Positivity-preserving and symmetry-preserving Lagrangian schemes for compressible Euler equations in cylindrical coordinates

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Abstract

For a Lagrangian scheme defined in the cylindrical coordinates, two important issues are whether the scheme can maintain spherical symmetry (symmetry-preserving) and whether the scheme can maintain positivity of density and internal energy (positivity-preserving). While there were previous results in the literature either for symmetry-preserving in the cylindrical coordinates or for positivity-preserving in cartesian coordinates, the design of a Lagrangian scheme in cylindrical coordinates, which is high order in one-dimension and at least second order in two-dimensions, and can maintain both spherical symmetry-preservation and positivity-preservation simultaneously, is challenging. In this paper we design such a Lagrangian scheme and provide numerical results to demonstrate its good behavior.

Keywords: Lagrangian method; cylindrical coordinates; symmetry-preserving; positivity-preserving; compressible flows

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1 Introduction

There are two different and typical frameworks to describe the motion of fluid flow, namely the Eulerian framework and the Lagrangian framework. When we mention the latter framework, we refer to the kinematic description which considers a time dependent reference frame that follows the fluid motion. Lagrangian methods are more suitable for problems involving interfaces between materials or free surfaces and are widely applied in many fields of multi-material flow simulations such as in astrophysics or in inertial confinement fusion (ICF).

In these applications, there often exist spherical-symmetric models such as sphere-shape capsules. When such models are simulated by Lagrangian methods in cylindrical coordinates, it is a critical and challenging issue to preserve the spherical symmetry in the cylindrical coordinate system which is distinct from that symmetry. For example, in the simulation of an implosion problem with strong compressions, the preservation of spherical symmetry is very important, since the small deviation from spherical symmetry due to numerical errors may be amplified by Rayleigh-Taylor or other instabilities which may potentially produce unpredictably large errors. Earlier strategies to design schemes in two-dimensional cylindrical coordinates to preserve spherical symmetry often sacrifice momentum and energy conservation, or at least momentum conservation. In [3], a cell-centered Lagrangian scheme was developed based on the control volume discretization. By discretizing the source term in the momentum equation compatibly, the scheme was designed to preserve one-dimensional spherical symmetry in a two-dimensional cylindrical geometry using an equal-angel-zoned grid without losing conservation. Based on the first order control volume scheme of Maire in [12], Cheng and Shu applied the methodology in [3] to obtain the spherical symmetry property. This modified scheme can keep several good properties, such as symmetry, conservation and the geometric conservation law (GCL). In order to get higher than first-order symmetry-preserving schemes, Cheng and Shu in [6] presented a second-order cell-centered Lagrangian scheme for solving Euler equations of compressible gas dynamics in cylindrical
coordinates. This scheme not only preserves symmetry but also preserves the conservation for mass, momentum and total energy as well as the GCL.

Another important issue in computational fluid dynamics is the positivity-preserving property. As in a conservative approximation to Euler equations where density, momentum and total energy are solved directly, the kinetic energy is computed from mass and momentum and then subtracted from the total energy to provide internal energy. Therefore, at high Mach numbers, the internal energy appears as a small difference of two large quantities, and is prone to large percentage errors. It may easily become negative numerically which may lead to nonlinear instability and a failure of the numerical scheme. To overcome this difficulty, many first order positivity-preserving Eulerian schemes were developed in earlier years, for instance, the classical Godunov scheme [8], Lax-Friedrichs scheme [16, 20], the modified HLLE scheme [8] and the HLLC scheme [1] and so on. Some of them are designed also up to second order accuracy. Recently, Zhang and Shu proposed a general framework of high-order positivity-preserving Eulerian schemes such as the Runge-Kutta discontinuous Galerkin (RKDG) methods and the weighted essentially non-oscillatory (WENO) finite volume schemes in [20, 21, 22].

Compared with Eulerian methods, positivity-preserving Lagrangian schemes are less investigated. The pioneering work on this issue includes the positivity-preserving Godunov-type scheme based on the modified HLL Riemann solver [14], and the positive and entropic schemes [9]. In [5], Cheng and Shu constructed high order positivity-preserving Lagrangian schemes in one- and in two-dimensional spaces by developing an HLLC Riemann solver and applying the Zhang-Shu positivity-preserving framework. More recently, cell-centered positivity-preserving Lagrangian schemes for compressible flows in both one-dimensional and two-dimensional spaces were presented by Vilar et al in [18, 19] relying on the two-state solver. We remark that these schemes are designed in the cartesian coordinates and for problems without source terms. For equations on non-cartesian coordinates and with source terms, positivity-preserving is more difficult to achieve. This is especially the case when
symmetry-preserving must also be taken into consideration.

In this paper, we will focus on designing high order cell-centered Lagrangian schemes which can achieve positivity-preserving and symmetry-preserving properties simultaneously. This is not a straightforward combination of the symmetry-preserving technique in [3, 6] and the positivity-preserving technique in [5, 18, 19], since the design of one technique must ensure that the other property is not lost. In the one-dimensional case, for the positivity-preserving property, we make an additional time step constraint by controlling the change rate of the control volume to achieve this goal with any definition of positive acoustic impedance, mainly following [5, 18, 19, 21]. For the extension to two-dimensions, the design and analysis are similar, however the positivity-preserving limiter must be carefully applied in order not to affect the spherical symmetry preservation when computed on an equal-angle-zoned grid. For this purpose, our scheme is based on the work of Cheng and Shu in [6] and makes a careful balance between the original symmetry-preserving framework and the new positivity-preserving modification, in order to make sure one does not affect the performance of the other. The final scheme thus has both symmetry-preserving and positivity-preserving properties, as well as the GCL and conservation properties.

The remainder of this paper is organized as follows: In Section 2, we first formulate the compressible Euler equations in cylindrical coordinates, describe the two-state Riemann solver, and then design the first order and high order positivity-preserving Lagrangian schemes in this one-dimensional case. In Section 3, we show how to extend the positivity-preserving technique to two-dimensional cylindrical coordinates without destroying the original spherical symmetry preservation. In Section 4, one- and two-dimensional numerical examples are given to verify the performance of our positivity-preserving and symmetry-preserving Lagrangian schemes. In Section 5, we will make some concluding remarks.
2 One-dimensional case

2.1 Governing equations

The Euler equation for gas dynamics in one-dimensional cylindrical coordinates can be given by the following integral form in the Lagrangian framework

\[
\begin{align*}
\frac{d}{dt} \int_{\Omega(t)} \rho r dr &= 0, \\
\frac{d}{dt} \int_{\Omega(t)} \rho u r dr &= -\int_{\Gamma(t)} p ndl + \int_{\Omega} p dr, \\
\frac{d}{dt} \int_{\Omega(t)} \rho E r dr &= -\int_{\Gamma(t)} p u \cdot ndl,
\end{align*}
\tag{2.1}
\]

where \( r > 0 \) denotes the radial direction, \( \rho \) is density, \( u \) is velocity, \( p \) is pressure and \( E \) is specific total energy, \( \Gamma(t) \) is the boundary of the control volume \( \Omega(t) \) and \( n \) denotes the unit outward normal to \( \Gamma(t) \). The system (2.1) presents the conservation of mass, momentum and total energy.

The set of equations is completed by the additional equation of state (EOS), which has the following general form

\[ p = p(\rho, e), \]

with the specific internal energy \( e = E - \frac{1}{2} u^2 \). The sound speed for the fluid flow is defined as \( a = \sqrt{\frac{p}{\rho}} \).

The geometric conservation law (GCL) means that the rate of change of a Lagrangian control volume should be computed consistently with the node motion, which can be formulated as

\[
\frac{d}{dt} \int_{\Omega(t)} dV = \int_{\Gamma(t)} u \cdot ndl. \tag{2.2}
\]

This property should also hold in the fully-discretized scheme.

2.2 First-order scheme

Let \( I_i = [r_{i-\frac{1}{2}}, r_{i+\frac{1}{2}}] \) be the cell, \( A_i = \int_{I_i} dr \) be the area of the cell and \( m_i = \int_{I_i} \rho r dr \) be the mass in the cell \( I_i \), which keeps a constant value during the time marching according
to the first equation in (2.1). Then we introduce the cell averaged value in the cell $I_i$ as $\bar{U}_i = (\bar{\rho}_i, \bar{\mathbf{u}}_i, \bar{E}_i)^T$, in which the averaged values of density, velocity and total energy are defined as follows

$$\bar{\rho}_i = \frac{1}{V_i} \int_{I_i} \rho r dr, \quad \bar{\mathbf{u}}_i = \frac{1}{m_i} \int_{I_i} \rho \mathbf{u}r dr, \quad \bar{E}_i = \frac{1}{m_i} \int_{I_i} \rho E r dr,$$

(2.3)

where $V_i = \int_{I_i} r dr$ denotes the volume of the cell obtained by rotating the cell $I_i$ around the origin of coordinate (without the $2\pi$ factor).

Based on these notations, we can rewrite the system in (2.1) in the following control volume formulation

$$\begin{align*}
\frac{d}{dt} \bar{\rho}_i &= \frac{m_i}{V_i}, \\
\frac{d}{dt} \bar{\mathbf{u}}_i &= -\int_{\partial I_i} p n r dl + \int_{I_i} p dr, \\
\frac{d}{dt} \bar{E}_i &= -\int_{\partial I_i} p \mathbf{u} \cdot n r dl.
\end{align*}$$

(2.4)

Notice that (2.4) is satisfied by the exact solution of the partial differential equations (PDEs) (2.1) and is not a scheme yet. However, when the point values on the right-hand side in (2.4) are approximated using the cell averages (2.3), it will become a scheme which evolves these cell averages as well as moves the mesh. Moreover, we can get the fully discrete finite volume Lagrangian scheme, using Euler forward time discretization to simplify the description

$$\begin{align*}
\bar{\rho}_i^{n+1} &= \frac{m_i}{V_i^n}, \\
\bar{\mathbf{u}}_i^{n+1} &= \bar{\mathbf{u}}_i^n - \frac{\Delta t^n}{m_i} (r_{i+\frac{1}{2}} p_{i+\frac{1}{2}}^* - r_{i-\frac{1}{2}} p_{i-\frac{1}{2}}^*) + \frac{\Delta t^n}{m_i} A_i P_s, \\
\bar{E}_i^{n+1} &= \bar{E}_i^n - \frac{\Delta t^n}{m_i} (r_{i+\frac{1}{2}} p_{i+\frac{1}{2}}^* u_{i+\frac{1}{2}}^* - r_{i-\frac{1}{2}} p_{i-\frac{1}{2}}^* u_{i-\frac{1}{2}}^*),
\end{align*}$$

(2.5)

where $P_s$ is the approximation of the source term, particularly in the first order case it can be taken as $\bar{p}_i^n$. The intercell values $p_{i+\frac{1}{2}}^*$ and $u_{i+\frac{1}{2}}^*$ are the pressure and velocity at the node $r_{i+\frac{1}{2}}$, respectively, obtained from an exact or approximate Riemann solver by giving the left and right states, which are $\bar{U}_i^n$ and $\bar{U}_{i+1}^n$ in the first order case. The scheme (2.5) is not complete without the time integration of the trajectory equation, which enables us to
advance in the time the grid position, the cell area and volume as

\[
\begin{cases}
r_i^{n+1} = r_i^n + \Delta t u_i^* + \frac{1}{2}, \\
A_i^{n+1} = r_i^{n+1} - r_i^{n+1}, \\
V_i^{n+1} = \frac{1}{2} A_i^{n+1} (r_i^{n+1} + r_i^{n+1})
\end{cases}
\]  

(2.6)

Notice that the definition of \( \rho_i^{n+1} \) by the first equation in (2.5) ensures the GCL property.

Thus the numerical scheme relies on the choice of the numerical flux, which is generally obtained by exactly or approximately solving the Riemann problem at the cell interface \( r_i^{n+1} \) with the given left and right states respectively. The resolution of such problems has been addressed in many references. In the following subsection, we will present an approximate Riemann solver, which is used widely in the Lagrangian framework and is generally referred to as the two-state Riemann solver.

### 2.3 Two-state Riemann solver

We present here an approximate Riemann solver associated with a simple Riemann problem [13, 18]. The data of the initial discontinuity are specified by \( U_L = (\rho_L, u_L, E_L)^\top \) on the left side and \( U_R = (\rho_R, u_R, E_R)^\top \) on the right side. At later time, the initial discontinuity is solved by a system of waves which consists of a contact discontinuity, a leftward wave and a rightward wave, with the left and right wavespeeds, \( \tilde{z}_L \) and \( \tilde{z}_R \) being some local approximations of the acoustic impedance. If we denote the left and right intermediate states as \( U_-^* \) and \( U_+^* \), the Rankine-Hugoniot relations enable us to fully determine the intermediate states

\[
\begin{align*}
\frac{1}{\rho_-} &= \frac{1}{\rho_L} + \frac{u_-^* - u_L}{z_L}, \\
\frac{1}{\rho_+} &= \frac{1}{\rho_R} + \frac{u_+^* - u_R}{z_R}, \\
\frac{1}{E_-} &= E_L - \frac{z_L}{p_L} u_-^* - \frac{p_L u_L}{z_L}, \\
\frac{1}{E_+} &= E_R + \frac{z_R}{p_R} u_+^* - \frac{p_R u_R}{z_R}.
\end{align*}
\]  

(2.7)

Besides, these R-H relations tell us the pressure and the velocity are continuous across a contact discontinuity, i.e. \( u_-^* = u_+^* = u^* \) and \( p_-^* = p_+^* = p^* \), so we obtain the associated numerical fluxes

\[
u^* = \frac{\tilde{z}_L u_L + \tilde{z}_R u_R}{\tilde{z}_L + \tilde{z}_R} - \frac{1}{\tilde{z}_L + \tilde{z}_R} (p_R - p_L) \]  

(2.8a)
which, by the construction of the solver itself, preserves contact discontinuities. In the particular case where \( \tilde{z}_L = \tilde{z}_R = \tilde{z} \), one recovers the one-state solver. Different choices on the wave speeds \( \tilde{z}_L \) and \( \tilde{z}_R \) are possible and yield different properties. The simplest one, and certainly the most widely used, is the acoustic approximation where the wave speeds are set to be the left and right acoustic impedances, i.e. \( \tilde{z}_L = \rho_L a_L \) and \( \tilde{z}_R = \rho_R a_R \). In this particular case, this two-state solver is nothing but the Godunov acoustic solver.

Relations in (2.8), derived as the solution of the two-state approximate Riemann problem, enable us to end the construction of the first-order finite volume scheme (2.5), by defining the numerical flux \( p^*_{i+\frac{1}{2}} \) and \( u^*_{i+\frac{1}{2}} \) at the intercell \( r_{i+\frac{1}{2}} \). In this definition, the left and right wavespeeds are denoted respectively by \( \tilde{z}_{i+\frac{1}{2}}^- \) and \( \tilde{z}_{i+\frac{1}{2}}^+ \).

### 2.4 First-order positivity-preserving scheme

For the desired property, we define the set of admissible states by

\[
G = \left\{ \mathbf{U} = \begin{pmatrix} \rho \\ u \\ E \end{pmatrix}, \quad \rho > 0 \text{ and } e = E - \frac{1}{2} |u|^2 > 0 \right\}. \tag{2.9}
\]

**Lemma 2.1.** The set of admissible states \( G \) is a convex set, referring to [5, 18, 21].

The scheme (2.5) is called positivity-preserving if \( \{ \mathbf{U}_i^n \in G, i = 1, \ldots, N \} \) implies \( \{ \mathbf{U}_i^{n+1} \in G, i = 1, \ldots, N \} \). For simplicity, we define \( \Delta r_i^n = r_{i+\frac{1}{2}}^n - r_{i-\frac{1}{2}}^n \). By adding and subtracting \( \frac{\Delta \rho_i^n}{m_i} \Delta r_i^n \bar{p}_i^n \) in the second equation of (2.5), the scheme (2.5) can be rewritten as

\[
\mathbf{U}_i^{n+1} = \begin{pmatrix} \frac{p_i^{n+1}}{u_i^{n+1}} \\ \frac{u_i^{n+1}}{E_i^{n+1}} \end{pmatrix} = \begin{pmatrix} \frac{m_i}{\bar{v}_i^n} - \frac{\Delta \rho_i^n}{m_i} (r_{i+\frac{1}{2}} p_{i+\frac{1}{2}}^* - r_{i-\frac{1}{2}} p_{i-\frac{1}{2}}^*) + \frac{\Delta \rho_i^n}{m_i} \Delta r_i^n \bar{p}_i^n + \frac{\Delta \rho_i^n}{m_i} (A_i P_s - \Delta r_i^n \bar{p}_i^n) \\ \frac{E_i^n - \frac{\Delta \rho_i^n}{m_i} (r_{i+\frac{1}{2}} p_{i+\frac{1}{2}}^* u_{i+\frac{1}{2}}^* - r_{i-\frac{1}{2}} p_{i-\frac{1}{2}}^* u_{i-\frac{1}{2}}^*)}{\bar{v}_i^n} \end{pmatrix}

= \frac{1}{2} \mathbf{H} + \frac{1}{2} \mathbf{W} \tag{2.10}
\]
where
\[
H = \begin{pmatrix} \frac{m_i V_{n+1}^i}{\bar{u}_i^n} - \frac{2\Delta t^n}{m_i} (r_{i+\frac{1}{2}}^i p_{i+\frac{1}{2}}^* - r_{i-\frac{1}{2}}^i p_{i-\frac{1}{2}}^*) + \frac{2\Delta t^n}{m_i} \Delta r_{n+1}^i p_{n+1}^i \\ \frac{E_{n+1}^i}{\bar{u}_i^n} - \frac{2\Delta t^n}{m_i} (r_{i+\frac{1}{2}}^i p_{i+\frac{1}{2}}^* u_{i+\frac{1}{2}}^* - r_{i-\frac{1}{2}}^i p_{i-\frac{1}{2}}^* u_{i-\frac{1}{2}}^*) \end{pmatrix},
\]
\[
W = \begin{pmatrix} \frac{m_i V_{n+1}^i}{\bar{u}_i^n} + \frac{2\Delta t^n}{m_i} (A_i P_s - \Delta r_{n+1}^i p_{n+1}^i) \\ \frac{E_{n+1}^i}{\bar{u}_i^n} \end{pmatrix}.
\]

(2.11)

It is obvious that \( \mathbf{U}_{n+1}^i \) is a convex combination of \( H \) and \( W \). To ensure that \( \mathbf{U}_{n+1}^i \) can preserve the positivity property, we could consider the sufficient condition that both \( H \) and \( W \) are positivity-preserving.

