Modeling Magma Dynamics with a Mixed Fourier Collocation - Discontinuous Galerkin Method

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Abstract. A high-order discretization consisting of a tensor product of the Fourier collocation and discontinuous Galerkin methods is presented for numerical modeling of magma dynamics. We solve an advection-reaction-compaction system, consisting of two hyperbolic equations and one elliptic equation. The high-order solution basis allows for accurate and efficient representation of wave-like behavior that is predicted from linear theory. New insights into the processes of melt generation and segregation are revealed from time-dependent simulations of the model system.

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1 Introduction

The generation and segregation of melt from the earth’s interior is a fundamental, but only partially understood, process. Melt is generated by adiabatic decompressional melting during upwelling in the Earth’s upper mantle. Segregation of melt involves two-phase flow in which low viscosity melt percolates through a much more viscous solid matrix [3]. During its upward migration, melt generated in the deeper part of the upwelling mantle will interact both thermally and chemically with the overlying mantle. Precise understanding of the evolution of melt is constrained by physical evidence based upon field observations at ophiolites, where sections of the Earth’s mantle have been uplifted and exposed at the surface. The key observation from these outcrops is the local depletion of a soluble mineral, orthopyroxene (opx). This observation has been interpreted as the localization of porous melt flow into high porosity channels that have lead

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to the depletion of the soluble mineral. The main objective of the present study is to develop a high-order numerical model that can be used to study the interaction between the melt and solid mantle during melt migration. Unlike previous studies we explicitly track the evolution of the soluble mineral (opx) to understand the formation of the observed opx-depletions and their spatial and temporal relation to localized melt flow in regions of high permeability.

A fundamental geologic observation that gives insight into the existing theory of melt transport is that the chemical composition of erupted basalt is not in equilibrium with residual mantle at low pressure, particularly at diverging plate boundaries. Additionally, it is observed that the mantle is chemically and lithologically heterogeneous. To preserve the geochemical signature developed at depth, melt must rise from depths of at least 30 km to the surface without extensive re-equilibration with the surrounding mantle. One mechanism for this involves reactive infiltration instability, where highly porous regions form due to a feedback between melt flow and dissolution of a soluble mineral in the solid. It has been suggested that high-permeability dunite channels act as conduits for focused flow, where melt may efficiently segregate from its source region while still maintaining its geochemical signature at depth. For further discussion we refer to [9] and the references therein.

Previous numerical study [12] utilizing a low-order finite difference scheme demonstrates localization of the melt flow into high-porosity channels. However, this work does not explicitly account for the soluble mineral opx, whose presence is integral to the formation of dunite channels, nor does it consider the effects of upwelling or a bulk viscosity dependent upon the porosity. Further numerical study incorporating upwelling of the mantle is presented in [11], although this work does not discuss the numerical model used. In this paper, a high-order accurate numerical model is presented following the physical model presented in [6], which includes upwelling, a porosity-dependent bulk viscosity, and an additional equation to track the opx abundance. We assume a formulation of local chemical equilibrium. Linear analysis of our system predicts the emergence of wave-like features; in addition to the balancing between advection, reaction and compaction, these features present a formidable challenge in numerical modeling. We address this challenge by developing a high-order numerical method that provides accurate resolution at a reasonable cost. Numerical simulations confirm the existence of porosity waves, and also demonstrate that melt channels are not necessarily spatially or temporally correlated with dunite channels.

2 Simulation Equations

The simplified governing equations for our model system and the non-dimensionalization have been introduced in [1, 6]. The system comprises two hyperbolic equations for the fluid and opx fractions, $\phi_f$ and $\phi_{opx}$, respectively, and an elliptic equation for the over-pressure in the fluid, $p$:
\[
\frac{\partial \phi_f}{\partial t} = -\frac{\gamma}{\delta} \frac{\partial \phi_f}{\partial z} + p \phi_f - \Gamma_{\text{opx}} 
\] \quad (2.1)

\[
\frac{\partial \phi_{\text{opx}}}{\partial t} = -\frac{\gamma}{\delta} \frac{\partial \phi_{\text{opx}}}{\partial z} + \eta \Gamma_{\text{opx}} 
\] \quad (2.2)

\[
-\nabla \cdot \left[ \delta \phi_f \nabla p \right] + \delta \phi_f = -\frac{\partial \phi_f^3}{\partial z}. 
\] \quad (2.3)

The system is closed with Darcy’s law which yields the fluid velocity \( \mathbf{V}_f \) relative to the uniform solid upwelling via the algebraic equation

\[
\phi_f \mathbf{V}_f - \gamma \phi_f \mathbf{n}_z = -\phi_f^2 (\delta \nabla p - \mathbf{n}_z). 
\] \quad (2.4)

Here, we have introduced \( \mathbf{n}_z \), the unit vector in the vertical dimension. The rate of mass transfer, \( \Gamma_{\text{opx}} \), is proportional to the Darcy flux and is computed algebraically as

\[
\Gamma_{\text{opx}} = -\phi_f \left( \mathbf{V}_f \cdot \mathbf{n}_z \right) \frac{1}{1 - \delta z} \mathcal{I}_{\phi_{\text{opx}}}
\] 

\[
= -\frac{1}{1 - \delta z} \left( \gamma \phi_f - \delta \phi_f^3 \frac{\partial p}{\partial z} + \phi_f^3 \right) \mathcal{I}_{\phi_{\text{opx}}}. 
\] \quad (2.5)

We have introduced the indicator function \( \mathcal{I}_{\phi_{\text{opx}}} \), which returns 1 if \( \phi_{\text{opx}} \) is positive and zero otherwise. In this way, mass transfer exists only if the soluble mineral opx is present.

