Superconvergence of the local discontinuous Galerkin method for linear fourth order time dependent problems in one space dimension

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Abstract

In this paper, we investigate the superconvergence property of the local discontinuous Galerkin (LDG) methods for solving one dimensional linear time dependent fourth order problems. We prove that the error between the LDG solution and a particular projection of the exact solution achieves \((k + \frac{3}{2})\)-th order superconvergence when polynomials of degree \(k\) \((k \geq 1)\) are used. Numerical experiments are shown to validate the theoretical results.

Key words: local discontinuous Galerkin method, superconvergence, fourth order problems, error estimates

1 Introduction

In this paper, we are interested in the superconvergence property of the local discontinuous Galerkin (LDG) methods for a class of one dimensional linear fourth order problems formulated as

\[ u_t + \alpha u_x + \beta u_{xx} + u_{xxxx} = 0, \]  

(1.1)

where \(\alpha\) and \(\beta\) are arbitrary constants. Note that, for \(\beta > 0\), there is an anti-diffusion term \(\beta u_{xx}\) in the equation, which is however dominated by the higher order diffusion
term $u_{xxxx}$. The general problem (1.1) includes the following linear time dependent biharmonic equation

$$u_t + u_{xxxx} = 0,$$

(1.2)

and the linearized Cahn-Hilliard equation

$$u_t + u_{xx} + u_{xxxx} = 0$$

(1.3)
as its special cases.

The discontinuous Galerkin (DG) method is a class of finite element methods, using discontinuous piecewise polynomials as the solution and the test spaces. It was first introduced by Reed and Hill [18] for solving first order steady state linear hyperbolic conservation laws, and later developed by Cockburn et al. [11, 10, 9, 12] for solving time dependent nonlinear equations. Motivated by the successful numerical experiments of Bassi and Rebay [3] for the compressible Navier-Stokes equations, the LDG methods were developed for solving nonlinear convection diffusion equations [13] containing second order spatial derivatives, in which $L^2$ stability and a sub-optimal $L^2$ error estimates were obtained for linear equations with smooth solutions. Later, the LDG methods were generalized to solve various PDEs involving higher order derivatives. For KdV type equations containing third order derivatives, an LDG method was developed in [25] where a sub-optimal error estimate was proved for the linear case, and more recently, optimal $L^2$ error estimate was obtained in [24]. In [26, 21, 22, 20], the LDG techniques were developed for solving other types of high order PDEs including the time dependent biharmonic equations, the fully nonlinear $K(n, n, n)$ equations, the Kuramoto-Sivashinsky type equations, the Cahn-Hilliard type equations and so on. In [15], optimal error estimates of the LDG method for the linear biharmonic equation and linearized Cahn-Hilliard type equations were obtained in one dimension and in multidimensions for Cartesian and triangular meshes. For more details of the DG and LDG methods, we refer to the lecture notes [8] and review papers [14] and [23].
Apart from the LDG methods mentioned above, there are also other finite element methods in the literature for solving fourth order time dependent problems. For example, in [16], Elliott and Zheng applied a conforming finite element method to the Cahn-Hilliard equation and obtained optimal error estimates in $L^2$ and $L^\infty$ norms provided the approximate solution is bounded in $L^\infty$ and the polynomial degree $k \geq 3$. In [17], Feng and Prohl applied a mixed finite element method for solving Cahn-Hilliard equation on quasi-uniform triangular meshes, and obtained an optimal error estimate under minimum regularity assumptions on the initial data and the domain.

In [2, 1], Adjerid et al. showed that the LDG solution is superconvergent at Radau points for solving convection or diffusion dominant time dependent equations. Based on Fourier analysis, Cheng and Shu in [4] and [5] proved superconvergence of the DG and LDG solutions towards a particular projection of the exact solution in the case of piecewise linear polynomials on uniform meshes for the linear conservation law and heat equation, respectively. The results were later improved, using a different technique, in [6] for arbitrary nonuniform regular meshes and schemes of any order. In this paper, we follow the approach in [6] to obtain the superconvergence property of the LDG method for a class of fourth order problems. An important motivation for studying such superconvergence is to set a firm theoretical foundation for the excellent behavior of DG and LDG methods for long time simulations, which have been repeatedly observed by practitioners. Indeed, if superconvergence for the error between the DG or LDG solution and a particular projection of the exact solution of the order $(k + \frac{3}{2})$, with linear growth in time, can be shown for polynomials of degree $k$, then the error between the numerical solution and the exact solution does not grow for a long time $t = O\left(\frac{1}{\sqrt{h}}\right)$ where $h$ is the mesh size [6]. The generalization from first and second order equations in [6] to the fourth order equation in this paper involves several technical difficulties, including the estimate of different combinations of the LDG solution and auxiliary variables which approximate derivatives of different orders, and the design and analysis of a special operator.
to guarantee the superconvergence property of the initial condition.

This paper is organized as follows. In Section 2, we define the LDG scheme for the fourth order time dependent problems, state the main results, and present the details of the proof for the superconvergence property. In Section 3, numerical examples are displayed to demonstrate the theoretical results. Concluding remarks and comments on future work are given in Section 4. The proofs for some of technical lemmas are collected in the appendix.

2 LDG scheme for fourth order problems

We consider the following linear fourth order equations

\[ u_t + \alpha u_x + \beta u_{xx} + u_{xxxx} = 0 \]  \hspace{1cm} (2.1a)

with initial condition

\[ u(x, 0) = u_0(x) \]  \hspace{1cm} (2.1b)

and periodic boundary conditions

\[ u(0, t) = u(2\pi, t). \]  \hspace{1cm} (2.1c)

We would like to remark that the assumption of periodic boundary conditions is for simplicity only and not essential.

2.1 The LDG scheme

We assume the following mesh to cover the computational domain \( I = [0, 2\pi] \), consisting of cells \( I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \), for \( 1 \leq j \leq N \), where

\[ 0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi. \]

The cell center is denoted by \( x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2 \). We also set \( \Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} \) and \( h = \max_j \Delta x_j \). We denote by \( u^-_{j+\frac{1}{2}} \) and \( u^+_{j+\frac{1}{2}} \) the values of \( u \) at the discontinuity point
$x_{j+\frac{1}{2}}$, from the the left cell, $I_j$, and from the right cell, $I_{j+1}$, respectively. The following piecewise polynomials space is chosen as the finite element space:

$$V_h^k = \{ v : v|_{I_j} \in P^k(I_j), j = 1, \cdots, N \},$$

where $P^k(I_j)$ denotes the set of polynomials of degree up to $k$ defined on the cell $I_j$. Note that functions in $V_h$ are allowed to have discontinuities across element interfaces.

In order to construct the LDG scheme, firstly we introduce some auxiliary variables approximating various order derivatives of the solution and rewrite the equation (2.1a) into a first order system

$$u_t + (\alpha u + \beta q + r)_x = 0, \quad r - p_x = 0, \quad p - q_x = 0, \quad q - u_x = 0.$$

Then the semi-discrete LDG scheme is defined as follows: Find $u_h, q_h, p_h, r_h \in V_h^k$, such that

$$\int_{I_j} (u_h)_t \rho dx - \int_{I_j} \alpha u_h \rho_x dx + \alpha \tilde{u}_h \rho^-|_{j+\frac{1}{2}} - \alpha \tilde{u}_h \rho^+|_{j-\frac{1}{2}} - \int_{I_j} \beta q_h \rho_x dx + \beta \tilde{q}_h \rho^-|_{j+\frac{1}{2}} - \beta \tilde{q}_h \rho^+|_{j-\frac{1}{2}} = 0, \quad (2.2a)$$

$$\int_{I_j} r_h \eta dx + \int_{I_j} p_h \eta_x dx - \hat{p}_h \eta^-|_{j+\frac{1}{2}} + \hat{p}_h \eta^+|_{j-\frac{1}{2}} = 0, \quad (2.2b)$$

$$\int_{I_j} p_h \xi dx + \int_{I_j} q_h \xi_x dx - \hat{q}_h \xi^-|_{j+\frac{1}{2}} + \hat{q}_h \xi^+|_{j-\frac{1}{2}} = 0, \quad (2.2c)$$

$$\int_{I_j} q_h \psi dx + \int_{I_j} u_h \psi_x dx - \hat{u}_h \psi^-|_{j+\frac{1}{2}} + \hat{u}_h \psi^+|_{j-\frac{1}{2}} = 0 \quad (2.2d)$$

hold for any $\rho, \psi, \xi, \eta \in V_h^k$, where $\tilde{u}_h$ is the upwind flux depending on the sign of $\alpha$. Without loss of generality we assume $\alpha \geq 0$ and $\tilde{u}_h = u_h^-$, and choose the alternating fluxes for the diffusion terms as

$$\hat{u}_h = u_h^-, \quad \hat{q}_h = q_h^+, \quad \hat{p}_h = p_h^-, \quad \hat{r}_h = r_h^+.$$

(2.3)

### 2.2 Notations and auxiliary results

To prove the superconvergence property of the LDG method, we would like to introduce the following notations, definitions and useful lemmas.
2.2.1 Notations for the DG discretization

First, we use $[\xi] = \xi^+ - \xi^-$ to denote the jump of the function $\xi$ at each cell boundary point. For the linear problems discussed in this paper, we introduce the DG discretization operator $\mathcal{D}$ as in [24]: for each cell $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}],$

$$\mathcal{D}_{I_j}(\xi; \eta; \hat{\xi}) = -\int_{I_j} \xi \eta dx + \hat{\xi} \eta^-|_{j+\frac{1}{2}} - \hat{\xi} \eta^+|_{j-\frac{1}{2}}.$$  

We also use the notation

$$\mathcal{D}(\xi; \eta; \hat{\xi}) = \sum_j \mathcal{D}_{I_j}(\xi; \eta; \hat{\xi}).$$

Using the definition of the operator, we have the following lemmas, whose proof is straightforward, see [24].

**Lemma 2.1.** [24] Choosing different numerical fluxes, the DG discretization operator satisfies the following equalities

\begin{align*}
\mathcal{D}(\xi, \eta; \xi^-) + \mathcal{D}(\eta, \xi; \eta^+) &= 0, \quad (2.4a) \\
\mathcal{D}(\xi, \eta; \xi^+) + \mathcal{D}(\eta, \xi; \eta^-) &= 0, \quad (2.4b) \\
\mathcal{D}(\xi, \eta; \xi^-) + \mathcal{D}(\eta, \xi; \eta^+) &= -\sum_j [\xi]_{j+\frac{1}{2}} [\eta]_{j+\frac{1}{2}}, \quad (2.4c) \\
\mathcal{D}(\xi, \eta; \xi^+) + \mathcal{D}(\eta, \xi; \eta^-) &= \sum_j [\xi]_{j+\frac{1}{2}} [\eta]_{j+\frac{1}{2}}, \quad (2.4d) \\
\mathcal{D}(\xi, \xi; \xi^-) &= \frac{1}{2} \sum_j [\xi]_{j+\frac{1}{2}}^2, \quad (2.4e) \\
\mathcal{D}(\xi, \xi; \xi^+) &= -\frac{1}{2} \sum_j [\xi]_{j+\frac{1}{2}}^2. \quad (2.4f)
\end{align*}

**Lemma 2.2.** By integration by parts, we also have

\begin{align*}
\mathcal{D}_{I_j}(\xi; \eta; \xi^-) &= \int_{I_j} \xi \eta dx + [\xi] \eta^-|_{j-\frac{1}{2}}, \quad (2.5) \\
\mathcal{D}_{I_j}(\xi; \eta; \xi^+) &= \int_{I_j} \xi \eta dx + [\xi] \eta^+|_{j+\frac{1}{2}}. \quad (2.6)
\end{align*}

For the definition and properties of the DG discretization operator for nonlinear problems, we refer to [27].
2.2.2 Projections and interpolation properties

In what follows, we define two special projections $P_h^\pm$ into $V_h$, which are commonly used in the analysis of DG methods. For any given function $u \in H^1(I)$ and arbitrary subinterval $I_j = [x_j - \frac{1}{2}, x_j + \frac{1}{2}]$, the special projections of $u$, denoted by $P_h^+ u$ and $P_h^- u$, are the unique functions in the finite element space $V_h^k$ satisfying, for each $j$,

$$\int_{I_j} (P_h^+ u(x) - u(x)) \rho(x) dx = 0, \quad \forall \rho \in P^{k-1}, \quad (P_h^+ u)_{j - \frac{1}{2} +} = u(x_{j - \frac{1}{2}}); \quad (2.7)$$

$$\int_{I_j} (P_h^- u(x) - u(x)) \rho(x) dx = 0, \quad \forall \rho \in P^{k-1}, \quad (P_h^- u)_{j + \frac{1}{2} -} = u(x_{j + \frac{1}{2}}). \quad (2.8)$$

For the special projections mentioned above, we have, by the standard approximation theory [7], that

$$\| P_h^\pm u(\cdot) - u(\cdot) \|_{L^2} \leq C h^{k+1}, \quad (2.9)$$

where here and below $C$ is a positive constant (which may have a different value in each occurrence) depending solely on $u$ and its derivatives but independent of $h$. In particular, in (2.9), $C = C' \| u \|_{k+1}$, where $\| u \|_{k+1}$ is the standard Sobolev $(k + 1)$ norm and $C'$ is a constant independent of $u$.