In [18, 19], the authors proposed two different approaches to achieve the positivity preservation. The first way is to modify the Dukowicz wavespeed \( \tilde{z}_{i+\frac{1}{2}} \) by replacing the constant \( \Gamma \) with \( \tilde{\Gamma} \) to ensure the intermediate states \( \mathbf{U}_{i+\frac{1}{2}}^{*,*} \in G \), which makes the work more complicated and the constraint on the time step more strict. The second one is to directly consider positivity for \( \mathbf{U}_{n+1}^i \), without insisting that \( \mathbf{U}_{i+\frac{1}{2}}^{*,*} \in G \). This also puts an additional constraint on the time step but it then works with any positive definition of the wave speeds \( \tilde{z}_{i+\frac{1}{2}} \).

We adopt the second approach and put a constraint on the time step for our scheme (2.5) to achieve our desired property. Before that, we would also want to obey the classical CFL condition, which reads as
\[
\Delta t^n \leq \lambda \min_i \frac{\Delta r_{n+1}^i}{a_i^n + |\bar{u}_i^n|} = \Delta t_1,
\]
where \( \lambda = 0.5 \) is Courant number, and \( a_i^n \) is the sound speed determined by cell averages. The CFL condition has guaranteed that the two waves starting from the interface \( r_{i-\frac{1}{2}}^i \) and \( r_{i+\frac{1}{2}}^i \) will not interact with each other within the time \( \Delta t^n \), which implies the area of each cell \( A_i > 0 \), then \( V_i > 0 \), and this can ensure the positivity of density based on the first relation in the scheme (2.5).

Motivated by [18], we put an additional constraint on the time step as
\[
\Delta t^n \leq \min_i \frac{\sigma V_{n+1}^i}{|r_{i+\frac{1}{2}}^i u_{i+\frac{1}{2}}^* - r_{i-\frac{1}{2}}^i u_{i-\frac{1}{2}}^*|} = \min_i \frac{\sigma m_i}{|r_{i+\frac{1}{2}}^i u_{i+\frac{1}{2}}^* - r_{i-\frac{1}{2}}^i u_{i-\frac{1}{2}}^*|} = \Delta t_2
\]

(2.13)
where $0 < \sigma < 1$, then we can ensure that the volume changes at most by a factor $\sigma$ from $t^n$ to $t^{n+1}$ and hope to achieve positive internal energy by determining the factor $\sigma$.

Now we assess the required condition to ensure the internal energy $e(H) = E(H) - \frac{1}{2}(u(H))^2 > 0$. From (2.7), we know that

$$u^*_{i+\frac{1}{2}} = \bar{p}^n_i - \frac{p^*_{i+\frac{1}{2}} - \bar{p}^n_i}{\bar{z}^-_{i+\frac{1}{2}}} \quad \text{and} \quad u^*_{i-\frac{1}{2}} = \bar{u}^n_i - \frac{p^*_{i-\frac{1}{2}} - \bar{p}^n_i}{\bar{z}^+_{i-\frac{1}{2}}},$$

which leads to

$$p^*_{i+\frac{1}{2}} = \bar{p}^n_i + (\bar{u}^n_i - u^*_{i+\frac{1}{2}})\bar{z}^-_{i+\frac{1}{2}} \quad \text{and} \quad p^*_{i-\frac{1}{2}} = \bar{p}^n_i + (\bar{u}^n_i - u^*_{i-\frac{1}{2}})\bar{z}^+_{i-\frac{1}{2}}. \quad (2.14)$$

Finally, we have

$$e(H) = E(H) - \frac{1}{2}(u(H))^2 = \bar{E}^n_i - \frac{2\Delta t^n}{m_i} \left( r_{i+\frac{1}{2}}p^*_{i+\frac{1}{2}}u^*_{i+\frac{1}{2}} - r_{i-\frac{1}{2}p^*_{i-\frac{1}{2}}u^*_{i-\frac{1}{2}}} \right)$$

$$- \frac{1}{2} \left( \bar{u}^n_i - \frac{2\Delta t^n}{m_i} \left( r_{i+\frac{1}{2}}p^*_{i+\frac{1}{2}} - r_{i-\frac{1}{2}}p^*_{i-\frac{1}{2}} \right) + \frac{2\Delta t^n}{m_i} \Delta r_i p^*_{i} \right)^2. \quad (2.15)$$

By plugging (2.14) into (2.15), $e(H)$ can be rewritten as

$$e(H) = \bar{E}^n_i - \frac{2\Delta t^n}{m_i} \bar{p}^n_i \left( r_{i+\frac{1}{2}} \bar{u}^n_{i+\frac{1}{2}} - r_{i-\frac{1}{2}} \bar{u}^n_{i-\frac{1}{2}} \right)$$

$$+ \frac{2\Delta t^n}{m_i} r_{i+\frac{1}{2}} \bar{z}^-_{i+\frac{1}{2}} u^2_{i+\frac{1}{2}} + \frac{2\Delta t^n}{m_i} r_{i-\frac{1}{2}} \bar{z}^+_{i-\frac{1}{2}} w^2_{i-\frac{1}{2}}$$

$$- 2 \left( \frac{\Delta t^n}{m_i} \right)^2 \left( r_{i+\frac{1}{2}} \bar{z}^-_{i+\frac{1}{2}} w_{i+\frac{1}{2}} + r_{i-\frac{1}{2}} \bar{z}^+_{i-\frac{1}{2}} w_{i-\frac{1}{2}} \right)^2 \quad (2.16)$$

$$= A_i \bar{E}^n_i + B_i$$

where $A_i$ and $B_i$ are denoted as follows

$$A_i = 1 - \frac{2\Delta t^n}{\bar{E}^n_i} \cdot \frac{\bar{p}^n_i}{m_i} \left( r_{i+\frac{1}{2}} \bar{u}^n_{i+\frac{1}{2}} - r_{i-\frac{1}{2}} \bar{u}^n_{i-\frac{1}{2}} \right), \quad (2.17)$$

$$B_i = \frac{2\Delta t^n}{m_i} \left[ r_{i+\frac{1}{2}} \bar{z}^-_{i+\frac{1}{2}} w^2_{i+\frac{1}{2}} + r_{i-\frac{1}{2}} \bar{z}^+_{i-\frac{1}{2}} w^2_{i-\frac{1}{2}} - \Delta t^n \left( r_{i+\frac{1}{2}} \bar{z}^-_{i+\frac{1}{2}} w_{i+\frac{1}{2}} + r_{i-\frac{1}{2}} \bar{z}^+_{i-\frac{1}{2}} w_{i-\frac{1}{2}} \right)^2 \right], \quad (2.18)$$

with $w_{i\pm\frac{1}{2}} = u^*_{i\pm\frac{1}{2}} - \bar{p}^n_i$. 

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Based on these formulations, if we can prove $B_i \geq 0$, then a sufficient condition to ensure $e > 0$ is to have $A_i > 0$. One can easily check that if $\sigma$ in the time step constraint (2.13) satisfies $\sigma \leq \frac{1}{2} \frac{|\tau_i^m|}{|\bar{\tau}_i^m|}$, then $A_i$ will be strictly positive. Now let us rewrite $B_i$ as

$$\begin{align*}
B_i &= \frac{2\Delta t^n}{m_i} \left[ r_{i+\frac{1}{2}} \bar{z}_{i+\frac{1}{2}} + r_{i-\frac{1}{2}} \bar{z}_{i-\frac{1}{2}} - \frac{\Delta t^n}{m_i} \left( r_{i+\frac{1}{2}} \bar{z}_{i+\frac{1}{2}} + r_{i-\frac{1}{2}} \bar{z}_{i-\frac{1}{2}} \right) \right] \\
&= \frac{2\Delta t^n}{m_i} \begin{pmatrix} w_{i+\frac{1}{2}} \\
\bar{w}_{i+\frac{1}{2}} 
\end{pmatrix}^T M_i \begin{pmatrix} w_{i+\frac{1}{2}} \\
\bar{w}_{i+\frac{1}{2}} 
\end{pmatrix}.
\end{align*}$$

(2.19)

where $M_i$ reads as follows

$$M_i = \begin{pmatrix}
  r_{i+\frac{1}{2}} \bar{z}_{i+\frac{1}{2}}^+ & 1 - \frac{\Delta t^n}{m_i} r_{i+\frac{1}{2}} \bar{z}_{i+\frac{1}{2}}^+ \\
-\frac{\Delta t^n}{m_i} r_{i-\frac{1}{2}} \bar{z}_{i-\frac{1}{2}}^+ & r_{i+\frac{1}{2}} \bar{z}_{i+\frac{1}{2}}^-
\end{pmatrix} \begin{pmatrix}
  1 - \frac{\Delta t^n}{m_i} r_{i+\frac{1}{2}} \bar{z}_{i+\frac{1}{2}}^- \\
-\frac{\Delta t^n}{m_i} r_{i-\frac{1}{2}} \bar{z}_{i-\frac{1}{2}}^- & r_{i+\frac{1}{2}} \bar{z}_{i+\frac{1}{2}}^-
\end{pmatrix}.
$$

It can be easily proven that $M_i$ is positively semi-definite for all $i$ if and only if

$$\Delta t^n \leq \min_i \frac{m_i}{r_{i+\frac{1}{2}} \bar{z}_{i+\frac{1}{2}}^+ + r_{i+\frac{1}{2}} \bar{z}_{i+\frac{1}{2}}^-} = \Delta t_3.$$

Hence we have obtained another time step restriction for the scheme (2.5). In particular, for the acoustic solver case, i.e. $\bar{z}_{i+\frac{1}{2}}^+ = \bar{z}_{i+\frac{1}{2}}^- = \bar{P}_i^m \bar{\tau}_i^m$, this condition writes $\Delta t^n \leq \frac{\Delta r_i^n}{\bar{\tau}_i^m}$, which can be recovered by the classic CFL condition (2.12).

Now, let us try to find the sufficient condition for $W \in G$. Similarly, we can have

$$e(W) = E(W) - \frac{1}{2} (u(W))^2$$

$$= \bar{E}_i - \frac{1}{2} \left( \bar{\tau}_i^m + \frac{2\Delta t^n}{m_i} (A_i P_s - \Delta r_i^n \bar{P}_i^n) \right)^2$$

(2.21)

$$= \bar{\tau}_i^m - \frac{2\Delta t^n}{m_i} (A_i P_s - \Delta r_i^n \bar{P}_i^n) \bar{\tau}_i^m - 2 \left( \frac{\Delta t^n}{m_i} \right)^2 (A_i P_s - \Delta r_i^n \bar{P}_i^n)^2.$$

This is a quadratic inequality $e(W) \geq 0$, which is guaranteed if

$$\Delta t^n \leq \min_i \left( -\frac{m_i \bar{\tau}_i^m}{2\mu} + \frac{m_i \bar{\tau}_i^m}{2|\mu|} \sqrt{(|\bar{\tau}_i^m|)^2 + 2\bar{\tau}_i^n} \right) = \Delta t_4.$$

(2.22)

with $\mu = A_i P_s - \Delta r_i^n \bar{P}_i^n$. In particular, when we choose $P_s$ as $\bar{P}_i^n$ for the first order approximation of the source term, the condition (2.22) always holds for any $\Delta t^n$. 

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Theorem 2.2. For the first order Lagrangian scheme (2.5) with the numerical fluxes defined in (2.7)-(2.8) and any general positive wavespeeds definition, assuming \( \mathbf{U}_i^n \in G \), \( \mathbf{U}_i^{n+1} \) is ensured to be in the admissible set \( G \) under the following time step constraint condition

\[
\Delta t^n \leq \min(\Delta t_1, \Delta t_2, \Delta t_3, \Delta t_4),
\]

where \( \Delta t_1 \) is the classical CFL condition defined in (2.12), \( \Delta t_2 \) reads as that in (2.13) with \( \sigma \leq \min(1, \frac{1}{2} \frac{\rho_i^n}{|p_i^n|}) \), \( \Delta t_3 \) is defined in (2.20) and \( \Delta t_4 \) is defined in (2.22) with \( \mu = A_i P_s - \Delta r_i^n \frac{p_i^n}{p_i} \).

Now, let us make a summary about the first-order positivity-preserving scheme (2.5) and give the algorithm flowchart for it at each time step:

1. Assuming \( \mathbf{U}_i \in G \) at the \( n \)-th time level, compute the numerical fluxes \( p_{i+\frac{1}{2}}^n \) and \( u_{i+\frac{1}{2}}^n \) for all \( i \) by (2.8).
2. Compute the time step \( \Delta t^n \) by (2.23).
3. Update the position of each cell vertex, and obtain the area and volume of each cell by (2.6).
4. Compute the averaged values \( \mathbf{U}_i \) at the \( (n+1) \)-th time level based on the scheme (2.5).

2.5 High-order scheme

Now, we consider a general high-order finite volume Lagrangian scheme which has the same general form as the first order Lagrangian scheme (2.5), where \( P_s \) is the approximation with high order accuracy of the source term \( \frac{1}{A_i} \int_{I_i} p dr \). For instance, if we perform a third order scheme, we can compute \( P_s \) using the Gauss-Lobatto quadrature rule

\[
P_s = \frac{1}{6} p_{i-\frac{1}{2}}^n + \frac{2}{3} p_i^n + \frac{1}{6} p_{i+\frac{1}{2}}^n.
\]

For a high-order accurate spacial discretization, by the information of the corresponding cell-average variables from the cell \( I_i \) and its neighboring cells, we apply the techniques of essentially non-oscillatory (ENO) reconstruction and local characteristic decomposition [10].
to obtain reconstruction polynomials, through which we can determine the values at the cell interfaces \( \{ U_{i+\frac{1}{2}}^\pm \} \). Then the numerical fluxes \( p_{i+\frac{1}{2}}^* \) and \( u_{i+\frac{1}{2}}^* \) for all \( i \) can be computed by the Riemann solver stated in (2.7)-(2.8).

In this section, we consider two different ways of ENO reconstruction. The first approach is standard, treating the cell averages as integrals on the usual control volumes. This is simply a standard reconstruction which can be made arbitrarily high order accurate by increasing the polynomial degree. The second approach is non-standard, treating the cell averages as integrals on the cell area. This interpretation of the cell averages already commits a second order error, hence the second approach would yield at most a second order scheme regardless of the reconstruction polynomial degree. However, this second approach would be needed in the two-dimensional case in order to preserve symmetry. More details of these two approaches of reconstructions will be given below.

### 2.5.1 Reconstruction on the control volumes

In this case, we treat the cell averages as integrals on the usual control volumes. We perform the ENO reconstruction on the conserved variables \((\rho, \rho u, \rho E)^\top\) with a set of polynomials \(\{ W_i(r) = (\rho_i(r), M_i(r), E_i(r))^\top \} \) of degree \( k \geq 1 \) in the cell \( I_i \) for each \( i \), which implies

\[
\bar{\rho}_i = \frac{1}{V_i} \int_{I_i} \rho_i(r) r dr = \frac{m_i}{V_i},
\]

\[
\bar{M}_i = \frac{1}{V_i} \int_{I_i} M_i(r) r dr = \frac{\int_{I_i} M_i(r) r dr}{m_i} \cdot \frac{m_i}{V_i} = \bar{\rho}_i \cdot \bar{u}_i,
\]

\[
\bar{E}_i = \frac{1}{V_i} \int_{I_i} E_i(r) r dr = \frac{\int_{I_i} E_i(r) r dr}{m_i} \cdot \frac{m_i}{V_i} = \bar{\rho}_i \cdot \bar{E}_i.
\]

(2.24)

If we use polynomials of degree \( k \) in the reconstruction process, we obtain a \((k + 1)\)-th order scheme.

### 2.5.2 Reconstruction on the areas

In order to facilitate extension to the two-dimensional case with symmetry consideration, we also consider a second approach, treating the cell averages as integrals on the cell area. That
is, we perform the ENO reconstruction on the variables \((\rho, u, E)^\top\) with a set of polynomials 
\[ \{ \mathbf{U}_i(r) = (\rho_i(r), u_i(r), E_i(r))^\top \} \]
of degree \(k \geq 1\) in the cell \(I_i\) for each \(i\), which satisfies
\[
\begin{align*}
\bar{\rho}_i &= \frac{1}{\Delta r_i} \int_{I_i} \rho_i(r) dr, \\
\bar{u}_i &= \frac{1}{\Delta r_i} \int_{I_i} u_i(r) dr, \\
\bar{E}_i &= \frac{1}{\Delta r_i} \int_{I_i} E_i(r) dr.
\end{align*}
\] (2.25)

This interpretation of the cell averages already commits a second order error, as it can be readily verified that, at least when we are away from the origin \(r = 0\), we have
\[
\begin{align*}
\frac{1}{V_i} \int_{I_i} \rho(r) r dr &= \frac{1}{\Delta r_i} \int_{I_i} \rho(r) dr + O(\Delta r_i^2), \\
\frac{1}{V_i} \int_{I_i} M(r) r dr &= \frac{m_i}{V_i} \frac{1}{\Delta r_i} \int_{I_i} u(r) dr + O(\Delta r_i^2), \\
\frac{1}{V_i} \int_{I_i} E(r) r dr &= \frac{m_i}{V_i} \frac{1}{\Delta r_i} \int_{I_i} E(r) dr + O(\Delta r_i^2).
\end{align*}
\]

Hence the second approach would yield at most a second order scheme regardless of the reconstruction polynomial degrees.

### 2.6 High-order positivity-preserving schemes

In this section, we focus on how to design the high order scheme (2.5) to be positivity-preserving. We discuss this in the contexts of the two different approaches of reconstructions.