We assume periodicity in the horizontal dimension, whereas boundary conditions in the vertical dimension are given as

\[
\phi_f(x,0,t) = 1, \quad \phi_{\text{opx}}(x,0,t) = 1, \\
p(x,0,t) = -1 - \gamma \quad \frac{\partial p}{\partial z}(x,z_{\text{max}},t) = 0. 
\] \quad (2.6)

The initial-boundary value problem (2.1)-(2.6) is governed by the dimensionless parameters \( \delta, \gamma, \) and \( \eta \). \( \delta \) is the solubility gradient of the equilibrium concentration of the soluble mineral in the fluid, while the upwelling rate \( \gamma \) is the ratio of the characteristic solid to fluid velocity. Unless otherwise noted, we choose standard values for these parameters to be \( 10^{-2} \). The mantle fertility rate is \( \eta \), representing the ratio of fluid to opx at inflow, and is taken to be 0.266.

Assuming a positive opx fraction, mass transfer remains present, and one-dimensional analysis [6] of the system (2.1)-(2.6) yields the approximate steady states

\[
\bar{\phi}_f = \left( 1 + q_0 \frac{\delta z}{1 - \delta z} \right)^{1/3} \\
\bar{\phi}_{\text{opx}} = 1 - \frac{\eta q_0 \delta z}{\gamma} \frac{1 - \delta z}{1 - \delta z} \\
\bar{p} = - \frac{q_0}{(1 - \delta z)^2}. 
\] \quad (2.7)
where $q_0 = 1 + \gamma$. Solving for $\phi_{opx} = 0$, the location $z^*$ of opx exhaustion is

$$z^* = \frac{\gamma / \delta}{\eta q_0 + \gamma}$$  \hspace{1cm} (2.8)

As reaction is everywhere zero above this location, $z^*$ is a natural upper bound on domain height. To allow for the possible presence of opx at the top of the computational box we typically select a domain height of 75% of this value,

$$z_{\text{max}} = 0.75 z^* \approx 2.69$$  \hspace{1cm} (2.9)

3 Numerical Discretization

To solve the system (2.1)-(2.6) we employ a nodal discontinuous Galerkin (DG) scheme [8] in the vertical dimension and a Fourier collocation (FC) scheme [7], also known as a pseudospectral scheme, in the horizontal dimension. Time is discretized with an explicit, fourth-order Runge-Kutta (RK) scheme. The rationale behind these choices are as follows. The high-order nature of the spatial discretization provides the necessary accuracy to efficiently resolve the dynamics of the problem, which involves a balance of advection, reaction, and compaction. Analysis in [6] predicts the emergence of wave-like features, for which high-order methods are well-suited to resolve. The assumption of periodic boundary conditions in the horizontal direction is easily satisfied by the FC scheme, where solutions are represented by periodic basis functions. An explicit method is sufficient for time stepping and simple to implement, requiring only function evaluations of the nonlinear right-hand-sides to (2.1)-(2.2). Unlike previous numerical studies [12], our model assumes local chemical equilibrium and thus explicit time stepping is not restricted by a large reaction rate constant. The terms limiting time integration therefore are the advection and reaction terms in (2.1) and are typically of comparable magnitude.

Proceeding by a method-of-lines approach, the spatial operations in the hyperbolic equations (2.1)-(2.2) are discretized to produce a system of differential-algebraic equations that are a function of time only. Time integration is performed with the five-stage, fourth-order RK method as detailed in [5]. Each inner RK stage is concluded by discretizing the elliptic equation (2.3) with the updated value of $\phi_f$, and solving the resulting linear system with an iterative method.

3.1 Selecting a mesh

Before detailing the numerical schemes we first introduce a mesh

$$\left(x_i, z_j^k\right) \in X \times Z.$$  \hspace{1cm} (3.1)

An even number $M$ of equidistant horizontal points are chosen,

$$X = \{0 = x_0, x_1, \ldots, x_{M-1}\}, \quad x_i = \frac{i}{M} x_{\text{max}}.$$  \hspace{1cm} (3.2)
These grid points yield the explicitly defined interpolating polynomials \( l_{i}^{FS} \),
\[
    l_{i}^{FS}(x) = \frac{1}{M} \sin \left( \frac{\pi M}{x_{\text{max}}} (x-x_{i}) \right) \cot \left( \frac{\pi}{x_{\text{max}}} (x-x_{i}) \right),
\]  
(3.3)
as well as the discrete differentiation operator [7]
\[
    (D^{x})_{ij} = \left. \frac{d}{dx} l_{i}^{FS}(x) \right|_{x_{j}} = \left\{ \begin{array}{cc}
    (-1)^{i+j} \frac{\pi}{x_{\text{max}}} (x_{j}-x_{i}), & i \neq j \\
    0, & i = j.
    \end{array} \right.
\]  
(3.4)