In the proofs of the error estimates, the following inverse and trace properties are needed: For any $q \in V_h^k$, there exists a positive constant $C$ independent of $h$, such that

$$\| q \|_{\Gamma} \leq C h^{-\frac{1}{2}} \| q \|_{L^2}, \quad (2.10)$$

$$\| \partial x q \|_{L^2} \leq C h^{-1} \| q \|_{L^2}, \quad (2.11)$$

where $\| q \|_{\Gamma}$ is the usual $L^2$ norm on the cell interfaces of the mesh.

2.2.3 Functionals related to the $L^2$ norm

To get the superconvergence property of the method, two functionals related to the $L^2$ norm of a function on $I_j$ are needed as defined in [6]:

$$B_j^*(f) = \int_{I_j} f(x) \frac{x - x_{j - 1/2}}{h_j} \frac{d}{dx} \left( f(x) \frac{x - x_j}{h_j} \right) dx,$$
The functionals defined above have the following properties, which are essential to the proof of the superconvergence.

**Lemma 2.3.** [6] For any function \( f(x) \in C^1 \) on \( I_j \),

\[
\mathcal{B}_j^{-}(f) = \frac{1}{4h_j} \int_{I_j} f^2(x) dx + \frac{f^2(x_{j+1/2})}{4}, \\
\mathcal{B}_j^{+}(f) = -\frac{1}{4h_j} \int_{I_j} f^2(x) dx - \frac{f^2(x_{j-1/2})}{4}.
\]

The proof of this lemma is straightforward, see [6].

### 2.2.4 Initial condition

To obtain the superconvergence property of the method, the initial condition of the numerical scheme should be chosen carefully to be compatible with the superconvergence error estimate. To this end, we define an operator \( P_h^* \) as follows: For any function \( u \), \( P_h^* u \in V_h^k \), and suppose \( q_h, p_h, r_h \in V_h^k \) are the unique solutions (with given \( P_h^* u \)) to

\[
\int_{I_j} r_h \eta dx + \int_{I_j} p_h \eta_x dx - p_h^* \eta^-|_{j+\frac{1}{2}} + p_h^* \eta^+|_{j-\frac{1}{2}} = 0, \tag{2.14a}
\]

\[
\int_{I_j} p_h \xi dx + \int_{I_j} q_h \xi_x dx - q_h^* \xi^-|_{j+\frac{1}{2}} + q_h^* \xi^+|_{j-\frac{1}{2}} = 0, \tag{2.14b}
\]

\[
\int_{I_j} q_h \psi dx + \int_{I_j} P^*_h u \psi x dx - (P^*_h u)^- \psi^-|_{j+\frac{1}{2}} + (P^*_h u)^- \psi^+|_{j-\frac{1}{2}} = 0 \tag{2.14c}
\]

for any \( \psi, \xi, \eta \in V_h^k \), then we require

\[
\int_{I_j} ((P^-_h u - P^*_h u) - (P^+_h q - q_h) + (P^+_h r - r_h)) \rho dx = 0 \tag{2.15}
\]

for any \( \rho \in P^{k-1} \) on \( I_j \) and

\[
(P^-_h u - P^*_h u)^- = (P^+_h q - q_h)^+ - (P^+_h r - r_h)^+ \quad \text{at} \quad x_{j-1/2}. \tag{2.16}
\]

For the regular mesh considered in this paper, we denote \( \lambda = \max_j \Delta x_j / \min_j \Delta x_j \), which is a constant during mesh refinements. As to the operator defined above, we have the following lemma.
Lemma 2.4. $P_h^* u$ exists and is unique. Moreover, there holds the error estimate

$$
\|P_h^- u - P_h^* u\| \leq C(\lambda, \|u\|_{k+4})h^{k+3/2}.
$$

The proof of this lemma is given in the appendix.

We would like to remark that the purpose for introducing the operator $P_h^*$ is only theoretical: it is needed for the technical proof of superconvergence. In actual numerical computation, we have observed that we can use the usual $L^2$ projection of $u$ as the initial condition and still observe superconvergence, see the numerical experiments in Section 3. Of course, if the standard $L^2$ projection is used for the initial condition, then the superconvergence result does not hold at $t = 0$ and also for small $t$. For later time, the dissipativity in the PDE and the numerical scheme seems to help to recover the superconvergence performance.

2.3 Main results

Before we state the main results, we would like to introduce the following notations

\[
\begin{align*}
\varepsilon_u &= u - u_h = (u - P_h^- u) + (P_h^- u - u_h) = \varepsilon_u + \tilde{\varepsilon}_u \\
\varepsilon_q &= q - q_h = (q - P_h^+ q) + (P_h^+ q - q_h) = \varepsilon_q + \tilde{\varepsilon}_q \\
\varepsilon_p &= p - p_h = (p - P_h^- p) + (P_h^- p - p_h) = \varepsilon_p + \tilde{\varepsilon}_p \\
\varepsilon_r &= r - r_h = (r - P_h^+ r) + (P_h^+ r - r_h) = \varepsilon_r + \tilde{\varepsilon}_r.
\end{align*}
\]

For the case $\alpha \geq 0$, we have the following error estimates.

Theorem 2.5. Let $u, p = u_{xx}$ be the exact solution of the fourth order problem (2.1). Let $u_h, p_h$ be the LDG solution of (2.2) when the diffusion alternating fluxes (2.3) are used. We choose the initial condition as $u_h(\cdot, 0) = P_h^* u_0$. For regular triangulations of $I = [0, 2\pi]$, if the finite element space $V_h^k$ with $k \geq 1$ is used, then there holds the following error estimate

\[
\|\varepsilon_u(\cdot, t)\|_{L^2}^2 + \int_0^t \|\varepsilon_p(\cdot, t)\|_{L^2}^2 dt \leq C e^{2C_1 t} h^{2k+3},
\]

(2.18)
and, in particular,

\[ \| \bar{e}_u(\cdot, t) \|_{L^2} \leq C e^{C_1 t} h^{k+3/2}, \]

where \( C = C(\alpha, \beta, \lambda, \|u\|_{k+4}, \|u_t\|_{k+4}, \|u_{tt}\|_{k+4}, \|u_{ttt}\|_{k+1}) \) and here and below \( C_1 = C_1(\alpha, \beta) \geq 0. \)

**Remark 2.1.** For the case \( \alpha \leq 0 \), we can choose \( \tilde{u}_h = u_h^+ \) and take the diffusion alternating fluxes as

\[ \hat{u}_h = u_h^+, \quad \hat{q}_h = q_h^-, \quad \hat{p}_h = p_h^+, \quad \hat{r}_h = r_h^-. \]  

(2.19)

Theorem 2.5 still holds in this case with the obvious change of the projections.

For the case \( \alpha = \beta = 0 \), equation (2.1a) reduces to the biharmonic equation (1.2), and we have the following result.

**Theorem 2.6.** Let \( u, p = u_{xx} \) be the exact solution of the fourth order problem (1.2). Let \( u_h, p_h \) be the LDG solution of (2.2) when \( \alpha = \beta = 0 \) and diffusion alternating fluxes (2.3) are used. We choose the initial condition as \( u_h(\cdot, 0) = P_h^* u_0 \). For regular triangulations of \( I = [0, 2\pi] \), if the finite element space \( V_h^k \) with \( k \geq 1 \) is used, then there holds the following error estimate

\[ \| \bar{e}_u(\cdot, t) \|_{L^2}^2 + \int_0^t \| \bar{e}_p(\cdot, t) \|_{L^2}^2 dt \leq C(1 + t)^2 h^{2k+3}, \]

and, in particular,

\[ \| \bar{e}_u(\cdot, t) \|_{L^2} \leq C(1 + t) h^{k+3/2}, \]

where \( C = C(\lambda, \|u\|_{k+4}, \|u_t\|_{k+4}, \|u_{tt}\|_{k+4}, \|u_{ttt}\|_{k+1}) \).

The proof of this theorem is similar to that for the previous theorem, except that we need to carefully evaluate and estimate several terms to obtain a linear growth bound without employing the Gronwall’s inequality. The detailed proof is given in the appendix.

The case with the fluxes (2.19) is the same with the obvious change of the projections.
Notice that, for the general cases including the anti-diffusive case $\beta > 0$, the exponential growth of the constant with respect to time in Theorem 2.5 is expected, as the exact solution may have such growth in time for small wave numbers.

2.4 Proof of Theorem 2.5

By using the DG discretization operator, the LDG scheme (2.2) with the fluxes (2.3) can be written as

\[
\int_{I_j} (u_h)_t \rho dx + \alpha D_{I_j}(u_h, \rho; u_h^-) + \beta D_{I_j}(q_h, \rho; q_h^+) + D_{I_j}(r_h, \rho; r_h^+) = 0, \tag{2.20a}
\]

\[
\int_{I_j} r_h \eta dx - D_{I_j}(p_h, \eta; p_h^-) = 0, \tag{2.20b}
\]

\[
\int_{I_j} p_h \xi dx - D_{I_j}(q_h, \xi; q_h^+) = 0, \tag{2.20c}
\]

\[
\int_{I_j} q_h \psi dx - D_{I_j}(u_h, \psi; u_h^-) = 0 \tag{2.20d}
\]

for any $\rho, \psi, \xi, \eta \in V^k_h$. Since the exact solutions $u, q = u_x, p = u_{xx}, r = u_{xxx}$ also satisfy the scheme (2.2), we have therefore the error equations

\[
\int_{I_j} (e_u)_t \rho dx + \alpha D_{I_j}(e_u, \rho; e_u^-) + \beta D_{I_j}(e_q, \rho; e_q^+) + D_{I_j}(e_r, \rho; e_r^+) = 0, \tag{2.21a}
\]

\[
\int_{I_j} e_r \eta dx - D_{I_j}(e_p, \eta; e_p^-) = 0, \tag{2.21b}
\]

\[
\int_{I_j} e_p \xi dx - D_{I_j}(e_q, \xi; e_q^+) = 0, \tag{2.21c}
\]

\[
\int_{I_j} e_q \psi dx - D_{I_j}(e_u, \psi; e_u^-) = 0 \tag{2.21d}
\]

which, by the properties of the projections $P_h^-$ and $P_h^+$, (2.7) and (2.8), is
for any \( \rho, \psi, \xi, \eta \in V_h^k \). Taking \( (\rho, \psi, \xi, \eta) = (\bar{e}_u, -\bar{e}_r, \bar{e}_p, \bar{e}_q) \) in (2.21), adding them up and summing over all \( j \), we obtain

\[
\int_I (\bar{e}_u t) \bar{e}_u dx + \int_I \bar{e}_u^2 dx + \int_I (\bar{e}_u t) \bar{e}_u dx + \int_I \bar{e}_p \bar{e}_p dx + \int_I \bar{e}_r \bar{e}_r dx + \int_I \bar{e}_q \bar{e}_q dx - \int_I \bar{e}_q \bar{e}_r dx + \alpha D(\bar{e}_u, \bar{e}_u; \bar{e}_u^-) + \beta D(\bar{e}_r, \bar{e}_u; \bar{e}_r^+) + D(\bar{e}_u, \bar{e}_r; \bar{e}_u^-) - D(\bar{e}_q, \bar{e}_r; \bar{e}_q^-) - D(\bar{e}_p, \bar{e}_q; \bar{e}_p^-) = 0.
\]

