Stability analysis and error estimates of arbitrary Lagrangian-Eulerian discontinuous Galerkin method coupled with Runge-Kutta time-marching for linear conservation laws

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Abstract: In this paper, we discuss the stability and error estimates of the fully discrete schemes for linear conservation laws, which consists of the arbitrary Lagrangian-Eulerian discontinuous Galerkin methods in space and explicit total variation diminishing Runge-Kutta (TVD-RK) up to third order accuracy in time. The scaling arguments and the standard energy analysis are the key technique used in our work. We present a rigorous proof to obtain stability for the three fully discrete schemes under suitable CFL conditions. With the help of the reference cell, the error equations are easy to establish and we derive the quasi-optimal error estimates in space and optimal convergence rates in time. For the Euler-forward scheme with piecewise constant elements, the second order TVD-RK method with piecewise linear elements and the third order TVD-RK scheme with any order polynomials, the usual CFL condition is needed, while for other cases, stronger time step restriction is needed for the results to hold true. More precisely, the Euler-forward scheme needs $\tau \leq \rho h^2$ and the second order TVD-RK needs $\tau \leq h^{\frac{2}{3}}$ for higher order polynomials in space, where $\tau$ and $h$ are the time and maximum space step, respectively, and $\rho$ is a positive constant independent of $\tau$ and $h$.

Keywords: arbitrary Lagrangian-Eulerian discontinuous Galerkin methods, Runge-Kutta methods, stability, error estimates, conservation laws

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1 Introduction

In this paper, we consider the stability analysis and error estimates of the arbitrary Lagrangian-Eulerian discontinuous Galerkin (ALE-DG) method coupled with Runge-Kutta time-marching for one-dimensional linear conservation laws

\[
\begin{align*}
  u_t + (\beta u)_x &= 0, \quad (x, t) \in [a, b] \times (0, T], \\
  u(x, 0) &= u_0(x), \quad x \in [a, b]
\end{align*}
\]

(1.1)

with periodic boundary condition. We only pay attention to the smooth solution of (1.1).

The discontinuous Galerkin (DG) method is a class of finite element methods, in which the basis functions are completely discontinuous, piecewise polynomials. Reed and Hill introduced the first DG method to solve the neutron equation [19] and later, Cockburn, Shu et al. extended the method to Runge-Kutta DG (RKDG) for nonlinear conservation laws in a series of papers [5, 4, 2, 6]. The DG method has a wide range of applications owing to some advantages like parallelization capability, the strong stability and high-order accuracy, and so on. We refer to [20, 8, 3, 12, 7] and the references therein for more references of the DG methods.

For the theoretical analysis of the fully discrete DG method, Zhang and Shu have done a lot of work for conservation laws [22, 23, 24, 25, 26], where the time discretization is the explicit second or third order total variation diminishing Runge-Kutta (TVD-RK) method. For the smooth solutions, they obtained (quasi)-optimal error estimates for both the second and third order TVD-RK time-marching with periodic boundary condition and suitable CFL conditions. The stability of the third order TVD-RK (TVD-RK3) was shown in [24]. They also considered the inflow boundary condition as well as the discontinuous initial data [25, 26]. Moreover, Erik, Ern and Fern [10] analyzed the explicit RK schemes in combination with stabilized finite elements method for first-order linear partial differential equation system and established a sub-optimal error estimates for smooth solution, which presented a unified analysis for several high-order symmetrically stabilized finite element methods encountered in the literature. We refer to [15, 21] for the energy analysis, which is the main technique for all the work listed above.

However, all the analysis listed above are considered on the static grids. The ALE-DG method discussed here is a moving mesh DG method and the grid moving methodology belongs to the class of arbitrary Lagrangian-Eulerian (ALE) methods [9], which allows the motion of the mesh to be like either the Lagrangian or the Eulerian description of motion and should satisfy the geometric conservation law (GCL). The significance of the GCL has been analyzed by Guillard and Farhat [11]. There have been works about the implementation and applications of the ALE-DG method in the literature, e.g., [16, 17, 18, 14]. Klingenberg, Schmücke, and Xia developed an ALE-DG method for one-dimensional conservation law [13], where local affine linear mappings connecting the cells for the current and next time level are defined and yield the time-dependent
approximation space. They show that the ALE-DG satisfies the GCL for any Runge-Kutta method and is efficient for the conservation law. They also show that the ALE-DG method shares many good properties of the DG method defined on static grids, e.g., the $L^2$ stability, the local maximum principle, high order accuracy, and so on.

The main purpose of our work is to study the stability and the error estimates for the ALE-DG method combined with the explicit Runge-Kutta time-marching schemes, in which the Euler-forward, the second order TVD-RK (TVD-RK2) and third order TVD-RK3 are considered. Compared with the work on the static grids, the process of our analysis is similar but more complicated. Owing to the time-dependent functional space, the scaling arguments play an important role in this work. With the energy estimates, we prove that all three fully discrete schemes are stable under suitable CFL conditions. More precisely, for the Euler-forward scheme with $P^0$ (piecewise constant) elements, the TVD-RK2 scheme with $P^1$ (piecewise linear) elements and the TVD-RK3 approach with any order polynomials in space, in the usual CFL condition is needed, while for the Euler-forward scheme with $P^k$ elements for $k \geq 1$ requires $\tau \leq \rho h^2$ and the TVD-RK2 approach with $P^k$ elements for $k \geq 2$ needs $\tau \leq h^{\frac{4}{3}}$ for the results to hold true. Here $\tau$ and $h$ are the time and maximum spatial mesh sizes, respectively, and $\rho$ is a positive constant independent of $\tau$ and $h$. To best understand the error equations, we reformulate the Eq. (1.1) in terms of a suitable coordinate transformation. Then we proceed to obtain the quasi-optimal error estimates in space and optimal convergence rates in time under the same CFL condition as the stability. To the best of our knowledge, the above results are the first for high order ALE methods without smoothness assumptions on mesh movements and without the need of remapping.

The organization of our paper is as follows. In Section 2, we list some notations adopted throughout the paper. The semi-discrete ALE-DG scheme for the linear conservation law is given in Section 3, where we also show some properties of the scheme. Section 4 presents the stability of the ALE-DG scheme in combination with the explicit RK time-marching up to the third order. The error estimates for the three corresponding fully discrete schemes are proven in Section 5. We conclude our results in Section 6.

2 Notations

In this section, we will introduce some notations adopted throughout the paper.

2.1 Notations for the distribution of the mesh

Let $\Omega = [a, b]$. In order to describe the semi-discrete ALE-DG scheme of Eq. (1.1), we first introduce some notations for the distribution of the mesh. Assume that the mesh generating points $\{x_{j-\frac{1}{2}}^n\}_{j=1}^N$ are given at any time level $t_n$, $n = 0, \cdots, M$, and
the points $x^n_j$ and $x^{n+1}_j$ are connected by time-dependent straight lines
\[ x_{j-\frac{1}{2}}(t) := x^n_{j-\frac{1}{2}} + \omega_{j-\frac{1}{2}}(t-t_n), \quad \forall t \in [t_n, t_{n+1}], \quad (2.1) \]
where
\[ \omega_{j-\frac{1}{2}} := \frac{x^{n+1}_{j-\frac{1}{2}} - x^n_{j-\frac{1}{2}}}{t_{n+1} - t_n}. \quad (2.2) \]
Note that for any time point $t$, the first point $x_{\frac{1}{2}}(t)$ and the last point $x_{N+\frac{1}{2}}(t)$ stay the same for compactly supported problem and could move with the same speed $\frac{d}{dt} x_{\frac{1}{2}}(t) = \frac{d}{dt} x_{N+\frac{1}{2}}(t)$ for the periodic boundary problems. Then the straight lines (2.1) provide the time-dependent cells
\[ K_j(t) := [x_{j-\frac{1}{2}}(t), x_{j+\frac{1}{2}}(t)], \quad \forall t \in [t_n, t_{n+1}] \quad \text{and} \quad j = 1, \ldots, N. \]
The length of each cell $K_j(t)$ is denoted by $\Delta_j(t) := x_{j+\frac{1}{2}}(t) - x_{j-\frac{1}{2}}(t)$. Moreover, we set $h(t) := \max_{1 \leq j \leq N} \Delta_j(t)$ and $h := \max h(t)$. We assume that the mesh is quasi-uniform in the sense that $h \leq C \Delta_j(t)$ for $j = 1, 2, \ldots, N$, where $C$ is a positive constant and independent of $h$. In addition, the grid velocity field for all $t \in [t_n, t_{n+1}]$ and $x \in K_j(t)$ is given by
\[ \omega(x, t) = \omega_{j+\frac{1}{2}} \frac{x - x_{j-\frac{1}{2}}(t)}{\Delta_j(t)} + \omega_{j-\frac{1}{2}} \frac{x_{j+\frac{1}{2}}(t) - x}{\Delta_j(t)}, \quad (2.3) \]
which yields that, for $x \in K_j(t)$,
\[ \partial_x \omega(x, t) = \frac{\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}}{\Delta_j(t)} = \frac{\Delta_j'(t)}{\Delta_j(t)}. \quad (2.4) \]
We assume that $\Delta_j(t)$ and $\omega(x, t)$ satisfy the following properties:

(\omega 1): For all $t \in [t_n, t_{n+1}]$, $n = 0, \ldots, M - 1$, there holds
\[ \Delta_j(t) = (\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}})(t - t_n) + \Delta_j(t_n) > 0; \quad (2.5) \]
Note that this property is guaranteed by the quasi-uniformity assumption for the meshes.

(\omega 2): There exists a constant $C_w \geq 0$, independent of $h$, such that,
\[ \max_{(x,t) \in [a,b] \times [0,T]} |\omega(x, t)| \leq C_w; \quad (2.6) \]

(\omega 3): There exists a constant $C_{wx} \geq 0$, independent of $h$, such that,
\[ \max_{(x,t) \in [a,b] \times [0,T]} |\partial_x \omega(x, t)| \leq C_{wx}. \quad (2.7) \]
For any $K_j(t)$, we define the following time-dependent linear mapping

$$\chi_j : [-1, 1] \rightarrow K_j(t), \quad \chi_j(\xi, t) = \frac{\Delta_j(t)}{2}(\xi + 1) + x_{j-\frac{1}{2}}(t), \quad (2.8)$$

which yields a characterization of the grid velocity

$$\partial_t(\chi_j(\xi, t)) = \omega(\chi_j(\xi, t), t), \quad \forall (\xi, t) \in [-1, 1] \times [t_n, t_{n+1}].$$

For simplicity, we denote $K^*_j \equiv K_j(t_n)$ and $\Delta^*_j \equiv \Delta_j(t_n)$, for any $n = 1, \cdots, M$.

### 2.2 Notations for function space and norms

For any $t \in [t_n, t_{n+1}]$, the finite element space is defined by

$$V_h(t) := \{ v \in L^2(\Omega) : v(\chi_j(\xi, t)) \in P^k([-1, 1]), \quad j = 1, 2, \cdots, N \},$$

where $P^k([-1, 1])$ denotes the space of polynomials of degree at most $k$ on $[-1, 1]$. We denote the inner product over the interval $K_j(t)$ and the associated norm by

$$(v, r)_{K_j(t)} = \int_{K_j(t)} vr dx, \quad \|v\|_{K_j(t)} = \sqrt{(v, v)_{K_j(t)}}.$$ We also use the usual notations of Sobolev space. Let $H^s(D)$ be the Sobolev space on sub-domain $D \subset \Omega$, which is equipped with the norm $\| \cdot \|_{H^s(D)}$ for any integer $s \geq 0$. Then we define the broken Sobolev space

$$H^1_h(t) := \{ v : v(\chi_j(\xi, t)) \in H^1([-1, 1]), \quad j = 1, 2, \cdots, N \},$$

which contains the finite element space. Moreover, the left and right limits of $v$ at the point $x_{j-\frac{1}{2}}(t)$ are denoted by $v^-_{j-\frac{1}{2}}$ and $v^+_{j-\frac{1}{2}}$, respectively, where

$$v^\pm_{j-\frac{1}{2}} = \lim_{\varepsilon \to 0^\pm} v(x_{j-\frac{1}{2}}(t) \pm \varepsilon, t).$$

Thus the cell average and the jump at the point $x_{j-\frac{1}{2}}(t)$ are defined by

$$\|v\|_{j\frac{1}{2}} = \frac{1}{2} \left( v^+_{j-\frac{1}{2}} + v^-_{j-\frac{1}{2}} \right), \quad \|v\|_{j\frac{1}{2}} = v^+_{j\frac{1}{2}} - v^-_{j\frac{1}{2}}.$$ Summing over all the element, we denote

$$(v, r)(t) = \sum_{j=1}^{N} (v, r)_{K_j(t)}, \quad \|v\|^2(t) = \sum_{j=1}^{N} \|v\|^2_{K_j(t)}, \quad \|v\|^2(t) = \sum_{j=1}^{N} \|v\|^2_{j\frac{1}{2}}.$$ Let $\Gamma_h(t)$ be the union of all element interface points and define the $L^2$-norm on $\Gamma_h(t)$ by

$$\|v\|_{\Gamma_h(t)} = \left[ \sum_{j=1}^{N} \left( |v^+_{j\frac{1}{2}}|^2 + |v^-_{j+\frac{1}{2}}|^2 \right) \right]^{1/2}.$$
2.3 Notations for coordinate transformation

In the following, we will introduce some notations for coordinate transformations, which are often used in our stability analysis. For simplicity, we only consider the uniform partition of the time interval $[0, T]$, namely $\{t_n = n\tau\}_{n=0}^M$ with the time step $\tau$ and $M\tau = T$. For the three different time stages $t_n$, $t_{n+\frac{1}{2}}$, and $t_{n+1}$, the corresponding spatial variables in the $j$-th cell are denoted by $x_j^n$, $x_{j+\frac{1}{2}}^n$, and $x_j^{n+1}$ for $1 \leq j \leq N$. With the time-dependent linear mapping (2.8), we can define the coordinate transformations, $\forall j = 1, \cdots, N$,

\[
K_j^n \mapsto K_j^{n+\frac{1}{2}}, \quad x_j^{n+\frac{1}{2}} = \frac{\Delta_j^{n+\frac{1}{2}}}{\Delta_j^{n}}(x_j^n - x_{j-\frac{1}{2}}^n) + x_{j-\frac{1}{2}}^{n+\frac{1}{2}}, \tag{2.9}
\]

\[
K_j^n \mapsto K_j^{n+1}, \quad x_j^{n+1} = \frac{\Delta_j^{n+1}}{\Delta_j^{n}}(x_j^n - x_{j-\frac{1}{2}}^n) + x_{j-\frac{1}{2}}^{n+1}, \tag{2.10}
\]

\[
K_j^{n+\frac{1}{2}} \mapsto K_j^{n+1}, \quad x_j^{n+1} = \frac{\Delta_j^{n+\frac{1}{2}}}{\Delta_j^{n}}(x_j^{n+\frac{1}{2}} - x_{j-\frac{1}{2}}^{n+\frac{1}{2}}) + x_{j-\frac{1}{2}}^{n+1}. \tag{2.11}
\]

Thus for arbitrary function $\phi(x_j^n, t_n)$, $\varphi(x_j^{n+\frac{1}{2}}, t_{n+\frac{1}{2}})$ and $\psi(x_j^{n+1}, t_{n+1})$, we define the representations after the coordinate transformations as

\[
\hat{\phi}(x_j^{n+1}, t_{n+1}) := \phi(x_j^n, t_n), \quad \overline{\phi}(x_j^{n+\frac{1}{2}}, t_{n+\frac{1}{2}}) := \phi(x_j^n, t_n), \tag{2.12}
\]

\[
\hat{\varphi}(x_j^{n+1}, t_{n+1}) := \varphi(x_j^{n+\frac{1}{2}}, t_{n+\frac{1}{2}}), \quad \overline{\varphi}(x_j^{n+\frac{1}{2}}, t_{n+\frac{1}{2}}) := \varphi(x_j^n, t_n), \tag{2.13}
\]

\[
\hat{\psi}(x_j^n, t_n) := \psi(x_j^{n+1}, t_{n+1}), \quad \overline{\psi}(x_j^{n+\frac{1}{2}}, t_{n+\frac{1}{2}}) := \psi(x_j^{n+1}, t_{n+1}). \tag{2.14}
\]