Defining the vertical grid points \( z_{k}^{j} \in \mathbb{Z} \) requires more effort. \( Z \) is comprised of \( K \) (interval) elements \( I^{k} \),
\[
    Z = \bigcup_{k=1}^{K} I^{k}, \quad I^{k} = \left\{ z_{0}^{k}, z_{1}^{k}, \ldots, z_{N}^{k} \right\},
\]  
(3.5)
where the value chosen for \( N \) determines the local order of approximation. For each element a set of grid points is introduced,
\[
    z_{N}^{k-1} = z_{0}^{k} < z_{1}^{k} < \cdots < z_{K}^{k} = z_{0}^{k+1}, \quad z_{N}^{0} = 0, z_{N}^{K+1} = z_{\text{max}}.
\]  
(3.6)
It is noted that at cell boundaries the solution is doubly-defined. Although in theory any choice of grid points may suffice, in practice one would prefer a set that is optimal for computation. In particular, for higher \( N \) the choice of equidistant grid points leads to poorly conditioned operators. Instead, we use the Gauss-Lobatto points which lead to well-conditioned operators. While these points cannot be defined explicitly, they can be computed in a pre-processing step; see [8] for an efficient implementation in MATLAB.

Similar to the notation (3.3)-(3.4) introduced for the FC scheme, we define interpolating polynomials \( l_{j}^{DG,k} \) on the grid points (3.6) and a corresponding differentiation matrix
\[
    (D^{z})_{m,n} = \left. \frac{d l_{j}^{DG,k}}{dz} \right|_{z=z_{m}^{k}}, \quad 0 \leq m,n \leq N.
\]  
(3.7)

With the definition by (3.1), the solution variables \( u \in \{ \phi_{f}, \phi_{opx}, p \} \) are discretized as
\[
    u \approx u^{h} = \bigoplus_{k=1}^{K} u_{h}^{k}(x,z,t), \quad u_{h}^{k}(x,z,t) = \sum_{i=0}^{M-1} \sum_{j=0}^{N} u_{h}^{k}(x_{i},z_{j}) l_{j}^{FS}(x) l_{i}^{DG,k}(z).
\]  
(3.8)

On the rectangular patch \( X \times I^{k} \), representing the entire domain width and vertical section corresponding to DG element \( k \), solutions are represented as the two-dimensional array
\[
    u_{h}^{k}(x,z,t) = \begin{bmatrix}
        u_{0,0}^{k} & u_{0,1}^{k} & u_{0,M-1}^{k} \\
        u_{1,0}^{k} & u_{1,1}^{k} & \cdots & u_{1,M-1}^{k} \\
        \vdots & \vdots & \ddots & \vdots \\
        u_{N,0}^{k} & u_{N,1}^{k} & \cdots & u_{N,M-1}^{k}
    \end{bmatrix}
\]  
(3.9)
where \( u_{i,j}^{k} = u_{h}^{k}(x_{i},z_{j},t) \). Therefore, discrete \( z \) operators act on columns from the left while \( x \) operators act on rows from the right.
3.2 Full numerical method

Having defined a mesh, the full numerical method is detailed, beginning with the porosity evolution (2.1); the other equations quickly follow thereafter.

On the rectangular patch $X \times I^k$, the residual $R_h^k$ is formed by substituting the approximate solution(s)

$$u_h^k = \left\{ (\phi_f)^k_h, (\phi_{opx})^k_h, p^k_h \right\}$$

into (2.1),

$$R_h^k(x,z,t) = \frac{\partial (\phi_f)^k_h}{\partial t} + \gamma \frac{\partial (\phi_f)^k_h}{\partial z} - p^k_h (\phi_f)^k_h + (\Gamma_{opx})^k_h$$

(3.11)

where $(\Gamma_{opx})^k_h$ is $\Gamma_{opx}$ evaluated on the grid points.

It is how the residual (3.11) is chosen to vanish that distinguishes the two schemes. Horizontally, the collocation approach simply requires the residual to vanish on the grid points. That is,

$$For \ all \ (j,k): R_h^k(x_i,z_j^k,t) = 0, 1 \leq i \leq M. \hspace{1cm} (3.12)$$

Vertically, the Galerkin method requires the residual is $L^2$-orthogonal to the solution basis $V_N^k$ of polynomials of degree $N$ whose support lies in element $I^k$. The DG interpolating polynomials span this basis:

$$V_N^k = \left\{ l_{ij}^{DG,k} : 0 \leq j \leq N \right\}, k = 1,2,...,K. \hspace{1cm} (3.13)$$