Before that, let us consider the \(K\)-point Legendre Gauss-Lobatto quadrature rule in the interval \(I_i\), which is exact for integrals of polynomials with degree up to \(2K - 3\), and we denote these quadrature points in \(I_i\) as
\[
S_i = \{ \frac{r_{i-\frac{1}{2}}}{2} = r_i^1, r_i^2, \ldots, r_i^{K-1}, r_i^K = r_{i+\frac{1}{2}} \} \tag{2.26}
\]
Let \(\omega_\alpha\) be the quadrature weights such that \(\omega_\alpha > 0, \alpha = 1, \ldots, K, \omega_1 = \omega_K, \text{ and } \sum_{\alpha=1}^{K} \omega_\alpha = 1\).

#### 2.6.1 Reconstruction on the control volumes

Assuming that we perform the ENO reconstruction as in (2.24), we get a polynomial vector 
\[ \{ \mathbf{W}_i(r) = (\rho_i(r), M_i(r), E_i(r))^\top \} \]
of degree \(k \geq 1\).
We choose $K$ to be the smallest integer satisfying $2K - 3 \geq k$, then the $K$-point Legendre Gauss-Lobatto quadrature rule is exact for the integrals involved in the reconstruction, hence we have

$$
\bar{W}_i = \sum_{\alpha=1}^{K} \omega_{\alpha} W_{\alpha i} r_i^\alpha \Delta r_i = (\omega_1 W_{i-\frac{1}{2}}^+ r_{i-\frac{1}{2}} + \bar{\omega}_i^* W_i^* + \omega_K W_{i+\frac{1}{2}}^- r_{i+\frac{1}{2}}) \Delta r_i
$$

with

$$
\bar{\omega}_i^* = \sum_{\alpha=2}^{K-1} \omega_{\alpha} r_i^\alpha, \quad W_i^* = \frac{1}{\omega_i^*} \sum_{\alpha=2}^{K-1} \omega_{\alpha} W_{\alpha i} r_i^\alpha = \frac{\bar{W}_i}{\Delta r_i} - \omega_1 W_{i-\frac{1}{2}}^+ r_{i-\frac{1}{2}} - \omega_K W_{i+\frac{1}{2}}^- r_{i+\frac{1}{2}},
$$

(2.27)

which implies

$$
\bar{p}_i = \frac{1}{V_i} \int_{I_i} \rho_i(r) r dr = \frac{1}{V_i} \sum_{\alpha=1}^{K} \omega_{\alpha} m_{\alpha i},
$$

$$
\bar{u}_i = \frac{1}{m_i} \int_{I_i} \mathcal{M}_i(r) r dr = \frac{1}{m_i} \sum_{\alpha=1}^{K} \omega_{\alpha} m_{\alpha i} u_{\alpha i},
$$

$$
\bar{E}_i = \frac{1}{m_i} \int_{I_i} \mathcal{E}_i(r) r dr = \frac{1}{m_i} \sum_{\alpha=1}^{K} \omega_{\alpha} m_{\alpha i} E_{\alpha i},
$$

(2.28)

where $m_{\alpha i} = \rho_{\alpha i} r_i^\alpha \Delta r_i^n$ with

$$
\rho_{\alpha i} = \rho_i(r_i^\alpha), \quad u_{\alpha i} = \mathcal{M}_i(r_i^\alpha) / \rho_i(r_i^\alpha), \quad E_{\alpha i} = \mathcal{E}_i(r_i^\alpha) / \rho_i(r_i^\alpha).
$$

Now, let us first introduce the artificial numerical fluxes $p_i^*$ and $u_i^*$, which are computed from the left and right states $U_{i-\frac{1}{2}}^+$ and $U_{i+\frac{1}{2}}^-$. In order to keep accordance with them, we also need to define the artificial local wavespeeds relative to this term, $\bar{z}_{i\pm\frac{1}{2}}$, in the same way as $\bar{z}_{i\pm\frac{1}{2}}^\pm$. For example, if we take the Dukowicz definition of $\bar{z}_{i\pm\frac{1}{2}}$ [7],

$$
\bar{z}_{i\pm\frac{1}{2}} = \rho_{i\pm\frac{1}{2}} (a_{i\pm\frac{1}{2}} + \Gamma |u_{i\pm\frac{1}{2}}^* - u_{i\pm\frac{1}{2}}^-|),
$$

then the artificial wavespeeds $\bar{z}_{i\pm\frac{1}{2}}^\pm, u_{i\pm\frac{1}{2}}$ read as

$$
\bar{z}_{i\pm\frac{1}{2}}^\pm = \rho_{i\pm\frac{1}{2}} (a_{i\pm\frac{1}{2}} + \Gamma |u_{i\pm\frac{1}{2}}^* - u_{i\pm\frac{1}{2}}^-|),
$$

where the artificial velocity $u_i^*$ and artificial pressure $p_i^*$ read as

$$
u_i^* = \frac{\bar{z}_{i-\frac{1}{2}}^+ u_i^+ - \bar{z}_{i+\frac{1}{2}}^- u_i^-}{\bar{z}_{i-\frac{1}{2}}^+ + \bar{z}_{i+\frac{1}{2}}^-} - \frac{1}{\bar{z}_{i-\frac{1}{2}}^+ + \bar{z}_{i+\frac{1}{2}}^-} (p_i^- - p_i^+),
$$

$$
p_i^* = \frac{\bar{z}_{i-\frac{1}{2}}^+ p_i^+ - \bar{z}_{i+\frac{1}{2}}^- p_i^-}{\bar{z}_{i-\frac{1}{2}}^+ + \bar{z}_{i+\frac{1}{2}}^-} - \frac{1}{\bar{z}_{i-\frac{1}{2}}^+ + \bar{z}_{i+\frac{1}{2}}^-} (u_i^- - u_i^+).
$$
If we take the acoustic impedance \( \tilde{Z}_{i}^{\pm} = \rho_{i}^{\pm} a_{i}^{\pm} \), there will be no difference, i.e. \( \tilde{Z}_{i}^{\pm} = \tilde{Z}_{i}^{\pm} \).

Then, by adding and subtracting \( \frac{\Delta t}{m_{i}} \Delta r_{i} \rho_{i}^{*} u_{i}^{*} \) in the second equation in (2.5) and \( \frac{\Delta t}{m_{i}} \Delta r_{i} \rho_{i}^{*} u_{i}^{*} \) in the third equation in (2.5) respectively, the scheme (2.5) becomes

\[
\mathbf{U}_{i}^{n+1} = \frac{1}{2} \mathbf{H} + \frac{1}{2} \mathbf{W} \tag{2.29}
\]

where

\[
\mathbf{H} = \left( \begin{array}{c}
\frac{m_{i}}{V_{i}^{n+1}} u_{i}^{n} - \frac{2\Delta t}{m_{i}} (r_{i-\frac{1}{2}} p_{i-\frac{1}{2}}^{*} - r_{i-\frac{1}{2}} p_{i-\frac{1}{2}}^{*}) + \frac{2\Delta t}{m_{i}} \Delta r_{i} \rho_{i}^{*} \\
\frac{E_{i}^{n} - \frac{2\Delta t}{m_{i}} (r_{i-\frac{1}{2}} p_{i-\frac{1}{2}}^{*} u_{i-\frac{1}{2}}^{*} - r_{i-\frac{1}{2}} p_{i-\frac{1}{2}}^{*} u_{i-\frac{1}{2}}^{*}) + \frac{2\Delta t}{m_{i}} \Delta r_{i} \rho_{i}^{*} u_{i}^{*}}{m_{i}}
\end{array} \right), \tag{2.30}
\]

and

\[
\mathbf{W} = \left( \begin{array}{c}
\frac{m_{i}}{V_{i}^{n+1}} u_{i}^{n} + \frac{2\Delta t}{m_{i}} \Delta r_{i} (P_{i} - \rho_{i}^{*}) \\
\frac{E_{i}^{n} - \frac{2\Delta t}{m_{i}} \Delta r_{i} \rho_{i}^{*} u_{i}^{*}}{m_{i}}
\end{array} \right). \tag{2.31}
\]

We notice that \( \mathbf{H} \) can be expressed as the following convex combination

\[
\mathbf{H} = \frac{m_{i}^{*}}{m_{i}} \hat{\mathbf{F}}^{*} + \frac{\omega_{1} m_{1i}}{m_{i}} \hat{\mathbf{F}}_{1} + \frac{\omega_{K} m_{Ki}}{m_{i}} \hat{\mathbf{F}}_{K} \tag{2.32}
\]

where \( m_{i}^{*} = \sum_{\alpha=2}^{K-1} \omega_{\alpha} m_{\alpha i} \), and \( \hat{\mathbf{F}}^{*} = \frac{1}{m_{i}^{*}} \left( \begin{array}{c}
\frac{m_{i}^{*}}{V_{i}^{n+1}} \\
\frac{m_{i}^{*}}{V_{i}^{n+1}}
\end{array} \right) \left( \begin{array}{c}
\sum_{\alpha=2}^{K-1} \omega_{\alpha} m_{\alpha i} u_{\alpha i} \\
\sum_{\alpha=2}^{K-1} \omega_{\alpha} m_{\alpha i} E_{\alpha i}
\end{array} \right)^{\top}
\)

\[
\hat{\mathbf{F}}_{1} = \left( \begin{array}{c}
\frac{m_{i}}{V_{i}^{n+1}} u_{i-\frac{1}{2}}^{+} - \frac{2\Delta t}{\omega_{1} m_{1i}} (r_{i-\frac{1}{2}} p_{i-\frac{1}{2}}^{*} - r_{i-\frac{1}{2}} p_{i-\frac{1}{2}}^{*}) \\
E_{i-\frac{1}{2}}^{+} - \frac{2\Delta t}{\omega_{1} m_{1i}} (r_{i-\frac{1}{2}} p_{i-\frac{1}{2}}^{*} u_{i-\frac{1}{2}}^{*} - r_{i-\frac{1}{2}} p_{i-\frac{1}{2}}^{*} u_{i-\frac{1}{2}}^{*})
\end{array} \right), \tag{2.33}
\]

\[
\hat{\mathbf{F}}_{K} = \left( \begin{array}{c}
\frac{m_{i}}{V_{i}^{n+1}} u_{i+\frac{1}{2}}^{+} - \frac{2\Delta t}{\omega_{K} m_{Ki}} (r_{i+\frac{1}{2}} p_{i+\frac{1}{2}}^{*} - r_{i+\frac{1}{2}} p_{i+\frac{1}{2}}^{*}) \\
E_{i+\frac{1}{2}}^{+} - \frac{2\Delta t}{\omega_{K} m_{Ki}} (r_{i+\frac{1}{2}} p_{i+\frac{1}{2}}^{*} u_{i+\frac{1}{2}}^{*} - r_{i+\frac{1}{2}} p_{i+\frac{1}{2}}^{*} u_{i+\frac{1}{2}}^{*})
\end{array} \right). \tag{2.34}
\]
We can see that $H$ can be expressed as a convex combination of three different terms in (2.32). Consequently, if these terms belong to the convex admissible set $G$, then so does $H$. Among these terms, $\hat{F}^*$ is only made of the contribution of the polynomial solution at the time level $n$ at some quadrature points. To ensure this quantity to be in $G$, a particular limitation will be designed in the later section.

We notice that $\hat{F}_1$ and $\hat{F}_K$ exactly mimic the first order scheme (2.5) but without the source term. Thus we would like to apply the similar analysis as that in the first-order scheme and then obtain $\hat{F}_1, \hat{F}_K \in G$ under the following condition

$$\Delta t^n \leq \min_i \left( \frac{\sigma_1 \Delta r^n_i}{|u_i^* - u_{i-1/2}^*|}, \frac{\sigma_2 \Delta r^n_i}{|u_{i+1/2}^* - u_i^*|} \right) = \Delta t_2,$$

$$\Delta t^n \leq \min_i \left( \frac{\rho_i^+ \Delta r^n_i}{\bar{z}^{+}_{i-rac{1}{2}} + \bar{z}^{-}_{i-rac{1}{2}}}, \frac{\rho_i^- \Delta r^n_i}{\bar{z}^{+}_{i+rac{1}{2}} + \bar{z}^{-}_{i+rac{1}{2}}} \right) = \Delta t_3,$$

with $\sigma_1 \leq \min \left( 1, \frac{\rho_i^+ \epsilon_i^+}{2\bar{p}_{i-rac{1}{2}}} \right), \sigma_2 \leq \min \left( 1, \frac{\rho_i^- \epsilon_i^-}{2\bar{p}_{i+rac{1}{2}}} \right)$.

As for $W$, the density is obviously positive. We only need to find a sufficient condition to ensure $e(W) = E(W) - \frac{1}{2}u^2(W) > 0$, which is formulated as

$$\Delta t^n \leq \min_i \left( -\frac{m_i \mu}{2 \Delta r^n_i (P_s - p_i^s)^2} + \frac{m_i}{2 \Delta r^n_i (P_s - p_i^s)^2} \sqrt{\mu^2 + 2\bar{e}_i^s (P_s - p_i^s)^2} \right) = \Delta t_4 \quad (2.36)$$

with $\mu = p_i^s u_i^* + (P_s - p_i^s) m_i^s$.

**Theorem 2.3.** If the numerical fluxes are determined by the two-state Riemann solver defined in (2.7)-(2.8) with any general positive wavespeeds definition, assume $\bar{\mathbf{U}}_i^n \in G$ and $U_{i-1/2}^+, U_{i+1/2}^- \in G$, then $\bar{\mathbf{U}}_i^{n+1}$ in the high order scheme (2.5) with the reconstruction in (2.24) is also in the admissible set $G$ if

$$\Delta t^n \leq \min(\Delta t_1, \Delta t_2, \Delta t_3, \Delta t_4) \quad (2.37)$$

where $\Delta t_1$ is the CFL condition in (2.12), $\Delta t_2, \Delta t_3, \Delta t_4$ are defined in (2.35)-(2.36).
2.6.2 Reconstruction on the area

In this case, we perform the ENO reconstruction on the variables \((\rho, u, E)^T\) with a set of polynomials \(\{U_i(r) = (\rho_i(r), u_i(r), E_i(r))^T\}\) of degree \(k \geq 1\) in the cell \(I_i\) for each \(i\). We now choose \(K\) to be the smallest integer satisfying \(2K - 3 \geq k\), then the \(K\)-point Legendre Gauss-Lobatto quadrature rule is exact for the reconstruction polynomials, which gives us

\[
\begin{align*}
\overline{\rho}_i &= \frac{1}{\Delta r_i} \int_{I_i} \rho(r) dr = \sum_{\alpha=1}^{K} \omega_{\alpha} \rho_{\alpha i}, \\
\overline{u}_i &= \frac{1}{\Delta r_i} \int_{I_i} u(r) dr = \sum_{\alpha=1}^{K} \omega_{\alpha} u_{\alpha i}, \\
\overline{E}_i &= \frac{1}{\Delta r_i} \int_{I_i} E(r) dr = \sum_{\alpha=1}^{K} \omega_{\alpha} E_{\alpha i}
\end{align*}
\tag{2.38}
\]

where \(\rho_{\alpha i} = \rho_i(r_{\alpha i}^\alpha)\), \(u_{\alpha i} = u_i(r_{\alpha i}^\alpha)\), and \(E_{\alpha i} = E_i(r_{\alpha i}^\alpha)\). From (2.38), we know that

\[
\overline{U}_i = \omega_1 \overline{U}_{i-\frac{1}{2}} + \omega^* \overline{U}_i + \omega_K \overline{U}_{i+\frac{1}{2}}
\]

where \(\omega^* = \sum_{\alpha=2}^{K-1} \omega_{\alpha}\) and

\[
U_i^* = \frac{1}{\omega^*} \sum_{\alpha=2}^{K-1} \omega_{\alpha} U_{\alpha i} = \frac{1}{\omega^*} (\overline{U}_i - \omega_1 \overline{U}_{i-\frac{1}{2}} - \omega_K \overline{U}_{i+\frac{1}{2}}).
\tag{2.39}
\]

Then the cell-average variable \(\overline{U}_i^{n+1}\) can be formulated as

\[
\begin{pmatrix}
\overline{\rho}_i^{n+1} \\
\overline{u}_i^{n+1} \\
\overline{E}_i^{n+1}
\end{pmatrix} = 
\begin{pmatrix}
\frac{m_i}{V_i^{n+1}} \\
\sum_{\alpha=1}^{K} \omega_{\alpha} u_{\alpha i} - \frac{\Delta t^n}{m_i} (r_{i-\frac{1}{2}} p_{i-\frac{1}{2}}^* - r_{i-\frac{1}{2}} p_{i-\frac{1}{2}}^*) + \frac{\Delta t^n}{m_i} A_i P_s \\
\sum_{\alpha=1}^{K} \omega_{\alpha} E_{\alpha i} - \frac{\Delta t^n}{m_i} (r_{i-\frac{1}{2}} p_{i-\frac{1}{2}}^* u_{i-\frac{1}{2}} - r_{i-\frac{1}{2}} p_{i-\frac{1}{2}}^* u_{i-\frac{1}{2}})
\end{pmatrix}
\tag{2.40}
\]

Similarly, by adding and subtracting \(\frac{\Delta t^n}{m_i} r_i p_i^*\) in the second equation in (2.40) and \(\frac{\Delta t^n}{m_i} r_i p_i^* u_i^*\) in the third equation in (2.40) respectively, the scheme (2.40) becomes

\[
\overline{U}_i^{n+1} = \omega^* \hat{F}^* + \omega_1 \hat{F}_1 + \omega_K \hat{F}_K
\tag{2.41}
\]
where \( \omega^* = \sum_{a=2}^{K-1} \omega_a \) and \( \hat{F}^* = \frac{1}{\omega} \left( \frac{m_i}{V_i^{n+1}}, \sum_{a=2}^{K-1} \omega_a u_{a,i}, \sum_{a=2}^{K-1} \omega_a E_{a,i} \right) \),

\[
\hat{F}_1 = \left( \frac{m_i}{V_i^{n+1}} \right) u_{i-\frac{1}{2}}^+ - \frac{\Delta t^n}{\omega_1 m_i} \left( r_i \frac{p_i^*}{\omega} r_i l_{i-\frac{1}{2}} p_{i-\frac{1}{2}}^* - \frac{r_i p_i^*}{\omega} p_{i-\frac{1}{2}}^* \right) + \frac{\Delta t^n}{\omega_1 m_i} \left( \frac{1}{2} A_i \right) P_s,
\]

(2.42)

\[
\hat{F}_K = \left( \frac{m_i}{V_i^{n+1}} \right) u_{i+\frac{1}{2}}^- - \frac{\Delta t^n}{\omega_K m_i} \left( r_i \frac{p_i^*}{\omega} r_i l_{i+\frac{1}{2}} p_{i+\frac{1}{2}}^* - \frac{r_i p_i^*}{\omega} p_{i+\frac{1}{2}}^* \right) + \frac{\Delta t^n}{\omega_K m_i} \left( \frac{1}{2} A_i \right) P_s.
\]

(2.43)

Notice that \( U_i^{n+1} \) still can be expressed as a convex combination of three different terms in (2.41), while \( \hat{F}_1 \) and \( \hat{F}_K \) are nothing but the same type as the first order scheme in (2.5). Consequently, we have the similar conclusion as stated below.

