,K} = \left( \sum_{K \in T_h} \|\nabla v\|^2_{0,K} \right)^{1/2}.
\]

(3.17)
4. Convergence for the periodic case. Let us analyze the multiscale method (3.11), or equivalently, (3.12) for the periodic case, e.g.,

\[
A^*(x) = A\left(\frac{x}{\epsilon}\right)
\]

where \(y \mapsto A(y)\) is a \(Y\)-periodic function with \(Y = (0,1)^n\).

We assume that the following inequality holds

\[
0 < h' < \epsilon < h < 1
\]

for the coarse mesh size \(h\), the period \(\epsilon\), and the local mesh size \(h'\).

4.1. Existence and uniqueness of multiscale finite element formulations. In the following we will establish some basic properties of the multiscale finite element formulation, beginning with

**Lemma 4.1.** When Assumption 3.1 is true, we have

\[
\|\nabla w^{\epsilon,h}\|_{L^\infty(K)} \leq C \quad \forall K \in T_h.
\]  

**Proof.** By the triangle inequality,

\[
\|\nabla w^{\epsilon,h}\|_{L^\infty(K)} \leq \|\nabla (w^{\epsilon,h} - \hat{w}^{\epsilon,h})\|_{L^\infty(K)} + \|\nabla (\hat{w}^{\epsilon} - \hat{w}^{\epsilon,h})\|_{L^\infty(K)} + \|\nabla \hat{w}^{\epsilon}\|_{L^\infty(K)}.
\]

We can bound the three terms as

\[
\|\nabla (w^{\epsilon,h} - \hat{w}^{\epsilon,h})\|_{L^\infty(K)} = \epsilon\|\nabla \theta^{\epsilon,K}\|_{L^\infty(K)},
\]

\[
\|\nabla (\hat{w}^{\epsilon} - \hat{w}^{\epsilon,h})\|_{L^\infty(K)} \leq C(h'/\epsilon)^{k'},
\]

\[
\|\nabla \hat{w}^{\epsilon}\|_{L^\infty(K)} = \epsilon\|\nabla \chi_i\|_{L^\infty(K)} \leq C.
\]

By Assumptions 3.1, the result follows. \(\Box\)

**Lemma 4.2.** When Assumption 3.1 is true, for \(u_h^\epsilon = J^{\epsilon,h}u_h\), we have

\[
\|u_h^\epsilon\|_{h,\Omega} \leq C\|\nabla u_h\|_{0,\Omega}.
\]  

**Proof.** It follows from Lemma 4.1 that

\[
\|u_h^\epsilon\|_{h,\Omega} \leq C \left( \sum_{K \in T_h} \|\nabla w^{\epsilon,h}\|_{0,K}^2 \right)^{1/2} \leq C \left( \sum_{K \in T_h} \|\nabla w^{\epsilon,h}\|_{L^\infty(K)}^2 \|\nabla u_h\|_{0,K}^2 \right)^{1/2} \leq C\|\nabla u_h\|_{0,\Omega},
\]

completing the result. \(\Box\)

By Lemma 4.2, we obtain the following result.

**Lemma 4.3.** When Assumption 3.1 is true, we have

\[
a_h^\epsilon(u_h^\epsilon, v_h) \leq C\|u_h^\epsilon\|_{h,\Omega}\|\nabla v_h\|_{0,\Omega} \quad \forall u_h^\epsilon \in V_h^\epsilon, v_h \in V_h
\]  

and

\[
a_h(u_h, v_h) \leq C\|\nabla u_h\|_{0,\Omega}\|\nabla v_h\|_{0,\Omega} \quad \forall u_h, v_h \in V_h
\]
Define $\tilde{A}_w^\epsilon$ as
\[
(\tilde{A}_w^\epsilon)_{ij} = \sum_{k=1}^{n} A_{ik}^\epsilon \frac{\partial \tilde{w}_j^\epsilon}{\partial x_k} = \sum_{k=1}^{n} A_{ik}^\epsilon \left( \delta_{kj} + \frac{\partial Y_j(y)}{\partial y_k} \right), \quad \text{or} \quad \tilde{A}_w^\epsilon = A^\epsilon [\nabla \tilde{w}^\epsilon].
\]