This orthogonality condition is enforced by multiplying the residual by a test function from $V_N^k$ and integrating. For each horizontal point $x_i$, we require

$$\int_{\mu} R_h(x_i,z,t) l_{ij}^{DG,k}(z) dz = 0, 0 \leq j \leq N, 1 \leq k \leq K. \hspace{1cm} (3.14)$$

Integration by parts is performed twice on the flux terms in (3.14) for the strong DG form; suppressing $h$ subscripts we write

$$\int_{\mu} \frac{\partial \phi_f}{\partial z} l_{ij}^{DG,k} dz - \int_{\mu} \phi_f l_{ij}^{DG,k} \frac{\partial}{\partial z} l_{ij}^{DG,k} dz + \int_{\partial \mu} (\phi_f)^* l_{ij}^{DG,k} dz$$

$$\rightarrow \int_{\mu} \frac{\partial \phi_f}{\partial z} l_{ij}^{DG,k} dz + \int_{\partial \mu} \left( (\phi_f)^* - \phi_f \right) l_{ij}^{DG,k} dz. \hspace{1cm} (3.15)$$

There are no continuity requirements on the solution across element boundaries, from which the discontinuous label in the method name originates. Rather, information is transferred across elements via the numerical flux $(\phi_f)^*$, which approximates $\phi_f$ on element boundaries. This flux is a consistent, monotone, Lipschitz-continuous function of all its arguments.
Applying the FC scheme (3.12) and DG scheme (3.14)-(3.15), the semi-discrete form of the porosity evolution equation (2.1) is

\[
\frac{d\phi_k^f}{dt} = -\frac{\gamma}{\delta} D_{z,k} \phi_k^f \\
+ \frac{\gamma}{\delta} \left( M^k \right)^{-1} \left[ \left( \phi_k^k \right) - \left( \phi_k^k \right)^* \right]_{j=0}^{j=N} \\
+ P^k \phi_j^f - \Gamma_{opx}^k
\]

The local mass and differentiation matrices, \( M_k \) and \( D_{z,k} \), are defined as

\[
\left( M^k \right)_{m,n} = \int_{I_k} I_{DG,k}^m I_{DG,k}^n dz, \\
\left( D_{z,k} \right)_{m,n} = \frac{dI_{DG,k}^m}{dz} \bigg|_{z=t_k}
\]

and operate on solution columns via matrix-matrix products. The remaining products are computed by entry-wise multiplication. Using notation (3.9), the interpolant \( I_{DG,k}^j \) takes the value 1 on all entries in row \( j \) and 0 elsewhere. At the cell intersection \( I_k \cap I_{k+1} = [z^k, z^{k+1}] \), we use an upwinding numerical flux,

\[
\left( \phi_j^k \right)^* = \phi_j^k \left( z_N^k \right).
\]

The opx evolution equation (2.2) is discretized in a similar fashion,

\[
\frac{d\phi_{opx}^k}{dt} = -\frac{\gamma}{\delta} D_{z,k} \phi_{opx}^k \\
+ \frac{\gamma}{\delta} \left( M^k \right)^{-1} \left[ \left( \phi_{opx}^k \right) - \left( \phi_{opx}^k \right)^* \right]_{j=0}^{j=N} \\
+ \eta \Gamma_{opx}^k
\]

As before, the numerical flux is computed by upwinding. Note that evaluation of the reaction rate \( \Gamma_{opx}^k \) in (3.16) and (3.19) involves computing the derivative \( \partial p / \partial z \), for which we apply the DG derivative.

Lastly, the elliptic equation (2.3) is a modified Helmholtz equation, discretized by introducing the auxiliary variable \( q = \partial p / \partial z \) to reduce the equation into a system of first order equations. To accommodate the non-zero Dirichlet boundary condition \( p(z=0) = p_0 \), we substitute \( p = \hat{p} / \delta + p_0 \) so that \( \hat{p}(z=0) = 0 \), yielding a modified right hand side \( g = -\delta p_0 \phi_f - \partial \phi^{3}_{opx} / \partial z \) to (2.3). The local discretization becomes

\[
-\mathcal{M}^{z,k} \left[ \left( \phi_j^k \left( \hat{p}^k \right)^T \right) \left( \hat{D}^{x,k} \right)^T + \hat{D}^{z,k} \left( \hat{q}^k \right)^T - \hat{p}^k \phi_j^k \right] \\
+ \left[ I_{DG,k}^j \left( q^k - \left( q^k \right)^* \right) \right]_{j=0}^{j=N} = \mathcal{M}^{z,k} \delta^k, \\
q^k = \hat{D}^{z,k} \hat{p}^k - \left( \mathcal{M}^{z,k} \right)^{-1} \left[ I_{DG,k}^j \left( \hat{p}^k - \left( \hat{p}^k \right)^* \right) \right]_{j=0}^{j=N}.
\]

(3.20)
Horizontal derivatives are computed with the Fourier differentiation matrix $D^x$ as given in (3.4).