In what follows, we omit the label $(x_j^n, t_n)$, $(x_j^{n+1}, t_{n+1})$ and $(x_j^{n+\frac{1}{2}}, t_{n+\frac{1}{2}})$ if it is clear in which cell, i.e., $\hat{\phi}$ is defined in $K_j^{n+1}$. Moreover, from (2.4) and the assumption (2.5), we have

\[
\frac{\Delta_j^n}{\Delta_j^{n+1}} = 1 - s_2 > 0, \quad \frac{\Delta_j^{n+1}}{\Delta_j^n} = 1 + s_1 > 0, \tag{2.15}
\]

\[
\frac{\Delta_j^n}{\Delta_j^{n+\frac{1}{2}}} = 1 - \frac{s_3}{2} > 0, \quad \frac{\Delta_j^{n+\frac{1}{2}}}{\Delta_j^n} = 1 + \frac{s_1}{2} > 0, \tag{2.16}
\]

\[
\frac{\Delta_j^{n+\frac{1}{2}}}{\Delta_j^{n+1}} = 1 - \frac{s_2}{2} > 0, \quad \frac{\Delta_j^{n+1}}{\Delta_j^{n+\frac{1}{2}}} = 1 + \frac{s_3}{2} > 0, \tag{2.17}
\]

where

\[
s_1 = \tau \omega_s(t_n), \quad s_2 = \tau \omega_s(t_{n+1}), \quad s_3 = \tau \omega_s(t_{n+\frac{1}{2}}). \tag{2.18}
\]
and \( \omega_x(t) \equiv \partial_x \omega(x, t) \) is given by (2.4). Note that
\[
\frac{\Delta_j^n}{\Delta_j^{n+1}} \cdot \frac{\Delta_j^{n+1}}{\Delta_j^n} = 1, \quad \frac{\Delta_j^n}{\Delta_j^{n+\frac{1}{2}}} \cdot \frac{\Delta_j^{n+\frac{1}{2}}}{\Delta_j^n} = 1, \quad \frac{\Delta_j^{n+\frac{1}{2}}}{\Delta_j^{n+1}} \cdot \frac{\Delta_j^{n+1}}{\Delta_j^{n+\frac{1}{2}}} = 1,
\]
we have
\[
s_1 = s_2 + \left( \frac{s_1}{2} \right), \quad s_3 = s_2 + \left( \frac{s_2}{2} \right). \tag{2.19}
\]
In the end, we present some properties, which are frequently used in our analysis. For any function \( v_h^n \in V_h(t_n) \), the scaling arguments and the assumption (2.7) of \( \omega_x \) indicate that,
\[
\| v_h^n \|^2_{K_j^n} = \frac{\Delta_j^n}{\Delta_j^{n+1}} \| v_{h_1}^n \|^2_{K_j^{n+1}} = \frac{1}{(1-s_2)} \| v_{h_1}^n \|^2_{K_j^{n+1}} \leq (1+C_{w_x} \tau) \| v_{h_1}^n \|^2_{K_j^{n+1}}, \tag{2.20}
\]
\[
\| v_{h_1}^n \|^2_{K_j^{n+1}} = \frac{\Delta_j^{n+1}}{\Delta_j^n} \| v_h^n \|^2_{K_j^n} = \frac{1}{(1-s_1)} \| v_h^n \|^2_{K_j^n} \leq (1+C_{w_x} \tau) \| v_h^n \|^2_{K_j^n}. \tag{2.21}
\]
Similarly, we also have
\[
\| v_{h_1}^{n+\frac{1}{2}} \|^2_{K_j^{n+\frac{1}{2}}} \leq (1+C_{w_x} \frac{\tau}{2}) \| v_h^n \|^2_{K_j^{n+1}}, \quad \| v_{h_1}^{n+\frac{1}{2}} \|^2_{K_j^{n+\frac{1}{2}}} \leq (1+C_{w_x} \frac{\tau}{2}) \| v_h^n \|^2_{K_j^{n+\frac{1}{2}}}. \tag{2.22}
\]

### 2.4 Projections and inverse properties

In this section, we will present two types of projections. The \( L^2 \) projection \( P_h \) and Gauss-Radau projections \( P_h^{\pm} \) into \( V_h(t) \), which are often used to derive the quasi-optimal and optimal \( L^2 \) error bounds of the DG methods. For a function \( u \in L^2(\Omega) \), the \( L^2 \) projection is defined by
\[
(P_h u, v)_{K_j(t)} = (u, v)_{K_j(t)}, \quad \forall v \in V_h(t). \tag{2.23}
\]
For \( k \geq 1 \) and \( v(\cdot, t) \in P_{k-1}([-1, 1]) \), the Gauss-Radau projections are defined by
\[
(P^- h u, v)_{K_j(t)} = (u, v)_{K_j(t)}, \quad P^- h u \left( x_{j+\frac{1}{2}}^{-1} (t) \right) = u \left( x_{j+\frac{1}{2}}^{-1} (t) \right), \tag{2.24}
\]
\[
(P^+ h u, v)_{K_j(t)} = (u, v)_{K_j(t)}, \quad P^+ h u \left( x_{j-\frac{1}{2}}^{+1} (t) \right) = u \left( x_{j-\frac{1}{2}}^{+1} (t) \right).
\]
Let \( Q_h u \) be either \( P_h u \) or \( P_h^{\pm} u \). Suppose \( u \in H^{k+1}(\Omega) \), then by a standard scaling argument, it is easy to show (c.f. [1]) for both projections that
\[
\| \eta \| (t) + h^{1/2} \| \eta \|_{H_{K_j(t)}} + h \| \partial_x \eta \| (t) \leq C h^{k+1}, \tag{2.25}
\]
where \( \eta = Q_h u - u \) and the positive constant \( C \) depends on \( u \) and its derivatives, but it is independent of \( h \). Finally, we present the well-known inverse properties of the
finite element space $V_h(t)$. For any $v \in V_h(t)$, there exists positive constants $\mu_1$ and $\mu_2$, independent of $v$ and $h$, such that

$$h\|v_x\|(t) \leq \mu_1\|v\|(t), \quad h^2\|v\|_{\Gamma_h(t)} \leq \mu_2\|v\|(t). \tag{2.26}$$

In the following, we denote $\mu = \max\{\mu_1, \mu_2^2\}$. For more details of the inverse property, we refer the reader to [1].

3 Semi-discrete ALE-DG method

3.1 ALE-DG scheme

To derive the semi-discrete ALE-DG method, we first list the following lemma, which has been proven in [13].

**Lemma 1.** Let $u$ be a sufficiently smooth function in any cell $K_j(t)$. Then for all $v \in V_h(t)$, there holds the transport equation

$$\frac{d}{dt}(u, v)_{K_j(t)} = (\partial_t u, v)_{K_j(t)} + (\partial_x(u), v)_{K_j(t)}, \quad \forall j = 1, \ldots, N. \tag{3.1}$$

Next, multiply the Eq.(1.1) by a test function $v \in V_h(t)$ and apply the integration by parts as well as the transport equation (3.1), we obtain the semi-discrete ALE-DG method for arbitrary $K_j(t), t \in [t_n, t_{n+1}]:$ find $u_h \in V_h(t)$ such that for all test functions $v \in V_h(t)$, we have

$$\frac{d}{dt}(u_h, v)_{K_j(t)} = (g(\omega, u_h), v_x)_{K_j(t)} - \hat{g}(\omega, u_h)_{j+\frac{1}{2}} v_{j+\frac{1}{2}}^- + \hat{g}(\omega, u_h)_{j-\frac{1}{2}} v_{j-\frac{1}{2}}^+, \tag{3.2}$$

where $g(\omega, u_h) = (\beta - \omega)u_h$ and the numerical fluxes $\hat{g}(\omega, u_h)_{j+\frac{1}{2}}$ can be chosen as the Lax-Friedrichs flux, for $j = 1, \ldots, N$,

$$\hat{g}(\omega, u_h)_{j+\frac{1}{2}} = (\beta - \omega)_{j+\frac{1}{2}} u_h_{j+\frac{1}{2}} - \frac{\alpha}{2} u_h_{j+\frac{1}{2}}, \quad \alpha = \max_{x \in \Omega} |\beta - \omega|. \tag{3.3}$$

For simplicity, we define the ALE-DG spatial operator $\mathcal{A}$ as

$$\mathcal{A}(v, r)(t) = \sum_{j=1}^{N} \mathcal{A}(v, r)_{K_j(t)}, \quad \forall v, r \in H_h^1(t), \tag{3.4}$$

where

$$\mathcal{A}(v, r)_{K_j(t)} = -\frac{1}{2} (\beta - \omega) r_x \Big|_{K_j(t)} + \hat{g}(\omega, v)_{j+\frac{1}{2}} r_{j+\frac{1}{2}}^- - \hat{g}(\omega, v)_{j-\frac{1}{2}} r_{j-\frac{1}{2}}^+, \tag{3.5}$$

and $\hat{g}(\omega, v)_{j+\frac{1}{2}}$ is the Lax-Friedrichs flux defined by (3.3). Then by the above notations, the semi-discrete ALE-DG scheme (3.2) can be rewritten as

$$\frac{d}{dt}(u_h, v)_{K_j(t)} = -\mathcal{A}(u_h, v)_{K_j(t)}, \quad \forall v \in V_h(t).$$
3.2 The properties of the ALE-DG scheme

In this subsection, we shall present some properties of the operator $\mathcal{A}$ defined by (3.4), which implies the properties of the ALE-DG spatial discretization.

**Lemma 2** (Boundedness of the operator $\mathcal{A}$). Suppose $\mathcal{A}$ are defined by (3.4), then for any $v, r \in V_h(t)$ and $t \in [t_n, t_{n+1}]$, we have

$$|\mathcal{A}(v, r)(t)| \leq 3\alpha \mu h^{-1} \|v\|(t) \|r\|(t),$$

(3.6)

$$|\mathcal{A}(v, r)(t)| \leq \left( \alpha \|v_x\|(t) + C_{wx} \|v\|(t) + \sqrt{2} \alpha \mu h^{-\frac{1}{2}} \|v\|(t) \right) \|r\|(t).$$

(3.7)

Moreover, for the piecewise linear case, i.e., $k = 1$ in the finite element space $V_h(t)$, there holds

$$|\mathcal{A}(v, r - P_0^h r)(t)| \leq \left( (\mu + 1) C_{wx} \|v\|(t) + \sqrt{2} \alpha \mu h^{-\frac{1}{2}} \|v\|(t) \right) \|r - P_0^h r\|(t),$$

(3.8)

where $P_0^h r$ denotes the $L^2$ projection of $r$ onto the piecewise constant finite element space.

**Proof.** By the periodic boundary condition, we first obtain

$$\mathcal{A}(v, r)(t) = - \left( (\beta - \omega)v, r_x \right)(t) - \sum_{j=1}^{N} \hat{g}(\omega, v)_{j+\frac{1}{2}} [r]_{j+\frac{1}{2}}. \quad (3.9)$$

The definition (3.3) yields,

$$\hat{g}(\omega, v)_{j+\frac{1}{2}} \leq \alpha (|v^+_j| + |v^-_j|).$$

Then sum over all $j$ to get

$$\sum_{j=1}^{N} \hat{g}(\omega, v)_{j+\frac{1}{2}}^2 \leq 2 \sum_{j=1}^{N} \alpha^2 (|v^+_j|^2 + |v^-_j|^2) \leq 2\alpha^2 \|v\|^2_{H(h(t))}. \quad (3.10)$$

In addition, we have the following estimates

$$\sum_{j=1}^{N} [r]_{j+\frac{1}{2}}^2 \leq 2 \sum_{j=1}^{N} \left( |r^+_j|^2 + |r^-_j|^2 \right) = 2 \|r\|^2_{H(h(t))}, \quad (3.11)$$

$$\sum_{j=1}^{N} \|r\|_{j+\frac{1}{2}}^2 \leq \frac{1}{2} \sum_{j=1}^{N} \left( |r^+_j|^2 + |r^-_j|^2 \right) = \frac{1}{2} \|r\|^2_{H(h(t))}. \quad (3.12)$$

Thus we can obtain the first inequality (3.6),

$$|\mathcal{A}(v, r)(t)| \leq \alpha \|v\|(t) \|r_x\|(t) + \left( \sum_{j=1}^{N} \hat{g}(\omega, v)_{j+\frac{1}{2}}^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^{N} [r]_{j+\frac{1}{2}}^2 \right)^{\frac{1}{2}}.$$
\[ \leq \alpha \mu_1 h^{-1} \|v\|(t) \|r\|(t) + 2\alpha \|v\| r, \| r, \| r, \| r, \| r, \]
\[ \leq 3\alpha \mu h^{-1} \|v\|(t) \|r\|(t). \]

Here we use the Cauchy-Schwarz inequality as well as the inverse property (2.26). To obtain the second inequality (3.7), we integrate (3.5) by parts and sum over all \( j \),

\[
\mathcal{A}(v, r)(t) = \left( (\beta - \omega)v_x, r \right)(t) + \left( \partial_x (\beta - \omega)v, r \right)(t)
+ \sum_{j=1}^{N} (\beta - \omega_{j+\frac{1}{2}}) \|v\|_{j+\frac{1}{2}} \|r\|_{j+\frac{1}{2}} + \sum_{j=1}^{N} \frac{\alpha}{2} \|v\|_{j+\frac{1}{2}} \|r\|_{j+\frac{1}{2}},
\]

(3.13)

where

\[
\mathcal{B}(v, r)(t) = \left( (\beta - \omega)v_x, r \right)(t) + \sum_{j=1}^{N} (\beta - \omega_{j+\frac{1}{2}}) \|v\|_{j+\frac{1}{2}} \|r\|_{j+\frac{1}{2}} + \sum_{j=1}^{N} \frac{\alpha}{2} \|v\|_{j+\frac{1}{2}} \|r\|_{j+\frac{1}{2}}. \]

Here we use the fact that the quantity \( \omega_x(t) \) defined by (2.4) only depends on \( t \). By (3.12) and the similar arguments to estimate (3.6), we get

\[
|\mathcal{B}(v, r)(t)| \leq \alpha \|v_x\|(t) \|r\|(t) + \sqrt{2} \alpha \|v\|(t) \|r\| r_h(t)
\leq \left( \alpha \|v_x\|(t) + \sqrt{2} \alpha \mu h^{-\frac{1}{2}} \|v\|(t) \right) \|r\|(t),
\]

(3.14)

which yields the desired result (3.7),

\[
|\mathcal{A}(v, r)(t)| \leq \left( \alpha \|v_x\|(t) + C_{wx} \|v\|(t) + \sqrt{2} \alpha \mu h^{-\frac{1}{2}} \|v\|(t) \right) \|r\|(t).
\]

Here we use the property (2.7) of \( \partial_x \omega(x, t) \). Finally, we analyze the inequality (3.8). By the property of the piecewise constant \( L^2 \) projection,

\[
(r - P_h^0 r, v_x)_{K_j(t)} = 0, \quad \forall v(\chi_j(\cdot, t)) \in P^1([-1, 1]),
\]

we have

\[
\left( (\beta - \omega)v_x, r - P_h^0 r \right)_{K_j(t)} = \left( (\omega_{j+\frac{1}{2}} - \omega)v_x, r - P_h^0 r \right)_{K_j(t)},
\]

which yields

\[
\left( (\beta - \omega)v_x, r - P_h^0 r \right)(t) \leq C_{wx} h \|v_x\|(t) \|r - P_h^0 r\|(t)
\leq \mu C_{wx} \|v\|(t) \|r - P_h^0 r\|(t).
\]

Henceforth, replacing \( r \) with \( r - P_h^0 r \) in (3.13) and by similar arguments, we obtain

\[
|\mathcal{A}(v, r)(t)| \leq \left( \mu + 1 \right) C_{wx} \|v\|(t) + \sqrt{2} \alpha \mu h^{-\frac{1}{2}} \|v\|(t) \right) \|r - P_h^0 r\|(t).
\]
Lemma 3. Suppose $A$ are defined by (3.4) and $P_h v$ is the $L^2$ projection defined by (2.23). Denote $\eta = P_h v - v$, then for any $v \in H_h^1(t), r \in V_h(t)$ and $t \in [t_n, t_{n+1}]$, we have

$$|A(\eta, r)(t)| \leq \mu C_{wx} \|\eta\|_h(t) r(t) + \sqrt{2} \alpha \|\| \Gamma_{h,t} \| r(t), \tag{3.15}$$

$$|A(\eta, r)(t)| \leq \mu C_{wx} \|\eta\|_h(t) r(t) + 2\alpha \mu h^{-\frac{3}{2}} \|\| \Gamma_{h,t} \| r(t). \tag{3.16}$$

Proof. From (3.9) and the definition of the $L^2$ projection (2.23), we have

$$A(\eta, r)(t) = -\left( (\beta - \omega) \eta, r_x \right)(t) - \sum_{j=1}^{N} \hat{g}(\omega, \eta)_{j+\frac{1}{2}} \| r \|_{j+\frac{1}{2}}$$

$$= \left( (\omega - \omega_{j-1}) \eta, r_x \right)(t) - \sum_{j=1}^{N} \hat{g}(\omega, \eta)_{j+\frac{1}{2}} \| r \|_{j+\frac{1}{2}}$$

$$\leq C_{wx} h \|\eta\|_h(t) r(t) + \left( \sum_{j=1}^{N} \hat{g}(\omega, \eta)_{j+\frac{1}{2}} \right)^{\frac{1}{2}} \| r \|_{j+\frac{1}{2}}$$

$$\leq \mu C_{wx} \|\eta\|_h(t) r(t) + \sqrt{2} \alpha \|\| \Gamma_{h,t} \| r(t).$$

Here we use the Cauchy-Schwarz inequality, the inverse inequality (2.26) as well as the estimates (3.10). It is easy to obtain (3.16) from (3.15) by using the estimates (3.11) and the inverse property (2.26).