Matrix $\tilde{A}_w^\epsilon$ is divergence free and its average in $Y$ is the constant homogenized matrix $A^*$ [14]:
\[
\langle \tilde{A}_w^\epsilon \rangle_Y = A^* \quad \text{and} \quad \nabla \cdot \tilde{A}_w^\epsilon = 0. \quad (4.6)
\]

**Theorem 4.4.** Suppose $(h'/\epsilon)^{k'}$ is small enough and (4.1) holds. When Assumption 3.1 is true, there exists a constant $C > 0$, independent of $\epsilon$ and $h$, such that
\[
b_h(v_h, v_h) \geq C\|v_h\|_{1, \Omega}^2, \quad \forall v_h \in V_h \quad (4.7)
\]

and
\[
\sup_{v \in V_h} \frac{b_h(u_h^\epsilon, v)}{\|v\|_{1, \Omega}} \geq C\|u_h^\epsilon\|_{h, \Omega}, \quad \forall u_h^\epsilon \in V_h^\epsilon. \quad (4.8)
\]

where $u_h^\epsilon = J^{\epsilon, h} u_h$.

**Proof.** By a simple decomposition, we have
\[
b_h(v_h, v_h) = \sum_{K \in T_h} (A_w^\epsilon \nabla v_h, \nabla v_h)_K \quad = \sum_{K \in T_h} (\tilde{A}_w^\epsilon \nabla v_h, \nabla v_h)_K + \sum_{K \in T_h} ((A_w^\epsilon - \tilde{A}_w^\epsilon) \nabla v_h, \nabla v_h)_K. \quad (4.9)
\]

First, we get a bound for the term $\sum_{K \in T_h}(\tilde{A}_w^\epsilon \nabla v_h, \nabla v_h)_K$ using an argument similar to Lemma 3.2 of [14].

By standard homogenization theory, see e.g. [9, 16], there exist positive constants $C_1$ and $C_2$, such that
\[
C_1|\xi|^2 \leq \xi^t A^* \xi \leq C_2|\xi|^2, \quad \text{for any vector } \xi \in \mathbb{R}^n. \quad (4.10)
\]

Furthermore
\[
(\tilde{A}_w^\epsilon \nabla v_h, \nabla v_h)_K = (A^* \nabla v_h, \nabla v_h)_K + ((\tilde{A}_w^\epsilon - A^*) \nabla v_h, \nabla v_h)_K. \quad (4.11)
\]

Divide $K$ into
\[
K = \left( \bigcup_{Y_k \subset K} Y_k \right) \bigcup K',
\]

where $Y_k$ is a periodic cell of $A(\frac{x}{\epsilon})$, and $K'$ is the difference between $K$ and the union of all $Y_k$ in $K$. Since $\langle \tilde{A}_w^\epsilon - A^* \rangle_{Y_k} = 0$, for the second term on the right-hand side of (4.11)
\[
((\tilde{A}_w^\epsilon - A^*) \nabla v_h, \nabla v_h)_K = ((\tilde{A}_w^\epsilon - A^*) \nabla v_h, \nabla v_h)_{K'} + \sum_{Y_k \subset K} ((\tilde{A}_w^\epsilon - A^*) \nabla v_h, \nabla v_h)_{Y_k}. \quad (4.12)
\]
For the first term of the right-hand side of (4.12) we have
\[
((\tilde{A}_w^\epsilon - A^\epsilon) \nabla v_h, \nabla v_h)_K \leq \max ((\tilde{A}_w^\epsilon - A^\epsilon)) ||\nabla v_h||^2_{0,K'} \leq C(\frac{\epsilon}{h}) \max ((\tilde{A}_w^\epsilon - A^\epsilon)) ||\nabla v_h||^2_{0,K}.
\]
(4.13)

For the second term of the right-hand side of (4.12), recalling that radius of \(Y_k\) is \(\epsilon\),
we recover
\[
((\tilde{A}_w^\epsilon - A^\epsilon) \nabla v_h, \nabla v_h)_{Y_k} \leq \int_{Y_k} (\nabla v_h - c)(\tilde{A}_w^\epsilon - A^\epsilon)(\nabla v_h - c)dx
\leq C\epsilon^2 \max ((\tilde{A}_w^\epsilon - A^\epsilon)) ||\nabla v_h||^2_{0,Y_k}.
\]
So by inverse estimates,
\[
\sum_{Y_k \subset K} ((\tilde{A}_w^\epsilon - A^\epsilon) \nabla v_h, \nabla v_h)_{Y_k} \leq C\epsilon^2 \max ((\tilde{A}_w^\epsilon - A^\epsilon)) ||\nabla v_h||^2_{0,K}
\leq C(\epsilon/h)^2 \max ((\tilde{A}_w^\epsilon - A^\epsilon)) ||\nabla v_h||^2_{0,K}.
\]