For the numerical fluxes $(\hat{p}^k)^*$ and $(q^k)^*$ we use a stabilized central flux [2], generally given on the intersection of cells $I_k \cap I_{k+1} = [z_N^k, z_0^{k+1}]$ as

\[
\begin{align*}
(\hat{p})^* &= \{\{\hat{p}\}\} \\
(q)^* &= \{\{q\}\} - \tau [\hat{p}]
\end{align*}
\] (3.21)

where $\{\{u\}\} = 1/2\left(u(z_N^k) + u(z_0^{k+1})\right)$ is the average solution at the cell interface while $|u| = u(z_0^{k+1}) - u(z_N^k)$ is the jump across the interface. $\tau$ is a positive number, whose role is to penalize jumps in $\hat{p}$. Different values of $\tau$ affect the accuracy of the scheme and conditioning of the linear system; greater discussion and analysis can be found in [8]. We select $\tau = 1$. Boundary conditions are enforced through the numerical flux by setting $\hat{p}(z_0^N) = -\hat{p}(z_1^0)$ and $q(z_0^{K+1}) = q(z^{K+1}_N)$.

Assembling (3.20) over all DG elements, we are left to solve a linear system of size $D$-by-$D$, where $D = (N+1) \times K \times M$. This system is nonsymmetric due to the permeability $\phi_3$. As a solver, we use a matrix-free implementation of the BiCGStab iterative method [10]. Solving the Helmholtz equation is the main bottleneck in computation. However, current multithreading capabilities available in MATLAB allow for reasonably fast transient simulations on desktop machines of size $D \sim 50,000$.

Finally, we note that with a nodal DG scheme, integrals are computed via quadrature on the $N+1$ Gauss-Lobatto grid points, which is exact for polynomials up to order $2N-1$. For higher-order, nonlinear terms that exceed this order, aliasing errors may lead to numerical instability. A standard, robust way to maintain stability is to post-process solutions at each time step with a filter (equivalent to adding artificial dissipation [8]). To this end, we apply an exponential filter on the porosity in both the vertical and horizontal dimensions. This in effect damps only the highest solution modes, and retains accuracy while preventing numerical instability. Implementation details for such a filter can be found in [7] and [8].

### 3.3 Convergence study

The efficiency of a high-order method is apparent when considering the relatively rapid rate of convergence. This is demonstrated in Fig. 1.

To verify the convergence of the FC scheme we conduct transient simulations of (2.1)-(2.5) with standard parameter choices on a domain width $x_{\text{max}}=1$ with the boundary condition $\phi_f(x,z=0,t) = 1 + 0.05 \exp \left[-100 (x - 0.5)^2\right]$ and take the vertical extension of this throughout the domain as an initial condition. This setup guarantees a smooth transient solution with energy in all representable horizontal wavenumbers. We conduct simulations with varying number of $M$ horizontal grid points up to a terminal time $t=0.5$, with
6 DG elements of order 10 in the vertical dimension. As an “exact” solution, we choose a well-resolved case of $M = 128$. In Fig. 1 it is seen that the $L^2$-error decays exponentially as a function of $M$, verifying spectral convergence of the scheme.

To verify the convergence of the DG scheme we take the same parameter and domain setup as in the previous case, and assume exact, steady solutions

$$\phi^e_f(x,z) = \exp(0.04z)(1 + 0.1\cos(2\pi x)), \quad (3.22)$$

$$p^e(x,z) = (p_0 + 0.1\cos(2\pi x))z\sin(0.75\pi z) + 0.1z\cos(2\pi x). \quad (3.23)$$

Substituting (3.22)-(3.23) into (2.1) and (2.3) yields a residual for each equation that is in turn added back to the equations, guaranteeing an exact solution as prescribed. We integrate this system to steady state with very small time steps. From general analysis of the DG method [8] we expect the error of the scheme to diminish as a polynomial function of the size of the element, $K^{-1}$. This is confirmed in Fig. 1, where it is seen that the $L^2$-error trends as $K^{-(N+1)}$. Moreover, exponential convergence is observed as a function of the DG order $N$.

4 Linear Stability Analysis of the Physical System

We will use a linear stability analysis of the non-dimensionalized equations (2.1)-(2.6) to verify the results of our discretization and to gain insight into the dynamical behavior of the system. In [6] this analysis is described in depth; the numerical discretization used in this analysis is discussed here. Observing that the opx fraction evolution equation (2.2) is decoupled from the melt evolution (2.1) and compaction rate (2.3) equations, we
may consider the simpler two-variable \( u \in \{ \phi_f, p \} \) system. Accordingly, we linearize each around the one-dimensional base states (2.7)

\[
u(x,z,t) = \bar{u} + \epsilon \bar{u} e^{ikt + \sigma t},
\]

(4.1)

where \( \epsilon \ll 1 \) is a small number, and \( k \in [0, \infty) \) is the horizontal perturbation wavenumber. We seek eigenpairs \((\sigma, \tilde{u}(z))\) representing the dominant growth rate and vertical solution mode, respectively. The boundary conditions on the perturbations are

\[
\tilde{\phi}_f(0) = \tilde{p}(0) = \frac{\partial \tilde{p}}{\partial z} \bigg|_{z=z_{\text{max}}} = 0
\]

(4.2)