Lemma 4. For any $v, r \in H_h^1(t)$ and $t \in [t_n, t_{n+1}]$, we have

$$A(v, r)(t) + A(r, v)(t) = \sum_{j=1}^{N} \alpha [v]_{j+\frac{1}{2}} [r]_{j+\frac{1}{2}} - \sum_{j=1}^{N} \omega_x(t)(v, r) K_j(t), \tag{3.17}$$

$$A(v, v)(t) = \sum_{j=1}^{N} \alpha \| v \|_{j+\frac{1}{2}}^2 - \sum_{j=1}^{N} \omega_x(t) \| v \|_{K_j(t)}^2. \tag{3.18}$$

Proof. With the representation of $A(v, r)(t)$ in (3.9) and integration by parts, we can easily obtain

$$A(v, r)(t) + A(r, v)(t) = -\left( (\beta - \omega) v, r_x \right)(t) - \sum_{j=1}^{N} \hat{g}(\omega, v)_{j+\frac{1}{2}} \| r \|_{j+\frac{1}{2}}$$

$$- \left( (\beta - \omega) r, v_x \right)(t) - \sum_{j=1}^{N} \hat{g}(\omega, r)_{j+\frac{1}{2}} \| v \|_{j+\frac{1}{2}}$$

$$= - \sum_{j=1}^{N} \omega_x(t)(v, r) K_j(t) + \sum_{j=1}^{N} \alpha [v]_{j+\frac{1}{2}} [r]_{j+\frac{1}{2}}$$

$$- \sum_{j=1}^{N} (\beta - \omega_{j+\frac{1}{2}}) v_{j+\frac{1}{2}}^+ r_{j+\frac{1}{2}}^- + \sum_{j=1}^{N} (\beta - \omega_{j-\frac{1}{2}}) v_{j-\frac{1}{2}}^- r_{j-\frac{1}{2}}^+$$

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\[ -\sum_{j=1}^{N} (\beta - \omega_{j + \frac{1}{2}})(\{v\}_{j + \frac{1}{2}} + \{r\}_{j + \frac{1}{2}}) \]

\[ = -\sum_{j=1}^{N} \omega_x(t)(v, r)_{K_j(t)} + \sum_{j=1}^{N} \alpha[v]_{j + \frac{1}{2}}[r]_{j + \frac{1}{2}}. \]

Here in the last step we use the periodic boundary condition and \( [v]_0 + [r]_0 = v^+ + v^- - v^+ - v^- \). It is clear that (3.17) implies (3.18) if \( r = v \).

It is worth pointing out that the properties of \( A(v, r) \) in Lemmas 2-4 are similar to those in Zhang and Shu [24] developed for the static grids, which play very important roles in obtaining stability.

\section{Stability analysis for linear conservation laws}

In this section, we would like to analyze the stability of three fully discrete schemes, that is, the ALE-DG method coupled with Euler-forward, TVD-RK2 and TVD-RK3 time-marching scheme. In what follows, we denote the approximation of \( u_h(t_n) \) by \( u^n_h \).

\subsection{First order scheme}

The ALE-DG with the Euler-forward scheme is given in the following form: find \( u^{n+1}_h \in V_h(t_{n+1}) \), such that for any \( \hat{v}^n_h \equiv v_h(x, t^n) \in V_h(t_n) \) and \( 1 \leq j \leq N \), there holds

\[ (u^{n+1}_h, \hat{v}^n_h)_{K^{n+1}_j} = (u^n_h, v^n_h)_{K^n_j} - \tau A(u^n_h, v^n_h)_{K^n_j}. \]  

Here \( \hat{v}^n_h \) is defined by (2.12).

\textbf{Theorem 5.} Let \( u^{n+1}_h \) be the numerical solution of the fully discrete scheme (4.1), then we have for any \( n \), that

\[ \| u^{n+1}_h \|^2(t_{n+1}) \leq (1 + C\tau)\| u^n_h \|^2(t_n), \]

under the CFL condition

\[ \alpha^2 \mu^2 \tau h^{-2} \leq \frac{1}{9}. \] 

In particular, for the piecewise constant finite element space, \( V_h(t) = \{v(x_j, \cdot, t) \in P^0([-1, 1]) \} \), we have the strong stability

\[ \| u^{n+1}_h \|(t_{n+1}) \leq \| u^n_h \|(t_n), \]

with the usual CFL condition

\[ \alpha \mu^2 \tau h^{-1} \leq \frac{1}{4}. \]
Here $\alpha$ is defined by (3.3), $\mu$ is the inverse constant (2.26), and $C$ is a positive constant depending solely on $C_{wx}$.

Proof. To analyze the stability of the scheme (4.1), we need first to obtain the energy identity. Take $v_h^n = u_h^n$ in the scheme (4.1) to yield

$$ (u_h^{n+1}, u_h^n)_{K_j} = \|u_h^n\|_{K_j}^2 - \tau A(u_h^n, u_h^n)_{K_j}. $$

(4.4)

By the scaling argument with (2.15), we have

$$ \|\tilde{u}_h^n\|_{K_j}^2 = \frac{\Delta_j^{n+1}}{\Delta_j^n} \|u_h^n\|_{K_j}^2 = (1 + s_1)\|u_h^n\|_{K_j}^2. $$

(4.5)

Noting that

$$ (u_h^{n+1}, u_h^n)_{K_j} = \frac{1}{2}\|u_h^{n+1}\|_{K_j}^2 + \frac{1}{2}\|\tilde{u}_h^n\|_{K_j}^2 - \frac{1}{2}\|u_h^{n+1} - \tilde{u}_h^n\|_{K_j}^2, $$

we get the energy identity by summing up (4.4) over $j$,

$$ \frac{1}{2}\|u_h^{n+1}\|^2(t_{n+1}) - \frac{1}{2}\|u_h^n\|^2(t_n) = \frac{1}{2}\|u_h^{n+1} - \tilde{u}_h^n\|^2(t_{n+1}) - \sum_{j=1}^N s_1\|u_h^n\|_{K_j}^2 - \tau A(u_h^n, u_h^n)(t_n) $$

$$ = \frac{1}{2}\|u_h^{n+1} - \tilde{u}_h^n\|^2(t_{n+1}) - \frac{s_2}{2}\|u_h^n\|^2(t_n). $$

(4.6)

Here in the last step we use the property (3.18) of $A$ as well as the definition (2.18) of $s_1$. Next, we only need to analyze the first term of the right hand in (4.6). Apply the scaling arguments and (2.15) again to get

$$ (u_h^n, v_h^n)_{K_j} = \frac{\Delta_j^n}{\Delta_j^{n+1}}(u_h^n, v_h^n)_{K_j} = (1 - s_2)(u_h^n, v_h^n)_{K_j}, $$

(4.7)

and

$$ A(u_h^n, v_h^n)_{K_j} = A(u_h^n, v_h^n)_{K_j}. $$

(4.8)

It implies the equivalent form of (4.1),

$$ (u_h^{n+1} - \tilde{u}_h^n, v_h^n)_{K_j} = -s_2(u_h^n, \tilde{u}_h^n)_{K_j} - \tau A(u_h^n, v_h^n)_{K_j}. $$

(4.9)

**Pk case.** Take the test function $\tilde{v}_h^n = u_h^{n+1} - \tilde{u}_h^n$ in (4.9) and sum all over $j$ to obtain

$$ \|u_h^{n+1} - \tilde{u}_h^n\|^2(t_{n+1}) = -\sum_{j=1}^N s_2(u_h^n, u_h^{n+1} - \tilde{u}_h^n)_{K_j} - \tau A(u_h^n, u_h^{n+1} - \tilde{u}_h^n)(t_{n+1}). $$

(4.10)
Using the Cauchy-Schwarz inequality and the boundedness (3.6) of the operator $A$, we have

$$\|u_h^{n+1} - \hat{u}_h^n\|^2(t_{n+1}) \leq (C_{wx}\tau + 3\alpha\mu\tau h^{-1})\|\hat{u}_h^n\|\|u_h^{n+1} - \hat{u}_h^n\|(t_{n+1}).$$

Here we use the fact that $|s_2| \leq C_{wx}\tau$. Then divide both sides of the above inequality by $\|u_h^{n+1} - \hat{u}_h^n\|(t_{n+1})$ to get,

$$\|u_h^{n+1} - \hat{u}_h^n\|(t_{n+1}) \leq (C_{wx}\tau + 3\alpha\mu\tau h^{-1})\|\hat{u}_h^n\|(t_{n+1}),$$

which yields that

$$\frac{1}{2}\|u_h^{n+1} - \hat{u}_h^n\|^2(t_{n+1}) \leq (C_{wx}\tau^2 + 9\alpha^2\mu^2\tau^2 h^{-2})\|\hat{u}_h^n\|^2(t_{n+1}).$$

Under the CFL condition (4.2), we obtain the following the inequality,

$$\frac{1}{2}\|u_h^{n+1} - \hat{u}_h^n\|^2(t_{n+1}) \leq (C_{wx}\tau^2 + \tau)\|\hat{u}_h^n\|^2(t_{n+1}) \leq C\tau\|u_h^n\|^2(t_n).$$

Here the last step uses the scaling argument (4.5) and $|s_1| \leq C_{wx}\tau$ as well as $\tau \leq 1$. Consequently, the energy identity (4.6) implies that

$$\frac{1}{2}\|u_h^{n+1}\|^2(t_{n+1}) - \frac{1}{2}\|u_h^n\|^2(t_n) \leq C\tau\|u_h^n\|^2(t_n).$$

**P⁰ case.** Apply the equivalent form (3.13) of the operator $A$ to rewrite (4.9),

$$(u_h^{n+1} - \hat{u}_h^n, \hat{v}_h^n)_{K_{j+1}^{n+1}} = -\tau B(\hat{u}_h^n, \hat{v}_h^n)_{K_j^{n+1}},$$

due to the fact that $s_2 = \tau\omega_x(t_{n+1})$. Since the finite element space is piecewise constant, we have $\partial_xv_h^n = 0$, which is not available for $k \geq 1$. Take the test function $\hat{v}_h^n = u_h^{n+1} - \hat{u}_h^n$ in the above equality and use the boundedness (3.14) of the operator $B$ to yield,

$$\|u_h^{n+1} - \hat{u}_h^n\|(t_{n+1}) \leq \sqrt{2}\alpha\mu\tau h^{-\frac{1}{2}}\|u_h^n\|(t_n).$$

(4.11)

Here we use the fact $\hat{u}_h^n(x^{n+1}, t_{n+1}) = u_h^n(x^n, t_n)$. So that (4.11) leads to

$$\frac{1}{2}\|u_h^{n+1} - \hat{u}_h^n\|^2(t_{n+1}) \leq \alpha^2\mu^2\tau h^{-1}\|u_h^n\|^2(t_n).$$

(4.12)

If

$$\alpha^2\mu^2\tau^2 h^{-1} - \frac{\tau}{4} \leq 0, \text{ that is, } \alpha\mu^2\tau h^{-1} \leq \frac{1}{4},$$

we finish the proof by combining (4.6) and (4.12) together,

$$\frac{1}{2}\|u_h^{n+1}\|^2(t_{n+1}) - \frac{1}{2}\|u_h^n\|^2(t_n) \leq 0.$$
4.2 Second order scheme

The ALE-DG with TVD-RK2 scheme is given in the following form: find \( u_h^1, u_h^{n+1} \in V_h(t_{n+1}), \) such that for any \( v_h^n \equiv v_h(x, t_n) \in V_h(t_n) \) and \( 1 \leq j \leq N, \) there hold

\[
(u_h^1, \hat{v}_h^n)_{K_j}^{n+1} = (u_h^n, v_h^n)_{K_j}^n - \tau A(u_h^n, v_h^n)_{K_j}^n,
\]

\[
(u_h^{n+1}, \hat{v}_h^n)_{K_j}^{n+1} = \frac{1}{2}(u_h^n, v_h^n)_{K_j}^n + \frac{1}{2} (u_h^1, \hat{v}_h^n)_{K_j}^{n+1} - \frac{\tau}{2} A(u_h^1, \hat{v}_h^n)_{K_j}^{n+1}.
\]

Here \( \hat{v}_h^n \) is defined by (2.12).

**Theorem 6.** Let \( u_h^{n+1} \) be the numerical solution of the fully discrete scheme (4.13), then for any \( n, \) there holds

\[
\|u_h^{n+1}\|^2 (t_{n+1}) \leq (1 + C\tau)\|u_h^n\|^2 (t_n),
\]

under the CFL condition

\[
\tau h^{-4/3} \leq \sqrt[3]{\frac{4}{81(\alpha \mu)^4}}.
\]

In particular, for the piecewise linear finite element space, \( V_h(t) = \{ v(\chi_j(\cdot, t)) \in P^1([-1, 1]) \}, \) we just need the usual CFL condition,

\[
\alpha \mu \tau h^{-1} \leq \min \{ \frac{1}{32 \mu}, \frac{1}{\sqrt{16 \mu}} \}. \tag{4.14}
\]

Here \( \alpha \) is defined by (3.3), \( \mu \) is the inverse constant (2.26), and \( C \) is a positive constant depending solely on \( C_{wx} \) and \( \mu. \)

**Proof.** Rewrite the scheme (4.13) by the coordinate transformations (2.10), such that all of the terms are in the same cell \( K_j^{n+1}, \)

\[
(u_h^1, \hat{v}_h^n)_{K_j}^{n+1} = (1 - s_2) (u_h^n, \hat{v}_h^n)_{K_j}^{n+1} - \tau A(u_h^n, \hat{v}_h^n)_{K_j}^{n+1},
\]

\[
(u_h^{n+1}, \hat{v}_h^n)_{K_j}^{n+1} = \frac{1}{2} (u_h^n, \hat{v}_h^n)_{K_j}^{n+1} + \frac{1}{2} (u_h^1, \hat{v}_h^n)_{K_j}^{n+1} - \frac{\tau}{2} A(u_h^1, \hat{v}_h^n)_{K_j}^{n+1}.
\]

Here we use (4.7) and (4.8). By taking \( \hat{v}_h^n = \frac{1}{2} u_h^n, u_h^1 \) in the above equalities, respectively, and adding them together, we have

\[
\frac{1}{2} \|u_h^{n+1}\|^2_{K_j}^{n+1} - \frac{1}{2} \frac{1}{2}\|u_h^n\|_{K_j}^{n+1} = \frac{1}{2} \|u_h^{n+1} - u_h^n\|^2_{K_j}^{n+1} + \frac{s_2}{4} \|u_h^n - \hat{v}_h^n\|_{K_j}^{n+1} + \frac{s_2}{4} \|u_h^n - \hat{v}_h^n\|_{K_j}^{n+1} - \frac{s_2}{4} \|u_h^n - \hat{v}_h^n\|_{K_j}^{n+1}
\]

\[
-\frac{s_2}{4} \|u_h^n - \hat{v}_h^n\|_{K_j}^{n+1} - \frac{\tau}{2} A(u_h^n, \hat{v}_h^n)_{K_j}^{n+1} - \frac{\tau}{2} A(u_h^1, u_h^1)_{K_j}^{n+1}.
\]
Noticing that
\[ \| u_h^n \|_{K_j}^2 = (1 - s_2) \| u_h^n \|_{K_j}^2, \quad s_2 = \tau \omega_x(t_{n+1}), \quad (4.17) \]
we obtain the energy identity by summing (4.16) over \( j \) and using the property (3.18) of \( \mathcal{A} \),
\[
\frac{1}{2} \| u_h^{n+1} \|^2(t_{n+1}) - \frac{1}{2} \| u_h^n \|^2(t_n) = \frac{1}{2} \| u_h^{n+1} - u_h^n \|^2(t_{n+1}) + \sum_{j=1}^{N} \frac{s_2}{4} \| u_h^1 - u_h^n \|_{K_j}^2
- \frac{\tau}{4} \alpha \| u_h^n \|^2(t_n) - \frac{\tau}{4} \alpha \| u_h^1 \|^2(t_{n+1}). \quad (4.18)
\]
In order to obtain the stability, we just need to analyze the first two terms of the right hand in the above equality. From (4.15), it is straightforward to get
\[
(u_h^n - \hat{u}_h^n - \hat{v}_h^n)_{K_j} = -\tau \mathcal{A}(\hat{u}_h^n, \hat{v}_h^n)_{K_j} - s_2(\hat{u}_h^n, \hat{v}_h^n)_{K_j}, \quad (4.19)
\]
\[
(u_h^{n+1} - u_h^n - u_h^n)_{K_j} = -\tau \mathcal{A}(u_h^n - u_h^n, u_h^{n+1} - u_h^n). \quad (4.20)
\]
**\( P^k \) case.** Take the test function \( \hat{v}_h^n = u_h^{n+1} - u_h^n \) in (4.20) and sum up all over \( j \) to yield,
\[
\| u_h^{n+1} - u_h^n \|^2(t_{n+1}) = -\tau \mathcal{A}(u_h^n - u_h^n, u_h^{n+1} - u_h^n)(t_{n+1}). \quad (4.21)
\]
Using the boundedness (3.6) of \( \mathcal{A} \), we have
\[
\| u_h^{n+1} - u_h^n \|(t_{n+1}) \leq \frac{3}{2} \alpha \mu \tau h^{-1} \| u_h^n - u_h^n \|(t_{n+1}). \quad (4.22)
\]
Then by the similar arguments, taking the test function \( \hat{v}_h^n = u_h^n - u_h^n \) in (4.19) and using the boundedness (3.6) lead to
\[
\| u_h^n - u_h^n \|(t_{n+1}) \leq (C_{wx} \tau + 3 \alpha \mu \tau h^{-1}) \| u_h^n \|(t_{n+1}). \quad (4.23)
\]
Denote \( \lambda = \alpha \mu \tau h^{-1} \). Combine (4.22) and (4.23) to get that
\[
\frac{1}{2} \| u_h^{n+1} - u_h^n \|^2(t_{n+1}) \leq \frac{9}{4} \lambda^2 (C_{wx}^2 \tau^2 + 9 \lambda^2) \| u_h^n \|^2(t_{n+1}).
\]
If
\[
\frac{81}{4} \lambda^4 \leq \tau, \quad \text{that is,} \quad \tau h^{-\frac{4}{3}} \leq \sqrt[3]{\frac{4}{81(\alpha \mu)^4}}, \quad (4.24)
\]
then we have
\[
\frac{1}{2} \| u_h^{n+1} - u_h^n \|^2(t_{n+1}) \leq \left( \frac{C_{wx}^2}{2} \tau^2 + \tau \right) \| u_h^n \|^2(t_{n+1})
\]
Here we also use the Cauchy-Schwarz inequality and $\tau \leq 1$. Under the CFL condition (4.24), the estimates (4.23) turns out to be

$$
\|u_h^1 - \hat{u}_h^n\|^2(t_{n+1}) \leq 2(C_{wx}^2 \tau^2 + 2\sqrt{\tau})\|\hat{u}_h^n\|^2(t_{n+1}),
$$

which indicates that

$$
\sum_{j=1}^{N} \frac{s_2}{4} \|u_h^1 - \hat{u}_h^n\|^2_{K_{j+1}} \leq \frac{C_{wx}}{2} \tau (C_{wx}^2 \tau^2 + 2\sqrt{\tau})\|\hat{u}_h^n\|^2(t_{n+1})
$$

$$
\leq C\tau \|u_h^n\|^2(t_n).
$$

Thus we combine (4.18), (4.25) and (4.26) to obtain

$$
\frac{1}{2} \|u_h^{n+1}\|^2(t_{n+1}) - \frac{1}{2} \|u_h^n\|^2(t_n) \leq C\tau \|u_h^n\|^2(t_n).
$$