Summering up, using that by (4.1), \(\frac{\epsilon}{h} < 1\), we have
\[
C_1 ||\nabla v_h||^2_{0,K} \leq \sum_{K \in T_h} (\tilde{A}_w^\epsilon \nabla v_h, \nabla v_h)_K \leq C_2 ||\nabla v_h||^2_{0,K}, \quad \forall v_h \in V_h, K \in T_h.
\]
(4.14)

Let us now consider the second term on the right-hand side of (4.9). On an element \(K\),
\[
\tilde{A}_w^\epsilon - A_w^\epsilon = A^\epsilon([\nabla \tilde{w}^\epsilon] - [\nabla \tilde{w}^{\epsilon,h}]) = A^\epsilon([\nabla \tilde{w}^\epsilon] - [\nabla \tilde{w}^{\epsilon,h}]) + A^\epsilon([\nabla \tilde{w}^{\epsilon,h}]),
\]
The two terms on the right-hand side of (4.15) can be bounded by
\[
||A^\epsilon([\nabla \tilde{w}^\epsilon] - [\nabla \tilde{w}^{\epsilon,h}])||_{L^\infty(K)} \leq \epsilon ||A^\epsilon \nabla \theta^{\epsilon,K}||_{L^\infty(K)} \leq \epsilon C ||\nabla \theta^{\epsilon,K}||_{L^\infty(K)}.
\]
and
\[
||A^\epsilon([\nabla \tilde{w}^{\epsilon,h}])||_{L^\infty(K)} \leq \epsilon (h')^k |\tilde{w}^{\epsilon,h}|_{W^{\epsilon,k+1}(K)} \leq \epsilon (h'/\epsilon)^k.
\]

Hence
\[
\|\tilde{A}_w^\epsilon - A_w^\epsilon\|_{L^\infty(K)} \leq C(\epsilon + (h'/\epsilon)^k).
\]
(4.16)

Therefore, if Assumption 3.1 is true and \(\epsilon/h\) is bounded, the second term on the right-hand side of (4.9) can be bounded by the first term of the right-hand side of (4.9), provided \((h'/\epsilon)^k\) is small enough. This proves (4.7).

By Lemma 4.2, with \(v_h = u_h\)
\[
b_h(u_h,u_h) \geq C||\nabla u_h||^2_{0,\Omega} \geq C||\nabla u_h||_{1,\Omega}^2 \geq C||u_h||^2_{h,\Omega},
\]
completing the proof of (4.8).

**Remark 4.5.** For the non-oversampling case \(S = K\), we may still assume that
the coercivity results for the above Lemma is true for small enough \(\epsilon\). Naturally, the constant will be smaller than in the oversampling case, but we may still assume it is bounded away from zero. The inf-sup condition is much worse in the non-oversampling case and may blow up, since the bound \(||u_h||_{h,\Omega} \leq C||u_h||_{1,\Omega}^2\) may not be valid.
Theorem 4.6. With the same assumption as in Theorem 4.4, there exists a constant $C > 0$, such that
\[ a_h(u_h, u_h) \geq C \| u_h \|_{1, \Omega}^2, \quad \forall u_h \in V_h, \quad (4.17) \]
and
\[ \sup_{v \in V_h} \frac{a_h'(u_h', v)}{\| v \|_{1, \Omega}} \geq C \| u_h' \|_{h, \Omega}, \quad \forall u_h' \in V_h'. \quad (4.18) \]