**Theorem 2.4.** If the numerical fluxes are determined by the two-state Riemann solver defined in (2.7)-(2.8) with any general positive wavespeeds definition, assume \( U_i^n \in G \) and \( U_{i+1/2}^+, U_{i-1/2}^- \in G \), then \( U_i^{n+1} \) in the high order scheme (2.40) is also in the admissible set \( G \) if

\[
\Delta t^n \leq \min(\Delta t_1, \Delta t_2, \Delta t_3, \Delta t_4, \Delta t_5)
\]

(2.44)

where \( \Delta t_1 \) is the CFL condition in (2.12) and

\[
\Delta t_2 = \omega_1 \min_i \left( \frac{2 \sigma_1 m_i}{\rho_{i-\frac{1}{2}} \left| r_i \frac{u_i^*}{\omega} r_i l_{i-\frac{1}{2}} u_{i-\frac{1}{2}}^* \right|}, \frac{2 \sigma_2 m_i}{\rho_{i+\frac{1}{2}} \left| r_i \frac{u_i^*}{\omega} r_i l_{i+\frac{1}{2}} u_{i+\frac{1}{2}}^* \right|} \right),
\]

\[
\Delta t_3 = \omega_1 \min_i \left( \frac{2 m_i}{r_i \frac{\tilde{u}_i^+}{\omega} + r_i \frac{\tilde{u}_i^-}{\omega}}, \frac{2 m_i}{r_i \frac{\tilde{u}_i^+}{\omega} + r_i \frac{\tilde{u}_i^-}{\omega}} \right).
\]
with $\sigma_1 \leq \min \left(1, \frac{\rho_i^+ + e_i^+}{2|\rho_i^+ + \frac{1}{2}|}\right)$, $\sigma_2 \leq \min \left(1, \frac{\rho_i^- + e_i^-}{2|\rho_i^- + \frac{1}{2}|}\right)$ as well as

$$
\Delta t_4 = \omega_1 \min_i \left(\frac{-m_i u_{i-\frac{1}{2}}^+}{\mu_1} + \frac{m_i}{|\mu_1|} \sqrt{(u_{i-\frac{1}{2}}^+)^2 + 2e_{i-\frac{1}{2}}^+}\right),
$$

$$
\Delta t_5 = \omega_1 \min_i \left(\frac{-m_i u_{i+\frac{1}{2}}^-}{\mu_2} + \frac{m_i}{|\mu_2|} \sqrt{(u_{i+\frac{1}{2}}^-)^2 + 2e_{i+\frac{1}{2}}^-}\right),
$$

where $\mu_1 = A_i P_s - \Delta r_n i_{-\frac{1}{2}}$ and $\mu_2 = A_i P_s - \Delta r_n i_{+\frac{1}{2}}$.

So far, we have proved that assuming $\bar{U}_i^n, U_i^*, U_i^{+}_{i-\frac{1}{2}}$, and $U_i^{-}_{i+\frac{1}{2}}$ lie in the admissible set $G$, there exists a time step ensuring the new numerical solution $\bar{U}_i^{n+1} \in G$. To ensure the required assumptions, we need to make use of the particular limiter introduced in [5, 18, 19, 21].

### 2.7 Positivity-preserving limiter

In the previous section 2.5, we presented two different approaches to perform ENO reconstruction. In order to ensure the positivity property for the high order scheme (2.5), we need to make sure that the values of reconstruction polynomials at Gauss-Lobatto quadrature points should be positive. To achieve that, we need to make some modification to the reconstructed polynomials.

At the time level $n$, assume the polynomial reconstruction in the cell $I_i$ is the same as that we presented in the previous section 2.5.1, i.e $W_i(r)$ with degree $k$, and the cell average of $W_i(r)$ is $\bar{W}_i = (\bar{\rho}_i, \bar{\rho}_i \bar{u}_i, \bar{\rho}_i \bar{E}_i)^T$ according to the definition of cell-average variable $\bar{U}_i$. Under the assumption $\bar{U}_i^* = (\bar{\rho}_i^*, \bar{\mu}_i^*, \bar{E}_i^*) \in G$, we would like to modify the reconstruction polynomial $W_i(r)$ with a constant $\theta_i$ into another polynomial $\tilde{W}_i(r)$ as follows

$$
\tilde{W}_i(r) = \bar{W}_i + \theta_i (W_i(r) - \bar{W}_i),
$$

where $\theta_i \in [0, 1]$ is to be determined, such that $U(\tilde{W}_i(r)) \in G$ for all $r$ located in the Gauss-Lobatto quadrature points set $S_i$ defined in (2.26). In fact, we only need to require the three points $W_{i-\frac{1}{2}}^+, W_{i+\frac{1}{2}}^-, W_i^*$ to be in $G$, where $W_i^* = (\rho_i^*, \bar{M}_i^*, \bar{E}_i^*)^T$ is defined in (2.27). Referring to [5, 18, 21], the specific implementation can be taken as follows:
First, let us enforce the admissibility of the density. Choose a small number \( \varepsilon \) such that \( \bar{\rho}_i \geq \varepsilon \) for all \( i \). In practice, we usually take \( \varepsilon = 10^{-13} \). For each cell \( I_i \), compute

\[
\bar{\rho}_i^1(r) = \bar{\rho}_i + \theta_i^1(\rho_i(r) - \bar{\rho}_i), \quad \theta_i^1 = \min \left\{ 1, \frac{\bar{\rho}_i - \varepsilon}{\rho_i - \rho_{i-\frac{1}{2}}^*}, \frac{\bar{\rho}_i - \varepsilon}{\rho_i - \rho_{i+\frac{1}{2}}^*}, \frac{\bar{\rho}_i - \varepsilon}{\rho_i - \rho_{i-\frac{1}{2}}^*} \right\}.
\] (2.46)

Second, enforce the positivity of the internal energy \( e = E - \frac{1}{2} |u|^2 \) for the cells. Define \( \tilde{W}_i(r) = (\tilde{\rho}_i(r), \mathcal{M}_i(r), \mathcal{E}_i(r))^\top \). For each cell \( I_i \), if \( \min\left( e(\tilde{W}_{i-\frac{1}{2}}^+), e(\tilde{W}_i^*), e(\tilde{W}_{i+\frac{1}{2}}^-) \right) \geq 0 \) set \( \theta_i^2 = 1 \); otherwise,

\[
\theta_i^2 = \min \left\{ \frac{e(\tilde{W}_i)}{e(\tilde{W}_i) - e(\tilde{W}_{i-\frac{1}{2}}^+)}, \frac{e(\tilde{W}_i)}{e(\tilde{W}_i) - e(\tilde{W}_i^*)}, \frac{e(\tilde{W}_i)}{e(\tilde{W}_i) - e(\tilde{W}_{i+\frac{1}{2}}^-)} \right\}.
\]

Then we get the limited polynomial

\[
\tilde{W}_i(r) = \tilde{W}_i + \theta_i^2(\tilde{W}_i(r) - \tilde{W}_i).
\] (2.47)

It is easy to check that the cell average of \( \tilde{W}_i(r) \) over \( I_i \) is not changed and is still \( \bar{W}_i^{\text{nl}} \), and \( \tilde{W}_{i-\frac{1}{2}}^+, \tilde{W}_{i+\frac{1}{2}}^-, \tilde{W}_i^* \in G \). Moreover, the particular limiter does not destroy the high order accuracy in smooth regions.

If we perform the polynomial reconstruction by taking the integral on the area, we get another polynomial vector \( U_i(r) \), and we can also apply the above limiter on it similarly by replacing \( W_i(r) \) with \( U_i(r) \) and replacing \( W_i^* \) with \( U_i^* \) defined in (2.39).

### 2.8 High-order time discretization

To get a global high-order scheme, we make use of the SSP Runge-Kutta type method. For instance, the third-order algorithm is written as follows

Step 1,
\[ r_{i+\frac{1}{2}}^{n+1} = r_{i+\frac{1}{2}}^{n} + \Delta t^n u_i^{n+\frac{1}{2}}, \]
\[ A_i^{n+1} = r_{i+\frac{1}{2}}^{n+1} - r_{i-\frac{1}{2}}^{n+1}, \quad V_i^{n+1} = \frac{1}{2} A_i^{n+1} \left( r_{i+\frac{1}{2}}^{n+1} + r_{i-\frac{1}{2}}^{n+1} \right), \]
\[ \overline{p}_i^{n+1} = \frac{m_i}{V_i^{n+1}}, \quad (2.50) \]

Step 2,
\[ r_{i+\frac{1}{2}}^2 = \frac{3}{4} r_{i+\frac{1}{2}}^1 + \frac{1}{4} \left( r_{i+\frac{1}{2}}^1 + \Delta t^n u_i^{(1),*} \right), \]
\[ A_i^2 = r_{i+\frac{1}{2}}^2 - r_{i-\frac{1}{2}}^2, \quad V_i^2 = \frac{1}{2} A_i^2 \left( r_{i+\frac{1}{2}}^2 + r_{i-\frac{1}{2}}^2 \right), \]
\[ \overline{p}_i^2 = \frac{m_i}{V_i^2}, \quad (2.49) \]

Step 3,
\[ r_{i+\frac{1}{2}}^{n+1} = \frac{1}{3} r_{i+\frac{1}{2}}^{n} + \frac{2}{3} \left( r_{i+\frac{1}{2}}^{n} + \Delta t^n u_i^{(2),*} \right), \]
\[ A_i^{n+1} = r_{i+\frac{1}{2}}^{n+1} - r_{i-\frac{1}{2}}^{n+1}, \quad V_i^{n+1} = \frac{1}{2} A_i^{n+1} \left( r_{i+\frac{1}{2}}^{n+1} + r_{i-\frac{1}{2}}^{n+1} \right), \]
\[ \overline{p}_i^{n+1} = \frac{m_i}{V_i^{n+1}}, \quad (2.50) \]

At each step, the positivity-preserving limiter is performed to modify the polynomial. Notice that the SSP Runge-Kutta schemes are convex combinations of Euler forward time stepping, thus they are conservative, stable and positivity-preserving when the Euler forward time stepping is conservative, stable and positivity-preserving.

Based on the above description, we can summarize the algorithm flowchart of the third-order positivity-preserving Lagrangian scheme (2.5) as follows.

1. Perform the polynomial reconstruction to get \( W_i(r) \) or \( U_i(r) \) at the time level \( n \) from
the cell average $\mathbf{U}_i^n \in G, i = \{1, \ldots, N\}$ by applying the techniques of ENO reconstruction with local characteristic decomposition.

(2) Use the positivity-preserving limiter on $W_i(r)$ or $U_i(r)$ to get $\tilde{W}_i(r)$ or $\tilde{U}_i(r)$ such that

$\tilde{W}_{i+\frac{1}{2}}, \tilde{W}_{i-\frac{1}{2}}, \tilde{W}_i^* \in G$ or $\tilde{U}_{i+\frac{1}{2}}, \tilde{U}_{i-\frac{1}{2}}, \tilde{U}_i^* \in G$.

(3) Compute the intercell values $U_{i+\frac{1}{2}}^+$ and $U_{i+\frac{1}{2}}^-$ from the modified polynomial obtained from (2), then the numerical flux $u^*$ and $p^*$ can be determined by (2.7)-(2.8).

(4) Compute the time step $\Delta t^n$ by (2.37) or (2.44), depending on which way of the reconstruction is used.

(5) Update the position of each cell vertex, the area and volume of each cell and cell-average variables by (2.48) to get $r_{i+\frac{1}{2}}^{(1)}$ and $\mathbf{U}_i^{(1)}$.

(6) Based on $r_{i+\frac{1}{2}}^{(1)}$ and $\mathbf{U}_i^{(1)}$, repeat the above steps 1-3 to get the numerical flux, the vertex velocity, the area and volume of the cell for the second step of the third-order SSP Runge-Kutta method.

(7) Update the position of each cell vertex, the area and volume of each cell and cell-average variables by (2.49) to get $r_{i+\frac{1}{2}}^{(2)}$ and $\mathbf{U}_i^{(2)}$.

(8) Based on $r_{i+\frac{1}{2}}^{(2)}$ and $\mathbf{U}_i^{(2)}$, repeat the above steps 1-3 to get the numerical flux, the vertex velocity, the area and volume of the cell for the third step of the third-order SSP Runge-Kutta method.

(9) Update the position of each cell vertex, the area and volume of each cell and cell-average variables by (2.50) to get $r_{i+\frac{1}{2}}^{n+1}$ and $\mathbf{U}_i^{n+1}$.

3 Two-dimensional case

In this section, we focus on the compressible Euler system in the two-dimensional cylindrical coordinates. Its specific integral form in the Lagrangian framework can be described as
follows
\[
\begin{align*}
\frac{d}{dt} \iint_{\Omega(t)} \rho r dz dr &= 0, \\
\frac{d}{dt} \iint_{\Omega(t)} \rho u_z r dz dr &= -\int_{\Gamma(t)} \rho n_z r dl, \\
\frac{d}{dt} \iint_{\Omega(t)} \rho u_r r dz dr &= -\int_{\Gamma(t)} \rho n_r r dl + \int_{\Omega(t)} p dz dr, \\
\frac{d}{dt} \iint_{\Omega(t)} \rho E r dz dr &= -\int_{\Gamma(t)} pu \cdot n r dl
\end{align*}
\]  

(3.1)

where \(z\) and \(r\) are the axial and radial directions respectively. \(u = (u_z, u_r)\) where \(u_z, u_r\) are the velocity components in the \(z\) and \(r\) directions respectively, and \(n = (n_z, n_r)\) is the unit outward normal to the boundary \(\Gamma(t)\) in the \(z-r\) coordinates.

Similarly, the geometric conservation law in the two-dimensional case can be described as
\[
\frac{d}{dt} \iint_{\Omega(t)} dV = \int_{\Gamma(t)} u \cdot n r dl.
\]  

(3.2)

3.1 First-order scheme

At first, let us make some notations referring to [4, 6]. The 2D spacial domain \(\Omega\) is partitioned into quadrilateral computational cells, each of which is denoted by \(\Omega_c\) with the unique index \(c\). The boundary of the cell \(\partial \Omega_c\) is \(\partial \Omega_c\). Each vertex of the grid has its own unique index \(p\) and the counterclockwise ordered list of the vertices of \(\Omega_c\) is denoted by \(p(c)\). \(V_c\) and \(A_c\) denote the volume and the area of the cell \(\Omega_c\) respectively. There should be a remark that \(V_c\) is obtained by rotating this cell around the azimuthal \(z\)-axis (without the \(2\pi\) factor), which can be formulated as \(V_c = \iint_{\Omega_c} r dz dr\).

Besides, similar to (2.3), here we also define the cell averages of density, velocity and total energy as follows:
\[
\begin{align*}
\overline{\rho}_c &= \frac{1}{V_c} \iint_{\Omega_c} \rho r dz dr, \\
\overline{u_z}_c &= \frac{1}{m_c} \iint_{\Omega_c} \rho u_z r dz dr, \\
\overline{u_r}_c &= \frac{1}{m_c} \iint_{\Omega_c} \rho u_r r dz dr, \\
\overline{E}_c &= \frac{1}{m_c} \iint_{\Omega_c} \rho E r dz dr,
\end{align*}
\]  

(3.3)

where \(m_c = \iint_{\Omega_c} \rho r dz dr\) is the mass in the cell \(\Omega_c\), which keeps a constant during the time marching according to the first equation in (3.1). Based on these notations, we can rewrite
the system (3.1) in the following form

\[
\begin{align*}
\rho_c &= \frac{m_c}{V_c}, \\
\frac{d}{dt} \vec{u}_c &= -\int_{\partial \Omega_c} P_n z rdl, \\
\frac{d}{dt} \vec{u}_c &= -\int_{\partial \Omega_c} P_n x rdl + \iint_{\Omega_c} P d z d r, \\
\frac{d}{dt} \vec{E}_c &= -\int_{\partial \Omega_c} P u \cdot n r dl.
\end{align*}
\] (3.4)

We denote the coordinates and velocity of the vertex \( p \) as \((z_p, r_p)\) and \( \vec{u}_p = (u^z_p, u^r_p) \) respectively. \( l_{pp}^+ \) and \( l_{pp}^- \) stand for the length of the edges \([p, p^+]\) and \([p^-, p]\), and \( n_{pp}^+ \) and \( n_{pp}^- \) are the corresponding unit outward normals, where \( p^- \) and \( p^+ \) are the neighboring vertices of the vertex \( p \), see Figure 3.1.