**P¹ case.** Denote $z = u_h^1 - \hat{u}_h^n$. Using the boundedness (3.7) of $A$ in (4.21), we get

$$
\|u_h^{n+1} - u_h^1\|(t_{n+1}) \leq \frac{\tau}{2} \alpha \|z\|(t_{n+1}) + \frac{C_{wx}}{2} \tau \|z\|(t_{n+1}) + \frac{\sqrt{2}}{\mu} \alpha \mu \tau h^{-\frac{3}{2}} \|z\|(t_{n+1}). \tag{4.27}
$$

Now we will analyze $\|z_x\|(t_{n+1})$. Let $y = z - P_h^0 z$, where $P_h^0 z$ denotes the $L^2$ projection of $z$ onto the piecewise constant finite element space. It follows from the property of the $L^2$ projection and the identity (4.19) as well as the boundedness (3.8) of $A$,

$$
\|y\|^2(t_{n+1}) = \sum_{j=1}^{N} (z - P_h^0 z, y)_{K_{j+1}} = \sum_{j=1}^{N} (z, y)_{K_{j+1}}
$$

$$
= -\tau A(\hat{u}_h^n, y)(t_{n+1}) - \sum_{j=1}^{N} s_2(\hat{u}_h^n, y)_{K_{j+1}}
$$

$$
\leq (\mu + 2)C_{wx} \tau \|\hat{u}_h^n\|(t_{n+1}) + \sqrt{2} \alpha \mu \tau h^{-\frac{3}{2}} \|\hat{u}_h^n\|(t) \|y\|(t_{n+1}).
$$

Here we also use the Cauchy-Schwarz inequality and $|s_2| \leq C_{wx} \tau$ for the last step. Divide both sides of the above inequality by $\|y\|(t_{n+1})$ to yield,

$$
\|y\|(t_{n+1}) \leq (\mu + 2)C_{wx} \tau \|\hat{u}_h^n\|(t_{n+1}) + \sqrt{2} \alpha \mu \tau h^{-\frac{3}{2}} \|\hat{u}_h^n\|(t).
$$

Noting that $\partial_x (P_h^0 z) = 0$ and the inverse property (2.26), we get

$$
\|z_x\|(t_{n+1}) = \|y_x\|(t_{n+1}) \leq \mu h^{-1} \|y\|(t_{n+1}),
$$

which implies that

$$
\|z_x\|(t_{n+1}) \leq (\mu + 2)C_{wx} \mu \tau h^{-1} \|\hat{u}_h^n\|(t_{n+1}) + \sqrt{2} \alpha \mu^2 \tau h^{-\frac{3}{2}} \|\hat{u}_h^n\|(t).
$$
In addition, it is clear to observe that
\[
\|z\|(t_{n+1}) \leq \sqrt{2}(\|u_h^n\|(t_n) + \|u_h^1\|(t_{n+1})).
\]
Thus the estimates (4.23) and (4.27) gives that,
\[
\|u_h^{n+1} - u_h^1\|(t_{n+1}) \leq \frac{\alpha}{2} \tau \left((\mu + 2)C_{wx} \mu \tau h^{-1} \|u_h^n\|(t_{n+1}) + \sqrt{2} \alpha \mu^2 \tau h^{-\frac{3}{2}} \|u_h^n\|(t_n)\right) + \frac{C_{wx}}{2} \tau \left(C_{wx} \tau + 3 \alpha \mu \tau h^{-1}\right) \|u_h^n\|(t_{n+1}) + \frac{\sqrt{2}}{2} \alpha \mu \tau h^{-\frac{3}{2}} \|z\|(t_{n+1})
\]
\[
\leq C_1 \tau \|u_h^n\|(t_{n+1}) + C_2 \|u_h^n\|(t_n) + C_3 \|u_h^n\|(t_{n+1}),
\]
where
\[
C_1 = \frac{(\mu + 5)\lambda + C_{wx} \tau}{2} C_{wx}, \quad C_2 = \alpha \mu \tau h^{-\frac{1}{2}} + \frac{\sqrt{2}}{2} \alpha \mu^2 \tau^2 h^{-\frac{3}{2}}, \quad C_3 = \alpha \mu \tau h^{-\frac{1}{2}},
\]
and \(\lambda = \alpha \mu \tau h^{-1}\). Squaring the above inequality, we have
\[
\frac{1}{2} \|u_h^{n+1} - u_h^1\|^2(t_{n+1}) \leq C_2^2 \|u_h^n\|^2(t_n) + 2C_3^2 \|u_h^n\|^2(t_{n+1}) + 2C_1^2 \tau^2 \|u_h^n\|^2(t_{n+1}).
\]
If we let
\[
C_2^2 \leq \frac{\alpha}{8} \tau, \quad 2C_3^2 \leq \frac{\alpha}{8} \tau,
\]
or furthermore,
\[
2\alpha^2 \mu^2 \tau^2 h^{-1} \leq \frac{\alpha}{16} \tau, \quad \alpha^4 \mu^4 \tau^4 h^{-3} \leq \frac{\alpha}{16} \tau, \quad \text{that is,} \quad \lambda \leq \min\{\frac{1}{32 \mu}, \frac{1}{\sqrt{16 \mu}}\}, \quad (4.28)
\]
then
\[
\frac{1}{2} \|u_h^{n+1} - u_h^1\|^2(t_{n+1}) \leq \frac{\alpha}{8} \tau \|u_h^n\|^2(t_n) + \frac{\alpha}{8} \tau \|u_h^n\|^2(t_{n+1}) + C \tau \|u_h^n\|^2(t_n). \quad (4.29)
\]
Here we use the property (2.21) and \(\tau \leq 1\). Owing to the choice of the CFL condition (4.28) and the estimates (4.23), we obtain
\[
\sum_{j=1}^{N} \frac{s_2}{4} \|u_h^1 - \hat{u}_h^n\|^2_{\nu_j^{n+1}} \leq \frac{C_{wx}}{4} \tau \left(C_{wx} \tau + 3 \lambda\right)^2 \|u_h^n\|^2(t_{n+1})
\]
\[
\leq C \tau \|u_h^n\|^2(t_n). \quad (4.30)
\]
Finally, we finish the proof by combining (4.18), (4.29) and (4.30),
\[
\frac{1}{2} \|u_h^{n+1}\|^2(t_{n+1}) - \frac{1}{2} \|u_h^n\|^2(t_n) \leq C \tau \|u_h^n\|^2(t_n).
\]
\(\square\)
4.3 Third order scheme

The ALE-DG with TVD-RK3 scheme is given in the following form: find $u_h^1, u_h^{n+1} \in V_h(t_{n+1})$ and $u_h^n \in V_h(t_{n+\frac{1}{2}})$, such that for any $v_h^n \equiv v_h(x, t_n) \in V_h(t_n)$ and $1 \leq j \leq N$, there hold

$$
(u_h^1, \tilde{u}_h^n)_{K_j^{n+1}} = (u_h^n, \tilde{u}_h^n)_{K_j^n} - \tau A(u_h^n, \tilde{u}_h^n)_{K_j^n},
$$

$$
(u_h^2, \tilde{u}_h^n)_{K_j^{n+\frac{1}{2}}} = \frac{3}{4}(u_h^n, \tilde{u}_h^n)_{K_j^n} + \frac{1}{4}(u_h^n, \tilde{u}_h^n)_{K_j^{n+1}} - \frac{1}{4}\tau A(u_h^n, \tilde{u}_h^n)_{K_j^{n+1}},
$$

$$
(u_h^{n+1}, \tilde{v}_h^n)_{K_j^{n+1}} = \frac{1}{3}(u_h^n, v_h^n)_{K_j^n} + \frac{2}{3}(u_h^n, \tilde{v}_h^n)_{K_j^{n+1}} - \frac{2}{3}\tau A(u_h^n, \tilde{v}_h^n)_{K_j^{n+\frac{1}{2}}},
$$

where $\tilde{v}_h^n, \overline{v}_h^n$ are defined by (2.12). In this subsection, we are going to obtain the $L^2$-norm stability for the fully discrete scheme (4.31). Similar to the second order case, we first rewrite the scheme (4.31) by the coordinate transformations (2.10)-(2.11), such that all of the terms are in the same cell $K_j^{n+1}$,

$$
(u_h^1, \tilde{u}_h^n)_{K_j^{n+1}} = (\tilde{u}_h^n, \tilde{u}_h^n)_{K_j^{n+1}} - \tau A(\tilde{u}_h^n, \tilde{u}_h^n)_{K_j^{n+1}} - s_2(\tilde{u}_h^n, \tilde{u}_h^n)_{K_j^{n+1}},
$$

$$
(\tilde{u}_h^2, \tilde{u}_h^n)_{K_j^{n+\frac{1}{2}}} = \frac{3}{4}(\tilde{u}_h^n, \tilde{u}_h^n)_{K_j^n} + \frac{1}{4}(\tilde{u}_h^n, \tilde{u}_h^n)_{K_j^{n+1}} - \frac{1}{4}\tau A(\tilde{u}_h^n, \tilde{u}_h^n)_{K_j^{n+1}} + \frac{s_2}{2}(\tilde{u}_h^n - \tilde{u}_h^n, \tilde{u}_h^n)_{K_j^{n+1}},
$$

$$
(\tilde{u}_h^{n+1}, \tilde{v}_h^n)_{K_j^{n+1}} = \frac{1}{3}(\tilde{u}_h^n, \tilde{v}_h^n)_{K_j^n} + \frac{2}{3}(\tilde{u}_h^n, \tilde{v}_h^n)_{K_j^{n+1}} - \frac{2\tau}{3} A(\tilde{u}_h^n, \tilde{v}_h^n)_{K_j^{n+1}} - \frac{s_2}{3}(\tilde{u}_h^n + \tilde{u}_h^n, \tilde{v}_h^n)_{K_j^{n+1}}.
$$

Here we use the scaling arguments and (2.15)-(2.17). Next, for the convenience of the analysis, we define

$$
E_1 = u_h^1 - \tilde{u}_h^n, \quad E_2 = 2\tilde{u}_h^n - u_h^1 - \tilde{u}_h^n, \quad E_3 = u_h^{n+1} - 2u_h^n + \tilde{u}_h^n.
$$

Then we can achieve the following identities by a direct calculation,

$$
(E_1, \tilde{u}_h^n)_{K_j^{n+1}} = -s_2(\tilde{u}_h^n, \tilde{u}_h^n)_{K_j^{n+1}} - \tau A(\tilde{u}_h^n, \tilde{u}_h^n)_{K_j^{n+1}},
$$

$$
(E_2, \tilde{u}_h^n)_{K_j^{n+1}} = -s_2(\tilde{u}_h^n - \tilde{u}_h^n, \tilde{u}_h^n)_{K_j^{n+1}} - \frac{\tau}{2} A(E_1, \tilde{u}_h^n)_{K_j^{n+1}},
$$

$$
(E_3, \tilde{u}_h^n)_{K_j^{n+1}} = -s_2(\tilde{u}_h^n - \tilde{u}_h^n, \tilde{u}_h^n)_{K_j^{n+1}} - \frac{\tau}{3} A(E_2, \tilde{u}_h^n)_{K_j^{n+1}}.
$$

For the last identity, we rewrite $E_3 = (u_h^{n+1} - \frac{1}{3}\tilde{u}_h^n - \frac{2}{3}\tilde{u}_h^n) - \frac{4}{3}(\tilde{u}_h^n - \tilde{u}_h^n)$, and

$$
(\tilde{u}_h^n - \tilde{u}_h^n, \tilde{u}_h^n)_{K_j^{n+1}} = -s_2(\tilde{u}_h^n, \tilde{u}_h^n)_{K_j^{n+1}} + \frac{s_2}{2}(\tilde{u}_h^n, \tilde{u}_h^n)_{K_j^{n+1}} - \frac{\tau}{4} A(u_h^1 + \tilde{u}_h^n, \tilde{u}_h^n)_{K_j^{n+1}}.
$$

In the following, we will present the stability for the fully discrete scheme (4.31).

**Theorem 7.** Let $u_h^{n+1}$ be the numerical solution of the fully discrete scheme (4.31), then we have for any $n$, that

$$
\|u_h^{n+1}\|^2(t_{n+1}) \leq (1 + C\tau)\|u_h^n\|^2(t_n),
$$

where $C$ is a constant.
Taking the test functions $\hat{v}_h^n = u_h^n, 4u_h^1, \text{and } 6u_h^2$ in the identities (4.32), respectively, and adding them together, we have

$$\int_{K_{j+1}} F dx^{n+1} = -s_2 ||u_h^n||^2_{K_{j+1}} - 2s_2 ||u_h^2||^2_{K_{j+1}} + s_2 R_1 - \tau R_2,$$

where

$$F = -2u_h^1 u_h^n - \left(\hat{u}_h^n\right)^2 + 4u_h^2 u_h^n - \left(u_h^1\right)^2 + 6u_h^{n+1} u_h^2 - 2u_h^n u_h^n - 4\left(u_h^2\right)^2,$$

$$R_1 = 2(\hat{u}_h^2, u_h^1 - u_h^n)_{K_{j+1}^n} - 3(\hat{u}_h^n, u_h^1)_{K_{j+1}^n},$$

$$R_2 = A(u_h^n, u_h^n)_{K_{j+1}^n} + A(u_h^n, u_h^1)_{K_{j+1}^n} + 4A(u_h^2, u_h^2)_{K_{j+1}^n}.$$

Noting that

$$F = 3\left(\left(u_h^{n+1}\right)^2 - \left(u_h^n\right)^2\right) - E_2^2 - 3(u_h^{n+1} - u_h^n)E_3,$$

we get the following identity by summing over all $j$,

$$3||u_h^{n+1}||^2(t_{n+1}) - 3||u_h^n||^2(t_{n+1}) = ||E_2||^2 (t_{n+1}) + 3(u_h^{n+1} - u_h^n, E_3) (t_{n+1}) + \sum_{j=1}^N s_2 R_1$$

$$- \sum_{j=1}^N s_2 \left|\hat{u}_h^n\right|^2_{K_{j+1}^n} - 2 \sum_{j=1}^N s_2 \left|\hat{u}_h^n\right|^2_{K_{j+1}^n} - \tau \sum_{j=1}^N R_2. \tag{4.37}$$

Denote each line of the right hand in the above equality by $\Theta_1, \Theta_2$, respectively. The definitions (4.33), the identities (4.34) and the property (3.17)-(3.18) of $A$ yield,

$$\Theta_1 = ||E_2||^2 + 3(E_3, E_1 + E_2 + E_3) + \sum_{j=1}^N s_2 R_1$$

$$= - ||E_2||^2 + 2(E_2, E_2) + 3(E_3, E_1) + 3(E_3, E_2) + 3||E_3||^2 + \sum_{j=1}^N s_2 R_1$$

$$= - ||E_2||^2 + 3||E_3||^2 - \tau A(E_1, E_2) - \tau A(E_2, E_1) - \tau A(E_2, E_2)$$

$$+ \sum_{j=1}^N s_2 (u_h^n - u_h^2, E_2 + 3E_1)_{K_{j+1}^n} + \sum_{j=1}^N s_2 R_1$$

under the CFL condition

$$\alpha \mu \tau h^{-1} \leq \frac{1}{3}, \tag{4.36}$$

where $\alpha$ is defined by the Lax-Friedrichs numerical flux (3.3), $\mu$ is the inverse constant (2.26), and $C$ is a positive constant depending solely on $C_{wx}$.