Using the property of the operator \( D \) in Lemma 2.1, we thus have

\[
\int_I (\bar{e}_u t) \bar{e}_u dx + \int_I \bar{e}_u^2 dx + \int_I (\bar{e}_u t) \bar{e}_u dx + \int_I \bar{e}_p \bar{e}_p dx + \int_I \bar{e}_r \bar{e}_r dx - \int_I \bar{e}_q \bar{e}_r dx + \frac{\alpha}{2} \sum_j [\bar{e}_u]_{j+\frac{1}{2}}^2 + \beta D(\bar{e}_r, \bar{e}_u; \bar{e}_r^+) = 0.
\]  

(2.22)

By taking \( \xi = \bar{e}_u \) in (2.21c) and summing over all \( j \), we get

\[
D(\bar{e}_r, \bar{e}_u; \bar{e}_r^+) = \int_I \bar{e}_p \bar{e}_u dx.
\]  

(2.23)

Combing (2.22) and (2.23), we arrive at

\[
\frac{1}{2} \frac{d}{dt} \|\bar{e}_u\|^2_{L^2} + \|\bar{e}_p\|^2_{L^2} \leq - \int_I (\bar{e}_u t) \bar{e}_u dx - \int_I \bar{e}_p \bar{e}_p dx - \int_I \bar{e}_r \bar{e}_r dx + \int_I \bar{e}_q \bar{e}_r dx - \beta \int_I \bar{e}_p \bar{e}_u dx.
\]  

(2.24)

It follows from the Cauchy-Schwarz inequality that

\[
\frac{1}{2} \frac{d}{dt} \|\bar{e}_u\|^2_{L^2} + \|\bar{e}_p\|^2_{L^2} \leq \int_I (\bar{e}_u t) \bar{e}_u dx + \int_I \bar{e}_p \bar{e}_p dx + \int_I \bar{e}_r \bar{e}_r dx + \int_I \bar{e}_q \bar{e}_r dx + \|\bar{e}_q \bar{e}_r dx\| + \beta \|\bar{e}_p \bar{e}_u dx\| + \frac{\beta^2}{2} \|\bar{e}_u\|^2_{L^2}.
\]  

(2.25)

On the other hand, using Lemma 2.2, equation (2.21) can be rewritten as

\[
\int_{I_j} (\bar{e}_u t) \rho dx + \alpha D_{I_j}(\bar{e}_u, \rho; \bar{e}_u^-) + \int_{I_j} (\bar{e}_r + \beta \bar{e}_q) x \rho dx + [\bar{e}_r + \beta \bar{e}_q] \rho^- j_{\frac{1}{2}} = 0, \quad (2.26a)
\]

\[
\int_{I_j} \bar{e}_p \eta dx - \int_{I_j} (\bar{e}_p) x \eta dx - [\bar{e}_p] \eta^+ j_{-\frac{1}{2}} = 0, \quad (2.26b)
\]

\[
\int_{I_j} \bar{e}_u \xi dx - \int_{I_j} (\bar{e}_u) x \xi dx - [\bar{e}_u] \xi^- j_{\frac{1}{2}} = 0, \quad (2.26c)
\]

\[
\int_{I_j} \bar{e}_q \psi dx - \int_{I_j} (\bar{e}_u) x \psi dx - [\bar{e}_u] \psi^+ j_{-\frac{1}{2}} = 0. \quad (2.26d)
\]

Denote

\[
\bar{e}_u = r_j + d_j(x) (x - x_j)/h_j, \quad \bar{e}_q = b_j + s_j(x)(x - x_j)/h_j,
\]

12
\[ \tilde{e}_p = v_j + w_j(x)(x - x_j)/h_j, \quad \tilde{e}_r = l_j + g_j(x)(x - x_j)/h_j, \]

where \( r_j, b_j, v_j, l_j \) are constants and \( d_j(x), s_j(x), w_j(x), g_j(x) \in P^{k-1} \). First, taking \( \psi = d_j(x)(x - x_{j-1/2})/h_j \) in (2.26d), and using the definition of \( B_j^- \), we have

\[
g \int_{I_j} e_q d_j(x)(x - x_{j-1/2})/h_j dx - B_j^-(d_j) = 0.\]

By the property of \( B_j^- \) in Lemma 2.3, we obtain

\[
g \int_{I_j} d_j^2(x)dx \leq 4 \int_{I_j} e_q d_j(x)(x - x_{j-1/2})dx.\]

Defining piecewise polynomials \( d(x) \) and \( \phi_1(x) \), such that \( d(x) = d_j(x) \) and \( \phi_1(x) = x - x_{j-1/2} \) on \( I_j \), and summing the above inequality over \( j \), we get

\[
\|d\|_{L^2} \leq 4\|e_q\|_{L^2}\|\phi_1\|_{L^\infty} \leq 4h\|e_q\|_{L^2}, \tag{2.27}
\]

where we have used the fact that \( \|\phi_1\|_{L^\infty} = h \). Similarly, taking \( \xi = s_j(x)(x - x_{j+1/2})/h_j \) in (2.26c), \( \eta = w_j(x)(x - x_{j-1/2})/h_j \) in (2.26b) and using the definition of \( B_j^- \) and \( B_j^+ \), we have

\[
g \int_{I_j} e_r w_j(x)(x - x_{j-1/2})/h_j dx - B_j^-(w_j) = 0,
\]

\[
g \int_{I_j} e_p s_j(x)(x - x_{j+1/2})/h_j dx - B_j^+(s_j) = 0.
\]

By the properties of \( B_j^- \) and \( B_j^+ \) in Lemma 2.3, we obtain

\[
g \int_{I_j} w_j^2(x)dx \leq 4 \int_{I_j} e_r w_j(x)(x - x_{j-1/2})dx,
\]

\[
g \int_{I_j} s_j^2(x)dx \leq -4 \int_{I_j} e_p s_j(x)(x - x_{j+1/2})dx.
\]

Defining piecewise polynomials \( w(x), s(x) \) and \( \phi_2(x) \), such that \( w(x) = w_j(x), s(x) = s_j(x) \) and \( \phi_2(x) = x - x_{j+1/2} \) on \( I_j \), and summing the above inequality over \( j \), we get

\[
\|w\|_{L^2} \leq 4\|e_r\|_{L^2}\|\phi_1\|_{L^\infty} \leq 4h\|e_r\|_{L^2}, \tag{2.28}
\]

\[
\|s\|_{L^2} \leq 4\|e_p\|_{L^2}\|\phi_2\|_{L^\infty} \leq 4h\|e_p\|_{L^2} \tag{2.29}
\]
where we have used the fact that \( \| \phi_1 \|_{L^\infty} = \| \phi_2 \|_{L^\infty} = h \). Then, letting \( \rho = (g_j(x) + \beta s_j(x))(x - x_{j+1/2})/h_j \) in (2.26a), we have

\[
\int_{I_j} (e_u)_t \rho dx - \alpha \left[ \int_{I_j} \bar{e}_u \rho_x dx + \bar{e}_u \rho^+ |_{j-\frac{1}{2}} \right] + \int_{I_j} (\bar{e}_r + \beta \bar{e}_q)_x \rho dx = 0,
\]

which can be written as

\[
\int_{I_j} (e_u)_t (g_j(x) + \beta s_j(x))(x - x_{j+1/2})/h_j dx - \alpha R^1_j + R^2_j = 0,
\]

where

\[
R^1_j = \int_{I_j} \bar{e}_u ((g_j(x) + \beta s_j(x))(x - x_{j+1/2})/h_j)_x dx - \left[ r_{j-1} + \frac{1}{2} d_{j-1}(x_{j-\frac{1}{2}}) \right] (g_j(x_{j-\frac{1}{2}}) + \beta s_j(x_{j-\frac{1}{2}}))
\]

\[
= \int_{I_j} d_j(x) \frac{x - x_j}{h_j} ((g_j(x) + \beta s_j(x))(x - x_{j+1/2})/h_j)_x dx
\]

\[
+ \left[ r_j - r_{j-1} - \frac{1}{2} d_{j-1}(x_{j-\frac{1}{2}}) \right] (g_j(x_{j-\frac{1}{2}}) + \beta s_j(x_{j-\frac{1}{2}}))
\]

and

\[
R^2_j = B^+_j (g_j + \beta s_j) = -\frac{1}{4h_j} \int_{I_j} (g_j(x) + \beta s_j(x))^2 dx - \frac{1}{4} (g_j(x_{j-\frac{1}{2}}) + \beta s_j(x_{j-\frac{1}{2}}))^2.
\]

Therefore,

\[
\int_{I_j} (g_j(x) + \beta s_j(x))^2 dx \leq 4 \int_{I_j} (e_u)_t (g_j(x) + \beta s_j(x))(x - x_{j+1/2}) dx - 4 \alpha h_j R^1_j.
\]

Summing up the above inequality over all \( j \), we arrive at

\[
\| g + \beta s \|_{L^2}^2 \leq 4 \| (e_u)_t \|_{L^2} \| g + \beta s \|_{L^2} \| \phi_2 \|_{L^\infty} + 4 \alpha \sum_j h_j R^1_j.
\] (2.30)

Taking \( \psi = 1 \) in (2.26d), we get

\[
r_j - r_{j-1} - \frac{1}{2} d_{j-1}(x_{j-\frac{1}{2}}) = \int_{I_j} e_q dx - \frac{1}{2} d_j(x_{j+\frac{1}{2}}).
\]

So, the term \( h_j R^1_j \) can be formulated as

\[
h_j R^1_j = \int_{I_j} d_j(x)(x - x_j)(g_j(x) + \beta s_j(x))/h_j dx
\]
\[+ \int_{I_j} d_j(x)(x - x_j)(g'_j(x) + \beta s'_j(x))(x - x_{j+1/2})/h_j dx\]
\[+ h_j \left[ \int_{I_j} e_q dx - \frac{1}{2} d_j(x_{j+1/2}) \right] (g_j(x_{j-1/2}) + \beta s_j(x_{j-1})).\]

By the inverse and trace inequalities (2.10) and (2.11), we have the following estimate
\[
\left| \sum_j h_j R_j^i \right| \leq C(k) \|g + \beta s\|_{L^2} (\|d\|_{L^2} + h \|e_q\|_{L^2}).
\] (2.31)

where \(k\) is the degree of polynomials in the finite element space \(V_h^k\). Combining (2.27), (2.30), (2.31) together and recalling that \(\|\phi_2\|_{L^\infty} = \|x - x_{j+1/2}\|_{L^\infty} = h\), we conclude
\[
\|g + \beta s\|_{L^2} \leq C(\alpha, k) h (\|(e_u)_t\|_{L^2} + \|e_q\|_{L^2}).
\] (2.32)

Thus,
\[
\|g\|_{L^2} \leq \|g + \beta s\|_{L^2} + \|\beta\|_{L^2} \leq C(\alpha, \beta, k) h (\|(e_u)_t\|_{L^2} + \|e_q\|_{L^2} + \|e_p\|_{L^2}).
\] (2.33)