Proof. For $k = 1$, $a_h$ and $b_h$ are equivalent so we only consider the case of $k \geq 2$. Note that
\[ \| u^{\epsilon, h} - x \|_{L^\infty(K)} \leq \| u^{\epsilon, h} - x \|_{L^\infty(K)} + \| u^{\epsilon, h} - \tilde{w}^\epsilon \|_{L^\infty(K)} + \| \tilde{w}^\epsilon - x \|_{L^\infty(K)} \]
\[ \leq \epsilon \| \theta^{\epsilon, K} \|_{L^\infty(K)} + C \left( \frac{h'}{h} \right)^{k' + 1} + C \epsilon \leq C \left( \epsilon + \left( \frac{h'}{h} \right)^{k' + 1} \right). \quad (4.19) \]

By the above bound and an inverse estimate, we recover
\[ c_h(u_h, v_h) = \sum_{K \in T_h} (A'[\nabla u_h])(w^{\epsilon, h} - x, \nabla v_h)_K \]
\[ \leq C \sum_{K \in T} \| w^{\epsilon, h} - x \|_{L^\infty(K)} \frac{1}{h} \| \nabla u_h \|_{0, K} \| \nabla v_h \|_{0, K} \]
\[ \leq C \left( \frac{h'}{h} \right)^{k' + 1} \| \nabla u_h \|_{0, K} \| \nabla v_h \|_{0, K}. \quad (4.20) \]

The results follow from Theorem 4.4. \( \Box \)

Remark 4.7. It is tempting to use the formulation based on the bilinear form $b$ only. Suppose $u_{h,a}$ is the solution of (3.12) and $u_{h,b} \in V_h$ is the solution of
\[ b_h(u_{h,b}, v_h) = (f, v_h) \quad \forall v_h \in V_h. \]
Let $z_h = u_{h,a} - u_{h,b}$, then
\[ C \| z_h \|_{1, \Omega}^2 \leq b_h(z_h, z_h) = -c_h(u_{h,a}, z_h) \leq C \left( \frac{h'}{h} \right)^{k' + 1} \| \nabla u_{h, a} \|_{1, \Omega} \| z_h \|_{1, \Omega}. \]

For $k \geq 2$, the difference between the two solutions may as bad as of order $\frac{\epsilon + \left( \frac{h'}{h} \right)^{k' + 1}}{h}$. Hence, the method based on the $b$ form alone is not recommended.

4.2. Convergence proof of multiscale finite element methods. Let us first recall a slightly altered standard result

Lemma 4.8. (Generalized Cea’s Lemma) Let $u^\epsilon$ be the solution of (2.1) and $u_h^\epsilon$ the solution of (3.11). Taking the assumptions of Theorem 4.4 to be true, there exists a constant $C$ such that
\[ \| u^\epsilon - u_h^\epsilon \|_{h, \Omega} \leq C \inf_{v_h^\epsilon \in V_h^\epsilon} \| u^\epsilon - v_h^\epsilon \|_{h, \Omega}. \quad (4.21) \]
Proof. Note that the following Galerkin orthogonality holds as
\[ a_h(u^\epsilon, v_h) = 0, \quad \forall v_h \in V_h. \]
Then by a standard argument, see e.g. Lemma 3.7 of [8], we have the desired result.
\[ \square \]

Remark 4.9. The constant \( C \) in the above Lemma depends on the inf-sup constant so we must ensure that the size of the oversampling domain \( S \) is big enough.

For a sufficiently smooth function \( v \) with meaningful nodal values, define \( \Pi_h \) to be the standard \( V_h \)-interpolation operator,
\[ \Pi_h v(x) = \sum_{\ell=1}^{N_h} v(n_\ell) \Phi^h_\ell(x) \in V_h, \]
where \( n_\ell \) is the node associated with the \( P_k \) finite elements. The operator \( \Pi_h^\epsilon \) is defined as
\[ \Pi_h^\epsilon v = J^\epsilon,h(\Pi_h v) = \sum_{\ell=1}^{N_h} v(n_\ell) \Phi^{\epsilon,h}_\ell(x) \in V_h^\epsilon. \]

Theorem 4.10. Let \( u^\epsilon \) be the solution of (2.1). Assume \( u^\epsilon_h \) is the solution of the Petrov-Galerkin formulation (3.11) with the over-sampling domain \( S \) satisfying Assumption 3.1. Then the following error estimate holds:
\[ \| u^\epsilon - u^\epsilon_h \|_{h,\Omega} \leq C \left( h^k \| f \|_0 + \sqrt{\epsilon} + \epsilon + \left( \frac{h'}{\epsilon} \right)^{k'} \right). \] (4.22)