![Figure 3.1: Notations of nodes and nodal variables](image)

In order to calculate the discrete gradient operator over the cell \( \Omega_c \), we need to denote two nodal pressures at each vertex \( p \) as \( \pi^c_p \) and \( \pi^c_{p^+} \), which can be seen in Figure 3.1. These two pressures are related to the two edges sharing the vertex \( p \). With these, we also need to define the half lengths and the unit outward normals of the edges connected to the vertex \( p \) in the following way [6]

\[
\begin{align*}
l_c^\perp &= \frac{1}{2} l_{pp}^-, \\
l_c^p &= \frac{1}{2} l_{pp}^+, \\
\vec{n}_c^\perp &= \frac{1}{2} \vec{n}_{pp}^-, \\
\vec{n}_c^p &= \frac{1}{2} \vec{n}_{pp}^+.
\end{align*}
\] (3.5)

Besides, the pseudo-radii \( r_c^\perp \) and \( r_c^p \) are defined as

\[
\begin{align*}
r_c^\perp &= \frac{1}{3} (2r_p + r_{p^-}), \\
r_c^p &= \frac{1}{3} (2r_p + r_{p^+})
\end{align*}
\] (3.6)

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by which the GCL in (3.2) can be rewritten as the following form referring to [6, 12].

$$\frac{d}{dt} V_c(t^n) = \sum_{p \in P(c)} (r_{p,c}^2 \mathbf{n}_{c,p}^t + r_{t,p}^2 \mathbf{n}_{c,p}^t) \cdot \mathbf{u}_p. \tag{3.7}$$

We will see that the formula is significant for our analysis of positivity preservation in the later section.

Similarly, we denote

$$z_c^p = \frac{1}{3} (2z_p + z_p^-), \quad z_c^p = \frac{1}{3} (2z_p + z_p^+),$$

$$\xi_c^p = \sqrt{(z_c^p)^2 + (r_c^p)^2}, \quad \xi_c^p = \sqrt{(z_c^p)^2 + (r_c^p)^2}. \tag{3.8}$$

Based on these notations, we can rewrite (3.4) as the following fully discrete finite volume Lagrangian scheme for the PDE system (3.1) in the two-dimensional case

$$\mathbf{U}^n_c = \left( \begin{array}{c} \frac{V_{c,n+1}}{V_{c,n}} \\ \frac{\bar{u}_{c,n} - \Delta t}{m_c} \sum_{p \in P(c)} (r_{p,c}^n \mathbf{u}_p \cdot \mathbf{n}_{c,p}^n + r_{t,p}^n \mathbf{u}_p \cdot \mathbf{n}_{c,p}^n) \\ \mathbf{E}_c^n - \frac{\Delta t}{m_c} \sum_{p \in P(c)} (r_{p,c}^n \mathbf{u}_p \cdot \mathbf{n}_{c,p}^n + r_{t,p}^n \mathbf{u}_p \cdot \mathbf{n}_{c,p}^n) \cdot \mathbf{u}_p \end{array} \right) \tag{3.9}$$

where $\mathbf{n}_{c,p} = (n_{c,z}^p, n_{c,r}^p)$ and $\mathbf{n}_{c,p}^t = (n_{c,z}^t, n_{c,r}^t)$.

If we denote $\mathbf{U}_c = (\rho_c^t, \mathbf{u}_c^t, E_c^t)^T$, then the nodal pressure $\pi_c^t$ and $\pi_c^t$ can be determined in the following way,

$$\pi_c^t = p_c^t - \bar{z}_c^t (\mathbf{u}_p - \bar{u}_c^t) \cdot \mathbf{n}_{c,p}^t,$$

$$\pi_c^t = p_c^t - \bar{z}_c^t (\mathbf{u}_p - \bar{u}_c^t) \cdot \mathbf{n}_{c,p}^t. \tag{3.10}$$

where $p_c^t$ and $p_c^t$ are the pressure values at the vertex $p$ which are computed from $\mathbf{U}_c$ and $\mathbf{U}_c$ respectively. Generally, $\mathbf{U}_c$ and $\mathbf{U}_c$ can be obtained by polynomial reconstruction, to be introduced in the next section for high-order accuracy. In particular, for the first-order approximation, $\mathbf{U}_c = \mathbf{U}_c = \bar{U}_c$, and $\bar{z}_c^t$ and $\bar{z}_c^t$ are the approximations of the acoustic impedance.

For the sake of the symmetry-preserving property, we limit the choices of $\bar{z}_c^t$ and $\bar{z}_c^t$ to be the Godunov acoustic solver for the two-dimensional case, i.e.

$$\bar{z}_c^t = \rho_c^t \alpha_c^t, \quad \bar{z}_c^t = \rho_c^t \alpha_c^t. \tag{3.11}$$
which is different from that in the one-dimensional case.

In order to ensure the scheme to be conservative for the momentum and total energy, the following condition needs to be satisfied [12],

\[
\sum_{c \in c(p)} (r_{pc}^c \pi_{pc}^c n_{pc}^c + r_{pc}^c \pi_{pc}^c n_{pc}^c) = 0, \tag{3.12}
\]

where \( c(p) \) is the set of cells sharing the vertex \( p \). Making use of (3.10) and (3.12), we can obtain

\[
\sum_{c \in c(p)} \left[ r_{pc}^c [p_{pc}^c n_{pc}^c - \tilde{z}_p^c (n_{pc}^c \otimes n_{pc}^c) (u_p - \tilde{u}_c^c)] + r_{pc}^c \pi_{pc}^c n_{pc}^c (u_p - \tilde{u}_c^c) \right] = 0, \tag{3.13}
\]

which leads to

\[
u_p = M_p^{-1} \sum_{c \in c(p)} \left[ r_{pc}^c [p_{pc}^c n_{pc}^c + \tilde{z}_p^c (n_{pc}^c \otimes n_{pc}^c) \tilde{u}_c^c] + r_{pc}^c \pi_{pc}^c [p_{pc}^c n_{pc}^c + \tilde{z}_p^c (n_{pc}^c \otimes n_{pc}^c) \tilde{u}_c^c] \right] \tag{3.14}
\]

where the matrix \( M_p \) reads as

\[
M_p = \sum_{c \in c(p)} M_{pc}, \tag{3.15}
\]

with \( M_{pc} = r_{pc}^c \pi_{pc}^c (n_{pc}^c \otimes n_{pc}^c) + r_{pc}^c \pi_{pc}^c (n_{pc}^c \otimes n_{pc}^c) \) being the projection matrix along the two normals \( n_{pc}^c \) and \( n_{pc}^c \).

Once the nodal velocity \( u_p \) at the vertex \( p \) has been determined, the cell vertex and the area and volume of the cell will be updated as follows [12],

\[
\begin{align*}
z_p^{n+1} &= z_p^n + \Delta t^n u_p^z, \\
r_p^{n+1} &= r_p^n + \Delta t^n u_p^r, \\
A_c^{n+1} &= \sum_{p \in p(c)} (r_{pc}^{c,n+1} l_{pc}^{c,n+1} n_{pc}^{c,r,n+1} + r_{pc}^{c,n+1} l_{pc}^{n+1} n_{pc}^{c,n+1}), \\
V_c^{n+1} &= \frac{1}{4} A_c^{n+1} \sum_{p \in p(c)} r_p^{n+1}.
\end{align*}
\tag{3.16}
\]

where \( u_p^z \) and \( u_p^r \) stand for the components of \( u_p \) in the \( z \) and \( r \) direction respectively.

### 3.2 First-order positivity- and symmetry-preserving scheme

In this section, we will discuss how to obtain the significant properties of positivity- and symmetry-preserving for the scheme (3.9).
Before that, let us recall the general CFL condition for the two-dimensional case, which is formulated as

\[
\Delta t^n \leq \lambda \min_c \frac{l_c^n}{a_c^n + |\bar{u}_c|} = \Delta t_1, \quad (3.17)
\]

where \( \lambda = 0.5 \) is Courant number, \( l_c^n \) is the length of the shortest edge of the cell \( \Omega_c \) and \( \bar{u}_c^n \) is the sound speed computed by the cell averages.

Now, we start with the preservation of positivity. To achieve that, we put an additional constraint on the time step \([19]\), which can be formulated as follows

\[
\Delta t^n \leq \min_c \frac{\sigma V_c^n}{\sum_{p \in \partial c} (r_c^p \bar{E}_p^c \bar{n}_p^c + r_c^p \bar{E}_p^{z} \bar{n}_p^{z}) \cdot \bar{u}_p} = \Delta t_2 \quad (3.18)
\]

with the factor \( \sigma \in (0, 1) \) to be determined.

Similarly, we define the admissible set \( G \) as

\[
G = \left\{ U = \begin{pmatrix} \rho \\ \mathbf{u} \\ E \end{pmatrix}, \ \rho > 0 \ \text{and} \ e = E - \frac{1}{2} | \mathbf{u} |^2 \right\}. \quad (3.19)
\]

Making use of the scheme (3.9), we can know that the density will be positive as long as the volume of the cell, \( V_c \), is positive, which can be ensured under the general CFL condition (3.17). Therefore, we will pay more attention to the positivity of the internal energy.

By adding and subtracting \( \frac{\Delta t^n}{m_c} \sum_{p \in \partial c} (r_c^p \bar{E}_p^c \bar{n}_p^c + r_c^p \bar{E}_p^{z} \bar{n}_p^{z}) \bar{p}_c^n \) and \( \frac{\Delta t^n}{m_c} \sum_{p \in \partial c} (r_c^p \bar{E}_p^c \bar{n}_p^c + r_c^p \bar{E}_p^{z} \bar{n}_p^{z}) \bar{p}_c^n \) in the second and third equations in (3.9) respectively, we can partition the system (3.9) into two parts, that is

\[
\mathbf{U}_c^{n+1} = \begin{pmatrix} \bar{p}_c^{n+1} \\ \bar{u}_c^{n+1} \\ \bar{E}_c^{n+1} \end{pmatrix} = \frac{1}{2} \mathbf{H} + \frac{1}{2} \mathbf{W}, \quad (3.20)
\]
\[
\begin{pmatrix}
\frac{m_c}{V_c^{n+1}} \\
\bar{u}_c^{x,n} - \frac{2\Delta t^n}{m_c} \sum_{p \in p(c)} (r_{p}^{c} c_p^{c,n} n_{p}^{c} + r_{p}^{c} c_p^{c,n} n_{n}^{c}) + \frac{2\Delta t^n}{m_c} \sum_{p \in p(c)} (r_{p}^{c} c_p^{c,n} n_{p}^{c} + r_{p}^{c} c_p^{c,n} n_{n}^{c}) \bar{P}_c^n \\
\bar{u}_c^{r,n} - \frac{2\Delta t^n}{m_c} \sum_{p \in p(c)} (r_{p}^{c} c_p^{c,n} n_{p}^{c} + r_{p}^{c} c_p^{c,n} n_{n}^{c}) + \frac{2\Delta t^n}{m_c} \sum_{p \in p(c)} (r_{p}^{c} c_p^{c,n} n_{p}^{c} + r_{p}^{c} c_p^{c,n} n_{n}^{c}) \bar{P}_c^n \\
\bar{E}_c^n - \frac{2\Delta t^n}{m_c} \sum_{p \in p(c)} (r_{p}^{c} c_p^{c,n} n_{p}^{c} + r_{p}^{c} c_p^{c,n} n_{n}^{c}) \cdot u_p
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\frac{m_c}{V_c^{n+1}} \\
\bar{u}_c^{x,n} - \frac{2\Delta t^n}{m_c} \sum_{p \in p(c)} (r_{p}^{c} c_p^{c,n} n_{p}^{c} + r_{p}^{c} c_p^{c,n} n_{n}^{c}) \bar{P}_c^n \\
\bar{u}_c^{r,n} - \frac{2\Delta t^n}{m_c} \sum_{p \in p(c)} (r_{p}^{c} c_p^{c,n} n_{p}^{c} + r_{p}^{c} c_p^{c,n} n_{n}^{c}) \bar{P}_c^n + \frac{2\Delta t^n}{m_c} A_c P_s \\
\bar{E}_c^n
\end{pmatrix}
\]

By this way, we express \( \bar{U}_c^{n+1} \) as a convex combination consisting of \( H \) and \( W \). Thus, assuming \( \bar{U}_c^n \in G \), if we are able to prove \( H, W \in G \), then we can ensure that \( \bar{U}_c^{n+1} \in G \).

By means of (3.10) and (3.21), we obtain

\[
e(H) = \bar{E}_c^{n+1} - \frac{1}{2}((\bar{u}_c^{x,n+1})^2 + (\bar{u}_c^{r,n+1})^2) = A_c \bar{u}_c^n + B_c
\]

where

\[
A_c = 1 - \frac{2\Delta t^n}{m_c} \bar{P}_c^n \sum_{p \in p(c)} (r_{p}^{c} c_p^{c,n} n_{p}^{c} + r_{p}^{c} c_p^{c,n} n_{n}^{c}) \cdot u_p \geq 1 - 2\sigma \frac{\bar{P}_c^n}{|P_c|},
\]

\[
B_c = \frac{2\Delta t^n}{m_c} \left( \sum_{p \in p(c)} (r_{p}^{c} c_p^{c,n} w_p^2 + r_{p}^{c} c_p^{c,n} w_p^2) - \frac{\Delta t^n}{m_c} \sum_{p \in p(c)} (r_{p}^{c} c_p^{c,n} w_p^2 + r_{p}^{c} c_p^{c,n} n_p^c + r_{p}^{c} c_p^{c,n} n_p^c) \right)^2
\]

with \( w_p = (u_p - \bar{u}_c^n) \cdot n_p^c \) and \( w_P = (u_P - \bar{u}_c^n) \cdot n_P^c \). Then, the sufficient condition to ensure \( e(H) > 0 \) is \( A_c > 0 \), \( B_c \geq 0 \). One can easily verify that if the factor \( \sigma \) in the time step constraint (3.18) satisfies \( \sigma \leq \frac{1}{2}\frac{|P_c|}{m_c} \), \( A_c \) will be positive. Then let us rewrite \( B_c \) into a
matrix-vector form as
\[ \mathcal{B}_c = \frac{2\Delta t^n}{m_c} w^T M_c w, \] (3.25)
where \( w = (w_1, w_\downarrow, w_2, w_\downarrow, w_3, w_\downarrow, w_4, w_\downarrow)^T. \) If we denote the node set \( Q_c = \{1, 2, 3, 3, 4, 4\}, \) the entries of the matrix \( M_c \) can be expressed explicitly as
\[
M_c^{qs} = \begin{cases} 
\frac{\Delta t^n}{m_c} r_{q=s}^c (1 - \frac{\Delta t^n}{m_c} r_{q=s}^c), & \text{if } q = s, \\
-\frac{\Delta t^n}{m_c} r_{q=s}^c r_{q=s}^c r_{q=s}^c (n_q^c \cdot n_s^c), & \text{if } q \neq s.
\end{cases}
\] (3.26)
with \( q, s \in Q_c. \)

We notice that \( M_c \) is a symmetric matrix and \( M_c \) is positively semi-definite if and only if
\[
\Delta t^n \leq \sum_{s \in Q_c} \frac{m_c}{r_{s}^c (n_q^c \cdot n_s^c)},
\] (3.27)
for any \( q \in Q_c. \) Acknowledging that \( |(n_q^c \cdot n_s^c)| \leq 1 \) always holds, we can ensure \( \mathcal{B}_c \geq 0 \) for all \( c \) under the following condition
\[
\Delta t^n \leq \min_c \frac{m_c}{\sum_{q \in Q_c} r_{q}^c (n_q^c \cdot n_s^c)} = \frac{m_c}{\sum_{p \in \mathcal{P}(c)} (r_{p}^c r_{p}^c + r_{p}^c r_{p}^c)} = \Delta t_3.
\] (3.28)

Now, let us try to find the sufficient condition for \( W \in G \) and we can get
\[
e(W) = e_c = -\frac{2\Delta t^n}{m_c} \bar{u}_c^t A_c P_s - \frac{1}{2} \left( \frac{2\Delta t^n}{m_c} \right)^2 (A_c P_s)^2 \\
- \frac{1}{2} \left( \frac{2\Delta t^n}{m_c} \right)^2 \left( \bar{P}_c \right)^2 \left| \sum_{p \in \mathcal{P}(c)} (r_{p}^c r_{p}^c n_p^c + r_{p}^c r_{p}^c n_p^c) \right|^2 \\
+ \frac{2\Delta t^n}{m_c} \sum_{p \in \mathcal{P}(c)} (r_{p}^c r_{p}^c n_p^c + r_{p}^c r_{p}^c n_p^c) \cdot \bar{u}_c \\
+ \left( \frac{2\Delta t^n}{m_c} \right)^2 A_c P_s \bar{P}_c \sum_{p \in \mathcal{P}(c)} (r_{p}^c r_{p}^c n_p^c + r_{p}^c r_{p}^c n_p^c r_p^c).
\] (3.29)

Recalling that in [12]
\[
\sum_{p \in \mathcal{P}(c)} (r_{p}^c r_{p}^c n_p^c + r_{p}^c r_{p}^c n_p^c) = 0, \\
\sum_{p \in \mathcal{P}(c)} (r_{p}^c r_{p}^c n_p^c + r_{p}^c r_{p}^c n_p^c r_p^c) = A_c,
\] (3.30)
we can rewrite $e(W)$ as
\[
e(W) = \varepsilon_c^n - 2 \left( \frac{\Delta t^n}{m_c} \right)^2 (A_c P_s - A_c \overline{P}_c^n)^2 - \frac{2 \Delta t^n}{m_c} (A_c P_s - A_c \overline{P}_c^n) \overline{w}_{c,n}.
\]
(3.31)

This is a quadratic inequality $e(W) \geq 0$, which is guaranteed by the following condition
\[
\Delta t^n \leq \min_c \left( -\frac{m_c \overline{w}_{c,n}}{2\mu} + \frac{m_c}{2|\mu|} \sqrt{(\overline{w}_{c,n})^2 + 2\overline{\mu}_c} \right) = \Delta t_4,
\]
(3.32)
with $\mu = A_c (P_s - \overline{P}_c^n)$.

**Theorem 3.1.** Consider the first-order scheme (3.9) based on the two-state Riemann solver in (3.10)-(3.15) for any positive definition of wavespeeds $\overline{z}_c$ and $\overline{z}_c$. Assume $\overline{U}_c^n \in G$, then $\overline{U}_c^{n+1}$ can also be in the set $G$ under the following time step constraints
\[
\Delta t^n \leq \min (\Delta t_1, \Delta t_2, \Delta t_3, \Delta t_4)
\]
(3.33)
where $\Delta t_1$ is the classic CFL condition (3.17), $\Delta t_2, \Delta t_3, \Delta t_4$ are defined in (3.18), (3.28), (3.32) respectively with $\sigma \leq \min \left( 1, \frac{\overline{P}_c \overline{u}_{c,n}}{2|\overline{u}_{c,n}|} \right)$ for $\Delta t_2$ and $\mu = A_c (P_s - \overline{P}_c^n)$ for $\Delta t_4$.