Now, we return to the error equation (2.25). Note that \((e_u)_t, \epsilon_q, \epsilon_p\) and \(\epsilon_r\) are orthogonal to any piecewise constants, then
\[
\left| \int_I (e_u)_t \epsilon_u dx \right| = \left| \sum_j \int_{I_j} (e_u)_t d_j(x)(x - x_j)/h_j \right| \leq \|(e_u)_t\|_{L^2} \|d\|_{L^2} \|\phi\|_{L^\infty},
\]
\[
\left| \int_I \epsilon_p \epsilon_p dx \right| = \left| \sum_j \int_{I_j} \epsilon_p w_j(x)(x - x_j)/h_j \right| \leq \|\epsilon_p\|_{L^2} \|w\|_{L^2} \|\phi\|_{L^\infty},
\]
\[
\left| \int_I \epsilon_r \epsilon_r dx \right| = \left| \sum_j \int_{I_j} \epsilon_r s_j(x)(x - x_j)/h_j \right| \leq \|\epsilon_r\|_{L^2} \|s\|_{L^2} \|\phi\|_{L^\infty},
\]
\[
\left| \int_I \epsilon_q \epsilon_r dx \right| = \left| \sum_j \int_{I_j} \epsilon_q g_j(x)(x - x_j)/h_j \right| \leq \|\epsilon_q\|_{L^2} \|g\|_{L^2} \|\phi\|_{L^\infty},
\]
\[
\left| \int_I \epsilon_p \epsilon_u dx \right| = \left| \sum_j \int_{I_j} \epsilon_p d_j(x)(x - x_j)/h_j \right| \leq \|\epsilon_p\|_{L^2} \|d\|_{L^2} \|\phi\|_{L^\infty},
\]

where \(\phi = (x - x_j)/h_j\). Then, equation (2.25) becomes
\[
\frac{1}{2} \frac{d}{dt} \|\epsilon_u\|_{L^2}^2 + \frac{1}{2} \|\epsilon_p\|_{L^2}^2 \leq \|\phi\|_{L^\infty} \left[ \|(e_u)_t\|_{L^2} \|d\|_{L^2} + \|\epsilon_p\|_{L^2} \|w\|_{L^2} + \|\epsilon_r\|_{L^2} \|s\|_{L^2}
\]
\[+ \|\epsilon_q\|_{L^2} \|g\|_{L^2} + |\beta| \|\epsilon_p\|_{L^2} \|d\|_{L^2} \right] + \frac{\beta^2}{2} \|\epsilon_u\|_{L^2}^2.
\]
Using the approximation property of the projections (2.9) and the fact that \( \|\phi\|_{L^\infty} = \frac{1}{2} \), we get
\[
\frac{1}{2} \frac{d}{dt} \|\bar{e}_u\|_{L^2}^2 + \frac{1}{2} \|\bar{e}_p\|_{L^2}^2 \leq Ch^{k+1} (\|d\|_{L^2} + \|s\|_{L^2} + \|w\|_{L^2} + \|g\|_{L^2}) + \frac{\beta^2}{2} \|\bar{e}_u\|_{L^2}^2, \tag{2.34}
\]
where \( C = C(\beta, \|u\|_{k+4}, \|u_t\|_{k+1}) \). Substituting (2.27), (2.28), (2.29) and (2.33) into (2.34), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\bar{e}_u\|_{L^2}^2 + \frac{1}{2} \|\bar{e}_p\|_{L^2}^2 \leq Ch^{k+2} (\|(e_u)_t\|_{L^2} + \|e_q\|_{L^2} + \|e_p\|_{L^2} + \|e_r\|_{L^2}) + \frac{\beta^2}{2} \|\bar{e}_u\|_{L^2}^2, \tag{2.35}
\]
where \( C = C(\alpha, \beta, \|u\|_{k+4}, \|u_t\|_{k+1}) \). Integrating the above inequality with respect to time and using the initial condition in Lemma 2.4, we obtain
\[
\frac{1}{2} \|\bar{e}_u(t)\|_{L^2}^2 + \frac{1}{2} \int_0^t \|\bar{e}_p(t)\|_{L^2}^2 dt \leq C e^{C_1 t} h^{2k+3}, \tag{2.36}
\]
where \( C = C(\alpha, \beta, \lambda, \|u\|_{k+4}, \|u_t\|_{k+1}) \).

To get the superconvergence result, we need the following lemma.

**Lemma 2.7.** Under the same condition as in Theorem 2.5, we have
\[
\|e_u\|_{L^2} + \|e_q\|_{L^2} \leq C e^{C_1 t} h^{k+1}, \tag{2.37}
\]
\[
\int_0^t (\|e_p\|_{L^2} + \|e_r\|_{L^2}) dt \leq C e^{C_1 t} h^{k+1}, \tag{2.38}
\]
where \( C = C(\alpha, \beta, \|u\|_{k+4}, \|u_t\|_{k+3}) \). Moreover, we have
\[
\int_0^t \|(e_u)_t\|_{L^2} dt \leq C e^{C_1 t} h^{k+1}, \tag{2.39}
\]
where \( C = C(\alpha, \beta, \|u\|_{k+4}, \|u_t\|_{k+4}, \|u_{tt}\|_{k+4}, \|u_{ttt}\|_{k+1}) \).

The proof of this lemma is given in the appendix. Using Lemma 2.7, we get from (2.36) that
\[
\frac{1}{2} \|\bar{e}_u(t)\|_{L^2}^2 + \frac{1}{2} \int_0^t \|\bar{e}_p(t)\|_{L^2}^2 dt \leq C e^{C_1 t} h^{2k+3} + \frac{\beta^2}{2} \int_0^t \|\bar{e}_u(t)\|_{L^2}^2 dt.
\]
A Gronwall’s inequality gives us the desired result
\[ \| \bar{e}_u(t) \|_{L^2}^2 + \int_0^t \| \bar{e}_p(t) \|_{L^2}^2 dt \leq C e^{2C_1 t} h^{2k+3}, \]
and in particular,
\[ \| \bar{e}_u(t) \|_{L^2} \leq C e^{C_1 t} h^{k+3/2}, \]
where \( C = C(\alpha, \beta, \lambda, \| u \|_{k+4}, \| u_t \|_{k+4}, \| u_{tt} \|_{k+4}, \| u_{ttt} \|_{k+1}) \) and \( C_1 = C_1(\alpha, \beta) > 0 \).

3 Numerical examples

In this section, we use some numerical experiments to demonstrate the superconvergence property of the LDG method for fourth order problems. Consider the following fourth order problem
\[
\begin{cases}
  u_t + u_x + u_{xx} + u_{xxxx} = 0, \\
  u(x, 0) = \sin(x), \\
  u(0, t) = u(2\pi, t).
\end{cases}
\] (3.1)

The exact solution to this problem is
\[ u(x, t) = \sin(x - t). \] (3.2)

Note that for problems containing high order derivatives, such as problem (3.1), the popular explicit nonlinearly stable high order TVD Runge-Kutta methods [19] will suffer from extremely small time step restriction due to the stiffness of the LDG spatial discretization operator. Thus, in the computation below, the second order implicit Crank-Nicholson time discretization is used. We consider both the special projection \( P_h^* \) and the usual \( L^2 \) projection of the initial condition as our numerical initial conditions and get similar results. Uniform meshes are used in the calculation.

Table 3.1 lists the numerical errors and their orders for \( k = 1 \) at different final time \( T \) when the special projection \( P_h^* u \) is used as the initial condition. From the table we conclude that, at any time, we can always observe third order accuracy for \( \bar{e}_u \) and \( \bar{e}_p \), indicating that the error estimate obtained in (2.18) is not optimal. Even though we
have derived an exponential growth result for $e_u$ and $e_p$, we can clearly observe that they actually grow linearly with respect to time for this particular example, which guarantees that the errors for $e_u$ and $e_p$ do not grow much with respect to time for a long time $t = O(\frac{1}{h})$. This is especially prominent for fine grids.

Table 3.1: $P^1$ polynomials on a uniform mesh of $N$ cells at different time $T$. $P_h^*$ projection of the initial condition.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$T = 1$</th>
<th>$T = 10$</th>
<th>$T = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L^2$ error</td>
<td>order</td>
<td>$L^2$ error</td>
</tr>
<tr>
<td>---</td>
<td>----:</td>
<td>--:</td>
<td>----:</td>
</tr>
<tr>
<td>$e_u$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>4.68E-04</td>
<td>-</td>
<td>3.04E-03</td>
</tr>
<tr>
<td>40</td>
<td>6.18E-05</td>
<td>2.92</td>
<td>3.87E-04</td>
</tr>
<tr>
<td>80</td>
<td>7.93E-06</td>
<td>2.96</td>
<td>4.90E-05</td>
</tr>
<tr>
<td>160</td>
<td>1.00E-06</td>
<td>2.98</td>
<td>6.15E-06</td>
</tr>
</tbody>
</table>

| $e_p$ |       |      |       |      |       |      |
| 20  | 4.27E-03 | -     | 5.25E-03 | -     | 2.96E-02 | -     |
| 40  | 1.06E-03 | 2.01  | 1.13E-03 | 2.22  | 3.90E-03 | 2.93  |
| 80  | 2.65E-04 | 2.00  | 2.70E-04 | 2.07  | 5.42E-04 | 2.85  |
| 160 | 6.64E-05 | 2.00  | 6.66E-05 | 2.02  | 8.89E-05 | 2.61  |

Table 3.2 lists the numerical errors and their orders for $k = 2$ at different final time $T$ when the special projection $P_h^* u$ is used as initialization. We can clearly see that both $\bar{e}_u$ and $\bar{e}_p$ achieve fourth order accuracy at $T = 1$. For longer time, for example $T = 10$ and $T = 50$, the orders seem also to converge to four, if we keep on refining the meshes. We also observe that the errors for both $\bar{e}_u$ and $\bar{e}_p$ do not grow much until the final time $T = 50$ we have run, especially for fine grids. For the case of $k = 3$, the results in Table 3.3 also demonstrate the superconvergence property of $\bar{e}_u$ and $\bar{e}_p$.

If we use the $L^2$ projection of the initial condition as our numerical initial condition instead, we also obtain the superconvergence results for $\bar{e}_u$ and $\bar{e}_p$ and observe little difference compared to the case when $P_h^* u$ is used as the numerical initial condition,
Table 3.2: $P^2$ polynomials on a uniform mesh of $N$ cells at different time $T$. $P_h^*$ projection of the initial condition.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$T = 1$</th>
<th>$T = 10$</th>
<th>$T = 50$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L^2$ error</td>
<td>order</td>
<td>$L^2$ error</td>
</tr>
<tr>
<td>10</td>
<td>6.93E-05</td>
<td>-</td>
<td>1.17E-04</td>
</tr>
<tr>
<td>20</td>
<td>4.23E-06</td>
<td>4.03</td>
<td>5.20E-06</td>
</tr>
<tr>
<td>40</td>
<td>2.63E-07</td>
<td>4.01</td>
<td>2.79E-07</td>
</tr>
<tr>
<td>80</td>
<td>1.64E-08</td>
<td>4.00</td>
<td>1.64E-08</td>
</tr>
</tbody>
</table>

indicating that the definition of the operator $P_h^*$ is only for the technical purpose in the proof and not essential to the computation, at least for this test case. See Tables 3.4-3.6.

We would like to mention that, apart from the superconvergence results for $e_u$ and $e_p$, we have also obtained similar superconvergence results for $e_q$ and $e_r$ in our numerical experiments, which are not listed here to save space.