Proof. Similar to the second order scheme (4.13), we need first obtain the energy identity.
where we have dropped the symbol \( t_{n+1} \) since all quantities are considered there and

\[
\Theta_{11} = (\hat{u}_h^n - \hat{u}_h^n, E_2 + 3E_1)_{K_{n+1}^n} + R_1 + (E_1, E_2)_{K_{n+1}^n} + \frac{1}{2}\|E_2\|_{K_{n+1}^n}^2
\]

\[
= -\frac{5}{2}\|u_h^n\|_{K_{n+1}^n}^2 - \frac{1}{2}\|u_h^n\|_{K_{n+1}^n}^2.
\]

On the other hand, the property (3.18) of \( A \) implies that

\[
\Theta_2 = -\frac{\alpha}{2}\tau\|u_h^n\|_F^2(t_n) - \frac{\alpha}{2}\tau\|u_h^n\|_F^2(t_{n+\frac{1}{2}}) - 2\alpha\tau\|u_h^n\|_F^2(t_{n+1})
\]

\[
- \sum_{j=1}^{N}s_2\|u_h^n\|_{K_{j}^{n+1}}^2 + \sum_{j=1}^{N}s_2\|u_h^n\|_{K_{j}^{n+1}}^2.
\]

Recalling the relationship (4.17), we add \( \Theta_1 \) and \( \Theta_2 \) to the equality (4.37) and obtain the energy identity

\[
3\|u_h^{n+1}\|_F^2(t_{n+1}) - 3\|u_h^n\|_F^2(t_n) = I_1 + I_2,
\]

where

\[
I_1 = -\|E_2\|_F^2(t_{n+1}) + 3\|E_3\|_F^2(t_{n+1}) - \alpha\tau\sum_{j=1}^{N}[\|E_1\|_{j+\frac{1}{2}} E_2]_{j+\frac{1}{2}} - \frac{\alpha}{2}\tau\|E_2\|_F^2(t_{n+1}),
\]

\[
I_2 = -\frac{\alpha}{2}\tau\|u_h^n\|_F^2(t_n) - \frac{\alpha}{2}\tau\|u_h^n\|_F^2(t_{n+\frac{1}{2}}) - 2\alpha\tau\|u_h^n\|_F^2(t_{n+1}).
\]

Denote the last two terms on the right hand of (4.39) by \( I_{11} \) and we have the following estimates by applying the Young’s inequality,

\[
I_{11} \leq \frac{\alpha}{4}\tau\|E_1\|_F^2(t_{n+1}) + \frac{\alpha}{2}\tau\|E_2\|_F^2(t_{n+1})
\]

\[
\leq \frac{\alpha}{2}\tau\|u_h^n\|_F^2(t_{n+1}) + \frac{\alpha}{2}\tau\|u_h^n\|_F^2(t_n) + \alpha\tau\|E_2\|_F^2(t_{n+1})
\]

\[
\leq \frac{\alpha}{2}\tau\|u_h^n\|_F^2(t_{n+1}) + \frac{\alpha}{2}\tau\|u_h^n\|_F^2(t_n) + \alpha\mu\tau h^{-1}\|E_2\|_F^2(t_{n+1}),
\]

where we use the estimates (3.11) for the second inequality and the last inequality uses the inverse inequality (2.26). Next, we will analyze \( 3\|E_3\|_F^2(t_{n+1}) \) with the identity (4.34). Denote \( \lambda = \alpha\mu\tau h^{-1} \) as before. Take the test function \( v_h^n = E_3 \) in the third equality of (4.34), sum over all \( j \) and use the boundedness (3.6) of \( A \) to derive,

\[
\|E_3\|_F^2(t_{n+1}) \leq \left( C_{wx}\tau\|\hat{u}_h^n - \hat{u}_h^n\|_{(t_{n+1})} + \lambda\|E_2\|_{(t_{n+1})}\right) \|E_3\|_{(t_{n+1})},
\]

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which implies that,

$$3\|\mathbb{E}_3\|^2(t_{n+1}) \leq 6\lambda^2\|\mathbb{E}_2\|^2(t_{n+1}) + 6C_{wx}^2\tau^2\|\tilde{u}_h^n - \tilde{u}_h^n\|^2(t_{n+1}).$$  \hspace{1cm} (4.42)

Here we use $|s_2| = |\omega_x(t_{n+1})\tau| \leq C_{wx}\tau$. Then from (4.39)-(4.42) and under the CFL condition (4.36), we get,

$$\mathbb{I}_1 + \mathbb{I}_2 \leq 6C_{wx}^2\tau^2\|\tilde{u}_h^n - \tilde{u}_h^n\|^2(t_{n+1}) - (1 - 6\lambda^2 - \lambda)\|\mathbb{E}_2\|^2(t_{n+1})$$

$$\leq 6C_{wx}^2\tau\|\tilde{u}_h^n - \tilde{u}_h^n\|^2(t_{n+1}),$$  \hspace{1cm} (4.43)

since $\tau \leq 1$. In addition, it is clear to obtain the following equality from (4.35),

$$(1 - \frac{s_2}{2})(\tilde{u}_h^n - \tilde{u}_h^n, \tilde{v}_h^n)_{K_j^{n+1}} = -\frac{s_2}{2}(\tilde{u}_h^n, \tilde{v}_h^n)_{K_j^{n+1}} - \frac{\tau}{4}\mathcal{A}(\tilde{u}_h^n + \tilde{u}_h^n, \tilde{v}_h^n)_{K_j^{n+1}},$$

which yields,

$$(\tilde{u}_h^n - \tilde{u}_h^n, \tilde{v}_h^n)_{K_j^{n+1}} = -\frac{s_2}{2}(\tilde{u}_h^n, \tilde{v}_h^n)_{K_j^{n+1}} - \frac{\tau}{4}(1 + \frac{s_3}{2})\mathcal{A}(\tilde{u}_h^n + \tilde{u}_h^n, \tilde{v}_h^n)_{K_j^{n+1}}$$

$$= -\frac{s_2}{2}(\tilde{u}_h^n, \tilde{v}_h^n)_{K_j^{n+1}} - \frac{\tau}{4}(1 + \frac{s_3}{2})\mathcal{A}(\tilde{E}_1, \tilde{v}_h^n)_{K_j^{n+1}}$$

$$- \frac{\tau}{2}(1 + \frac{s_3}{2})\mathcal{A}(\tilde{u}_h^n, \tilde{v}_h^n)_{K_j^{n+1}}.$$  \hspace{1cm} (4.44)

Here we use the relationship (2.19) between $s_2$ and $s_3$ for the first step. Taking the test function $\tilde{v}_h^n = \tilde{u}_h^n - \tilde{u}_h^n$ in the above equality, summing it over all elements, and applying the boundedness (3.6) of $\mathcal{A}$, we have

$$\|\tilde{u}_h^n - \tilde{u}_h^n\|(t_{n+1}) \leq \frac{C_{wx}}{2}\tau\|\tilde{u}_h^n\|(t_{n+1}) + \frac{3}{4}\lambda(1 + \frac{C_{wx}}{2}\tau)\|\mathcal{E}_1\|(t_{n+1}) + \frac{3}{2}\lambda(1 + \frac{C_{wx}}{2}\tau)\|\tilde{u}_h^n\|(t_{n+1})$$

$$\leq (\frac{1}{2} + 3\frac{C_{wx}}{4}\tau)\|\tilde{u}_h^n\|(t_{n+1}) + \frac{1}{4}(1 + \frac{C_{wx}}{2}\tau)\|\mathcal{E}_1\|(t_{n+1}).$$  \hspace{1cm} (4.45)

Here for the first step we use $|s_3| = |\omega_x(t_{n+1})\tau| \leq C_{wx}\tau$ and the last step uses the CFL condition (4.36). For the estimate of $\|\mathcal{E}_1\|(t_{n+1})$, we take $\tilde{v}_h^n = \mathcal{E}_1$ in the first equality of (4.34) and use the similar arguments to derive,

$$\|\mathcal{E}_1\|(t_{n+1}) \leq (C_{wx}\tau + 3\lambda)\|\tilde{u}_h^n\|(t_{n+1})$$

$$\leq (C_{wx}\tau + 1)\|\tilde{u}_h^n\|(t_{n+1}).$$  \hspace{1cm} (4.45)

As a result, we collect the estimates from (4.43) to (4.45) to achieve,

$$\mathbb{I}_1 + \mathbb{I}_2 \leq C_1\tau\|\tilde{u}_h^n\|^2(t_{n+1}),$$

where $C_1$ depends solely on $C_{wx}$. Recalling the energy identity (4.38), we will finish the proof,

$$3\|u_h^{n+1}\|^2(t_{n+1}) - 3\|u_h^n\|^2(t_n) \leq C_1\tau\|\tilde{u}_h^n\|^2(t_{n+1})$$

$$\leq C\tau\|u_h^n\|^2(t_n),$$

where we use the property (2.21) and $\tau \leq 1$. \hspace{1cm} \(\Box\)
5 Error estimates for linear conservation laws

In this section, we will present error estimates for the fully discrete schemes with the help of the stability analysis in the previous section. We begin with some preliminaries. To make it clear to construct the error equation, we first show the representation of Eq. (1.1) after a time-dependent coordinate transformation \( x = x(\xi, t) \) defined by (2.8). For simplicity, we denote \( \tilde{v}(\xi, t) = v(x(\xi, t), t) \) for any function \( v(x, t) \). Then by the chain rule,

\[
\tilde{u}_\xi = u_x x_\xi, \quad \tilde{u}_t = u_t + u_x x_t,
\]

where \( x_\xi = \frac{\Delta_j(t)}{2} \) and \( x_t = \tilde{\omega} \). Thus in the reference coordinates \((\xi, t)\), the Eq. (1.1) of \( K_j(t), t \in [t_n, t_{n+1}] \) becomes

\[
\tilde{u}_t + \frac{2}{\Delta_j(t)} (\beta - \tilde{\omega}) \tilde{u}_\xi = 0, \quad (\xi, t) \in [-1, 1] \times [t_n, t_{n+1}]. \tag{5.1}
\]

On the other hand, by (2.3), (2.8) and (2.5), we have

\[
\tilde{\omega}_\xi = \frac{1}{2}(\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}) = \frac{\Delta_j(t)}{2}. \tag{5.2}
\]

Combine (5.1) and (5.2) to derive,

\[
\partial_t \left( \tilde{u} \Delta_j(t) \right) + \partial_\xi \left( 2(\beta - \tilde{\omega}) \tilde{u} \right) = 0,
\]

which is equivalent to the form,

\[
\tilde{U}_t + (a\tilde{U})_\xi = 0, \quad \tilde{U}(\xi, t) = \tilde{u} \Delta_j(t), \quad a(\xi, t) = \frac{2(\beta - \tilde{\omega})}{\Delta_j(t)}. \tag{5.3}
\]

5.1 First order scheme

In this subsection, we would like to show the error estimates for the ALE-DG spatial discretization coupled with the Euler-forward time marching scheme. Following the idea of analyzing the error estimates for static mesh [24], we first present the error equation.

5.1.1 Error equation

Denote \( u^n = u(x, t_n) \) for any time level \( n \). To proceed with the error equation, we need the following lemma, which describes the local truncation error in time.

Lemma 8. Let \( u \) be the exact solution of Eq. (1.1). Suppose \( u \) is sufficiently smooth with bounded derivatives, then for any \( v^n_h \equiv v_h(x, t_n) \in V_h(t_n) \) and \( 1 \leq j \leq N \), there holds,

\[
(u^{n+1}, \tilde{v}_h^n)_{K_j} = (u^n, v^n_h)_{K_j} - \tau \mathcal{A}(u^n, v^n_h)_{K_j} + (\varepsilon^n_1, v^n_h)_{K_j}, \tag{5.4}
\]
where $\hat{v}_h^n$ is defined by (2.12), $\varepsilon^n_1$ is the local truncation error in time and $\|\varepsilon^n_1\|_{K_j^n} = O(\tau^2)$ for any $j$ and $n$.

**Proof.** By the Taylor expansion with Lagrange form of the remainder, we obviously have,

$$
\hat{U}(\xi, t + \tau) = \hat{U}(\xi, t) - \tau (a \hat{U}\xi)(\xi, t) + \frac{\tau^2}{2} \hat{U}_{tt}(\xi, t_1), \quad t_1 \in (t, t + \tau),
$$

where we use the definition of $\hat{U}$ in (5.3). Let $t = t_n$ and we still use the notation $t_1$ to stand for a fixed value between $t_n$ and $t_{n+1}$. Multiply the test function $\hat{v}_h^n \in P^k([-1, 1])$ on both sides of the above equation, and integrate by parts to yield,

$$
\int_{-1}^{1} \hat{U}^{n+1} \hat{v}_h^n d\xi = \int_{-1}^{1} \hat{U}^{n} \hat{v}_h^n d\xi - \tau \hat{A}(\hat{U}^{n}, \hat{v}_h^n) + \frac{\tau^2}{2} \int_{-1}^{1} \hat{U}_{tt}(\xi, t_1) \hat{v}_h^n d\xi
$$

where

$$
\hat{A}(\hat{U}^{n}, \hat{v}_h^n) = - \int_{-1}^{1} a^n \hat{U}^{n} \partial_\xi(\hat{v}_h^n) d\xi + a^n \hat{U}^{n} \hat{v}_h^n|_{\xi=1} - a^n \hat{U}^{n} \hat{v}_h^n|_{\xi=-1}.
$$

Noting that $x_\xi = \frac{\Delta_x(t)}{2}$, we can easily get, by the scaling arguments and (5.3),

$$
\int_{-1}^{1} \hat{U}^{n+1} \hat{v}_h^n d\xi = 2(u^{n+1}, \hat{v}_h^n)_{K_j^{n+1}}, \quad \int_{-1}^{1} \hat{U}^{n} \hat{v}_h^n d\xi = 2(u^n, v_h^n)_{K_j^n}.
$$

(5.5)

Owing to the smooth exact solution, we have $\|u\|_{j-\frac{1}{2}} = 0$ at each element boundary point, which implies that,

$$
\hat{A}(\hat{U}^{n}, \hat{v}_h^n) = 2A(u^n, v_h^n)_{K_j^n}.
$$

(5.6)

Moreover, from the definition (5.3) of $\hat{U}$ and $\Delta_j'(t) = \omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}$, we obtain,

$$
\hat{U}_t = \Delta_j(t) \hat{u}_t + \Delta_j'(t) \hat{u}, \quad \hat{U}_{tt} = \Delta_j(t) \hat{u}_{tt} + 2\Delta_j'(t) \hat{u}_t.
$$

It is inferred that

$$
\frac{\tau^2}{2} \int_{-1}^{1} \hat{U}_{tt}(\xi, t_1) \hat{v}_h^n d\xi = \frac{\tau^2}{2} \int_{-1}^{1} \left( \Delta_j \hat{u}_{tt} + 2\Delta_j' \hat{u}_t \right) (\xi, t_1) \hat{v}_h^n d\xi
$$

$$
= \frac{\Delta_j(t_1)}{\Delta_j^n} \tau^2 \left( \hat{u}_{tt}(x^n_j, t_n), v_h^n \right)_{K_j^n} + \frac{2\Delta_j'}{\Delta_j^n} \tau^2 \left( \hat{u}_t(x^n_j, t_n), v_h^n \right)_{K_j^n}
$$

$$
= 2(\varepsilon_1^n, v_h^n)_{K_j^n}.
$$

(5.7)

where $\hat{u}_{tt}(x^n_j, t_n) = u_{tt}(x, t_1), \hat{u}_t(x^n_j, t_n) = u_t(x, t_1)$ for $x^n_j \in K_j^n, x \in K_j(t_1)$ and

$$
\varepsilon^n_1 = \frac{\Delta_j(t_1)}{2\Delta_j^n} \tau^2 \hat{u}_{tt} + \frac{\Delta_j'}{\Delta_j^n} \tau^2 \hat{u}_t.
$$

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By the quantity (2.4), the assumption (2.5) and (2.7), there holds,
\[ \frac{\Delta_j(t_1)}{\Delta_j^n} = 1 + \omega_x(t_n)(t_1 - t_n) \leq 1 + C_{w_x}, \quad \frac{\Delta_j'}{\Delta_j^n} = \omega_x(t_n) \leq C_{w_x}, \]
which implies that
\[ \| \varepsilon^n \|_{K_j^n} \leq C \tau^2. \]
Here \( C \) is positive constant independent of \( h \) and \( \tau \). Finally, we combine (5.5), (5.6) and (5.7) to finish the proof,
\[ (u^{n+1}, \hat{v}^n_h)_{K_j^n+1} = (u^n, v^n_h)_{K_j^n} - \tau \mathcal{A}(u^n, v^n_h)_{K_j^n} + (\varepsilon^n_1, v^n_h)_{K_j^n}. \]

\[ \square \]

For the convenience of analysis, we introduce some notations. The error between the exact solution and the numerical solution of the first scheme (4.1) is denoted by \( e^n = u(x, t_n) - u^n_h \) for any stage \( n \). Let \( \zeta^n = P_h u^n - u^n_h \) and \( \eta^n = P_h u^n - u^n \), where \( P_h u^n \) is the \( L^2 \) projection defined by (2.23). Subtracting (5.4) for the first scheme (4.1), we obtain the error equation for any \( v^n_h \in V_h(t_n) \) and \( 1 \leq j \leq N \),
\[ (e^{n+1}, \hat{v}^n_h)_{K_j^n+1} = (e^n, \hat{v}^n_h)_{K_j^n} - \tau \mathcal{A}(e^n, \hat{v}^n_h)_{K_j^n} + (\varepsilon^n_1, \hat{v}^n_h)_{K_j^n}. \]
Moreover, \( e^n = \zeta^n - \eta^n \) yields that
\[ (\zeta^{n+1}, \hat{v}^n_h)_{K_j^n+1} = (\zeta^n, \hat{v}^n_h)_{K_j^n} - \tau \mathcal{A}(\zeta^n, \hat{v}^n_h)_{K_j^n} + \mathcal{H}_j(\eta, v^n_h), \tag{5.8} \]
where
\[ \mathcal{H}_j(\eta, v^n_h) = (\eta^{n+1}, \hat{v}^n_h)_{K_j^n+1} - (\eta^n, v^n_h)_{K_j^n} + \tau \mathcal{A}(\eta^n, v^n_h)_{K_j^n} + (\varepsilon^n_1, v^n_h)_{K_j^n}. \tag{5.9} \]
In the end, we present the following estimates for the projection error.