Now, let us consider a one-dimensional spherical symmetric problem simulated on an equal-angled polar grid, then we know the cell average $\overline{U}_c = (\overline{\rho}_c, \overline{u}_c, \overline{E}_c)^\top$ in the cell $\Omega_c$ are symmetric, which means $\overline{\rho}_c, \overline{E}_c$ and the component of $\overline{u}_c$ in the radial direction are the same in all cells with the same radial position, while the component of $\overline{u}_c$ in the angular direction is zero for all cells. According to [6], for the preservation of symmetry, there is a special requirement on the choice of the approximation $P_s$ for pressure in the source term, which should be determined as
\[
P_s = \frac{\xi_1 \pi_{\frac{1}{2}}^c + \xi_2 \pi_{\frac{3}{2}}^c + \xi_3 \pi_{\frac{3}{2}}^c + \xi_4 \pi_{\frac{3}{2}}^c}{\xi_1 + \xi_2 + \xi_3 + \xi_4}
\]
(3.34)
where $\pi_{\frac{1}{2}}, \pi_{\frac{3}{2}}, \pi_{\frac{3}{2}}$ and $\pi_{\frac{3}{2}}$ are the values of pressure related to the two radial edges of the cell $\Omega_c$. $\xi_1, \xi_2, \xi_3$ and $\xi_4$ are defined as (3.8).
Figure 3.2: The local \(\xi-\theta\) coordinates for the cell \(\Omega_c\).

**Theorem 3.2.** The first-order scheme (3.9) and (3.10) will keep both positivity and symmetry simultaneously when it is used for a spherical-symmetric problem on an equal-angle-zoned grid, if the time step \(\Delta t^n\) satisfies the constraints in Theorem 3.1 and \(P_s\) is taken as (3.34).

At the end of this section, we make a summary for our first order positivity- and symmetry-preserving Lagrangian scheme (3.9) for two-dimensional cylindrical coordinates and give the following algorithm flowchart.

1. Assuming \(U^n_c \in G\) at the time level \(n\), compute the nodal pressure \(\pi_c^e, \pi_c^p\) and velocity \(u_p\) by (3.10)-(3.15) for all cells.
2. Compute the pressure in the source term using (3.34).
3. Compute the time step \(\Delta t^n\) by (3.33).
4. Update the position of each cell vertex and then compute the area \(A_{c}^{n+1}\) and volume \(V_{c}^{n+1}\) of each cell by (3.16).
5. Compute the new averaged values \(\overline{U}_{c}^{n+1}\) by using the scheme (3.9).

### 3.3 High order scheme

For a high order accuracy, the values of \(U_c^e\) and \(U_c^p\) at the vertex \(p\) in the scheme (3.9) will not be the cell average \(\overline{U}_c\) any more, but can be obtained from the reconstruction polynomials.

Considering a high order scheme, with both positivity and symmetry preservation, we need to reconstruct polynomial function in each cell \(\Omega_c\) based on the cell-average information
of the cell $\Omega_c$ and its neighbors. Then, the values of $\pi^c_p, \pi^c_p$ can be determined as follows

$$
\pi^c_p = p^c_p - \bar{z}^c_p (u_p - u^c_p) \cdot n^c_p,
$$

$$
\pi^c_p = p^c_p - \bar{z}^c_p (u_p - u^c_p) \cdot n^c_p.
$$

(3.35)

The nodal velocity can be obtained as follows

$$
\mathbf{u}_p = M_p^{-1} \sum_{c \in \Omega(p)} \left[ r^c_{p,c} (n^c_p \otimes n^c_p) u^c_p + \bar{z}^c_p (n^c_p \otimes n^c_p) u^c_p + \bar{z}^c_p (n^c_p \otimes n^c_p) u^c_p \right] \quad (3.36)
$$

where the matrices $M_p$ reads as

$$
M_p = \sum_{c \in \Omega(p)} M_{pc}
$$

(3.37)

with $M_{pc} = r^c_{p,c} (n^c_p \otimes n^c_p) + \bar{z}^c_p (n^c_p \otimes n^c_p)$ being the projection matrix along the two normals $n^c_p$ and $n^c_p$.

Here $P_s$ is also determined as (3.34). Besides, if we simulate a spherically symmetric problem, we would hope to keep the symmetry property with the reconstructed polynomials, which put more restrictions on the reconstruction. We would first need to transform the cell averages of the variables in the neighboring cells which are involved in the reconstruction from the usual $(z, r)$ coordinates to the local polar coordinates $(\xi, \theta)$, where $\xi$ stands for the radial direction passing through the center of the edge and the origin, and $\theta$ refers to the angular direction, orthogonal to $\xi$ counter-clockwisely. Also, we perform the integral on the area rather than over the usual control volume to get the reconstruction polynomial [6], which can avoid the difficulty caused by different values of $r$ in different cells. However, this approach of reconstruction will limit the approximation to be at most second order accurate, regardless of reconstruction polynomial degrees. This is however not a restriction in the two-dimensional case as it is known that straight-edge quadrilateral based Lagrangian methods can be at most second order accurate anyway [2].

In this paper, we apply the same technique of reconstruction to get polynomials from the cell averages $\overline{U}_c = (\overline{\rho}_c, \overline{u}_c, \overline{E}_c)^T$ as that in [6]. Hence we will not give more details about it here.
After we perform the reconstruction on the local $\xi-\theta$ coordinates to obtain the polynomials, we transform them into the original $z-r$ coordinates for the calculation of the scheme (3.9). According to [6], after reconstruction along each edge, we get four linear polynomials in the cell $\Omega_c$,

$$\{U_{m,c}(z,r) = (\rho_{m,c}(z,r), u_{m,c}(z,r), E_{m,c}(z,r))^\top, m = 1 \ldots , 4\},$$

which satisfy

$$\int\int_{\Omega_c} U_{m,c}(z,r)dzdr = A_c \overline{U}_c. \quad (3.38)$$

Here we define the edge sequence $m, m = 1, \ldots , 4$ of the cell $\Omega_c$ as those connecting the vertices “1” and “2”, “2” and “3”, “3” and “4”, “4” and “1” respectively.

### 3.4 High order positivity- and symmetry-preserving scheme

Assume we have obtained the reconstruction polynomials $U_{m,c}(z,r)$ along each edge, and by using the relation in (3.38) we can get

$$\overline{U}_c = \frac{1}{4A_c} \int\int_{\Omega_c} \sum_{m=1}^{4} U_{m,c}(z,r)dzdr. \quad (3.39)$$

If we use a coordinate transformation to convert the cell $\Omega_c$ with the general quadrilateral shape in the $z-r$ coordinates to the square $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ in the $x-y$ coordinates (see Figure 3.3), we can define the set of Gauss-Lobatto quadrature points for the cell $\Omega_c$ to be $S_c = \{(z_{\alpha}, r_{\beta}), \alpha = 1, \ldots , K, \beta = 1, \ldots , K\}$, which are the pre-images under the coordinate transformation of the Gauss-Lobatto quadrature points in the square $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$. We require the Gauss-Lobatto quadrature rule to be exact for polynomials of degree $2k + 1$. This is because a polynomial of degree $k$ in the $z-r$ coordinates becomes a polynomial of degree $2k$ in the $x-y$ coordinates, since the Jacobian of the coordinate transformation is a bilinear function [5].
In fact, since the reconstruction polynomials are linear, we just need to apply the Simpson quadrature rule, in which the quadrature points consist of the cell vertices, the mid-points of each edge and the cell center, i.e. $K = 3$. $\omega_1 = \omega_3 = \frac{1}{6}, \omega_2 = \frac{2}{3}$. Based on these, we have

$$
\mathbf{U}_c = \frac{1}{A_c} \int \int_{\Omega_c} U_{m,c}(z, r) \, dz \, dr \\
= \frac{1}{A_c} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} U_{m,c}(g_{m,c}(x, y)) \left| \frac{\partial g_{m,c}(x, y)}{\partial (x, y)} \right| \, dx \, dy \\
= \frac{1}{A_c} \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \omega_{\alpha} \omega_{\beta} \left| \frac{\partial g_{m,c}(x, y)}{\partial (x, y)} \right|_{(x_{\alpha,\beta}, y_{\beta})} \, U_{m,c}(z_{\alpha}, r_{\beta}),
$$

(3.40)

where $U_{m,c}(z_{\alpha}, r_{\beta})$ is the value of $U_{m,c}(z, r)$ at the corresponding Gauss-Lobatto quadrature points. $\left| \frac{\partial g_{m,c}(x, y)}{\partial (x, y)} \right|_{(x_{\alpha,\beta}, y_{\beta})}$ is the Jacobian for the coordinate transformation [5]. Then if we denote $|J|_{m,c}^{\alpha,\beta} = \left| \frac{\partial g_{m,c}(x, y)}{\partial (x, y)} \right|_{(x_{\alpha,\beta}, y_{\beta})}$ and $U_{m,c}^{\alpha,\beta} = U_{m,c}(z_{\alpha}, r_{\beta})$, we can rewrite the integral (3.40) as
\[
\mathcal{U}_c = \frac{1}{A_c} \left( \omega_1 \omega_2 |J|_{1,c}^{3,2} U_{1,c}^c + \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \omega_\alpha \omega_\beta |J|_{1,c}^{\alpha,\beta} U_{\alpha,\beta}^1 U_{\alpha,\beta}^c + \omega_1^2 |J|_{1,c}^3 U_T^c + \omega_1^2 |J|_{1,c}^3 U_{2,c}^c \right)
\]

\[
= \frac{1}{A_c} (\omega_1^2 |J|_{1,c}^3 U_T^c + \omega_1^2 |J|_{1,c}^3 U_{2,c}^c + \omega_1^2 U_{1,c}^c)
\]

\[
\mathcal{U}_c = \frac{1}{A_c} \left( \omega_1 \omega_2 |J|_{2,c}^{3,3} U_{2,c}^c + \sum_{\alpha=1}^{3} \sum_{\beta=1}^{2} \omega_\alpha \omega_\beta |J|_{2,c}^{\alpha,\beta} U_{\alpha,\beta}^{2,c} + \omega_1^2 |J|_{2,c}^3 U_T^c + \omega_1^2 |J|_{2,c}^3 U_{3,c}^c \right)
\]

\[
= \frac{1}{A_c} (\omega_1^2 |J|_{2,c}^3 U_T^c + \omega_1^2 |J|_{2,c}^3 U_{3,c}^c + \omega_2^2 U_{2,c}^c)
\]

\[
\mathcal{U}_c = \frac{1}{A_c} \left( \omega_1 \omega_2 |J|_{3,c}^{1,2} U_{1,2}^c + \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \omega_\alpha \omega_\beta |J|_{3,c}^{\alpha,\beta} U_{\alpha,\beta}^{3,c} + \omega_1^2 |J|_{3,c}^3 U_T^c + \omega_1^2 |J|_{3,c}^3 U_{1,c}^c \right)
\]

\[
= \frac{1}{A_c} (\omega_1^2 |J|_{3,c}^3 U_T^c + \omega_1^2 |J|_{3,c}^3 U_{1,c}^c + \omega_3^3 U_{3,c}^c)
\]

\[
\mathcal{U}_c = \frac{1}{A_c} \left( \omega_1 \omega_2 |J|_{4,c}^{2,4} U_{2,4}^c + \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \omega_\alpha \omega_\beta |J|_{4,c}^{\alpha,\beta} U_{\alpha,\beta}^{4,c} + \omega_1^2 |J|_{4,c}^3 U_T^c + \omega_1^2 |J|_{4,c}^3 U_{1,c}^c \right)
\]

\[
= \frac{1}{A_c} (\omega_1^2 |J|_{4,c}^3 U_T^c + \omega_1^2 |J|_{4,c}^3 U_{1,c}^c + \omega_4^4 U_{4,c}^c)
\]

where

\[
\omega_1^1 = \frac{1}{A_c} \left( \omega_1 \omega_2 |J|_{1,c}^{3,2} + \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \omega_\alpha \omega_\beta |J|_{1,c}^{\alpha,\beta} \right),
\]

\[
\omega_2^1 = \frac{1}{A_c} \left( \omega_1 \omega_2 |J|_{2,c}^{3,3} + \sum_{\alpha=1}^{3} \sum_{\beta=1}^{2} \omega_\alpha \omega_\beta |J|_{2,c}^{\alpha,\beta} \right),
\]

\[
\omega_3^1 = \frac{1}{A_c} \left( \omega_1 \omega_2 |J|_{3,c}^{1,2} + \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \omega_\alpha \omega_\beta |J|_{3,c}^{\alpha,\beta} \right),
\]

\[
\omega_4^1 = \frac{1}{A_c} \left( \omega_1 \omega_2 |J|_{4,c}^{2,4} + \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \omega_\alpha \omega_\beta |J|_{4,c}^{\alpha,\beta} \right).
\]

Therefore, we have

\[
\mathcal{U}_c = \frac{1}{4A_c} \int \int_{R_c} \sum_{m=1}^{4} U_{m,c}(z, r) \, dz \, dr
\]

\[
= \frac{1}{4} \sum_{m=1}^{4} \omega_m^m U_{m,c} + \sum_{p \in p(c)} (\omega_p U_p^c + \omega_p^p U_p^c)
\]

(3.41)
where

\[
\omega_1 = \frac{1}{4A_{c}} \omega_1^2 |J|_{1,c}^{3,1}, \quad \omega_\tau = \frac{1}{4A_{c}} \omega_1^2 |J|_{1,c}^{3,1},
\]

\[
\omega_2 = \frac{1}{4A_{c}} \omega_1^2 |J|_{1,c}^{3,3}, \quad \omega_\tau = \frac{1}{4A_{c}} \omega_1^2 |J|_{2,c}^{3,3},
\]

\[
\omega_3 = \frac{1}{4A_{c}} \omega_1^2 |J|_{2,c}^{1,3}, \quad \omega_\tau = \frac{1}{4A_{c}} \omega_1^2 |J|_{3,c}^{1,3},
\]

\[
\omega_4 = \frac{1}{4A_{c}} \omega_1^2 |J|_{3,c}^{1,1}, \quad \omega_\tau = \frac{1}{4A_{c}} \omega_1^2 |J|_{4,c}^{1,1}.
\]

In fact, we do not need to know the values at the corresponding quadrature points except the nodes at the cell edges according to the Remarks 3.3 and 3.4 in [5], that is to say, we can directly express it as

\[
\begin{align*}
U^*_{1,c} &= \frac{1}{\omega_1^*} \left( A_c U_{c}^0 - 4 \omega_1 U_{c}^1 + 4 \omega_2 U_{c}^2 \right), \\
U^*_{2,c} &= \frac{1}{\omega_2^*} \left( A_c U_{c}^0 - 4 \omega_2 U_{c}^1 + 4 \omega_3 U_{c}^3 \right), \\
U^*_{3,c} &= \frac{1}{\omega_3^*} \left( A_c U_{c}^0 - 4 \omega_3 U_{c}^1 + 4 \omega_4 U_{c}^4 \right), \\
U^*_{4,c} &= \frac{1}{\omega_4^*} \left( A_c U_{c}^0 - 4 \omega_4 U_{c}^1 + 4 \omega_1 U_{c}^2 \right),
\end{align*}
\]  (3.42)

which is quite useful when implementing the positivity-preserving limiter.

Therefore, the scheme (3.9) in the high-order case can be rewritten as

\[
\overline{U}_{c}^{n+1} = \omega^* U_{c}^{n} + \sum_{p \in p(c)} \left( \omega_p \hat{F}_p + \omega_\tau \hat{F}_\tau \right) 
\]  (3.43)

where \( U_{c}^{n} = \frac{1}{4 \omega_{c}} \sum_{m=1}^{4} \omega_{c}^{m} U_{m,c}^{n} \) but the first component reads as \( \frac{m_c}{\omega V_{c}^{n+1}} \) with \( \omega^* = \frac{1}{4} \sum_{m=1}^{4} \omega_{c}^{m}, \omega = \omega^* + \sum_{p \in p(c)} \left( \omega_p + \omega_\tau \right) \),

\[
\hat{F}_p = \begin{pmatrix}
\frac{m_c}{\omega V_{c}^{n+1}} \\
\omega_{c}^{z} \frac{\Delta t^n}{\omega_{p} m_{c}} \nu_{p,c}^{z} c_{p,c}^{z} \\
\omega_{c}^{r} \frac{\Delta t^n}{\omega_{p} m_{c}} \nu_{p,c}^{r} c_{p,c}^{r} \\
\Delta t^n \frac{1}{\omega_{p} m_{c}} \left( \frac{1}{8} A_c \right) P_s \\
\frac{\Delta t^n}{\omega_{p} m_{c}} \nu_{p,c}^{c} c_{p,c}^{c} \cdot u_p
\end{pmatrix}
\]  (3.44)

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\[
\frac{m_c}{\omega V_{c}^{n+1}}\begin{pmatrix}
\frac{\Delta t^n}{\omega_p m_c} r_p^c p^c, c^z + r_p^c p^c, n_p^c c^z \\
\frac{\Delta t^n}{\omega_p m_c} r_p^c p^c, c^r + r_p^c p^c, n_p^c c^r + \frac{\Delta t^n}{\omega_p m_c} \left(\frac{1}{8} A_c\right) P_s \\
\frac{\Delta t^n}{\omega_p m_c} r_p^c p^c, n_p^c u_p c^r + \frac{\Delta t^n}{\omega_p m_c} \left(\frac{1}{8} A_c\right) P_s \\
\end{pmatrix}.
\] (3.45)

Obviously, \( \overline{U}_{c}^{n+1} \) is a convex combination consisting of \( U_{c}^* \) and \( \hat{F}^p, \hat{F}_p \), hence we only need to ensure \( U_{c}^* \) and \( \forall p \in p(c), \hat{F}^p, \hat{F}_p \) are in the admissible set \( G \). For each \( \hat{F}_p \), we hope to apply exactly the same techniques as those presented in the first order case to make sure \( \hat{F}_p \in G \) [19], similarly for \( \hat{F}_p \in G, \forall p \in p(c) \).