Table 3.3: $P^3$ polynomials on a uniform mesh of $N$ cells. $T = 1$. $P_h^*$ projection of the initial condition.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\tilde{e}_u$</th>
<th>$e_u$</th>
<th>$\tilde{e}_p$</th>
<th>$e_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L^2$ error</td>
<td>order</td>
<td>$L^2$ error</td>
<td>order</td>
</tr>
<tr>
<td>5</td>
<td>6.60E-05</td>
<td>-</td>
<td>5.26E-04</td>
<td>-</td>
</tr>
<tr>
<td>10</td>
<td>1.73E-06</td>
<td>5.25</td>
<td>3.30E-05</td>
<td>4.00</td>
</tr>
<tr>
<td>20</td>
<td>5.39E-08</td>
<td>5.01</td>
<td>2.06E-06</td>
<td>4.00</td>
</tr>
<tr>
<td>40</td>
<td>1.71E-09</td>
<td>4.98</td>
<td>1.29E-07</td>
<td>4.00</td>
</tr>
</tbody>
</table>
Table 3.4: $P^1$ polynomials on a uniform mesh of $N$ cells. $T = 1$. $L^2$ projection of the initial condition.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\bar{e}_u$</th>
<th>$\bar{e}_u$</th>
<th>$\bar{e}_p$</th>
<th>$\bar{e}_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L^2$ error</td>
<td>order</td>
<td>$L^2$ error</td>
<td>order</td>
</tr>
<tr>
<td>20</td>
<td>4.36E-04</td>
<td>--</td>
<td>4.26E-03</td>
<td>--</td>
</tr>
<tr>
<td>40</td>
<td>5.63E-05</td>
<td>2.95</td>
<td>1.06E-03</td>
<td>2.00</td>
</tr>
<tr>
<td>80</td>
<td>7.15E-06</td>
<td>2.98</td>
<td>2.66E-04</td>
<td>2.00</td>
</tr>
<tr>
<td>160</td>
<td>9.00E-07</td>
<td>2.99</td>
<td>6.64E-05</td>
<td>2.00</td>
</tr>
</tbody>
</table>

Table 3.5: $P^2$ polynomials on a uniform mesh of $N$ cells. $T = 1$. $L^2$ projection of the initial condition.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\bar{e}_u$</th>
<th>$\bar{e}_u$</th>
<th>$\bar{e}_p$</th>
<th>$\bar{e}_p$</th>
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<tbody>
<tr>
<td></td>
<td>$L^2$ error</td>
<td>order</td>
<td>$L^2$ error</td>
<td>order</td>
</tr>
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<td>6.90E-05</td>
<td>--</td>
<td>8.56E-04</td>
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<tr>
<td>20</td>
<td>4.23E-06</td>
<td>4.03</td>
<td>1.07E-04</td>
<td>3.00</td>
</tr>
<tr>
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<td>2.62E-07</td>
<td>4.01</td>
<td>1.34E-05</td>
<td>3.00</td>
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<tr>
<td>80</td>
<td>1.65E-08</td>
<td>3.99</td>
<td>1.67E-06</td>
<td>3.00</td>
</tr>
</tbody>
</table>

4 Concluding remarks

In this paper, we have studied the superconvergence property of the LDG method for linear fourth order time dependent problems. We prove that the error between the numerical solution and a particular projection of the exact solution achieves $(k + \frac{3}{2})$-th order superconvergence when polynomials of degree $k$ ($k \geq 1$) are used. Numerical experiments are also displayed to verify the theoretical results. Even though we consider only the one-dimensional case in this paper, similar results should hold for certain tensor product two dimensional cases, see [6] for related discussion for convection and second

Table 3.6: $P^3$ polynomials on a uniform mesh of $N$ cells. $T = 1$. $L^2$ projection of the initial condition.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\bar{e}_u$</th>
<th>$\bar{e}_u$</th>
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<td>--</td>
<td>5.25E-04</td>
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<td>3.30E-05</td>
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<td>4.79</td>
<td>1.29E-07</td>
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</tr>
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</table>
order diffusion equations.

Future work includes the study of superconvergence of the LDG method for high order wave equations, which is more difficult than diffusion equations since there is no control on the derivatives from the initial condition itself. Analysis of the LDG method for nonlinear equations also constitute future work.

A Appendix

In this appendix, we give the proofs for some of the technical lemmas and theorems.

A.1 The proof of Lemma 2.4

We will first prove the existence and uniqueness of $P_h^* u$.

When using the DG discretization operator $\mathcal{D}$, equation (2.14) can be written as

$$\int_{I_j} r_h \eta dx - \mathcal{D}_{I_j} (p_h, \eta; P_h^-) = 0, \quad (A.1a)$$

$$\int_{I_j} p_h \xi dx - \mathcal{D}_{I_j} (q_h, \xi; q_h^+) = 0, \quad (A.1b)$$

$$\int_{I_j} q_h \psi dx - \mathcal{D}_{I_j} (P_h^* u, \psi; (P_h^* u)^-) = 0, \quad (A.1c)$$

for any $\psi, \xi, \eta \in V_h^k$. Since the exact solutions $u, q = u_x, p = u_{xx}, r = u_{xxx}$ also satisfy scheme (A.1), we thus have the error equations

$$\int_{I_j} (r - r_h) \eta dx - \mathcal{D}_{I_j} (p - p_h, \eta; (p - p_h)^-) = 0, \quad (A.2a)$$

$$\int_{I_j} (p - p_h) \xi dx - \mathcal{D}_{I_j} (q - q_h, \xi; (q - q_h)^+) = 0, \quad (A.2b)$$

$$\int_{I_j} (q - q_h) \psi dx - \mathcal{D}_{I_j} (u - P_h^* u, \psi; (u - P_h^* u)^-) = 0 \quad (A.2c)$$

for any $\psi, \xi, \eta \in V_h^k$. Denote

$$u - P_h^* u = (u - P_h^- u) + (P_h^- u - P_h^* u) = \varepsilon_u + E_u$$

$$q - q_h = (q - P_h^+ q) + (P_h^+ q - q_h) = \varepsilon_q + E_q$$

$$p - p_h = (p - P_h^- p) + (P_h^- p - p_h) = \varepsilon_p + E_p$$
\[ r - r_h = (r - P^+_h r) + (P^+_h r - r_h) = \varepsilon_r + E_r. \]

By virtue of the properties of the projections \( P^+_h \) and \( P^-_h \), (2.7) and (2.8), equation (A.2) becomes

\[
\int_{I_j} (\varepsilon_r + E_r) \eta dx - D_{I_j}(E_p, \eta; E_p^-) = 0, \quad (A.3a)
\]
\[
\int_{I_j} (\varepsilon_p + E_p) \xi dx - D_{I_j}(E_q, \xi; E_q^+) = 0, \quad (A.3b)
\]
\[
\int_{I_j} (\varepsilon_q + E_q) \psi dx - D_{I_j}(E_u, \psi; E_u^-) = 0. \quad (A.3c)
\]

Also, conditions (2.15) and (2.16) are equivalent to

\[
\int_{I_j} (E_u - E_q + E_r) \rho dx = 0 \quad (A.4)
\]
for any \( \rho \in P^{k-1} \) on \( I_j \) and

\[
E^-_u = (E_q - E_r)^+ \quad \text{at } x_{j-1/2}. \quad (A.5)
\]

Note that (A.3), (A.4) and (A.5) are a linear system for \( E_u, E_q, E_p, E_r \in V_h^k \). To prove the existence and uniqueness of \( P^*_h u \), we need only to prove the uniqueness of \( E_u \), then

\[
P^*_h u = P^-_h u - E_u \quad \text{will exist and is unique.}
\]

Plugging conditions (A.4) and (A.5) into (A.3), we obtain

\[
\int_{I_j} (\varepsilon_r + E_r) \eta dx - D_{I_j}(E_p, \eta; E_p^-) = 0, \quad (A.6a)
\]
\[
\int_{I_j} (\varepsilon_p + E_p) \xi dx - D_{I_j}(E_q, \xi; E_q^+) = 0, \quad (A.6b)
\]
\[
\int_{I_j} (\varepsilon_q + E_q) \psi dx - D_{I_j}(E_u - E_r, \psi; (E_q - E_r)^+) = 0, \quad (A.6c)
\]

which is

\[
\int_{I_j} E_r \eta dx - D_{I_j}(E_p, \eta; E_p^-) = -\int_{I_j} \varepsilon_r \eta dx, \quad (A.7a)
\]
\[
\int_{I_j} E_p \xi dx - D_{I_j}(E_q, \xi; E_q^+) = -\int_{I_j} \varepsilon_p \xi dx, \quad (A.7b)
\]
\[
\int_{I_j} E_q \psi dx - D_{I_j}(E_u - E_r, \psi; (E_q - E_r)^+) = -\int_{I_j} \varepsilon_q \psi dx \quad (A.7c)
\]
for any \( \psi, \xi, \eta \in V_h^k \). Note that equation (A.7) is a linear system, hence the existence of \((E_q, E_p, E_r)\) follows by the uniqueness.

We claim that the solution \((E_q, E_p, E_r)\) to (A.7) is unique. Suppose both \((E_q^1, E_p^1, E_r^1)\) and \((E_q^2, E_p^2, E_r^2)\) satisfy equation (A.7), and denote \( g_q = E_q^1 - E_q^2, g_p = E_p^1 - E_p^2, g_r = E_r^1 - E_r^2 \), then (A.7) yields

\[
\begin{align*}
\int_{I_j} g_r \eta dx - D_{I_j}(g_p, \eta; g_p^-) &= 0, \\
\int_{I_j} g_p \xi dx - D_{I_j}(g_q, \xi; g_q^+) &= 0, \\
\int_{I_j} g_q \psi dx - D_{I_j}(g_q - g_r, \psi; (g_q - g_r)^+) &= 0
\end{align*}
\]

for any \( \psi, \xi, \eta \in V_h^k \). Now taking \((\psi, \xi, \eta) = (g_q - g_r, g_p, g_q)\) in (A.8), adding them up and summing over all \( j \), we get

\[
\|g_q\|_{L^2}^2 + \|g_p\|_{L^2}^2 - D(g_p, g_q; g_p^-) - D(g_q, g_p; g_q^+) - D(g_q - g_r, g_q - g_r; (g_q - g_r)^+) = 0.
\]

By the property of the operator \( D \), Lemma 2.1, we have

\[
\|g_q\|_{L^2}^2 + \|g_p\|_{L^2}^2 + \frac{1}{2} \sum_j [g_q - g_r]_{j+rac{1}{2}}^2 = 0,
\]

which implies \( g_q = g_p = 0 \) and further \( g_r = 0 \). We have thus proved the existence and uniqueness of \( E_q \) and \( E_r \), then conditions (A.4) and (A.5) lead to the existence and uniqueness of \( E_u \), and thus \( P_h^* u \).

We obtain the error estimate (2.17) in three steps.

**Step 1:** By Lemma 2.2, equation (A.6) can be rewritten as

\[
\begin{align*}
\int_{I_j} (\varepsilon_r + E_r) \eta dx - \int_{I_j} (E_p)_x \eta dx - [E_p] \eta^+|_{j-rac{1}{2}} &= 0, \\
\int_{I_j} (\varepsilon_p + E_p) \xi dx - \int_{I_j} (E_q)_x \xi dx - [E_q] \xi^-|_{j+rac{1}{2}} &= 0, \\
\int_{I_j} (\varepsilon_q + E_q) \psi dx - \int_{I_j} (E_q - E_r)_x \psi dx - [E_q - E_r] \psi^-|_{j+rac{1}{2}} &= 0.
\end{align*}
\]

Define \( E_q = b_j + s_j(x-x_j)/h_j, E_p = v_j + w_j(x-x_j)/h_j, E_r = l_j + g_j(x-x_j)/h_j \) on \( I_j \), where \( b_j, v_j, l_j \) are constants and \( s_j(x), w_j(x), g_j(x) \) \( \in P^{k-1} \). First, we let \( \psi = \)
(s_j(x) - g_j(x))(x - x_{j+1/2})/h_j in (A.9c) and get, by the definition of B_j^+

\[ \int_{I_j} (\varepsilon_q + E_q)(s_j(x) - g_j(x))(x - x_{j+1/2})/h_j dx - B_j^+(s_j - g_j) = 0. \]

Using the property of B_j^+ in Lemma 2.3, we have

\[ \int_{I_j} (\varepsilon_q + E_q)(s_j(x) - g_j(x))(x - x_{j+1/2})/h_j dx + \frac{1}{4h_j} \int_{I_j} (s_j(x) - g_j(x))^2 dx \]
\[ + \frac{(s_j(x_{j-1/2}) - g_j(x_{j-1/2}))^2}{4} = 0. \]

Thus,

\[ \int_{I_j} (s_j(x) - g_j(x))^2 dx \leq -4 \int_{I_j} (\varepsilon_q + E_q)(s_j(x) - g_j(x))(x - x_{j+1/2}) dx. \quad (A.10) \]