**Lemma 9.** Suppose \( u \) is sufficiently smooth with bounded derivatives, then there exists a positive constant \( C \) independent of \( h, \tau \) and \( n \), such that for any \( v^n_h \in H^k_1(t_n) \) and \( \forall n \leq M \),
\[ \| \eta^n \|_{(t_n)} + h^{1/2}\| \eta^n \|_{\Gamma_h(t_n)} + h\| \partial_x \eta^n \|_{(t_n)} \leq C h^{k+1}, \tag{5.10} \]
\[ (\eta^{n+1}, \hat{v}^n_h(t_n)) - (\eta^n, v^n_h(t_n)) \leq C \tau h^{k+1}\| v^n_h \|_{(t_n)}. \tag{5.11} \]

**Proof.** The estimate (5.10) can be obtained directly from (2.25). By the scaling arguments, the definition \( \eta^n = P_h u^n - u^n \) and (5.3), we get,
\[ (\eta^{n+1}, \hat{v}^n_h)_{K_j^n+1} - (\eta^n, v^n_h)_{K_j^n} = \frac{\Delta_j^{n+1}}{2} \int_{-1}^1 \eta^{n+1} \hat{v}^n_h d\xi - \frac{\Delta_j^n}{2} \int_{-1}^1 \eta^n \hat{v}^n_h d\xi \]

\[ 25 \]
\[
\frac{1}{2} \int_{-1}^{1} \left[ (P_h \mathcal{U}^{n+1} - \mathcal{U}^{n+1}) - (P_h \mathcal{U}^n - \mathcal{U}^n) \right] \hat{v}_h^n d\xi \\
= \frac{1}{2} \int_{-1}^{1} \left[ P_h(\mathcal{U}^{n+1} - \mathcal{U}^n) - (\mathcal{U}^{n+1} - \mathcal{U}^n) \right] \hat{v}_h^n d\xi \\
\leq C_1 h^2 \tau \| \partial^{k+2} (a\mathcal{U})(\xi, t_n) \|_{L^2([-1,1])} \| \hat{v}_h^n \|_{L^2([-1,1])}.
\]

Here we use the fact that the \( L^2 \) projection is linear in time on the static mesh as well as the exact solution is smooth enough. Noting that \( a\mathcal{U} = 2(\beta - \omega)\hat{u} \) from (5.3), we apply the scaling arguments again to derive,

\[
\| \partial^{k+2} (a\mathcal{U})(\xi, t_n) \|_{L^2([-1,1])}^2 = \int_{-1}^{1} \left( \partial^{k+2} (a\mathcal{U})(\xi, t_n) \right)^2 d\xi \\
= 4 \int_{K_j^n} \left( \partial^{k+2} ((\beta - \omega)u^n) \right)^2 (x_\xi)^{2k+3} dx \\
= 4 \left( \frac{\Delta_j^n}{2} \right)^{2k+3} \int_{K_j^n} \left( \partial^{k+2} ((\beta - \omega)u^n) \right)^2 dx,
\]

and

\[
\| \hat{v}_h^n \|_{L^2([-1,1])}^2 = \int_{-1}^{1} (\hat{v}_h^n)^2 d\xi = \frac{2}{\Delta_j^n} \int_{K_j^n} (v_h^n)^2 dx.
\]

Finally, the above estimates yield,

\[
(\eta^{n+1}, \hat{v}_h^n)(t_{n+1}) - (\eta^n, \hat{v}_h^n)(t_n) \leq C \tau h^{k+1} \| v_h^n \|_{L^2([-1,1])}.
\]

The proof is completed. \( \square \)

In the following, we use the notation \( C \) to stand for a generic positive constant independent of \( \tau, h, n \) and \( u_h^n \), but may depends on the exact solution \( u \), the mesh speed function \( \omega \) and the inverse constant \( \mu \) (2.26). It may have a different value in each occurrence.

### 5.1.2 Error estimates for the first order scheme

**Theorem 10.** Let \( u_h^n \) be the numerical solution of the fully discrete scheme (4.1) with Euler-forward time-marching, and \( u \) be the exact solution of Eq.(1.1). Suppose \( u \) is sufficiently smooth with bounded derivatives, then we have the following error estimate,

\[
\max_{n \tau \leq T} \| u(x, t_n) - u_h^n \|_{L^2} \leq C (h^{k+\frac{1}{2}} + \tau),
\]

under the CFL condition

\[
\alpha^2 \mu^2 \tau h^{-2} \leq \frac{1}{9}.
\]

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In particular, for the piecewise constant finite element space, $V_h(t) = \{v(\chi_j(\cdot, t)) \in P^0([-1, 1])\}$, we just need the usual CFL condition

$$\alpha \mu^2 \tau h^{-1} \leq \frac{1}{8}. \quad (5.13)$$

Here $\alpha$ is defined by (3.3), $\mu$ is the inverse constant (2.26). The positive constant $C$ is independent of $h$, $\tau$, $n$ and $u_h$.

Proof. Similar to the stability analysis, we take the test function $v^\alpha_h = \zeta^n$ in the error equation (5.8) to derive the energy identity for $\zeta^n$,

$$\frac{1}{2} \|\zeta^{n+1}\|^2(t_{n+1}) - \frac{1}{2} \|\zeta^n\|^2(t_n) = \frac{1}{2} \|\zeta^{n+1} - \hat{\zeta}^n\|^2(t_{n+1}) - \frac{\tau}{2} \alpha \|\zeta^n\|^2(t_n) + \sum_{j=1}^N \mathcal{H}_j(\eta, \zeta^n). \quad (5.14)$$

In the following, we will estimate $\|\zeta^{n+1} - \hat{\zeta}^n\|^2(t_{n+1})$ and $\sum_{j=1}^N \mathcal{H}_j(\eta, \zeta^n)$ separately. From the estimates (5.11) and Lemma 8, it is easy to conclude that,

$$(\eta^{n+1}, \hat{v}_h^n)(t_{n+1}) - (\eta^n, \hat{v}_h^n)(t_n) + (\varepsilon^n, \hat{v}_h^n)(t_n) \leq C(\tau h^{k+1} + \tau^2)\|v_h^n\|(t_n). \quad (5.15)$$

By the scaling arguments with (2.15), we have,

$$(\zeta^n, \hat{v}_h^n)_{K_j} = \frac{\Delta_j^n}{\Delta_j^{n+1}} (\hat{\zeta}^n, \hat{v}_h^n)_{K_j^{n+1}} = (1 - s_2)(\hat{\zeta}^n, \hat{v}_h^n)_{K_j^{n+1}}. \quad (5.15)$$

Then the error equation (5.8) can be rewritten as

$$(\zeta^{n+1} - \hat{\zeta}^n, \hat{v}_h^n)_{K_j^{n+1}} = -s_2(\hat{\zeta}^n, \hat{v}_h^n)_{K_j^{n+1}} - \tau \mathcal{A}(\hat{\zeta}^n, \hat{v}_h^n)_{K_j^n} + \mathcal{H}_j(\eta, \hat{v}_h^n). \quad (5.16)$$

Choose $\hat{v}_h^n = \zeta^{n+1} - \hat{\zeta}^n$ in the above equality and sum over all $j$ to obtain

$$\|\zeta^{n+1} - \hat{\zeta}^n\|^2(t_{n+1}) = -\sum_{j=1}^N s_2(\hat{\zeta}^n, \zeta^{n+1} - \hat{\zeta}^n)_{K_j^{n+1}} - \tau \mathcal{A}(\hat{\zeta}^n, \zeta^{n+1} - \hat{\zeta}^n)(t_{n+1})$$

$$+ \sum_{j=1}^N \mathcal{H}_j(\eta, \hat{\zeta}^{n+1} - \zeta^n), \quad (5.16)$$

where $\hat{\zeta}^{n+1}$ is defined by (2.14). Now we will divide the analysis into two cases like in the stability analysis.

**$P^k$ case.** The boundedness (3.15) of $\mathcal{A}$ firstly gives that

$$\mathcal{A}(\eta^n, \zeta^n)(t_n) \leq \mu c_{wx}\|\eta^n\|(t_n)\|\zeta^n\|(t_n) + \sqrt{2}\alpha\|\eta^n\|_{H_0(t_n)}\|\zeta^n\|(t_n)$$

$$\leq Ch^{k+1}\|\zeta^n\|(t_n) + C^h h^{k+\frac{1}{2}}\|\zeta^n\|(t_n)$$
Here we use the estimates (5.10) for the second step and Young’s inequality for the last step. Take \( v^n_h = \zeta^n \) in (5.15) to infer that
\[
\sum_{j=1}^{N} \mathcal{H}_j(\eta, \zeta^n) = (\eta^{n+1}, \hat{\zeta}^{n})(t_{n+1}) - (\eta^n, \zeta^n)(t_n) + \tau \mathcal{A}(\eta^n, \zeta^n)(t_n) + (\epsilon_1^n, \zeta^n)(t_n)
\]
\[
\leq 2\tau \|\zeta^n\|^2(t_n) + \frac{\alpha}{2} \tau \|\zeta^n\|^2(t_n) + C\tau h^{2k+1} + \tau^2).
\] (5.17)

Here we use the Young’s inequality again. As for the estimate of \( \|\zeta^{n+1} - \hat{\zeta}^n\|^2(t_{n+1}) \), we will use the equality (5.16). By the boundedness (3.6) of \( \mathcal{A} \) and (2.20), we have,
\[
\mathcal{A}(\eta^n, \hat{\zeta}^{n+1} - \zeta^n)(t_n) \leq \left( \mu C_{wx} \|\eta^n\|(t_n) + 2\alpha \mu h^{-\frac{1}{2}} \|\eta^n\|_{r_h(t_n)} \right) \|\hat{\zeta}^{n+1} - \zeta^n\|(t_n)
\]
\[
\leq C \left( h^{k+1} + \alpha \mu h^k \right) \|\zeta^{n+1} - \hat{\zeta}^n\|(t_{n+1}).
\]

Owing to \( \tau \leq 1 \), it follows from taking \( \bar{v}_h^n = \zeta^{n+1} - \hat{\zeta}^n \) in (5.15) and (2.20) that
\[
\sum_{j=1}^{N} \mathcal{H}_j(\eta, \zeta^{n+1} - \zeta^n) \leq C \left( \tau h^{k+1} + \alpha \mu \tau h^k + \tau^2 \right) \|\zeta^{n+1} - \zeta^n\|(t_{n+1}).
\]

In addition, by the boundedness (3.6) of \( \mathcal{A} \), we get,
\[
\tau \mathcal{A}(\zeta^n, \zeta^{n+1} - \hat{\zeta}^n)(t_{n+1}) \leq 3\alpha \mu h^{-1} \|\hat{\zeta}^n\|(t_{n+1}) \|\zeta^{n+1} - \hat{\zeta}^n\|(t_{n+1}).
\]

Recalling the equality (5.16), we obtain the following estimates by dividing both sides by \( \|\zeta^{n+1} - \hat{\zeta}^n\|(t_{n+1}) \),
\[
\|\zeta^{n+1} - \hat{\zeta}^n\|(t_{n+1}) \leq C_{wx} \tau \|\hat{\zeta}^n\|(t_{n+1}) + 3\alpha \mu h^{-1} \|\hat{\zeta}^n\|(t_{n+1}) + C(\tau h^{k+1} + \alpha \mu \tau h^k + \tau^2),
\]
which implies that
\[
\frac{1}{2} \|\zeta^{n+1} - \hat{\zeta}^n\|^2(t_{n+1}) \leq \left( C\tau^2 + (3\alpha \mu h^{-1})^2 \right) \|\hat{\zeta}^n\|^2(t_{n+1}) + C \left( \tau h^{k+1} + \alpha \mu \tau h^k + \tau^2 \right)^2
\]
\[
\leq C\tau \|\hat{\zeta}^n\|^2(t_{n+1}) + C \tau (h^{2k+2} + \tau^2)
\]
\[
\leq C\tau \|\zeta^n\|^2(t_n) + C\tau (h^{2k+2} + \tau^2).
\] (5.18)

Here we use the CFL condition (5.12), the relationship (2.21) and \( \tau \leq 1 \). Hence, we combine (5.14), (5.17) and (5.18) to derive
\[
\|\zeta^{n+1}\|^2(t_{n+1}) - \|\zeta^n\|^2(t_n) \leq C\tau \|\zeta^n\|^2(t_n) + C\tau (h^{2k+1} + \tau^2).
\]

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Summing over \( n \) and using Gronwall’s inequality, we obtain,
\[
\| \zeta^n \|^2(t_n) \leq C(h^{2k+1} + \tau^2), \quad n \leq M,
\]
if we choose the initial condition \( u_h(x, 0) = P_h u(x, 0) \). Finally, apply the estimate (5.10) to yield,
\[
\| e^n \|^2(t_n) \leq C(h^{2k+1} + \tau^2), \quad n \leq M.
\]

**P⁰ case.** Since the finite element space is piecewise constant, we have \( \partial_x v^n_h = 0 \), which indicates that
\[
\mathcal{A}(\eta^n, v^n_h)(t_n) = -\sum_{j=1}^{N} \hat{g}(\omega, \eta^n)_{j+\frac{1}{2}} \left[ v^n_h \right]_{j+\frac{1}{2}} 
\leq \sqrt{2} \alpha \| \eta^n \|_{r_h(t_n)} \left[ v^n_h \right]_{(t_n)} \tag{5.19}
\leq \frac{\alpha}{4} \left[ v^n_h \right]^2(t_n) + Ch^{2k+1}. \tag{5.20}
\]

Here we use the estimate (3.10) for the second step, Young’s inequality and the estimates (5.10) are used for the third step. Then recalling the definition (5.9) of \( H_j \) and taking \( v^n_h = \zeta^n \) in (5.15) and (5.20), we get
\[
\sum_{j=1}^{N} H_j(\eta, \zeta^n) \leq \tau \| \zeta^n \|^2(t_n) + \frac{\alpha}{4} \tau \left[ \zeta^n \right]^2(t_n) + C \tau(h^{2k+1} + \tau^2). \tag{5.21}
\]

On the other hand, choose \( v^n_h = \hat{\zeta}^{n+1} - \zeta^n \) in (5.19) to derive,
\[
\mathcal{A}(\eta^n, \hat{\zeta}^{n+1} - \zeta^n)(t_n) \leq \sqrt{2} \alpha \| \eta^n \|_{r_h(t_n)} \| \hat{\zeta}^{n+1} - \zeta^n \|(t_n)
\leq 2\alpha \mu h^{-\frac{1}{2}} \| \eta^n \|_{r_h(t_n)} \| \hat{\zeta}^{n+1} - \zeta^n \|(t_n)
\leq C\alpha \mu h^k \| \zeta^{n+1} - \hat{\zeta}^n \|(t_{n+1}),
\]
where the same reasons for obtaining (3.16) are used in the second step, and for the last step we use the estimates (5.10) and (2.20). It is inferred from taking \( v^n_h = \zeta^{n+1} - \hat{\zeta}^n \) in (5.15) and (2.20) that
\[
\sum_{j=1}^{N} H_j(\eta, \hat{\zeta}^{n+1} - \zeta^n) \leq C \left( \tau h^{k+1} + \alpha \mu \tau h^k + \tau^2 \right) \| \zeta^{n+1} - \hat{\zeta}^n \|(t_{n+1}).
\]
Moreover, by the boundedness (3.7) of \( \mathcal{A} \), we obtain,
\[
\tau \mathcal{A}(\hat{\zeta}^n, \zeta^{n+1} - \hat{\zeta}^n)(t_{n+1}) \leq \left( C_{wx} \tau \| \zeta^n \|(t_{n+1}) + \sqrt{2} \alpha \mu \tau h^{-\frac{1}{2}} \| \zeta^n \|(t_n) \right) \| \zeta^{n+1} - \hat{\zeta}^n \|(t_{n+1}).
\]
In light of the equality (5.16), divide both sides by \( \| \zeta^{n+1} - \hat{\zeta}^{n} \| (t_{n+1}) \) to yield,

\[
\| \zeta^{n+1} - \hat{\zeta}^{n} \| (t_{n+1}) \leq 2C_{wx} \tau \| \hat{\zeta}^{n} \| (t_{n+1}) + \sqrt{2 \alpha \mu \tau h^{-\frac{1}{2}}} \| \zeta^{n} \| (t_{n}) + C(\tau h^{k+1} + \alpha \mu \tau h^{k} + \tau^{2}).
\]

Under the CFL condition (5.13), the above estimate leads to

\[
\frac{1}{2} \| \zeta^{n+1} - \hat{\zeta}^{n} \|^{2} (t_{n+1}) \leq 2 \alpha^{2} \mu^{2} \tau^{2} h^{-1} \| \zeta^{n} \|^{2} (t_{n}) + C \tau^{2} \| \zeta^{n} \|^{2} (t_{n+1}) + C(\tau h^{k+1} + \alpha \mu \tau h^{k} + \tau^{2})^{2} \leq \frac{\alpha}{4} \tau \| \zeta^{n} \|^{2} (t_{n}) + C \tau \| \zeta^{n} \|^{2} (t_{n+1}) + C \tau (h^{2k+1} + \tau^{2}).
\]  

(5.22)

Here we also use (2.21). Consequently, the estimates (5.21)-(5.22) together with the energy identity (5.14) imply

\[
\| \zeta^{n+1} \|^{2} (t_{n+1}) - \| \zeta^{n} \|^{2} (t_{n}) \leq C \tau \| \zeta^{n} \|^{2} (t_{n}) + C \tau (h^{2k+1} + \tau^{2}).
\]

Thus by the same arguments as in the \( P^{k} \) case, we can obtain the desired results,

\[
\| e^{n} \|^{2} (t_{n}) \leq C (h^{2k+1} + \tau^{2}), \quad n \leq M.
\]

The proof is completed. \( \square \)

5.2 Second order scheme

In this subsection, we will present the error estimate for the fully discrete scheme (4.13) with TVD-RK2 time-marching method. Similar to the first order case, we need first obtain the error equation.