Proof. Let \( u^\ast \) be the exact solution of the homogenized problem (2.5) and choose \( v_h^\ast = \Pi_h^\epsilon u^\ast \). Then we have
\[ \| u^\epsilon - u_h^\ast \|_{h,\Omega} \leq C \| u^\epsilon - \Pi_h^\epsilon u^\ast \|_{h,\Omega}. \] (4.23)

Define two operators \( J^\ast \) and \( \tilde{J}^\ast \), as
\[ J^\ast v|_K = (v + (\tilde{w}^\ast - x) \cdot \nabla v)|_K, \quad \tilde{J}^\ast v|_K = (v + (\tilde{w}^{\epsilon,h} - x) \cdot \nabla v)|_K \]
on each \( K \in T_h \).
(4.24)

We bound (4.23) as
\[ \| u^\epsilon - \Pi_h^\epsilon u^\ast \|_{h,\Omega} \leq \| u^\epsilon - J^\ast u^\ast \|_{h,\Omega} + \| J^\ast (u^\ast - \Pi_h u^\ast) \|_{h,\Omega} + \| (J^\ast - \tilde{J}^\ast,h) \Pi_h u^\ast \|_{h,\Omega} + \| (\tilde{J}^\ast,h - J^\ast,h) \Pi_h u^\ast \|_{h,\Omega}. \] (4.25)

The first term on the right-hand side of (4.25) is bounded by \( C \sqrt{\epsilon} \) using Lemma 2.13 of [3].

By the definition of the broken \( H^1 \)-norm, the second term of the right-hand side of (4.25) is bounded as
\[ \| J^\ast (u^\ast - \Pi_h u^\ast) \|_{h,\Omega} = \left( \sum_{K \in T_h} \| \nabla (J^\ast|_K (u^\ast - \Pi_h u^\ast)) \|_{0,K}^2 \right)^{1/2}. \] (4.26)
On each element $K \in T_h$, by standard interpolation results of $P_k$ finite elements,
\[
\|\nabla (J'(u^* - \Pi_h u^*))\|_{0,K} \leq \|\nabla J'\|_{L^\infty(K)} \|\nabla (u^* - \Pi_h u^*)\|_{0,K} \\
\leq C h^k \|\nabla J'\|_{L^\infty(K)} \|\nabla (u^* - \Pi_h u^*)\|_{k+1,K}
\]
The third term of in the right-hand side of (4.25) is bounded by
\[
\|((J' - \tilde{J}'^{'\epsilon,h})\Pi_h u^*)_{h,\Omega} \leq \|\nabla \Pi_h u^*\|_{L^\infty(\Omega)}, \nabla h(\tilde{u}^\epsilon - \tilde{w}^\epsilon,h)\|_{0,\Omega}.
\]
and the fourth term of in the right-hand side of (4.25) is bounded by
\[
\|((\tilde{J}'^{'\epsilon,h} - J'^{'\epsilon,h})\Pi_h u^*)_{h,\Omega} \leq \|\nabla \Pi_h u^*\|_{L^\infty(\Omega)}\|\nabla h(w^{'\epsilon,h} - \tilde{w}^{'\epsilon,h})\|_{0,\Omega}.
\]
By (3.9), this proves the theorem. □

4.3. Estimates on the homogenized solutions. In the following we prove an estimate for the homogenized solution in the $H^1$-norm similar to Theorem 3.4 of [14].

THEOREM 4.11. Assume that $u^*$ belongs to $H^3(\Omega) \cap W^{2,\infty}(\Omega)$ and $\partial \Omega$ is $C^{0,1}$. Under Assumption 3.1, we have the following estimate for the solution $u_h$ of (3.12):
\[
\|u_h - u^*\|_{1,\Omega} \leq C_1 \epsilon + C_2 h^k + C_3 \epsilon |\ln h|^{1/2} + C_4 \left(\frac{h'^k}{\epsilon}\right)^{k'^k}.
\]