Define piecewise polynomials s(x), g(x) and \( \phi_2(x) \), such that \( s(x) = s_j(x), \) \( g(x) = g_j(x), \) \( \phi_2(x) = x - x_{j+1/2} \) on \( I_j \), and sum (A.10) over all \( j \), we get

\[ \|s - g\|_{L^2} \leq 4\|\varepsilon_q + E_q\|_{L^2}\|\phi_2\|_{L^\infty}. \]

By approximation results (2.9) and the fact that \( \|\phi_2\|_{L^\infty} = h \), we get

\[ \|s - g\|_{L^2} \leq 4h\|\varepsilon_q + E_q\|_{L^2} \leq C h^{k+2} + Ch\|E_q\|_{L^2}, \quad (A.11) \]

where \( C = C(\|u\|_{k+2}) \). Similarly, letting \( \xi = s_j(x)(x - x_{j+1/2})/h_j \) in (A.9b) and \( \eta = w_j(x)(x - x_{j-1/2})/h_j \) in (A.9a), and using the definition of \( B_j^- \) and \( B_j^+ \), we get

\[ \int_{I_j} (\varepsilon_r + E_r)w_j(x)(x - x_{j-1/2})/h_j dx - B_j^-(w_j) = 0, \]
\[ \int_{I_j} (\varepsilon_p + E_p)s_j(x)(x - x_{j+1/2})/h_j dx - B_j^+(s_j) = 0. \]

Using the properties of \( B_j^- \) and \( B_j^+ \) in Lemma 2.3, we have

\[ \int_{I_j} (\varepsilon_r + E_r)w_j(x)(x - x_{j-1/2})/h_j dx - \frac{1}{4h_j} \int_{I_j} w_j^2(x) dx - \frac{w_j^2(x_{j+1/2})}{4} = 0, \]
\[ \int_{I_j} (\varepsilon_p + E_p)s_j(x)(x - x_{j+1/2})/h_j dx + \frac{1}{4h_j} \int_{I_j} s_j^2(x) dx + \frac{s_j^2(x_{j+1/2})}{4} = 0. \]
Thus,
\[
\int_{I_j} w_j^2(x) dx \leq 4 \int_{I_j} (\varepsilon_r + E_r)w_j(x)(x - x_{j-1/2}) dx,
\]
\[
\int_{I_j} s_j^2(x) dx \leq -4 \int_{I_j} (\varepsilon_p + E_p)s_j(x)(x - x_{j+1/2}) dx.
\]
Define piecewise polynomials \(w(x)\) and \(\phi_1(x)\), such that \(w(x) = w_j(x), \phi_1(x) = x - x_{j-1/2}\) on \(I_j\), thus, \(\|\phi_1\|_{L^\infty} = h\), finally, we get
\[
\|w\|_{L^2} \leq 4\|\varepsilon_r + E_r\|_{L^2}\|\phi_1\|_{L^\infty} \leq 4h\|\varepsilon_r + E_r\|_{L^2} \leq C h^{k+2} + Ch\|E_r\|_{L^2}, \tag{A.12}
\]
\[
\|s\|_{L^2} \leq 4\|\varepsilon_p + E_p\|_{L^2}\|\phi_2\|_{L^\infty} \leq 4h\|\varepsilon_p + E_p\|_{L^2} \leq C h^{k+2} + Ch\|E_p\|_{L^2}, \tag{A.13}
\]
where \(C = C(\|u\|_{k+4})\).

**Step 2:** On one hand, taking \((\psi, \xi, \eta) = (E_q - E_r, E_p, E_q)\) in equation (A.7), adding them up and summing over all \(j\), we obtain
\[
\|E_q\|_{L^2}^2 + \|E_p\|_{L^2}^2 - \mathcal{D}(E_p, E_q; E_q^{-}) - \mathcal{D}(E_q, E_p; E_q^{+}) - \mathcal{D}(E_q - E_r, E_q - E_r; (E_q - E_r)^{+})
\]
\[
= - \int_I \varepsilon_r E_q dx - \int_I \varepsilon_p E_p dx - \int_I \varepsilon_q (E_q - E_r) dx.
\]
By the property of the operator \(\mathcal{D}\), Lemma 2.1, we have
\[
\|E_q\|_{L^2}^2 + \|E_p\|_{L^2}^2 + \frac{1}{2} \sum_j \|E_q - E_r\|_{L^2}^2 = - \int_I \varepsilon_r E_q dx - \int_I \varepsilon_p E_p dx - \int_I \varepsilon_q (E_q - E_r) dx,
\]
and thus
\[
\|E_q\|_{L^2}^2 + \|E_p\|_{L^2}^2 \leq \left| \int_I \varepsilon_r E_q dx + \int_I \varepsilon_p E_p dx + \int_I \varepsilon_q (E_q - E_r) dx \right|.
\]
Notice that \(\varepsilon_q, \varepsilon_p\) and \(\varepsilon_r\) are orthogonal to any constant, then
\[
\left| \int_I \varepsilon_r E_q dx \right| = \sum_j \int_{I_j} \varepsilon_r s_j(x)(x - x_j) / h_j dx \leq \|\varepsilon_r\|_{L^2} \|s\|_{L^2} \|\phi\|_{L^\infty},
\]
\[
\left| \int_I \varepsilon_p E_p dx \right| = \sum_j \int_{I_j} \varepsilon_p w_j(x)(x - x_j) / h_j dx \leq \|\varepsilon_p\|_{L^2} \|w\|_{L^2} \|\phi\|_{L^\infty},
\]
\[
\left| \int_I \varepsilon_q (E_q - E_r) dx \right| = \sum_j \int_{I_j} \varepsilon_q (s_j(x) - g_j(x))(x - x_j) / h_j dx \leq \|\varepsilon_q\|_{L^2} \|s - g\|_{L^2} \|\phi\|_{L^\infty},
\]

25
where \( \phi = (x - x_j)/h_j \). Therefore,

\[
\|E_q\|_{L^2}^2 + \|E_p\|_{L^2}^2 \leq \|\phi\|_{L^\infty}(\|\varepsilon_r\|_{L^2} s_{L^2} + \|\varepsilon_p\|_{L^2} \|w\|_{L^2} + \|\varepsilon_q\|_{L^2} \|s - g\|_{L^2}).
\]

From the approximation results (2.9) and employing \( \|\phi\|_{L^\infty} = \frac{1}{2} \), we conclude that

\[
\|E_q\|_{L^2}^2 + \|E_p\|_{L^2}^2 \leq C h^{k+1}(\|s\|_{L^2} + \|w\|_{L^2} + \|s - g\|_{L^2}). \tag{A.14}
\]

Combing (A.11)-(A.13) and (A.14), we arrive at

\[
\|E_q\|_{L^2}^2 + \|E_p\|_{L^2}^2 \leq C h^{2k+3} + C h^{k+2}(\|E_q\|_{L^2} + \|E_p\|_{L^2} + \|E_r\|_{L^2}), \tag{A.15}
\]

where \( C = C(\|u\|_{k+4}) \).

On the other hand, taking \((\psi, \xi, \eta) = (-E_p, E_q, E_r - E_q)\) in equation (A.7), adding them up and summing over all \( j \), we obtain

\[
\|E_r\|_{L^2}^2 + D(E_p, E_q - E_r; E_r^-) + D(E_q - E_r, E_p; (E_q - E_r)^+) - D(E_q, E_q; E_q^+) = -\int_I \varepsilon_r(E_r - E_q)dx - \int_I \varepsilon_p E_q dx + \int_I \varepsilon_q E_p dx + \int_I E_q E_r dx.
\]

By the property of the operator \( D \), Lemma 2.1, we have

\[
\|E_r\|_{L^2}^2 + \frac{1}{2} \sum_j [E_q]_{j+\frac{1}{2}}^2 = -\int_I \varepsilon_r(E_r - E_q)dx - \int_I \varepsilon_p E_q dx + \int_I \varepsilon_q E_p dx + \int_I E_q E_r dx,
\]

and thus

\[
\frac{1}{2} \|E_r\|_{L^2}^2 \leq \left|\int_I \varepsilon_r(E_r - E_q)dx\right| + \left|\int_I \varepsilon_p E_q dx\right| + \left|\int_I \varepsilon_q E_p dx\right| + \frac{1}{2} \|E_q\|_{L^2}^2.
\]

Notice that \( \varepsilon_q, \varepsilon_p \) and \( \varepsilon_r \) are orthogonal to any constant, then

\[
\left|\int_I \varepsilon_r(E_r - E_q)dx\right| = \left|\sum_j \int_{I_j} \varepsilon_r(g_j(x) - s_j(x))(x - x_j)/h_j dx\right| \leq \|\varepsilon_r\|_{L^2} \|g - s\|_{L^2} \|\phi\|_{L^\infty},
\]

\[
\left|\int_I \varepsilon_p E_q dx\right| = \left|\sum_j \int_{I_j} \varepsilon_p s_j(x)(x - x_j)/h_j dx\right| \leq \|\varepsilon_p\|_{L^2} \|s\|_{L^2} \|\phi\|_{L^\infty},
\]

\[
\left|\int_I \varepsilon_q E_p dx\right| = \left|\sum_j \int_{I_j} \varepsilon_q w_j(x)(x - x_j)/h_j dx\right| \leq \|\varepsilon_q\|_{L^2} \|w\|_{L^2} \|\phi\|_{L^\infty},
\]

(26)
we recall that $\phi = (x - x_j)/h_j$. Therefore,
\[
\frac{1}{2} \|E_r\|_{L^2}^2 \leq \|\phi\|_{L^\infty} (\|\varepsilon_r\|_{L^2} \|g - s\|_{L^2} + \|\varepsilon_p\|_{L^2} \|s\|_{L^2} + \|\varepsilon_q\|_{L^2} \|w\|_{L^2}) + \frac{1}{2} \|E_q\|_{L^2}^2.
\]
From the approximation results (2.9) and employing $\|\phi\|_{L^\infty} = \frac{1}{2}$, we conclude that
\[
\frac{1}{2} \|E_r\|_{L^2}^2 \leq Ch^{k+1} (\|g - s\|_{L^2} + \|s\|_{L^2} + \|w\|_{L^2}) + \frac{1}{2} \|E_q\|_{L^2}^2.
\]
Combining (A.11)-(A.13) and (A.16), we arrive at
\[
\frac{1}{2} \|E_r\|_{L^2}^2 \leq Ch^{2k+3} + Ch^{k+2} (\|E_q\|_{L^2} + \|E_p\|_{L^2} + \|E_r\|_{L^2}) + \frac{1}{2} \|E_q\|_{L^2}^2,
\]
where $C = C(\|u\|_{k+4})$. Then (A.15) and (A.17) produce
\[
\frac{1}{2} \|E_q\|_{L^2}^2 + \|E_p\|_{L^2}^2 + \frac{1}{2} \|E_r\|_{L^2}^2 \leq Ch^{2k+3} + Ch^{k+2} (\|E_q\|_{L^2} + \|E_p\|_{L^2} + \|E_r\|_{L^2}),
\]
which implies
\[
\|E_q\|_{L^2} + \|E_p\|_{L^2} + \|E_r\|_{L^2} \leq C(\|u\|_{k+4}) h^{k+3/2}.
\]

**Step 3:** Suppose that
\[
E_u = \sum_{n=0}^k a_n P_n \left( \frac{2(x - x_j)}{h_j} \right), \quad E_q - E_r = \sum_{n=0}^k b_n P_n \left( \frac{2(x - x_j)}{h_j} \right)
\]
on $I_j$, where $P_n(\cdot)$ denotes the $m$th order Legendre polynomial. By using the technique in [6], conditions (A.4) and (A.5) yield the relationship
\[
\|E_u\|_{L^2} \leq C(\lambda) \|E_q - E_r\|_{L^2}.
\]
We recall that $\lambda$ is the maximum of the ratio of arbitrary two different mesh sizes. A combination of (A.18) and (A.19) gives us a bound for $E_u$,
\[
\|E_u\|_{L^2} \leq C(\lambda)(\|E_q\|_{L^2} + \|E_r\|_{L^2}) \leq C(\lambda, \|u\|_{k+4}) h^{k+3/2}.
\]
A.2 The proof of Lemma 2.7