5.2.1 Error equation

To obtain the error equation, we introduce the reference functions, which are in parallel to the TVD-RK2 time discretization stages. Similar to the first order case, we consider on the reference cell. To be more specific, let \( \hat{U}^{(0)}(\xi, t) = \Delta_{j}(t) \hat{u}(\xi, t) \) be the exact solution of the equation (5.3) in the \( j \)-th cell, and

\[
\hat{U}^{(1)}(\xi, t) = \hat{U}^{(0)}(\xi, t) + \tau \hat{U}_{t}^{(0)}(\xi, t).
\]  

(5.23)

Then define

\[
\hat{u}^{(0)}(\xi, t) = \frac{1}{\Delta_{j}(t)} \hat{U}^{(0)}(\xi, t), \quad \hat{u}^{(1)}(\xi, t) = \frac{1}{\Delta_{j}(t + \tau)} \hat{U}^{(1)}(\xi, t).
\]

Denote \( u^{n,l} = u^{(l)}(x, t_{n}) = \hat{u}^{(l)}(\xi, t_{n}) \) for any time level \( n \) and \( l = 0, 1 \). Now we are ready to state the following lemma, which describes the local truncation error in time.
Lemma 11. Let \( u \) be the exact solution of Eq. (1.1). Suppose \( u \) is sufficiently smooth with bounded derivatives, then for any \( v_h^n \equiv v_h(x, t_n) \in V_h(t_n) \) and \( 1 \leq j \leq N \), there holds,

\[
(u^{n,1}, \tilde{v}_h^n)_{K_j} = (u^n, \tilde{v}_h^n)_{K_j} - \tau A(u^n, v_h^n)_{K_j}, \quad (5.24)
\]

\[
(u^{n+1,1}, \tilde{v}_h^n)_{K_j} + \frac{1}{2} (u^{n,1}, \tilde{v}_h^n)_{K_j} + \frac{1}{2} (u^{n+1}, \tilde{v}_h^n)_{K_j} + \frac{1}{2} \tau A(u^{n,1}, \tilde{v}_h^n)_{K_j} + (\tilde{v}_h^n, v_h^n)_{K_j}, \quad (5.25)
\]

where \( \tilde{v}_h^n \) is defined by (2.12), \( \varepsilon^n_2 \) is the local truncation error in time and \( \| \varepsilon^n_2 \|_{K_j} = O(\tau^3) \) for any \( j \) and \( n \).

Proof. By the Taylor expansion with Lagrange form of the remainder and the definitions of the reference functions (5.23), it is not difficult to derive,

\[
\begin{align*}
\bar{U}^{(1)}(\xi, t) &= \tilde{U}^{(0)}(\xi, t) - \tau (a\tilde{U}^{(0)})_{\xi}(\xi, t), \\
\bar{U}(\xi, t + \tau) &= \frac{1}{2} \bar{U}^{(0)}(\xi, t) + \frac{1}{2} \bar{U}^{(1)}(\xi, t) - \frac{\tau}{2} \left[ a(\xi, t + \tau) \tilde{U}^{(1)}(\xi, t) \right]_{\xi} + \varepsilon(\xi, t),
\end{align*}
\]

where

\[
\varepsilon(\xi, t) = \frac{\tau^3}{6} \tilde{U}_{ttt}(t_21) + \frac{\tau^3}{2} \left[ (a_t \tilde{U}_t)(t) + \frac{1}{2} a_{ttt}(t_21) \tilde{U}^{(1)}(t) \right]_{\xi}, \quad t_{21}, t_{22} \in (t, t + \tau).
\]

Recalling the definition of \( \tilde{U} \) and \( a \) in (5.3) as well as \( \Delta_j'(t) = \omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}} \), we have,

\[
\begin{align*}
\tilde{U}_t &= \Delta_j(t) \tilde{u}_t + \Delta_j'(t) \tilde{u}, \quad \tilde{U}_{ttt} = \Delta_j(t) \tilde{u}_{ttt} + 3 \Delta_j'(t) \tilde{u}_{tt}, \\
a_t &= -\frac{2 \Delta_j'}{\Delta_j^3(t)} (\beta - \tilde{\omega}), \quad a_{tt} = \frac{(2 \Delta_j')^2}{\Delta_j^3(t)} (\beta - \tilde{\omega}).
\end{align*}
\]

Let \( t = t_n \) and we still use the notations \( t_{21} \) and \( t_{22} \) to stand for fixed values between \( t_n \) and \( t_{n+1} \). The scaling arguments imply that

\[
\int_{-1}^1 \tilde{U}_{ttt}(t_{21}) \tilde{v}_h^n \, d\xi = \int_{-1}^1 \left[ \Delta_j \tilde{u}_{ttt} + 3 \Delta_j'(t) \tilde{u}_{tt} \right](t_{21}) \tilde{v}_h^n \, d\xi
\]

\[
= \frac{2 \Delta_j(t_{21})}{\Delta_j^n} (\tilde{u}_{ttt}, v_h^n)_{K_j^n} + \frac{6 \Delta_j'}{\Delta_j^n} (\tilde{u}_{tt}, v_h^n)_{K_j^n},
\]

where \( \tilde{u}_{ttt}(x_j^n, t_n) = u_{ttt}(x, t_{21}), \tilde{u}_{tt}(x_j^n, t_n) = u_{tt}(x, t_{21}) \) for \( x_j^n \in K_j^n, x \in K_j(t_{21}) \). Similarly,

\[
\int_{-1}^1 [a_t \tilde{U}_t](\xi) \tilde{v}_h^n \, d\xi = -\frac{2 \Delta_j'}{\Delta_j^n} \int_{-1}^1 [((\beta - \tilde{\omega}) \tilde{u}_t)_{\xi}](t_n) \tilde{v}_h^n \, d\xi - 2 \frac{\Delta_j'}{\Delta_j^n} \int_{-1}^1 [((\beta - \tilde{\omega}) \tilde{u})_{\xi}](t_n) \tilde{v}_h^n \, d\xi
\]

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\[ = -2\frac{\Delta_j'}{\Delta_j^n} \left( [\beta - \omega u]_x, v^n_h \right)_{K_j^n} - 2 \left( \frac{\Delta_j'}{\Delta_j^n} \right)^2 \left( [\beta - \omega u]_x, v^n_h \right)_{K_j^n}, \]

and

\[
\int_{-1}^1 [a_{tt}(t_{22}) \bar{U}^{(1)}(t_n)]_x \bar{v}^n_h d\xi = \frac{(2\Delta_j')^2}{\Delta_j^3(t_{22})} \int_{-1}^1 [(\beta - \bar{\omega})(\bar{U} + \tau \bar{U}_t)(t_n)]_x \bar{v}^n_h d\xi
\]

\[
= \frac{(2\Delta_j')^2 \Delta_j^n}{\Delta_j^3(t_{22})} \int_{-1}^1 [(\beta - \bar{\omega})(\bar{u} + \tau \bar{u}_t)]_x(t_n) \bar{v}^n_h d\xi
\]

\[
+ \frac{(2\Delta_j')^3}{\Delta_j^3(t_{22})} \tau \int_{-1}^1 [(\beta - \bar{\omega})\bar{u}]_x(t_n) \bar{v}^n_h d\xi
\]

\[
= \frac{(2\Delta_j')^2 \Delta_j^n}{\Delta_j^3(t_{22})} \left( [(\beta - \omega)u + \tau u_t]_x(t_n), v^n_h \right)_{K_j^n}
\]

\[
+ \frac{(2\Delta_j')^3}{\Delta_j^3(t_{22})} \tau \left( [(\beta - \omega)u]_x(t_n), v^n_h \right)_{K_j^n}.
\]

As a result, we know that

\[
\int_{-1}^1 \varepsilon(x, t_n) \bar{v}^n_h d\xi = 2(\varepsilon^n_2, v^n_h)_{K_j^n},
\]

where \( \varepsilon^n_2 \) can be obtained by the above analysis and

\[\|\varepsilon^n_2\|_{K_j^n} \leq C\tau^3.\]

Here we use the assumption that \( u \) is smooth enough, the quantity (2.4), the assumption (2.5)-(2.7) and the fact that \( \tau \leq 1 \). The positive constant \( C \) is independent of \( \tau, h \) and \( n \). Finally, by the same arguments as Lemma 8, we obtain the desired results (5.24)-(5.25). \( \square \)

As is customary in the error estimates, we introduce some notations. Denote the error at each stage by \( e^{n,0} = u^n - u^n_h \) and \( e^{n,1} = u^{n,1} - u^n_h \), where \( u^n_h \) and \( u^n_{1h} \) are the solutions of the fully discrete scheme (4.13). For simplicity, denote \( u^{n,0}_h = u_h \) and \( u^{n,1}_h = u^1_h \). Let

\[ \zeta^{n,l} = P_h u^{n,l} - u^n_h, \eta^{n,l} = P_h u^{n,l} - u^{n,l}_h, \quad l = 0, 1, \]

where \( P_h u^{n,l} \) is the \( L^2 \) projection defined by (2.23). Noting that \( e^{n,1} = \zeta^{n,1} - \eta^{n,1} \), we obtain the error equation for any \( v^n_h \in V_h(t_n) \) and \( 1 \leq j \leq N \) by Lemma 11 and the scheme (4.13),

\[
(\zeta^{n,1}, \bar{v}^n_h)_{K_j^{n+1}} = (\zeta^n, v^n_h)_{K_j^n} - \tau A(\zeta^n, v^n_h)_{K_j^n} + L^1_j(\eta, v^n_h), \quad (5.26)
\]

\[
(\zeta^{n+1}, \bar{v}^n_h)_{K_j^{n+1}} = \frac{1}{2}(\zeta^n, v^n_h)_{K_j^n} + \frac{1}{2}(\zeta^{n,1}, \bar{v}^n_h)_{K_j^{n+1}} - \frac{\tau}{2} A(\zeta^{n,1}, \bar{v}^n_h)_{K_j^{n+1}} + L^2_j(\eta, v^n_h), \quad (5.27)
\]

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with any three constants restricted by \( d \) and \( m \).

Suppose \( \text{important for our analysis.} \)

By the scaling arguments, we have

\[
(\zeta^n, v^n_h)_{K^n_j} = (1 - s_2)(\tilde{\zeta}^n, \tilde{v}^n_h)_{K^{n+1}_j}, \quad A(\zeta^n, v^n_h)_{K^n_j} = A(\tilde{\zeta}^n, \tilde{v}^n_h)_{K^{n+1}_j},
\]

which indicates by a direct calculation,

\[
\begin{align*}
(\zeta^{n+1} - \tilde{\zeta}^n, \tilde{v}^n_h)_{K^{n+1}_j} &= -s_2(\tilde{\zeta}^n, \tilde{v}^n_h)_{K^{n+1}_j} - \tau A(\tilde{\zeta}^n, \tilde{v}^n_h)_{K^{n+1}_j} + L^1_j(\eta, v^n_h),

(\zeta^{n+1} - \tilde{\zeta}^n, \tilde{v}^n_h)_{K^{n+1}_j} &= -\frac{\tau}{2} A(\zeta^{n+1} - \tilde{\zeta}^n, \tilde{v}^n_h)_{K^{n+1}_j} + L^2_j(\eta, v^n_h) - \frac{1}{2} L^1_j(\eta, v^n_h).
\end{align*}
\]

Next, we will list some estimates for the projection error. The analysis is the same as that in Lemma 9, thus we only present the results without the detailed proof.

**Lemma 12.** Suppose \( u \) is sufficiently smooth with bounded derivatives, then there exists a positive constant \( C \) independent of \( h, \tau \) and \( n \), such that for any \( v^n_h \in H^1_h(t_n) \) and \( \forall n \leq M \),

\[
\begin{align*}
\| \eta^{l,j} \|_{(t_{n+l})} + h^{1/2} \| \eta^{l,j} \|_{\Gamma_h(t_{n+l})} + h \| \partial_x \eta^{l,j} \|_{(t_{n+l})} &\leq C \tau h^{k+1}, \quad l = 0, 1,

d_1(\eta^{n+1}, \tilde{v}^n_h)(t_{n+1}) + d_2(\eta^{n+1}, \tilde{v}^n_h)(t_{n+1}) + d_3(\eta^n, v^n_h)(t_n) &\leq C \tau h^{k+1} \| v^n_h \|_{(t_n)},
\end{align*}
\]

with any three constants restricted by \( d_1 + d_2 + d_3 = 0 \).

Based on the above estimates, we can easily get the following estimates, which is important for our analysis.

**Lemma 13.** Suppose \( u \) is sufficiently smooth with bounded derivatives, then we have the following estimates, for \( m = 1, 2 \),

\[
\begin{align*}
\sum_{j=1}^N L^m_j(\eta, v^n_h) &\leq C(\tau h^{k+1} + \delta_{2m} \tau^3) \| v^n_h \|_{(t_n)} + C \alpha \tau h^{k+1} \| v^n_h \|_{(t_n)},

\sum_{j=1}^N L^m_j(\eta, v^n_h) &\leq C(\tau h^{k+1} + \delta_{2m} \tau^3 + \alpha \mu \tau h^k) \| v^n_h \|_{(t_n)},
\end{align*}
\]

where \( \delta_{2m} \) is the Kronecker symbol.

**Proof.** It is straightforward to obtain the desired results by a combination of the estimates in Lemma 3, Lemma 11 and Lemma 12. Here we also need use the properties (2.21). \( \square \)
5.2.2 Error estimate for the second order scheme

**Theorem 14.** Let \( u_h^n \) be the numerical solution of the fully discrete scheme (4.13) with TVD-RK2 time-marching, and \( u \) be the exact solution of Eq. (1.1). Suppose \( u \) is sufficiently smooth with bounded derivatives, then we have the following error estimate,

\[
\max_{n\tau \leq T} \| u(x, t_n) - u_h^n \| (t_n) \leq C(h^{k+\frac{1}{2}} + \tau^2),
\]

under the CFL condition

\[
\tau \leq \rho h^{\frac{4}{3}},
\]

with any given positive constant \( \rho \). In particular, for the piecewise linear finite element space, \( V_h(t) = \{ v(\chi_j(\cdot, t)) \in P^1([-1, 1]) \} \), we just need the usual CFL condition,

\[
\tau \leq \rho h.
\]

Here \( \rho \) is a suitable positive constant depends solely on \( \alpha \) and \( \mu \), where \( \alpha \) is defined by (3.3) and \( \mu \) is the inverse constant (2.26). The positive constant \( C \) is independent of \( h, \tau, n \) and \( u_h \).