Let us define all the corner nodes for each cell as a set \( Q_c = \{1, \overline{1}, 2, \overline{2}, 3, \overline{3}, 4, \overline{4}\} \) and assume the following formula always holds

\[
\sum_{q \in Q_c} r_q^c \pi_q^c a^c n_q^c = 0
\] (3.46)

where \( \pi_q^c a \) stands for some artificial pressure and its corresponding artificial velocity is \( u_c \) and \( \pi_q^c a, u_c \) are to be determined later.

By adding the artificial term, \( \hat{F}_p \) is changed into

\[
\hat{F}_p = \begin{pmatrix}
\frac{m_c}{\omega V_{c}^{n+1}} \\
\frac{\Delta t^n}{\omega_p m_c} r_p^c p^c, c^z + r_p^c p^c, n_p^c c^z \\
\frac{\Delta t^n}{\omega_p m_c} r_p^c p^c, c^r + r_p^c p^c, n_p^c c^r + \frac{\Delta t^n}{\omega_p m_c} \left(\frac{1}{8} A_c\right) P_s \\
\frac{\Delta t^n}{\omega_p m_c} r_p^c p^c, n_p^c u_p c^r + \frac{\Delta t^n}{\omega_p m_c} \left(\frac{1}{8} A_c\right) P_s \\
\end{pmatrix}.
\] (3.47)

We notice that the condition (3.46) can be rewritten as

\[
\sum_{q \in Q_c \setminus p} r_q^c \pi_q^c a^c n_q^c = -r_p^c \pi_p^c a^c n_p^c \cdot u_p.
\] (3.48)

To make \( \hat{F}_p \) defined in (3.47) mimicking the first order scheme (3.9), we introduce

\[
\pi_q^p = \begin{pmatrix}
\pi_q^c , & \text{if } q = p \\
\pi_q^c a, & \text{otherwise}
\end{pmatrix}
\] (3.49)
where \( \pi^q_p \) is also some artificial pressure and its corresponding artificial velocity is denoted as \( \mathbf{u}^q_p \). Then based on (3.48), \( \tilde{F}_p \) for arbitrary \( p \) finally reads

\[
\tilde{F}_p = \begin{pmatrix}
\frac{m_c}{\omega \tilde{V}_c^{n+1}} \\
\frac{\Delta t^n}{\omega_p m_c} \sum_{q \in Q_c} r_q^c \pi^q_p \mathbf{n}_q^c \\
\frac{\Delta t^n}{\omega_p m_c} \sum_{q \in Q_c} r_q^c \pi^q_p \mathbf{n}_q^c + \frac{\Delta t^n}{\omega_p m_c} (\frac{1}{8} A_c) P_s \\
\frac{\Delta t^n}{\omega_p m_c} \sum_{q \in Q_c} r_q^c \pi^q_p \mathbf{n}_q^c \cdot \mathbf{u}_p^q
\end{pmatrix}.
\]

If we apply the formula in (3.35) to the artificial numerical flux \( \pi^q_p \), we can get

\[
\sum_{q \in Q_c} r_q^c \pi^q_p \mathbf{n}_q^c = \sum_{q \in Q_c} r_q^c (p_q^c - \pi^q_p) (u_p^q - u_p^q) \cdot \mathbf{n}_q^c
\]

\[
= p_q^c \sum_{q \in Q_c} r_q^c \mathbf{n}_q^c - \sum_{q \in Q_c} r_q^c \pi^q_p (u_p^q - u_p^q) \cdot \mathbf{n}_q^c
\]

\[
= A_p \mathbf{e}_r - \sum_{q \in Q_c} r_q^c \pi^q_p (u_p^q - u_p^q) \cdot \mathbf{n}_q^c
\]

where \( \mathbf{e}_r = (0, 1)^T \), \( M^q_p = r_q^c \pi^q_p (\mathbf{n}_q^c \otimes \mathbf{n}_q^c) \). Making use of (3.47) and (3.48), the last relation above can be rewritten as

\[
(\pi^p - \pi^c) r_q^c \mathbf{n}_q^c = A_p \mathbf{e}_r - M^p_p (u_p - u_p^c) - \sum_{q \in Q_c \setminus p} M^q_p (u_p - u_p^c)
\]

\[
= A_p \mathbf{e}_r - M^p_p (u_p - u_p^c) - M^p_c (u_p - u_p^c)
\]

where \( M^p_p = \sum_{q \in Q_c \setminus p} M^q_p \). Finally, by means of (3.35), we can define the artificial pressure \( \pi^c_p \) as

\[
\pi^c_p r_q^c \mathbf{n}_q^c = p_q^c \mathbf{e}_r - M^c_p (u_p - u_p^c) - A_p \mathbf{e}_r,
\]

then the condition (3.48) makes us to determine the artificial velocity \( \mathbf{u}_p \) uniquely

\[
\mathbf{u}_p = \left( \sum_{p \in p(c)} (M^p_p + M^c_p) \right)^{-1} \sum_{p \in p(c)} \left( \left( M^p_p + M^c_p \right) u_p - p_q^c r_q^c \mathbf{n}_q^c - p_q^c \pi^q_p \mathbf{n}_q^c + A_c (p_q^c + p_q^c) \mathbf{e}_r \right).
\]

(3.52)
We can see that \( \hat{F}_p \) is nothing but the first order scheme defined in (3.9), so is \( \hat{F}_p \).

Therefore by using the same analysis as that in the first order case, we have the following conclusion.

**Theorem 3.3.** Consider the Lagrangian scheme (3.9) based on the two-state Riemann solver defined in (3.35) with any positive definition of \( \bar{z}_c^p \) and \( \bar{z}_c^p \). Assume \( \bar{U}_c^m \in G \) and \( \bar{U}_c^c \in G \) for all \( p \), then \( \bar{U}_c^{n+1} \in G \) under the following time step constraint

\[
\Delta t^n \leq \min (\Delta t_1, \Delta t_2, \Delta t_3, \Delta t_4, \Delta t_5)
\]

(3.53)

where \( \Delta t_1 \) is the CFL condition (3.17),

\[
\Delta t_2 = \min_{p,c} \left( \frac{2\omega_p m_c}{\sum_{q \in Q_c} r_q^c z_q^c p}, \frac{2\sigma_p \omega_p m_c}{\rho_p^c (u_p - u_c) \cdot (n_p^c)} \right),
\]

\[
\Delta t_3 = \min_{p,c} \left( \frac{2\omega_p m_c}{\sum_{q \in Q_c} r_q^c z_q^c p}, \frac{2\sigma_p \omega_p m_c}{\rho_p^c (u_p - u_c) \cdot (n_p^c)} \right),
\]

with \( \sigma_p \leq \min \left( 1, \frac{\rho_p^c e_p^c}{2|\mu_p^c|} \right), \sigma_p \leq \min \left( 1, \frac{\rho_p^c e_p^c}{2|\mu_p^c|} \right), \) and

\[
\Delta t_4 = \min_{p,c} \left( -\frac{\omega_p m_c u^c_p}{2 \mu_1} + \frac{\omega_p m_c}{2 |\mu_1|} \sqrt{(u^c_p)^2 + 2e_p^c} \right),
\]

\[
\Delta t_5 = \min_{p,c} \left( -\frac{\omega_p m_c u^c_p}{2 \mu_2} + \frac{\omega_p m_c}{2 |\mu_2|} \sqrt{(u^c_p)^2 + 2e_p^c} \right)
\]

where

\[
\mu_1 = \frac{1}{8} A_c P_s - A_c \pi_p^c,
\]

\[
\mu_2 = \frac{1}{8} A_c P_s - A_c \pi_p^c.
\]

### 3.5 Positivity-preserving limiter

To achieve the preservation of symmetry, we need to perform the polynomial reconstruction and positivity-preserving limitation along each edge and in the local \( \xi-\theta \) coordinates for each cell \( \Omega_c \) as that we presented before, that is to say, we first need to transform the polynomials in the \( z-r \) coordinates \( U_{m,c}(z, r) = (\rho_{m,c}(z, r), u_{m,c}(z, r), E_{m,c}(z, r))^\top \) into the polynomials defined in the \( \xi-\theta \) coordinates \( U_{m,c}(\xi, \theta) = (\rho_{m,c}(\xi, \theta), u_{m,c}(\xi, \theta), E_{m,c}(\xi, \theta))^\top \).
Under the assumption $\bar{U}_c \in G$, we would like to modify the polynomial reconstruction $U_{m,c}(\xi, \theta)$ with a constant $\theta_c$ into another polynomial $\tilde{U}_{m,c}(\xi, \theta)$ such that the values of $\tilde{U}_{m,c}(\xi, \theta)$ at its corresponding Gauss-Lobatto quadrature points can be set in $G$. The implementation is similar to that for the high-order scheme in the one-dimensional case, which can be described as the following modification on the reconstruction polynomial

$$
\tilde{U}_{m,c}(\xi, \theta) = U_c + \theta_c(U_{m,c}(\xi, \theta) - U_c),
$$

where $\theta_c \in [0, 1]$ is to be determined, such that $\tilde{U}_{m,c}(\xi, \theta) \in G$ for all $(\xi, \theta) \in S_c$.

In fact, we do not need to modify the values at all Gauss-Lobatto quadrature points, we only need to modify the values of $U_{m,c}(\xi, \theta)$ at the two nodes of its corresponding edge and $U^*_{m,c}$ defined in (3.42), which represents a lumped contribution from all other Gauss-Lobatto quadrature points.

First, let us enforce the admissibility of the density. Choose a small number $\varepsilon$ such that $\overline{p}_c \geq \varepsilon$ for all $c$. In practice, we usually take $\varepsilon = 10^{-13}$. For the four edges of each cell $\Omega_c$, compute

$$
\begin{align*}
\tilde{\rho}_{1,c}(\xi, \theta) &= \overline{\rho}_c + \theta_{1,c}(\rho_{1,c}(\xi, \theta) - \overline{\rho}_c), \quad \theta_{1,c} = \min \left\{ 1, \frac{\rho_c - \varepsilon}{\overline{\rho}_c - \rho_{1,c}}, \frac{\rho_c - \varepsilon}{\overline{\rho}_c - \rho_{2,c}}, \frac{\rho_c - \varepsilon}{\overline{\rho}_c - \rho_{1,c}} \right\}; \\
\tilde{\rho}_{2,c}(\xi, \theta) &= \overline{\rho}_c + \theta_{2,c}(\rho_{2,c}(\xi, \theta) - \overline{\rho}_c), \quad \theta_{2,c} = \min \left\{ 1, \frac{\rho_c - \varepsilon}{\overline{\rho}_c - \rho_{1,c}}, \frac{\rho_c - \varepsilon}{\overline{\rho}_c - \rho_{2,c}}, \frac{\rho_c - \varepsilon}{\overline{\rho}_c - \rho_{1,c}} \right\}; \\
\tilde{\rho}_{3,c}(\xi, \theta) &= \overline{\rho}_c + \theta_{3,c}(\rho_{3,c}(\xi, \theta) - \overline{\rho}_c), \quad \theta_{3,c} = \min \left\{ 1, \frac{\rho_c - \varepsilon}{\overline{\rho}_c - \rho_{3,c}}, \frac{\rho_c - \varepsilon}{\overline{\rho}_c - \rho_{2,c}}, \frac{\rho_c - \varepsilon}{\overline{\rho}_c - \rho_{3,c}} \right\}; \\
\tilde{\rho}_{4,c}(\xi, \theta) &= \overline{\rho}_c + \theta_{4,c}(\rho_{4,c}(\xi, \theta) - \overline{\rho}_c), \quad \theta_{4,c} = \min \left\{ 1, \frac{\rho_c - \varepsilon}{\overline{\rho}_c - \rho_{4,c}}, \frac{\rho_c - \varepsilon}{\overline{\rho}_c - \rho_{3,c}}, \frac{\rho_c - \varepsilon}{\overline{\rho}_c - \rho_{4,c}} \right\}.
\end{align*}
$$

Second, enforce the positivity of the internal energy $e$ for the cells. Define $\tilde{U}_{m,c}(\xi, \theta) =$
For the four edges of each cell \( \Omega_c \),

\[
\theta_{1,c}^2 = \min \left\{ \frac{e(U_c)}{e(U_c) - e(U_1^\ast)}, \frac{e(U_c)}{e(U_c) - e(U_2^\ast)}, \frac{e(U_c)}{e(U_c) - e(U_1^1_c)} \right\};
\]
\[
\theta_{2,c}^2 = \min \left\{ \frac{e(U_c)}{e(U_c) - e(U_T)}, \frac{e(U_c)}{e(U_c) - e(U_2^\ast)}, \frac{e(U_c)}{e(U_c) - e(U_2^{})} \right\};
\]
\[
\theta_{3,c}^2 = \min \left\{ \frac{e(U_c)}{e(U_c) - e(U_3^\ast)}, \frac{e(U_c)}{e(U_c) - e(U_3^{})} \right\};
\]
\[
\theta_{4,c}^2 = \min \left\{ \frac{e(U_c)}{e(U_c) - e(U_4^\ast)}, \frac{e(U_c)}{e(U_c) - e(U_4^{})} \right\};
\]

Then we get the following limited polynomial relative to each edge

\[
\tilde{U}_{m,c}(\xi, \theta) = U_c + \theta_{m,c}^2(\tilde{U}_{m,c}(\xi, \theta) - U_c).
\]  

(3.56)

After getting the limitation factors \( \theta_{m,c}^2 \) for \( m = 1, \ldots, 4 \), we can get the modified values at two node points along each edge of the cell \( \Omega_c \). Then we need to transform back into the \( z-r \) coordinates to update the time marching. Performing the limiter in the local \( \xi-\theta \) coordinates can ensure that the values at the Gauss-Lobatto points with the same radial position and different angular position are the same, thus the property of symmetry can be maintained.

Besides, it is easy to check that the cell average of \( \tilde{U}_{m,c}(\xi, \theta) \) over \( \Omega_c \) is not changed and is still \( U_c^a \), and \( \tilde{U}_{m,c}(\xi_\alpha, \theta_\beta) \in G \) for all relevant \( \alpha, \beta \) (including the lumped ones). Moreover, the particular limiter does not destroy the high order accuracy in smooth regions and can keep symmetry.

### 3.6 High order time discretization

To obtain a Lagrangian scheme with uniformly second order accuracy both in time and space, the time march stepping can be implemented by a second order strong stability preserving (SSP) Runge-Kutta type method, which is detailed in [6] .

At each step, the limiter operation is performed to modify the polynomial. Notice that the SSP Runge-Kutta schemes are convex combinations of Euler forward time stepping, thus
they are conservative, stable and positivity-preserving when the Euler forward time stepping is conservative, stable and positivity-preserving.

Based on these, we can summarize the algorithm flowchart of the second order positivity- and symmetry-preserving Lagrangian scheme (3.9) based on (3.35)-(3.37) in two-dimensional cylindrical coordinates as follows.

(1) Reconstruct the polynomials in the left and right cells along each edge at the time step \( n \) from the cell average \( \vec{U}_c^n = (\rho_c^n, u_n^c, E_n^c)^\top = (\rho_c^n, u_\xi^n, u_\theta^n, E_c^n)^\top \) and its neighbors by using the techniques of reconstruction described in [6], where \((\xi, \theta)\) is the local polar coordinates, that is, \( \xi \) is the radial direction passing through the center of the edge and the origin, and \( \theta \) is the angular direction which is counter-clockwisely orthogonal to \( \xi \). Here along each edge, we have three Gauss-Lobatto quadrature points, that is, two vertices and the middle point of the edge respectively. Then the left and right values of the corresponding variables at vertices along this edge can be obtained.

(2) In each cell \( \Omega_c \), perform the positivity-preserving limiter on the four sets of polynomials associated to the four edges as described in the section 3.5, and then get the modified values at the two nodes along the four edges of the cell \( \Omega_c \) respectively.

(3) Transform the above obtained modified values along each edge to the physical \((z, r)\) coordinates and then calculate the nodal pressure and velocity by using the algorithm described in by (3.35)-(3.37).

(4) Calculate the source term by (3.34) and the time step \( \Delta t^n \) by (3.53).

(5) Update the position of each cell vertex and the conserved variables by Stage 1 of the second order strong stability preserving (SSP) Runge-Kutta method described in [6].

(6) Based on the updated position of each cell vertex and the conserved variables obtained by Step 5, repeat the above steps 1-4 to get the nodal pressure, nodal velocity, numerical flux and source term for Stage 2 of the second order SSP Runge-Kutta method.

(7) Update the position of each cell vertex and the conserved variables by Stage 2 of the second order SSP Runge-Kutta method to get the position \((z_p^{n+1}, r_p^{n+1})\) and the new
conserved variables $\mathbf{U}^{n+1}_c$ at the time step $n + 1$.

4 Numerical examples

In this section, we choose several challenging numerical examples in one- and two-dimensional cylindrical coordinates to show the performance of our first order and high order positivity-preserving and symmetry-preserving Lagrangian schemes. The examples are performed on the ideal gas with $\gamma = 5/3$ and the initially equal-angled polar grid is used in the two-dimensional case unless otherwise stated. The Godunov acoustic solver is used for all numerical tests, i.e. $\tilde{z} = \rho a$. All these examples encounter the problem of negative internal energy if the usual high order Lagrangian scheme without applying the positivity-preserving limiter is used.

In order to use larger time steps to improve efficiency as much as possible, we do not restrict the time step as strictly as presented in previous theorems in our actual code. Instead, we take the time step $\Delta t^n$ using just the classical CFL condition, defined by (2.12) and (3.17) relative to the one- and two-dimensional case respectively, to march to the time level $n + 1$. If the internal density thus obtained is positive, we will continue to the next time step; otherwise, we will come back to the previous time level $n$, and take a smaller time step such as $\frac{1}{2} \Delta t^n$, and proceed as before. The theorems in the previous sections ensure that we only need to come back a finite number of times before we will obtain a positive internal energy.