If we can prove

\[ \|\tilde{e}_u\|_{L^2}^2 + \|\tilde{e}_q\|_{L^2}^2 + \int_0^t (\|\tilde{e}_p\|_{L^2}^2 + \|\tilde{e}_r\|_{L^2}^2)\,dt \leq C e^{C_1 t} h^{k+1}, \]  

(A.20)

with \( C = C(\alpha, \beta, \|u\|_{k+4}, \|u_t\|_{k+3}) \), then the estimates (2.37) and (2.38) in Lemma 2.7 will follow by the approximation error estimates (2.9) and triangle inequality. To this end, on one hand, we rewrite (2.24) into

\[ \frac{1}{2} \frac{d}{dt} \|\tilde{e}_u\|_{L^2}^2 + \|\tilde{e}_p\|_{L^2}^2 \leq - \int_I (\varepsilon_u) \tilde{e}_u dx - \int_I \varepsilon_\rho \tilde{e}_p dx - \int_I \varepsilon_r \tilde{e}_q dx + \int_I \varepsilon_q \tilde{e}_r dx - \beta \int_I \varepsilon_p \tilde{e}_p dx + \beta^2 \|\tilde{e}_u\|_{L^2}^2 + \frac{1}{4} \|\tilde{e}_p\|_{L^2}^2. \]  

(A.21)

On the other hand, taking the time derivative in (2.21d), letting \((\rho, \psi, \xi, \eta) = (-\tilde{e}_p, \tilde{e}_q, (\tilde{e}_u)_t, \tilde{e}_r)\) in (2.21), adding them up and summing over all \( j \), we obtain

\[ \frac{1}{2} \frac{d}{dt} \|\tilde{e}_q\|_{L^2}^2 + \|\tilde{e}_r\|_{L^2}^2 + \int_I (\varepsilon_q) \tilde{e}_q dx + \int_I \varepsilon_r \tilde{e}_r dx - \int_I (\varepsilon_u) \tilde{e}_p dx + \int_I \varepsilon_p (\tilde{e}_u)_t dx - \alpha D(\tilde{e}_u, \tilde{e}_p; \tilde{e}_u^-) \]

\[ - \beta D(\tilde{e}_q, \tilde{e}_p; \tilde{e}_q^-) = 0. \]

Using the property of the operator \( D \) in Lemma 2.1, we have

\[ \frac{1}{2} \frac{d}{dt} \|\tilde{e}_q\|_{L^2}^2 + \|\tilde{e}_r\|_{L^2}^2 + \int_I (\varepsilon_q) \tilde{e}_q dx + \int_I \varepsilon_r \tilde{e}_r dx - \int_I (\varepsilon_u) \tilde{e}_p dx + \int_I \varepsilon_p (\tilde{e}_u)_t dx - \alpha D(\tilde{e}_u, \tilde{e}_p; \tilde{e}_u^-) - \beta D(\tilde{e}_q, \tilde{e}_p; \tilde{e}_q^-) = 0. \]

(A.22)

By taking \( \psi = \tilde{e}_p \) in (2.21d) and summing over all \( j \), we get

\[ D(\tilde{e}_u, \tilde{e}_p; \tilde{e}_u^-) = \int_I \varepsilon_q \tilde{e}_p dx. \]

(A.23)

Using the property of the operator \( D \) in Lemma 2.1, and then taking \( \eta = \tilde{e}_q \) in (2.21b) and summing over all \( j \), we obtain

\[ D(\tilde{e}_q, \tilde{e}_p; \tilde{e}_q^-) = -D(\tilde{e}_p, \tilde{e}_q; \tilde{e}_p^-) = -\int_I \varepsilon_r \tilde{e}_q dx. \]

(A.24)

Plugging (A.23) and (A.24) into (A.22), then

\[ \frac{1}{2} \frac{d}{dt} \|\tilde{e}_q\|_{L^2}^2 + \frac{1}{2} \|\tilde{e}_r\|_{L^2}^2 \leq - \int_I (\varepsilon_q) \tilde{e}_q dx - \int_I \varepsilon_r \tilde{e}_r dx + \int_I (\varepsilon_u) \tilde{e}_p dx - \int_I \varepsilon_p (\tilde{e}_u)_t dx + \alpha \int_I \varepsilon_q \tilde{e}_p dx - \beta \int_I \varepsilon_q \tilde{e}_p dx + \frac{1}{4} \|\tilde{e}_p\|_{L^2}^2 + \left( \alpha^2 + \frac{\beta^2}{2} \right) \|\tilde{e}_q\|_{L^2}^2. \]

(A.25)
Combing (A.21) and (A.25), we arrive at
\[
\frac{1}{2} \frac{d}{dt} \| \tilde{e}_u \|^2_{L^2} + \frac{1}{2} \frac{d}{dt} \| \tilde{e}_q \|^2_{L^2} + \frac{1}{2} \| \tilde{e}_p \|^2_{L^2} + \frac{1}{2} \| \tilde{e}_r \|^2_{L^2} \leq \Lambda + \Theta + \beta^2 \| \tilde{e}_u \|^2_{L^2} + \left( \alpha^2 + \frac{\beta^2}{2} \right) \| \tilde{e}_q \|^2_{L^2},
\]
(A.26)
where
\[
\Lambda = - \int_I \varepsilon(t, \tilde{e}_u) \, dx
\]
and
\[
\Theta = - \int_I (\varepsilon_u \tilde{e}_u \, dx - \int_I \varepsilon_p \tilde{e}_p \, dx - \int_I \varepsilon_r \tilde{e}_r \, dx + \int_I \varepsilon_q \tilde{e}_q \, dx - \beta \int_I \varepsilon_p \tilde{e}_u \, dx

- \int_I (\varepsilon_q \tilde{e}_q \, dx - \int_I \varepsilon_r \tilde{e}_r \, dx + \int_I (\varepsilon_u \tilde{e}_u \, dx + \alpha \int_I \varepsilon_q \tilde{e}_q \, dx - \beta \int_I \varepsilon_r \tilde{e}_q \, dx.
\]
Using the approximation property of the projections (2.9), we get
\[
|\Theta| \leq C(\alpha, \beta, \| u \|_{k+4}, \| u_t \|_{k+2}) h^{k+1}(\| \tilde{e}_u \|_{L^2} + \| \tilde{e}_q \|_{L^2} + \| \tilde{e}_p \|_{L^2} + \| \tilde{e}_r \|_{L^2}).
\]
Integrating \( \Lambda \) with respect to time, we have, after integration by parts
\[
\int_0^t \Lambda \, dt = \int_0^t (\varepsilon_p(t) \tilde{e}_u \, dx - \int_I (\tilde{e}_u(t) \varepsilon_p(t) - \tilde{e}_u(0) \varepsilon_p(0)) \, dx.
\]
Thus, by the approximation results (2.9) and the choice of initial condition in Lemma 2.4, we conclude that
\[
\left| \int_0^t \Lambda \, dt \right| \leq C h^{k+1} \int_0^t \| \tilde{e}_u \|_{L^2} \, dt + \frac{1}{2} \| \tilde{e}_u(t) \|^2_{L^2} + C h^{2k+2},
\]
where \( C = C(\| u \|_{k+4}, \| u_t \|_{k+3}) \). Integrating (A.26) with respect to time, we obtain
\[
\frac{1}{4} \| \tilde{e}_u \|^2_{L^2} + \frac{1}{2} \| \tilde{e}_q \|^2_{L^2} + \frac{1}{2} \| \tilde{e}_p \|^2_{L^2} + \frac{1}{2} \| \tilde{e}_r \|^2_{L^2} \leq \frac{1}{2} \| \tilde{e}_u(0) \|^2_{L^2} + \frac{1}{2} \| \tilde{e}_q(0) \|^2_{L^2} + C h^{k+1} \int_0^t (\| \tilde{e}_u \|_{L^2} + \| \tilde{e}_q \|_{L^2} + \| \tilde{e}_p \|_{L^2} + \| \tilde{e}_r \|_{L^2}) \, dt + \left( \alpha^2 + \frac{\beta^2}{2} \right) \int_0^t \| \tilde{e}_q \|^2_{L^2} \, dt + C h^{2k+2},
\]
(A.27)
where \( C = C(\alpha, \beta, \| u \|_{k+4}, \| u_t \|_{k+3}) \). A Gronwall’s inequality and the estimates of initial condition (2.17) and (A.18) give us the error estimate (A.20).
To prove the estimate (2.39), we first need to get a bound for \((\bar{e}_u)_t(\cdot, 0)\). Using conditions (2.15) and (2.16), we have, at \(t = 0\),

\[
\mathcal{D}_{I_j}(\bar{e}_r, \rho; \bar{e}_r^+) = \mathcal{D}_{I_j}(\bar{e}_q, \rho; \bar{e}_q^+) - \mathcal{D}_{I_j}(\bar{e}_u, \rho; \bar{e}_u^-)
\]

for any \(\rho \in V_h^k\). Then (2.21a) becomes

\[
\int_{I_j} (e_u)_t \rho dx + (\alpha - 1) \mathcal{D}_{I_j}(\bar{e}_u, \rho; \bar{e}_u^-) + (\beta + 1) \mathcal{D}_{I_j}(\bar{e}_q, \rho; \bar{e}_q^+) = 0.
\]

It follows from (2.21c) and (2.21d) that, at \(t = 0\),

\[
\int_{I_j} (e_u)_t \rho dx + (\alpha - 1) \int_{I_j} e_q \rho dx + (\beta + 1) \int_{I_j} e_p \rho dx = 0
\]

for any \(\rho \in V_h^k\). Taking \(\rho = (\bar{e}_u)_t(\cdot, 0)\) and summing above equality over all \(j\), we get, at \(t = 0\),

\[
\| (\bar{e}_u)_t(\cdot, 0) \|_{L^2} \leq \| (\bar{e}_u)_t(\cdot, 0) \|_{L^2} + |\alpha - 1| \| e_q(\cdot, 0) \|_{L^2} + |\beta + 1| \| e_p(\cdot, 0) \|_{L^2}.
\]

Then, the approximation results (2.9) and estimates for the initial data in (A.18) give us

\[
\| (\bar{e}_u)_t(\cdot, 0) \|_{L^2} \leq C h^{k+1}, \quad (A.28)
\]

where \(C = C(\alpha, \beta, \| u \|_{k+4}, \| u_t \|_{k+1})\). Then, taking the time derivative in (2.21), letting

\((\rho, \psi, \xi, \eta) = ((\bar{e}_u)_t, -(\bar{e}_r)_t, (\bar{e}_p)_t, (\bar{e}_q)_t)\), adding them up and summing over all \(j\), we obtain

\[
0 = \int_I (\bar{e}_u)_t (\bar{e}_u)_t \rho \, dx + \int_I (\bar{e}_p)_t^2 \rho \, dx + \int_I (\varepsilon_u)_t (\bar{e}_u)_t \rho \, dx + \int_I (\varepsilon_p)_t (\bar{e}_p)_t \rho \, dx + \int_I (\varepsilon_r)_t (\bar{e}_q)_t \rho \, dx
\]

\[
- \int_I (\varepsilon_q)_t (\bar{e}_q)_t \rho \, dx + \alpha \mathcal{D}((\bar{e}_u)_t, (\bar{e}_u)_t; (\bar{e}_u)_t^-) + \beta \mathcal{D}((\bar{e}_q)_t, (\bar{e}_u)_t; (\bar{e}_q)_t^+) + \mathcal{D}((\bar{e}_r)_t, (\bar{e}_u)_t; (\bar{e}_r)_t^+)
\]

\[
+ \mathcal{D}((\bar{e}_u)_t, (\bar{e}_r)_t; (\bar{e}_u)_t^-) - \mathcal{D}((\bar{e}_q)_t, (\bar{e}_p)_t; (\bar{e}_q)_t^+) - \mathcal{D}((\bar{e}_p)_t, (\bar{e}_q)_t; (\bar{e}_p)_t^-).
\]