Proof. The triangle inequality yields
\[
\|u_h - u^*\|_{1,\Omega} \leq \|u_h - \Pi_h u^*\|_{1,\Omega} + \|\Pi_h u^* - u^*\|_{1,\Omega}.
\]
Denote $u_h - \Pi_h u^*$ by $v$, then
\[
C \|u_h - \Pi_h u^*\|^2 = a_h(u_h - \Pi_h u^*, v) = a_h(u_h - u^*, v) + a_h(u^*, \Pi_h u^*, v)
\]
The term $a_h(u^* - \Pi_h u^*, v)$ is handled by continuity of $a_h$ and the approximation property of $\Pi_h$.
Since
\[
a_h(u_h, v) = (f, v_h) = (A^*\nabla u^*, \nabla v) \quad \forall \, v_h \in V_h,
\]
for the term $a_h(u_h - u^*, v)$, we have
\[
a_h(u_h - u^*, v) = a_h(u_h, v) - a_h(u^*, v) = (A^*\nabla u^*, \nabla v) - a_h(u^*, v) = I_1 + I_2,
\]
where
\[
I_1 = (A^*\nabla u^*, \nabla v) - (\tilde{A}_w^*\nabla u^*, \nabla v),
\]
and
\[
I_2 = (\tilde{A}_w^*\nabla u^*, \nabla v) - \sum_{K \in T_h} (A^*\nabla (J'^{'\epsilon,h} u^*), \nabla v)_K.
\]
From the analysis of Theorem 3.4 in [14], we recall
\[
I_1 \leq \left( C_1 \epsilon + C_2 h^k + C_3 \epsilon |\ln h|^{1/2}\right) \|v\|_{1,\Omega}.
\]
The order of $h$ is $k$ instead of 1 as in Theorem 3.4 of [14] since we consider polynomials of degree $k$.

Now, consider the restriction of $I_2$ to an element $K$,

$$(\tilde{A}_w^c \nabla u^*, - A^c \nabla (J^{c,h}w^*), \nabla v)_K = I_3 + I_4,$$

with

$$I_3 = ((\tilde{A}_w^c - A_w^c)^c \nabla u^*, \nabla v)_K \leq C(\epsilon + (h'/\epsilon)^k')\|\nabla u^*_0\|\|\nabla v\|_{0,K}$$

and

$$I_4 = -A^c [\nabla v]^c (w^{c,h} - x), \nabla v)_K \leq C(\epsilon + (h'/\epsilon)^{k' + 1})\|\nabla u^*_0\|\|\nabla v\|_{0,K}.$$ Estimates (4.16) and (4.19) are used in the derivation of the bounds of $I_3$ and $I_4$. Summing up all the bounds yields the final estimate and completes the proof.

4.4. $L^2$ estimates for the multiscale finite element solutions. In the following, we derive two bounds for the $L^2$-norm estimates of $u^*$, the solution of (2.1), and $u_h^*$, the solution of (3.11). This is done under the assumptions of Theorem 4.4.

**Theorem 4.12.** Assume $u^*$ belongs to $H^3(\Omega) \cap W^{2,\infty}(\Omega)$ and $\partial \Omega$ is $C^{0,1}$. Under Assumption 3.1, we have the following estimate for the solution $u_h^*$ of (3.11):

$$\|u_h^* - u^*\|_{0,\Omega} \leq C\left(\epsilon + h^k + \epsilon |\ln h|^{1/2} + \left(\frac{h'}{\epsilon}\right)^{k'}\right).$$

**Proof.** The proof we have follows that of Theorem 3.5 of [14]. By Theorem 4.11 and triangle inequalities,

$$\|u_h^* - u^*\|_{0,\Omega} \leq \|u_h - u^*\|_{0,\Omega} + \|u_h^* - u^*\|_{0,\Omega}$$

$$\leq \|u_h - u^*\|_{0,\Omega} + \|((w^{c,h} - x)\nabla u_h\|_{0,\Omega} + C\epsilon$$

$$\leq \|u_h - u^*\|_{1,\Omega} + C(\epsilon + (h'/\epsilon)^{k' + 1})$$

$$\leq C\left(\epsilon + h^k + \epsilon |\ln h|^{1/2} + \left(\frac{h'}{\epsilon}\right)^{k'}\right).$$

Another error estimate in the $L^2$-norm is given by the standard Aubin-Nitsche trick and (4.22). The proof is skipped.