4.1 One-dimensional tests

Example 1. Accuracy test.

We first test the accuracy of our schemes on a free-expansion problem. The initial condition is taken as

$$\rho = 1, \quad u = 0, \quad p = 1 - r^4, \quad r \in [0, 1].$$

Free boundary condition is applied on the outer boundary. The errors and accuracy of the scheme at $t = 1$ are listed in Tables 4.1-4.3 which are measured on the interval $[\frac{1}{10}N, \frac{9}{10}N]$ to
remove the influence from the boundary. The percentage of the cells in which the positivity-preserving limiter has been performed is also listed in the table. We take the result of the third order positivity-preserving Lagrangian scheme with 10000 cells as our reference solution when computing the errors. From these tables, we can clearly see that the first order and third order positivity-preserving schemes with the positivity-preserving limiter have achieved the expected order of accuracy in both $L_1$ and $L_\infty$ norms for all the evolved conserved variables.

Table 4.1: Errors of the first order scheme in 1D cylindrical coordinates using $N$ initially uniform cells

<table>
<thead>
<tr>
<th>$N$</th>
<th>Norm</th>
<th>Density</th>
<th>order</th>
<th>Momentum</th>
<th>order</th>
<th>Energy</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>$L_1$</td>
<td>0.44E-2</td>
<td>0.45E-2</td>
<td>0.56E-2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_\infty$</td>
<td>0.15E-1</td>
<td>0.85E-2</td>
<td>0.18E-1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>$L_1$</td>
<td>0.23E-2</td>
<td>0.93</td>
<td>0.97</td>
<td>0.29E-2</td>
<td>0.93</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_\infty$</td>
<td>0.78E-2</td>
<td>1.00</td>
<td>0.86</td>
<td>0.92E-2</td>
<td>0.98</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>$L_1$</td>
<td>0.11E-2</td>
<td>0.96</td>
<td>1.00</td>
<td>0.15E-2</td>
<td>0.96</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_\infty$</td>
<td>0.39E-2</td>
<td>1.01</td>
<td>1.01</td>
<td>0.46E-2</td>
<td>1.00</td>
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</tr>
<tr>
<td>400</td>
<td>$L_1$</td>
<td>0.61E-3</td>
<td>0.96</td>
<td>0.56E-3</td>
<td>1.01</td>
<td>0.77E-3</td>
<td>0.98</td>
</tr>
<tr>
<td></td>
<td>$L_\infty$</td>
<td>0.19E-2</td>
<td>1.00</td>
<td>0.11E-2</td>
<td>1.08</td>
<td>0.23E-2</td>
<td>0.99</td>
</tr>
<tr>
<td>800</td>
<td>$L_1$</td>
<td>0.32E-3</td>
<td>0.93</td>
<td>0.27E-3</td>
<td>1.02</td>
<td>0.39E-3</td>
<td>0.99</td>
</tr>
<tr>
<td></td>
<td>$L_\infty$</td>
<td>0.97E-3</td>
<td>1.00</td>
<td>0.52E-3</td>
<td>1.05</td>
<td>0.11E-2</td>
<td>0.99</td>
</tr>
</tbody>
</table>

Table 4.2: Errors of the third order scheme with positivity-preserving limiter in 1D cylindrical coordinates using $N$ initially uniform cells

<table>
<thead>
<tr>
<th>$N$</th>
<th>Norm</th>
<th>Density</th>
<th>order</th>
<th>Momentum</th>
<th>order</th>
<th>Energy</th>
<th>order</th>
<th>limited cells(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>$L_1$</td>
<td>0.29E-5</td>
<td>0.28E-5</td>
<td>0.30E-5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_\infty$</td>
<td>0.75E-5</td>
<td>0.65E-5</td>
<td>0.59E-5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>$L_1$</td>
<td>0.39E-6</td>
<td>2.89</td>
<td>0.36E-6</td>
<td>2.94</td>
<td>0.40E-6</td>
<td>2.89</td>
<td>4.87</td>
</tr>
<tr>
<td></td>
<td>$L_\infty$</td>
<td>0.98E-6</td>
<td>2.93</td>
<td>0.80E-6</td>
<td>3.03</td>
<td>0.88E-6</td>
<td>2.73</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>$L_1$</td>
<td>0.42E-7</td>
<td>3.21</td>
<td>0.46E-7</td>
<td>2.98</td>
<td>0.49E-7</td>
<td>3.04</td>
<td>2.46</td>
</tr>
<tr>
<td></td>
<td>$L_\infty$</td>
<td>0.11E-6</td>
<td>3.07</td>
<td>0.81E-7</td>
<td>3.29</td>
<td>0.12E-6</td>
<td>2.79</td>
<td></td>
</tr>
<tr>
<td>400</td>
<td>$L_1$</td>
<td>0.47E-8</td>
<td>3.15</td>
<td>0.58E-8</td>
<td>2.98</td>
<td>0.59E-8</td>
<td>2.86</td>
<td>1.24</td>
</tr>
<tr>
<td></td>
<td>$L_\infty$</td>
<td>0.16E-7</td>
<td>2.83</td>
<td>0.10E-7</td>
<td>2.99</td>
<td>0.17E-7</td>
<td>2.86</td>
<td></td>
</tr>
<tr>
<td>800</td>
<td>$L_1$</td>
<td>0.54E-9</td>
<td>3.12</td>
<td>0.73E-9</td>
<td>3.00</td>
<td>0.70E-9</td>
<td>3.08</td>
<td>0.62</td>
</tr>
<tr>
<td></td>
<td>$L_\infty$</td>
<td>0.23E-8</td>
<td>2.81</td>
<td>0.15E-8</td>
<td>2.70</td>
<td>0.23E-8</td>
<td>2.90</td>
<td>0.31</td>
</tr>
</tbody>
</table>
Table 4.3: Errors of the third order scheme with positivity-preserving limiter in 1D cylindrical coordinates using \( N \) initially uniform cells (performing the reconstruction on the area)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Norm</th>
<th>Density order</th>
<th>Momentum order</th>
<th>Energy order</th>
<th>limited cells(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>( L_1 )</td>
<td>0.25E-4</td>
<td>0.18E-4</td>
<td>0.18E-4</td>
<td>4.87</td>
</tr>
<tr>
<td></td>
<td>( L_{\infty} )</td>
<td>0.60E-4</td>
<td>0.31E-4</td>
<td>0.46E-4</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>( L_1 )</td>
<td>0.63E-5</td>
<td>1.98</td>
<td>0.46E-5</td>
<td>2.01</td>
</tr>
<tr>
<td></td>
<td>( L_{\infty} )</td>
<td>0.15E-4</td>
<td>1.93</td>
<td>0.73E-5</td>
<td>2.10</td>
</tr>
<tr>
<td>200</td>
<td>( L_1 )</td>
<td>0.16E-5</td>
<td>1.98</td>
<td>0.11E-5</td>
<td>2.01</td>
</tr>
<tr>
<td></td>
<td>( L_{\infty} )</td>
<td>0.39E-5</td>
<td>1.99</td>
<td>0.17E-5</td>
<td>2.06</td>
</tr>
<tr>
<td>400</td>
<td>( L_1 )</td>
<td>0.41E-6</td>
<td>1.97</td>
<td>0.28E-6</td>
<td>2.00</td>
</tr>
<tr>
<td></td>
<td>( L_{\infty} )</td>
<td>0.99E-6</td>
<td>1.98</td>
<td>0.43E-6</td>
<td>2.00</td>
</tr>
<tr>
<td>800</td>
<td>( L_1 )</td>
<td>0.10E-6</td>
<td>1.96</td>
<td>0.72E-7</td>
<td>1.98</td>
</tr>
<tr>
<td></td>
<td>( L_{\infty} )</td>
<td>0.25E-6</td>
<td>1.99</td>
<td>0.11E-6</td>
<td>1.99</td>
</tr>
</tbody>
</table>

**Example 2.** Sedov blast wave in a cylindrical coordinate [17].

The initial computational domain is \([0, 1.125]\). The initial condition is

\[
\rho = 1, \quad u = 0,
\]

the specific internal energy \( e \) is \( 10^{-14} \) everywhere except in the cells connected to the origin where they share a total value of 0.244816. In fact, Sedov blast wave problem is targeted at simulating vacuum. But it is not practical to take internal energy to be zero, which will result in the immediate cease of the code. Hence we choose \( e \) to be small enough and the ideal gas is used with \( \gamma = 1.4 \). Reflective boundary condition is applied on the outer boundary. The analytical solution is a shock with a peak density of 6 at \( r = 1 \) and at time \( t = 1 \). The numerical results with our first and third order schemes using 100 cells at \( t = 1 \) are shown in Figure 4.1. We can see the position of the shock has been captured very accurately. Although the internal energy and pressure are quite small, the scheme still works very well.
Example 3. The Noh shock problem in a cylindrical coordinate system [15].

The Noh problem is a classic test problem which is widely used to validate the performance of Lagrangian schemes on strong discontinuities. The initial computational domain is [0,1]. The initial density is 1, the initial pressure is 0, and the initial velocity is directed toward the origin with magnitude 1. The analytic solution is a shock generated by bringing the cold gas to rest at the origin. The density behind the shock is 16, and the shock speed is 1/3. But in practical numerical simulation, we cannot take the pressure to be zero. In the literature, the pressure is usually chosen as large as $10^{-5}$. However, in this test, to verify the performance of the positivity-preserving property in our scheme, we choose the initial pressure as small as $10^{-13}$, which brings significant challenge to the scheme. In fact, the third order Lagrangian scheme fails to compute it without the positivity-preserving limiter, even with very small time steps. Figure 4.2 shows the results of our first order and third order schemes with 100 cells at $t = 0.6$. We can observe that the shock has been captured very well, and our scheme demonstrates its good performance when the pressure and internal energy tends to zero.

Figure 4.1: The results of the Sedov problem with 100 cells at $t = 1$. 
4.2 Two-dimensional tests

Example 1. We test the accuracy of the scheme (3.9) with the two-dimensional Riemann solver (3.35)-(3.37) on a free expansion problem. The initial computational domain is $[0, 1] \times [0, \pi/2]$ defined in the polar coordinates. At $t = 0$, we have

$$\rho = 1, \quad u_\xi = 0, \quad u_\theta = 0, \quad p = 1 - \xi^4,$$

where $\xi = \sqrt{z^2 + r^2}$.

We perform the test on an initially equal-angled polar grid as shown on the left of Figure 4.3. Free boundary condition is applied on the outer boundary. The final grid is also given on the right of Figures 4.3. We can clearly observe the symmetry-preserving property of the scheme in the figures. The errors and accuracy of the scheme on this grid at $t = 1$ are listed in Table 4.4 which is measured on the interval $[\frac{1}{10} K, \frac{9}{10} K] \times [\frac{1}{10} L, \frac{9}{10} L]$ to remove the influence from the boundary. The percentage of the cells in which the positivity-preserving limiter has been performed is also listed in the table. Here we take the result of the one-dimensional third order positivity-preserving Lagrangian scheme in the spherical coordinate with 10000 cells as our reference solution. From the table, we can see the expected second order accuracy in both $L_1$ and $L_\infty$ norms for all the evolved conserved variables.
Figure 4.3: The grid of the free expansion problem with $20 \times 20$ cells. Left: the initial grid; Right: the grid at $t = 1$.

Table 4.4: Errors of the scheme in 2D cylindrical coordinates for the free expansion problem using $K \times L$ initially equal-angled polar grid cells

<table>
<thead>
<tr>
<th>$K = L$</th>
<th>Norm</th>
<th>Density order</th>
<th>Momentum order</th>
<th>Energy order</th>
<th>limited cells(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$L_1$</td>
<td>0.15E-2</td>
<td>0.19E-2</td>
<td>0.20E-2</td>
<td>8.15</td>
</tr>
<tr>
<td></td>
<td>$L_\infty$</td>
<td>0.11E-1</td>
<td>0.39E-2</td>
<td>0.78E-2</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>$L_1$</td>
<td>0.63E-3</td>
<td>1.28</td>
<td>2.12</td>
<td>2.22</td>
</tr>
<tr>
<td></td>
<td>$L_\infty$</td>
<td>0.22E-2</td>
<td>2.24</td>
<td>0.75E-3</td>
<td>2.22</td>
</tr>
<tr>
<td>40</td>
<td>$L_1$</td>
<td>0.18E-3</td>
<td>1.81</td>
<td>2.16</td>
<td>1.96</td>
</tr>
<tr>
<td></td>
<td>$L_\infty$</td>
<td>0.41E-3</td>
<td>2.44</td>
<td>0.17E-3</td>
<td>1.93</td>
</tr>
<tr>
<td>80</td>
<td>$L_1$</td>
<td>0.44E-4</td>
<td>2.03</td>
<td>2.16</td>
<td>1.97</td>
</tr>
<tr>
<td></td>
<td>$L_\infty$</td>
<td>0.10E-3</td>
<td>2.00</td>
<td>0.42E-4</td>
<td>2.02</td>
</tr>
</tbody>
</table>

Example 2 (The spherical Sedov problem in a cylindrical coordinate system on the polar grid [17]).

The spherical Sedov blast wave problem in a cylindrical coordinate system is a commonly used example of a diverging shock wave. The initial computational domain is a $\frac{1}{4}$-circle region defined in the polar coordinates by $[0, 1.125] \times [0, \pi/2]$. The initial condition is,

$$
\rho = 1, \quad u_\xi = 0, \quad u_\theta = 0,
$$

the specific internal energy $e$ is 0 except in the cells connected to the origin where they share a total value of 0.2468. Here in the practical simulation, we take $e$ to be a smaller positive value, that is $10^{-14}$ which is demonstrated to bring much more challenge to the scheme. In fact, the
second order Lagrangian scheme without the positivity-preserving limiter fails to calculate. Reflective boundary condition is applied on the outer boundary. The analytical solution is a shock with a peak density of 4 at radius unity at time unity. The final grid and the surface of density obtained by the second order scheme (3.9) with (3.35)-(3.37) with $30 \times 30$ cells are displayed in Figures 4.4. The density and pressure as a function of the radial radius are also shown in Figures 4.4. We observe the expected symmetry in the plots of grid and density. The shock position and peak density obtained by the second order scheme coincide with the analytical solutions well, which demonstrates the good performance of the scheme in symmetry-preserving, positivity-preserving, non-oscillation and accuracy properties.

![Figure 4.4](image)

**Figure 4.4:** The results of the Sedov problem with $30 \times 30$ cells at $t = 1$. Left: grid; Second: density contour; Third: density vs radial radius; Right: pressure vs radial radius. Solid line: exact solution; Symbols: second order scheme.

**Example 3** (Implosion problem of Lazarus [11]).

In this self-similar implosion problem, initially a sphere of unit radius has the following condition,

$$
\rho = 1, \quad u_\zeta(t) = \frac{-\alpha f}{(1 - ft)^{1-\alpha}}, \quad u_\theta(t) = 0, \quad e = 10^{-14},
$$

where $\alpha = 0.6883545$, $f = 1 - \varepsilon t - \delta t^3$, $\varepsilon = 0.185$, $\delta = 0.28$.

We test the problem on a grid of $200 \times 30$ cells in the initial computational domain $[0, 1] \times [0, \pi/2]$ defined in the polar coordinates. The numerically converged result computed using a one-dimensional third-order Lagrangian code in the spherical coordinate with 10000 cells is used as a reference solution. We display the results of the second order scheme (3.9)
with ((3.35)-(3.37) in Figures 4.5. In the plots of grid and density contour, we notice the expected symmetry. In the plot of density and pressure as a function of the radial radius, we observe the non-oscillatory and positivity-preserving properties of the scheme.

Figure 4.5: The results of the Lazarus problem with $200 \times 30$ cells. Left: grid at $t = 0.8$; Middle: density contour at $t = 0.8$; Third: density vs radial radius at $t = 0.74, 0.8$; Right: pressure vs radial radius at $t = 0.74, 0.8$. Solid line: reference solution; Symbols: second order scheme.

Example 4 (Spherical Sedov problem on the Cartesian grid).

The spherical symmetry problem simulated on the initially rectangular grid is demonstrated to be much more challenging for a Lagrangian scheme due to the shock direction being not aligned with the grid line. In this example, we test the spherical Sedov blast wave problem in a cylindrical coordinate system on the initially rectangular grid. The initial computational domain is a $1.125 \times 1.125$ square consisting of $30 \times 30$ uniform cells. Its initial condition is

$$\rho = 1, \quad u_z = 0, \quad u_r = 0.$$  

The specific internal energy $e$ is $10^{-14}$ except in the cell connected to the origin where it has a value of 0.2468. Figures 4.6 show the results of our second order scheme (3.9) with (3.35)-(3.37). From the figures, we can observe the results of our second order scheme are positivity-preserving and symmetric even on this non-polar grid.
Figure 4.6: The results of the Sedov problem with $30 \times 30$ Cartesian cells at $t = 1.0$. Left: grid; Second: density contour; Third: density vs radial radius; Right: pressure vs radial radius. Solid line: exact solution; Symbols: second order scheme.

5 Conclusion

In this paper, we focus on the methodology to design positivity-preserving and symmetry preserving Lagrangian schemes in one- and two-dimensional cylindrical coordinates for solving compressible Euler equations with general equations of state. Firstly, we develop the first order and high order positivity-preserving Lagrangian schemes by using positivity-preserving limiter for Euler equations in one-dimensional cylindrical coordinates, which are performed based on the two-state Riemann solver [13, 18]. Then for two-dimensional cylindrical coordinate case, we consider the preservation of positivity and symmetry simultaneously (the spherical symmetry is considered when computed on an equal-angle-zoned initial grid). To achieve this goal, we need to discuss the mutual impact of these two properties on each other, then make a balance between them. The main idea is to perform the area based reconstruction in a local $\xi-\theta$ coordinate [6], and perform the positivity-preserving limiter also for such reconstructions and in such local coordinates. Our schemes also maintain other good properties such as conservation for mass, momentum and total energy and the geometric conservation law. Several numerical examples in cylindrical coordinates are given to demonstrate the good performance of the schemes in terms of accuracy, positivity-preserving, symmetry preserving, non-oscillation and robustness properties.
References