Using the property of the operator \(\mathcal{D}\), Lemma 2.1, we have

\[
\int_I (\bar{e}_u)_t (\bar{e}_u)_t \rho \, dx + \int_I (\bar{e}_p)_t^2 \rho \, dx + \int_I (\varepsilon_u)_t (\bar{e}_u)_t \rho \, dx + \int_I (\varepsilon_p)_t (\bar{e}_p)_t \rho \, dx + \int_I (\varepsilon_r)_t (\bar{e}_q)_t \rho \, dx
\]

\[
- \int_I (\varepsilon_q)_t (\bar{e}_q)_t \rho \, dx + \frac{\alpha}{2} \sum_j [(\bar{e}_u)_t]_{j+\frac{1}{2}}^2 + \beta \mathcal{D}((\bar{e}_q)_t, (\bar{e}_u)_t; (\bar{e}_q)_t^+) = 0. \quad (A.29)
\]
Note that
\[
\mathcal{D}((\bar{e}_q)_t, (\bar{e}_u)_t; (\bar{e}_q)_t^+) = \int_I (\epsilon_p)_t (\bar{e}_u)_t dx.  \tag{A.30}
\]

Combing (A.29) and (A.30), we arrive at
\[
\frac{1}{2} \frac{d}{dt} \|(\bar{e}_u)_t\|_{L^2}^2 + \|(\epsilon_p)_t\|_{L^2}^2 \leq -\int_I (\epsilon_u)_tt(\bar{e}_u)_t dx - \int_I (\epsilon_r)_tt(\bar{e}_q)_t dx - \int_I (\epsilon_q)_tt(\bar{e}_q)_t dx + \int_I (\epsilon_q)_tt(\bar{e}_r)_t dx - \beta \int_I (\epsilon_q)_t (\bar{e}_q)_t (\bar{e}_u)_t dx.  \tag{A.31}
\]

Integrating the above inequality with respect to time,
\[
\frac{1}{2} \|(\bar{e}_u)_t(t)\|_{L^2}^2 + \int_0^t \|(\epsilon_p)_t(t)\|_{L^2}^2 dt \leq \frac{1}{2} \|(\bar{e}_u)_t(0)\|_{L^2}^2 + \Upsilon + \Xi,  \tag{A.32}
\]
where
\[
\Upsilon = -\int_I \int_0^t (\epsilon_p)_tt(\bar{e}_p)_t dt dx - \beta \int_I \int_0^t (\bar{e}_p)_t (\bar{e}_u)_t dt dx
\]
and
\[
\Xi = -\int_I \int_0^t (\epsilon_u)_tt(\bar{e}_u)_t dt dx - \int_I \int_0^t (\epsilon_r)_tt(\bar{e}_q)_t dt dx + \int_I \int_0^t (\epsilon_q)_tt(\bar{e}_q)_t dt dx - \beta \int_I \int_0^t (\epsilon_q)_t (\bar{e}_q)_t (\bar{e}_u)_t dt dx.
\]

By the Cauchy-Schwarz inequality and approximation result (2.9), we obtain
\[
|\Upsilon| \leq \int_0^t \|(\epsilon_p)_t(t)\|_{L^2}^2 dt + \frac{1}{2} \int_0^t \|(\epsilon_p)_t\|_{L^2}^2 dt + \frac{\beta^2}{2} \int_0^t \|(\bar{e}_u)_t(t)\|_{L^2}^2 dt
\]
\[
\leq \int_0^t \|(\epsilon_u)_tt\|_{L^2} \|\bar{e}_u\|_{L^2} dt + \|(\epsilon_u)_tt\|_{L^2} \|\bar{e}_u(t)\|_{L^2} + \|(\epsilon_u)_tt(0)\|_{L^2} \|\bar{e}_u(0)\|_{L^2}
\]
\[
+ \int_0^t \|(\epsilon_r)_tt\|_{L^2} \|\bar{e}_q\|_{L^2} dt + \|(\epsilon_r)_tt\|_{L^2} \|\bar{e}_q(t)\|_{L^2} + \|(\epsilon_r)_tt(0)\|_{L^2} \|\bar{e}_q(0)\|_{L^2}
\]
\[
+ \int_0^t \|(\epsilon_q)_tt\|_{L^2} \|\bar{e}_r\|_{L^2} dt + \|(\epsilon_q)_tt\|_{L^2} \|\bar{e}_r(t)\|_{L^2} + \|(\epsilon_q)_tt(0)\|_{L^2} \|\bar{e}_r(0)\|_{L^2}
\]
\[
+ |\beta| \left[ \int_0^t \|(\epsilon_p)_tt\|_{L^2} \|\bar{e}_u\|_{L^2} dt + \|(\epsilon_p)_tt\|_{L^2} \|\bar{e}_u(t)\|_{L^2} + \|(\epsilon_p)_tt(0)\|_{L^2} \|\bar{e}_u(0)\|_{L^2} \right]
\]
\[
\leq Ch^{k+1} \int_0^t (\|\bar{e}_u\|_{L^2} + \|\bar{e}_q\|_{L^2} + \|\bar{e}_r\|_{L^2}) dt
\]
\[
+ Ch^{k+1} (\|\bar{e}_u\|_{L^2} + \|\bar{e}_q\|_{L^2} + \|\bar{e}_r\|_{L^2}) + Ch^{2k+5/2}  \tag{A.33}
\]
\[
\]
\[
31
\]
\[
\]
with $C = C(\beta, \|u\|_{k+4}, \|u_t\|_{k+4}, \|u_{tt}\|_{k+4}, \|u_{ttt}\|_{k+1})$, where we have used the approximation results (2.9) and estimates for the initial data (A.18) and (2.17) to obtain the last inequality. Plugging (A.33) and (A.34) into (A.32) and using estimates (A.20), (A.28), we conclude

$$
\frac{1}{2} \| \langle \bar{e}_u \rangle_t(t) \|^2_{L^2} \leq C e^{C_1 h^{k+2}} + C h^{k+1} \| \bar{e}_r(t) \|_{L^2} + \frac{\beta^2}{2} \int_0^t \| \langle \bar{e}_u \rangle_t(t) \|^2_{L^2} dt. \tag{A.35}
$$

Denote $\mathbb{E}(t) = \int_0^t \| \langle \bar{e}_u \rangle_t(t) \|^2_{L^2} dt$ and integrate over (A.35) with respect to time,

$$
\frac{1}{2} \mathbb{E}(t) \leq C e^{C_1 h^{k+2}} + \frac{\beta^2}{2} \int_0^t \mathbb{E}(s) ds.
$$

A Gronwall’s inequality gives us

$$
\mathbb{E}(t) = \int_0^t \| \langle \bar{e}_u \rangle_t(t) \|^2_{L^2} dt \leq C e^{2C_1 h^{k+2}}.
$$

Therefore,

$$
\int_0^t \| \langle \bar{e}_u \rangle_t(t) \|_{L^2} dt \leq C e^{C_1 h^{k+1}},
$$

where $C = C(\alpha, \beta, \|u\|_{k+4}, \|u_t\|_{k+4}, \|u_{tt}\|_{k+4}, \|u_{ttt}\|_{k+1})$ and $C_1 = C_1(\alpha, \beta) > 0$. Then estimate (2.39) follows by taking into account the approximation error estimates (2.9) and triangle inequality. This finishes the proof for Lemma 2.7.

### A.3 The proof of Theorem 2.6

To get the linear growth result for the case $\alpha = \beta = 0$, similarly to Lemma 2.7, we need to prove the following error estimate

$$
\| \bar{e}_u \|_{L^2} + \| \bar{e}_q \|_{L^2} \leq C(1 + t) h^{k+1}, \tag{A.36}
$$

$$
\int_0^t (\| \bar{e}_p \|_{L^2} + \| \bar{e}_r \|_{L^2}) dt \leq C(1 + t)^{3/2} h^{k+1}, \tag{A.37}
$$

where $C = C(\|u\|_{k+4}, \|u_t\|_{k+3})$. Notice that, if (A.36) and (A.37) hold, we can easily get a bound for $(\bar{e}_u)_t(t)$,

$$
\int_0^t \| (\bar{e}_u)_t(t) \|_{L^2} dt \leq C(1 + t)^2 h^{k+1}, \tag{A.38}
$$
and thus (2.36) will give us the desired result in Theorem 2.6 by combing with the approximation error estimates (2.9). First, let us prove the error estimate (A.36).

If $\alpha = \beta = 0$, then (A.27) reduces to
\[
\frac{1}{4}\|\bar{e}_u\|_{L^2}^2 + \frac{1}{2}\|\bar{e}_q\|_{L^2}^2 + \frac{1}{2}\int_0^t (\|\bar{e}_p\|_{L^2}^2 + \|\bar{e}_r\|_{L^2}^2)dt
\]
\[
\leq \frac{1}{2}\|\bar{e}_u(0)\|_{L^2}^2 + \frac{1}{2}\|\bar{e}_q(0)\|_{L^2}^2 + Ch^{k+1}\int_0^t (\|\bar{e}_u\|_{L^2}^2 + \|\bar{e}_q\|_{L^2}^2 + \|\bar{e}_p\|_{L^2}^2 + \|\bar{e}_r\|_{L^2}^2)dt + Ch^{2k+2},
\]
which, by using the bound for the initial error (2.17) and (A.18), is
\[
\frac{1}{4}\|\bar{e}_u\|_{L^2}^2 + \frac{1}{2}\|\bar{e}_q\|_{L^2}^2 + \frac{1}{2}\int_0^t (\|\bar{e}_p\|_{L^2}^2 + \|\bar{e}_r\|_{L^2}^2)dt
\]
\[
\leq Ch^{k+1}\int_0^t (\|\bar{e}_u\|_{L^2}^2 + \|\bar{e}_q\|_{L^2}^2)dt + Ch^{k+1}\int_0^t (\|\bar{e}_p\|_{L^2}^2 + \|\bar{e}_r\|_{L^2}^2)dt + Ch^{2k+2}.
\]

By using the Cauchy-Schwarz inequality, we get
\[
(\|\bar{e}_u\|_{L^2} + \|\bar{e}_q\|_{L^2})^2 \leq Ch^{k+1}\int_0^t (\|\bar{e}_u\|_{L^2} + \|\bar{e}_q\|_{L^2})dt + Ch^{2k+2},
\]
where $C = C(\|u\|_{k+4}, \|u_t\|_{k+3})$. Notice that, for the pure diffusion problem in Theorem 2.6, $\|u\|_{k+4}$ and $\|u_t\|_{k+3}$ are exponentially decaying with respect to time, thus, we can assume $C = C(\|u\|_{k+4}, \|u_t\|_{k+3}) \leq \tilde{C}$ with $\tilde{C}$ a positive constant independent of time.

Denote $\tilde{E}(t) = \|\bar{e}_u\|_{L^2} + \|\bar{e}_q\|_{L^2}$, then we have
\[
\tilde{E}^2(t) \leq \tilde{C}h^{k+1}\int_0^t \tilde{E}(s)ds + \tilde{C}h^{2k+2}.
\]
Define $z(t) = \tilde{C}h^{k+1}\int_0^t \tilde{E}(s)ds + \tilde{C}h^{2k+2}$, thus $\sqrt{z(0)} = \tilde{C}h^{k+1}$, and the above inequality gives us
\[
\tilde{E}(t) \leq \sqrt{z(t)}.
\]
Therefore,
\[
\frac{dz(t)}{dt} = \tilde{C}h^{k+1}\tilde{E}(t) \leq \tilde{C}h^{k+1}\sqrt{z(t)}.
\]
Integrating the above inequality with respect to time between 0 and $t$ and using the control for $\sqrt{z(0)}$ will give us a bound on $\tilde{E}(t)$,
\[
\tilde{E}(t) \leq \sqrt{z(t)} \leq \sqrt{z(0)} + \tilde{C}h^{k+1}t \leq \tilde{C}h^{k+1}(1 + t).
\]
Thus,
\[ \| \bar{e}_u \|_{L^2} + \| \bar{e}_q \|_{L^2} \leq \tilde{C} h^{k+1}(1 + t). \]

Finally, the error estimate (A.37) follows by combing (A.36) and (A.39). This completes the proof for Theorem 2.6.

References