Substituting (4.1) into (2.1), (2.3), and (2.5), and keeping only \( O(\epsilon) \) terms results in the linear system

\[
\begin{pmatrix}
A_{S\phi_f} & A_{Sp} \\
A_{P\phi_f} & A_{Pp}
\end{pmatrix}
\begin{pmatrix}
\tilde{\phi}_f \\
\tilde{p}
\end{pmatrix} = \sigma
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{\phi}_f \\
\tilde{p}
\end{pmatrix}
\]

(4.3)

with entries

\[
A_{S\phi_f} = -\frac{\gamma}{\delta} D + \bar{p} - \frac{\gamma - 3\hat{\phi}_f^2 (\delta \bar{p}' - 1)}{\delta z - 1}
\]

\[
A_{Sp} = \hat{\phi}_f - \frac{\delta \hat{\phi}_f^2}{1 - \delta z} D
\]

\[
A_{P\phi_f} = \delta \bar{p} - 3 \left[ \left( \delta \bar{p}' - 1 \right) \left( \hat{\phi}_f^2 D + \left( \hat{\phi}_f^2 \right)' \right) - \delta \hat{\phi}_f^3 \bar{p}'' \right]
\]

\[
A_{Pp} = \delta \left[ \hat{\phi}_f - 3 D \hat{\phi}_f \hat{\phi}_f^2 + (k^2 - D^2) \hat{\phi}_f^4 \right].
\]

(4.4)

The \( D \) operator is the (discontinuous Galerkin) derivative, comprised of the local differentiation matrix and numerical flux terms,

\[
D = D^{z,k} + \left( M_k \right)^{-1} \left[ \mathcal{M}^{DG,k} \left( u^k - (u^k)^* \right) \right]_{j=0}^N.
\]

(4.5)

Note that the numerical flux \((u^k)^*\) varies depending upon whether one is computing the derivative of \( \phi_f \) (upwinding) or \( p \) (central). Substituting \( \bar{p} = -(A_{Pp})^{-1} A_{P\phi_f} \hat{\phi}_f \) into the generalized eigenvalue problem (4.3) reduces the system to the standard eigenvalue problem

\[
\begin{pmatrix}
A_{S\phi_f} - A_{Sp} A_{Pp}^{-1} A_{P\phi_f}
\end{pmatrix} \hat{\phi}_f = \sigma \hat{\phi}_f.
\]

(4.6)

Eigenpairs \((\sigma, \hat{\phi}_f)\) are easily computed from the linear system (4.6) with standard methods.

The DG method shows its strength in two different ways when conducting the linear stability analysis (4.3)-(4.4). First, the high-order solution approximation allows for
accurate resolution of oscillatory eigenfunctions $\tilde{\phi}_f$. Second, the combination of element order/size $(N/K)$ refinement allows us to robustly compute eigenpairs with far fewer degrees of freedom than a low-order scheme would allow, thereby improving the conditioning of the left-hand-side operator in (4.6).

4.1 Two unstable regimes: channels and waves

In [6], stability of our model system is explored over $\gamma - \delta$ space, and it is observed that two distinct types of instability (i.e. Re($\sigma$) > 0) occur. We will refer to these as the “channel” and “wave” regimes, corresponding to the case of Im($\sigma$) = 0 and Im($\sigma$) $\neq$ 0, respectively. These names are motivated by the structure of the computed eigenfunction $\tilde{\phi}_f$ and the nature of the time-dependent behavior of the solution. Key observations from [6] include a boundary in $\gamma - \delta$ space separating these instabilities to be a function of the ratio $\gamma/\delta^2$, as well as the existence of a stable regime sandwiched between the two unstable regimes that expands with increasing $\eta$.

In Fig. 2 we examine results from the linear stability analysis with two different upwelling rates, $\gamma \in \{10^{-5}, 10^{-2}\}$. It is observed that with little upwelling, the growing solution modes are monotonic in $z$ and the dispersion relationship does not yield a preferred wavenumber. In this case, the computed eigenvalue is purely real, so small perturbations are expected to grow monotonically in time as well. However, in the case of moderately higher upwelling rate, the computed eigenfunctions are now oscillatory in $z$, and the eigenfunctions have a nonzero complex part. Consequentially, a wave-like variation in time is predicted. A peak in the dispersion relationship in the neighborhood of $k = 4.5$ indicates a preferred wavenumber for the given parameter settings. We will restrict ourselves to consider the wave regime case, as this value of $\gamma$ is more relevant to our physical setting.