**Theorem 4.13.** Under Assumption 3.1, we have the following estimate for the solution $u_h^*$ of (3.11):

$$\|u_h^* - u^*\|_{0,\Omega} \leq C\left(h^{k+1}\|f\|_0 + \sqrt{\epsilon} + \epsilon + \left(\frac{h'}{\epsilon}\right)^{k'}\right).$$

**Remark 4.14.** We have only established the results for the Petrov-Galerkin formulation with oversampling. If a standard Galerkin formulation is used, the multiscale finite element methods with the new basis functions is non-conforming, and the optimal convergence results similar to that in [11] can be shown. If oversampling is not used and standard Galerkin method is applied, we can still recover results similar to those of [3]. However, in both cases, these results are inferior to the ones obtained above.
5. Numerical Tests. We present in the following a number of computational tests to validate the analysis presented above. The main goal is to show that the analysis is sharp and to illustrate the interplay between the different parameters in the high-order multiscale finite element method.

We consider the numerical experiment in [3] with a smooth scalar conductivity tensor:

\[ A^\epsilon = a(x/\epsilon)I \]

where

\[ a(x/\epsilon) = \frac{1}{(2 + P \sin(2\pi x/\epsilon))(2 + P \sin(2\pi y/\epsilon))} \]

where \( P = 1.8 \) and \( f = 1 \). All experiments are done on the unit square domain \([0, 1] \times [0, 1] \), uniformly triangulated with a coarse mesh of size \( h \). Each coarse element is furthermore uniformly triangulated with the fine mesh of size \( h' < \epsilon/10 \) to recover the local solution.

In Figure 5.1, we plot \( w_1^{\epsilon,h}(x/\epsilon) - x \) and \( w_2^{\epsilon,h}(y/\epsilon) - y \) for \( \epsilon = 0.01 \) on one triangular element (solved by \( P_2 \) element). The oscillatory feature of the oscillatory function \( w^{\epsilon,h} \) is clearly observed.

5.1. Convergence of the homogenized solution. We first compare the numerical homogenized solution with the approximate homogenized solution. In this case, the homogenized conductivity is known exactly as \( A^* = 1/(2(4 - P^2)^{1/2}) \) [3].

In Figure 5.2 we show the convergence of the \( L^2 \) and the \( H^1 \) error of the homogenized solution \( u_{th} \), where \( N \) is the number of degrees of freedom in each dimension. We see the optimal order of convergence in \( H^1 \)-norm before the error caused by \( \epsilon \) begins to dominates. For the \( P_2 \) approximation, we also clearly observe the influence of oversampling on the convergence in Figure 5.2.

5.2. Convergence of the multiscale solution. To explore the convergence of the multiscale solution, we repeat the numerical experiments above but now compare the reference solution computed by a first order finite element obtained using a very fine mesh. In Figure 5.3, we show the convergence of the \( L^2 \) and the \( H^1 \) error of the multiscale solution \( u^{ms} \) for \( \epsilon = 0.01 \), where \( N \) is the number of degrees of freedom in each dimension. In the \( P_1 \) case we observe first order convergence in the \( H^1 \) error.

For the \( P_2 \) case, approximate 3-rd order convergence in the \( L^2 \) norm is observed when the oversampling approach is used. In Figure 5.4, we also show the influence of
Concluding remarks. In this paper, we develop a new high-order multiscale finite element method for elliptic problems with highly oscillating coefficients. A more explicit multiscale finite element space is constructed inspired by Allaire and Brizzi’s method [3]. Optimal error estimates are proved in the case of periodically oscillating coefficients.

The method developed in this paper is being applied by the authors to Helmholtz equations with oscillating coefficients, where high-order methods are naturally required.

Due to the nature of similarity of local problems, we are also developing reduced basis multiscale finite element methods for the methods developed in [3] and this paper to reduce the computation cost.
**Fig. 5.4.** Error $u - u_{h}^{\text{ms}}$ by $P_2$ approximation in 2D with oversampling for $\epsilon = 0.01, 0.02$

**Fig. 5.5.** Comparison of the error $u - u_{h}^{\text{ms}}$ using a 2nd order Galerkin (RG) and Petrov-Galerkin (PG) approximations with oversampling approach in 2D for $\epsilon = 0.01$.

Note that the oversampling technique is very expensive computationally, to reduce the cost, the authors are presently exploring an adaptive methods for the oversampling problem based on local a posteriori error estimates.
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