4.2 Verification of numerical discretization

Beyond offering a first-order predictive glimpse of the nonlinear system, we shall also use that the linear stability analysis provides a mechanism to verify our discretization while the solution fields are relatively close to the steady states (2.7). This is accomplished in the following manner: selecting a horizontal wavenumber perturbation $k$, we compute the eigenfunction $\tilde{\phi}_f$ and evolve the initial condition

$$\phi_f(x,z,0) = \tilde{\phi}_f + 0.01\tilde{\phi}_f \cos(kx).$$

(4.7)

A horizontal domain length $x_{max} = 2 \times 2\pi/k$ is chosen to ensure resolution of the given wavenumber. With the data stored from each time step we compute the (numerical) growth rate and angular frequency, using recurrence relationships (see [4], Appendix A). These numerical values are then compared to the predicted values, Re($\sigma$) and Im($\sigma$), respectively. In Fig. 2 this comparison is demonstrated for the standard case $\gamma = 10^{-2}$ over wavenumbers $0 \leq k < 8$. Numerical growth and angular frequency rates are computed at
Figure 2: Results from linear stability analysis (4.3)-(4.4). Two upwelling rates, $\gamma \in \{10^{-5}, 10^{-2}\}$, are compared. With smaller upwelling, eigenfunctions are monotonic, eigenvalues are purely real, and there is no preferred wavenumber selection. With greater amounts of upwelling, eigenfunctions are oscillatory, eigenvalues are complex, and a peak in the dispersion relationship reveals a strong wavenumber selection. Transient simulations from our numerical model are verified to match the behavior predicted by linear analysis.
the terminal time \( t = 1 \). Our numerical discretization is very well matched to the predicted behavior.

Next, we employ the numerical scheme to verify the preferred wavenumber trend evidenced by the peak in the dispersion relationship from Fig. 2. This is accomplished by evolving the initial condition

\[
\phi_f(x, z, 0) = \tilde{\phi}_f + 0.1 \tilde{\phi}_{f, 2} \cos(2x) + 0.01 \tilde{\phi}_{f, 4} \cos(4x)
\]

where \( \tilde{\phi}_{f, 2} \) and \( \tilde{\phi}_{f, 4} \) are the computed eigenfunctions from the \( k = 2 \) and \( k = 4 \) horizontal wavenumber perturbations, respectively (displayed in Fig. 2). Whereas \( \tilde{\phi}_{f, 2} \) is seeded with ten times the perturbation magnitude of \( \tilde{\phi}_{f, 4} \) we expect from the linear stability analysis the former to decay and the latter to grow exponentially in time. This transition – from wavenumber 2 to 4 – is presented in Fig. 3. With a horizontal domain length \( x_{\text{max}} = 2\pi \), two wave crests are dominant in the initial \( \phi_f \) field. Stepping forward in time, we observe a transition to four wave crests by time \( t = 5 \).

As a means of quantifying this transition, we compute the expected value \( <k> \) of the wavenumber,

\[
<k> = \sum_k k \frac{\int_0^{x_{\text{max}}} |E_k|^2 \, dz}{\sum_k \int_0^{x_{\text{max}}} |E_k|^2 \, dz}
\]

where \( E_k \) are the discrete Fourier coefficients of the zero-mean quantity \( \phi_f - \bar{\phi}_f \) and are easily computable via the FFT. The quantity \( |E_k|^2 \) is a measure of energy in wavenumber \( k \), and the fractional quantity in (4.9) acts as a discrete probability density. Plotted in Fig. 3 as a function of time, \( <k> \) smoothly transitions from the initially seeded \( k = 2 \) to the preferred \( k = 4 \).

5 Results From Numerical Simulations

Linear stability analysis provides a foundation from which we may begin to explore non-linear behavior of our model, yielding locations in parameter space that are conducive to physical instability. Here we present a few results from transient simulations of our numerical scheme.

5.1 Simulating the effects of mantle heterogeneity with a fixed boundary perturbation in porosity

While the linear stability analysis presented in Sec. 4 measures the response of the system to small, internal perturbations, we also observe interesting dynamical behavior in response to altering the inflow porosity boundary condition. Consider the evolution of the IBVP (2.1)-(2.6) with initial values \( \phi_f = \tilde{\phi}_f, \phi_{opx} = \tilde{\phi}_{opx} \), and modifying the porosity boundary condition as

\[
\phi_f(x, 0, t) = 1 + A \exp \left[ -50 (x / x_{\text{max}} - 0.5)^2 \right], A > 0.
\]
Figure 3: Wavenumber selection. Initially most of the energy is focused in the stable $k=2$ mode, with a small amount in the unstable $k=4$ mode. As predicted by linear stability analysis, the former decays while the latter emerges as the dominant feature.
We select a domain width $x_{\text{max}} = 1$ so that the minimum resolvable wavenumber is $k = 2\pi$, placing us in a regime stable to internal perturbation (see Fig. 2). Therefore we expect a steady-state solution to exist.

Fig. 4 presents results with the sustained boundary perturbation in porosity (5.1) for increasing amplitudes $A$. Transient simulations are performed until a terminal time $t = 3$, at which time steady state has been reached. A number of observations can be made from these simulations. Although the governing equations admit wave-like instabilities, the fixed porosity boundary perturbation admits a distinctly channel-like feature (top row). Moreover, the channel of melt bifurcates as a consequence of opx exhaustion. It is seen that larger perturbation amplitudes deepen the depth of the dunite channel (bottom row). Compacting (i.e. low-porosity) boundary layers are observed outside high-porosity melt channels. Greater analysis of this phenomenon and discussion as to its geological consequences will be reported in future work.

5.2 Interaction between channels and waves

Linear analysis of the problem predicts porosity waves (Fig. 2), and we have observed that a sustained boundary perturbation in the inflow melt fraction induces a channel of melt that bifurcates upon exhaustion of the opx fraction (Fig. 4). Next we consider the superpositioning of these effects, by conducting numerical experiments in the wave regime with a fixed boundary perturbation.

In Fig. 5 we show results from a simulation with the fixed boundary perturbation (5.1), amplitude $A = 0.15$ and a domain width $x_{\text{max}} = 4\pi/4.4 \approx 2.86$. This suffices to guarantee a physical instability that is representable with our numerical scheme. Results are shown starting at time $t = 3$, from which the initial effects of the boundary perturbation have fully evolved. At $t = 50$, wave features are noticeable in the porosity field. These amplify with time, shown later at $t = 65$ and $t = 75$, eventually overwhelming the channel. The waves induce secondary high-porosity regions, locally increasing the reaction rate ($\Gamma_{\text{opx}} \sim \phi^3$, per Eq. (2.5)) and thus creating secondary dunite channels.

5.3 Dependence Upon Horizontal Domain Size

With the assumption of periodic boundary conditions, the domain length $x_{\text{max}}$ is the largest representable wavelength in the horizontal dimension; in other words, we are constrained in which wavenumbers we are able to represent. With $M$ grid points, the observable wavenumbers are quantized into units $2\pi n / x_{\text{max}}$, with integer $n \leq M/2$. This is clearly an issue to be recognizant of when performing long-time integration, where predictions from linear theory are not necessarily valid.

To demonstrate this we consider a series of numerical experiments with varying domain sizes $x_{\text{max}} = n \ast 2\pi / 4.4, n \in \{4,5,6\}$ and a noisy initial input with energy in all representable wavenumbers. Similar to Fig. 3, the preferred wavenumber is initially selected
Figure 4: Steady-state solutions with sustained inflow porosity perturbation as given in (5.1). Top: fluid fraction. Bottom: opx fraction. From left to right, perturbation amplitude $A = \{0.05, 0.1, 0.2, 0.4 \}$. 
Figure 5: Evolution of fluid fraction (top) and opx fraction (bottom) in unstable wave regime with fixed boundary perturbation (5.1). Interaction between porosity waves, inherent to the system, and a central channel, induced by the boundary condition, is observed. Outflow dunite (opx-free) channels are shown to vary in time as strong porosity waves emerge.
according to linear theory. However, for $n = 5$ we observe a transition at later times to the next representable wavenumber.

6 Summary

In this paper, a high-order numerical scheme has been presented for the equations of magma dynamics described by the physical model presented in [6]. This model includes a porosity-dependent bulk viscosity term, a solid upwelling term and a hyperbolic equation to track the fraction of the dissolvable mineral opx. The latter quantity is used as a level set function, where upon exhaustion the mass transfer rate is taken to zero.

Our numerical implementation consists of a Fourier collocation method in the horizontal direction and a discontinuous Galerkin method in the vertical direction. Numerical error estimates match their theoretically expected values. A high-order method is well-suited to resolve the wave-like dynamical features which are predicted by linear analysis. Our discretization is verified to match transient behavior predicted by linear stability analysis; a strong wavenumber selection is observed in the presence of moderate upwelling rate.

Numerical experiments reveal new insight into the full, nonlinear system. Mantle heterogeneity, represented by a perturbation in the inflow melt fraction, appears to be an agent of dunite channel formation. In the event of depleted opx fraction, melt channels bifurcate.

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A Algorithm for computing oscillatory behavior

In [4] a method utilizing simple recurrence relationships is given for computing the linear behavior of a signal

$$f_n \simeq \exp[(\gamma + i\omega)n\Delta t]. \quad (A.1)$$

The signal $f_n$ is sampled at equally spaced time intervals $\Delta t$. The recurrence relationships yield values for the real numbers $\gamma$ and $\omega$, the growth rate and frequency, respectively, as the signal asymptotically approaches its dominant mode.

We extend the method of [4] to extract the numerical growth rate and angular frequency from transient simulations seeded with the dominant eigenfunction (numerically computed for each $k$),

$$\phi_f(x,z,t=0) = \tilde{\phi}_f + 0.01\tilde{\phi}_f \cos(kx). \quad (A.2)$$
Figure 6: Expected wavenumber $<k>$ as a function of time for varying horizontal domain length $x_{\text{max}}$. The assumption of periodic boundary conditions constrains which wavenumbers we are able to represent.
A horizontal domain length $x_{\text{max}}$ is selected to ensure periodicity of wavenumber $k$. Solutions are then computed with our numerical scheme up to a terminal time $t = 1$, taking equal time steps. In order to reduce the two-dimensional $\phi_f$ field to a single “signal” data point $f_n$, at each time step we:

1. Subtract the base state $\bar{\phi}_f$ to achieve an (approximately) zero-mean solution
2. Extract the energy $E_k(z)$ in the $k$-mode at each vertical level by taking the FFT across the horizontal dimension
3. Integrate $E_k$ over $z$

References