**Proof.** To derive the energy identity for \( \zeta^n \), we take the test function \( \hat{v}_h^n = \frac{1}{2} \zeta^n, \zeta^{n,1} \) in the error equation (5.26) and (5.27), respectively, and add them together to yield,

\[
\frac{1}{2} \| \zeta^{n+1} \|^2 (t_{n+1}) - \frac{1}{2} \| \zeta^n \|^2 (t_n) = \frac{1}{2} \| \zeta^{n+1} - \zeta^{n,1} \|^2 (t_{n+1}) + \sum_{j=1}^N \frac{s_j}{4} \| \zeta^{n,1} - \zeta^n \|^2 \kappa_{j+1}^{n+1} + \frac{\alpha}{4} \tau \| \zeta^n \|^2 (t_n) - \frac{\alpha}{4} \tau \| \zeta^{n,1} \|^2 (t_{n+1}) + \sum_{j=1}^N \mathcal{L}_j^4(\eta, \zeta^n, \zeta^{n,1}),
\]

where

\[
\mathcal{L}_j^4(\eta, \zeta^n, \zeta^{n,1}) = \frac{1}{2} \mathcal{L}_j^1(\eta, \zeta^n) + \mathcal{L}_j^2(\eta, \zeta^{n,1}).
\]

The following proof is decomposed into four steps.

**Step 1.** Bound on \( \sum_{j=1}^N \mathcal{L}_j^1 \). By the estimate (5.34) in Lemma 13, we get,

\[
\sum_{j=1}^N \mathcal{L}_j^1(\eta, \zeta^n, \zeta^{n,1}) \leq C\tau h^{k+1}(\| \zeta^n \| + \| \zeta^{n,1} \|) (t_n) + C\tau^3 \| \zeta^{n,1} \| (t_n)
\]

\[
+ C\alpha \tau h^{k+\frac{1}{2}}(\| \zeta^n \| + \| \zeta^{n,1} \|) (t_n)
\]

\[
\leq \tau \| \zeta^n \|^2 (t_n) + \tau \| \zeta^{n,1} \|^2 (t_{n+1}) + \frac{\alpha}{8} \tau \| \zeta^n \|^2 (t_n)
\]

\[
+ \frac{\alpha}{8} \tau \| \zeta^{n,1} \|^2 (t_{n+1}) + C\tau (h^{2k+1} + \tau^4),
\]

(5.39)
here we use the Young’s inequality and (2.20).

**Step 2.** Bound on $\|\zeta^{n,1} - \hat{\zeta}^n\|(t_{n+1})$. Take the test function $v_h^n = \zeta^{n,1} - \hat{\zeta}^n$ in the equality (5.30) and sum over all $j$ to yield,

$$
\|\zeta^{n,1} - \hat{\zeta}^n\|^2(t_{n+1}) \leq (C_w \tau + 3\alpha \mu \tau h^{-1})\|\hat{\zeta}^n\|(t_{n+1})\|\zeta^{n,1} - \hat{\zeta}^n\|(t_{n+1}) + C(\tau h^{k+1} + \alpha \mu \tau h^k)\|\zeta^{n,1} - \hat{\zeta}^n\|(t_n).
$$

Here we use the Cauchy-Schwarz inequality, the boundedness (3.6) of $\mathcal{A}$ and the estimate (5.35). We also use the fact that $s_2 = \omega_x(t_{n+1}) \tau$ and the assumption (2.7). By the property (2.20) and dividing both sides by $\|\zeta^{n,1} - \hat{\zeta}^n\|(t_{n+1})$, we have,

$$
\|\zeta^{n,1} - \hat{\zeta}^n\|(t_{n+1}) \leq (C_w \tau + 3\alpha \mu \tau h^{-1})\|\hat{\zeta}^n\|(t_{n+1}) + C(\tau h^{k+1} + \alpha \mu \tau h^k). \quad (5.40)
$$

**Step 3.** Bound on $\|\zeta^{n,1}\|(t_{n+1})$. Taking the test function $v_h^n = \zeta^{n,1}$ in the error equation (5.26) and following the same line as that in Step 2, we can easily get the boundedness of $\|\zeta^{n,1}\|(t_{n+1})$,

$$
\|\zeta^{n,1}\|(t_{n+1}) \leq C(1 + 3\alpha \mu \tau h^{-1})\|\zeta^n\|(t_n) + C(\tau h^{k+1} + \alpha \mu \tau h^k). \quad (5.41)
$$

**Step 4.** Bound on $\|\zeta^{n,1} - \zeta^{n,1}\|(t_{n+1})$. This step will be divided into two cases, the general $P^k$ case and the $P^1$ case, which is the same as in the stability analysis. Denote $\lambda = \alpha \mu \tau h^{-1}$ for simplicity.

**$P^k$ case.** Take the test function $v_h^n = \zeta^{n,1} - \zeta^{n,1}$ in the equality (5.31) and apply the estimate (3.6) of $\mathcal{A}$ as well as (5.35) to derive,

$$
\|\zeta^{n,1} - \zeta^{n,1}\|(t_{n+1}) \leq \frac{3}{2}\alpha \mu \tau h^{-1}\|\zeta^{n,1} - \hat{\zeta}^n\|(t_{n+1}) + C(\tau h^{k+1} + \alpha \mu \tau h^k + \tau^3)
\leq \frac{3}{2}\lambda(C_w \tau + 3\lambda)\|\hat{\zeta}^n\|(t_{n+1}) + C(1 + \frac{3}{2}\lambda)\tau h^{k+1} + \alpha \mu \tau h^k + \tau^3).
$$

Here we use the estimate (5.40). If the time-step satisfies $\lambda^4 \leq \rho \tau$ for any positive constant $\rho$, the above inequality indicates that

$$
\frac{1}{2}\|\zeta^{n,1} - \zeta^{n,1}\|^2(t_{n+1}) \leq C\tau\|\zeta^{n,1}\|^2(t_{n+1}) + C(\tau h^{2k+1} + \tau^5) \leq C\tau\|\zeta^n\|^2(t_n) + C(\tau h^{2k+1} + \tau^5), \quad (5.42)
$$

where we use (2.21) and $\tau \leq 1$. In addition, the estimates (5.40)-(5.41) turn out to be

$$
\|\zeta^{n,1} - \hat{\zeta}^n\|(t_{n+1}) \leq C\|\zeta^n\|(t_{n+1}) + Ch^{k+1} \leq C\|\zeta^n\|(t_n) + Ch^{k+1},
\|\zeta^{n,1}\|(t_{n+1}) \leq C\|\zeta^n\|(t_n) + Ch^{k+1},
$$

under the CFL condition (5.36). Then combine the energy identity (5.38), the estimates (5.39) and (5.42) to get,

$$
\frac{1}{2}\|\zeta^{n+1}\|^2(t_{n+1}) - \frac{1}{2}\|\zeta^n\|^2(t_n) \leq C\tau\|\zeta^n\|^2(t_n) + \tau\|\zeta^{n,1}\|^2(t_{n+1}) + C\tau(h^{2k+1} + \tau^4)
$$
\[ + \frac{C_{wx}}{4} \tau \| \zeta^{n,1} - \hat{\zeta}^{n} \|^2(t_{n+1}) \]
\[ \leq C \tau \| \zeta^{n} \|^2(t_{n}) + C \tau (h^{2k+1} + \tau^4). \]

Summing over \( n \), using Gronwall's inequality and choosing the initial condition \( u_h(x, 0) = P_h u(x, 0) \), we obtain,
\[ \| \zeta^n \|^2(t_{n}) \leq C(h^{2k+1} + \tau^4), \quad n \leq M. \]

Finally, apply the estimate (5.32) to yield,
\[ \| \epsilon^n \|^2(t_{n}) \leq C(h^{2k+1} + \tau^4), \quad n \leq M. \]

**P1 case.** Denote \( z = \zeta^{n,1} - \hat{\zeta}^n \). Choosing the test function \( \hat{v}_h^n = \zeta^{n+1} - \zeta^{n,1} \) in the equality (5.31) and using the estimate (3.7) of \( A \) as well as (5.35), we have,
\[ \| \zeta^{n+1} - \zeta^{n,1} \|(t_{n+1}) \leq \left( \frac{\alpha}{2} \tau \| z_x \|(t_{n+1}) + \frac{C_{wx}}{2} \tau \| z \|(t_{n+1}) + \frac{\sqrt{2}}{2} \alpha \mu \tau h^{-\frac{1}{2}} \| z \|(t_{n+1}) + C(\tau h^{k+1} + \alpha \mu \tau h^k + \tau^3). \right) \]

(5.43)

Now we analyze \( \| z_x \|(t_{n+1}) \). Similar to the stability analysis, let \( y = z - P_h^0 z \) and \( P_h^0 z \) is the \( L^2 \) projection into the piecewise constant space. By the property of the \( L^2 \) projection and taking \( \hat{v}_h^n = y \) in the equality (5.30), we get,
\[ \| y \|^2(t_{n+1}) = \sum_{j=1}^{N} (z, y)_{K_j} \leq \left( (\mu + 2)C_{wx} \tau \| \hat{\zeta}^{n} \|(t_{n+1}) + \sqrt{2} \alpha \mu \tau h^{-\frac{1}{2}} \| \zeta^{n} \|(t_{n}) \right) \| y \|(t_{n+1}) \]
\[ + C(\tau h^{k+1} + \alpha \mu \tau h^k) \| y \|(t_{n+1}). \]

(5.44)

Here we use the boundedness (3.8) of \( A \) and the estimate (5.35). Divide both sides of the above inequality by \( \| y \|(t_{n+1}) \) to obtain the estimate of \( \| y \|(t_{n+1}) \). In addition,
\[ \| z_x \|(t_{n+1}) = \| y_x \|(t_{n+1}) \leq \mu h^{-1} \| y \|(t_{n+1}). \]

(5.45)

Let \( \tau \leq \rho h \) with a positive constant \( \rho \) independent of \( \tau \) and \( h \), we will show the restriction of \( \rho \) in the following. Collect the estimates (5.43)-(5.45) to yield,
\[ \| \zeta^{n+1} - \zeta^{n,1} \|(t_{n+1}) \leq (1 + \frac{\sqrt{2}}{2} \alpha \mu \rho) \alpha \mu \tau h^{-\frac{1}{2}} \| \zeta^n \|(t_{n}) + \alpha \mu \tau h^{-\frac{1}{2}} \| \zeta^{n,1} \|(t_{n+1}) + \mathbb{L}, \]

where
\[ \mathbb{L} = C \rho \tau \| \hat{\zeta}^{n} \|(t_{n+1}) + \frac{C_{wx}}{2} \tau \| z \|(t_{n+1}) + C(1 + \frac{\alpha \mu}{2} \rho)(\tau h^{k+1} + \alpha \mu \tau h^k + \tau^3). \]

Hence,
\[ \frac{1}{2} \| \zeta^{n+1} - \zeta^{n,1} \|^2(t_{n+1}) \leq (1 + \frac{\sqrt{2}}{2} \alpha \mu \rho)^2 (\alpha \mu)^2 \tau^2 h^{-1} \| \zeta^n \|^2(t_{n}) \]
If
\[ 2(\alpha \mu)^2 \tau^2 h^{-1} \leq \frac{\alpha}{16} \tau, \quad (\alpha \mu)^4 \rho^2 \tau^2 h^{-1} \leq \frac{\alpha}{16} \tau, \quad 2(\alpha \mu)^2 \tau^2 h^{-1} \leq \frac{\alpha}{8}, \]
that is,
\[ \rho \leq \min\left\{ \frac{1}{32 \alpha \mu^2}, \frac{1}{\alpha^2 \sqrt{16}} \right\}, \]
then we have
\[ \frac{1}{2} \| \zeta^{n+1} - \zeta^{n,1} \|^2(\tau_{n+1}) \leq \frac{\alpha}{8} \tau \| \zeta^n \|^2(t_n) + \frac{\alpha}{8} \tau \| \zeta^{n,1} \|^2(t_{n+1}) + 2L^2. \] (5.46)

With the CFL condition (5.37) and the property (2.21), the estimates (5.40)-(5.41) turn out to be,
\[ \| \zeta \| (t_{n+1}) \leq C \| \zeta^n \| (t_n) + Ch^{k+1}, \] (5.47)
\[ \| \zeta^{n,1} \| (t_{n+1}) \leq C \| \zeta^n \| (t_n) + Ch^{k+1}, \] (5.48)
which implies that,
\[ L^2 \leq C \tau \| \zeta^n \|^2(t_n) + C \tau h^{2k+1} + \tau^5. \] (5.49)

In light of the energy identity (5.38), we combine the estimates (5.39) and (5.46)-(5.49) to obtain,
\[ \frac{1}{2} \| \zeta^{n+1} \|^2(\tau_{n+1}) - \frac{1}{2} \| \zeta^n \|^2(\tau_n) \leq C \tau \| \zeta^n \|^2(t_n) + C \tau \| \zeta^{n,1} \|^2(\tau_{n+1}) + \frac{C_w}{4} \tau \| \zeta \|^2(\tau_{n+1}) + C \tau h^{2k+1} + \tau^5 \leq C \tau \| \zeta^n \|^2(t_n) + C \tau h^{2k+1} + \tau^5. \]

In the end, by the same arguments as the general $P^k$ case, we can obtain the desired results,
\[ \| e^n \|^2(t_n) \leq C(h^{2k+1} + \tau^4), \quad n \leq M. \]
The proof is completed.

5.3 Third order scheme

In this subsection, we will present the error estimate for the fully discrete scheme (4.31) with TVD-RK3 time-marching method. We begin with the error equation.
5.3.1 Error equation

Similar to second order case, the reference functions will be introduced to obtain the error equation. Considering the equation (5.3), define \( \tilde{U}^{(0)}(\xi, t) = \Delta_j(t) \tilde{u}(\xi, t) \) as the exact solution, and

\[
\begin{align*}
\tilde{U}^{(1)}(\xi, t) &= \tilde{U}^{(0)}(\xi, t) - \tau (a \tilde{U}^{(0)})_{\xi}(\xi, t), \\
\tilde{U}^{(2)}(\xi, t) &= \frac{3}{4} \tilde{U}^{(0)}(\xi, t) + \frac{1}{4} \tilde{U}^{(1)}(\xi, t) - \frac{\tau}{4} \left( a(\xi, t + \tau) \tilde{U}^{(1)}(\xi, t) \right)_{\xi}.
\end{align*}
\]

Then let

\[
\begin{align*}
\hat{u}^{(0)} &= \frac{1}{\Delta_j(t)} \tilde{U}^{(0)}, \quad \hat{u}^{(1)} = \frac{1}{\Delta_j(t + \tau)} \tilde{U}^{(1)}, \quad \hat{u}^{(2)} = \frac{1}{\Delta_j(t + \frac{\tau}{2})} \tilde{U}^{(2)},
\end{align*}
\]

here we omit the same symbol \((\xi, t)\) on both sides of the above equalities. By the standard explicit TVD-RK3 time marching for the equation (5.3) and the same idea as that in Lemma 8 and Lemma 11, we can easily obtain the following lemma, which describes the local truncation error in time. Before doing that, denote \( u^{n,l} = u^{(l)}(x, t_n) = \hat{u}^{(l)}(\xi, t_n) \) for any time level \( n \) and \( l = 0, 1, 2 \).

**Lemma 15.** Let \( u \) be the exact solution of Eq. (1.1). Suppose \( u \) is sufficiently smooth with bounded derivatives, then for any \( v^0_h \equiv v_h(x, t_n) \in V_h(t_n) \) and \( 1 \leq j \leq N \), there holds,

\[
\begin{align*}
(u^{n+1}, \hat{v}^n_h &\rangle_{K_j} = (u^n, v^n_h)_{K_j}^{\varepsilon_3} - \tau A(u^n, v^n_h)_{K_j}, \\
(u^{n+1}, \overline{v}^n_h &\rangle_{K_j} = \frac{3}{4} (u^n, v^n_h)_{K_j}^{\varepsilon_3} + \frac{1}{4} (u^{n+1}, \hat{v}^n_h)_{K_j} - \frac{\tau}{4} A(u^{n+1}, \hat{v}^n_h)_{K_j}^{\varepsilon_3}, \\
(u^{n+1}, \hat{v}^n_h &\rangle_{K_j} = \frac{1}{3} (u^n, v^n_h)_{K_j}^{\varepsilon_3} + \frac{2}{3} (u^{n+1}, \overline{v}^n_h)_{K_j}^{\varepsilon_3} - \frac{2\tau}{3} A(u^{n+1}, \overline{v}^n_h)_{K_j}^{\varepsilon_3} + \varepsilon_3^n \rangle_{K_j},
\end{align*}
\]

where \( \hat{v}_h^n \) and \( \overline{v}_h^n \) are defined by (2.12), \( \varepsilon_3^n \) is the local truncation error in time and \( \| \varepsilon_3^n \|_{K_j} = O(\tau^4) \) for any \( j \) and \( n \).

Similar to the second order case, we denote the error at each stage by \( e^{n,l} = u^{n,l} - u^{n,l}_h \) for any \( n \) and \( l = 0, 1, 2 \), where \( u^{n,l}_h \equiv u^{(l)}_h \) with \( l = 1, 2 \) are the solutions of the fully discrete scheme (4.31) and \( u^{n,0}_h = u^{n}_h \). In addition, the error can be rewritten as \( e^{n,l} = \zeta^{n,l} - \eta^{n,l} \) with

\[
\begin{align*}
\zeta^{n,l} &= P_h u^{n,l} - u^{n,l}_h, \quad \eta^{n,l} = P_h u^{n,l} - u^{n,l}, \quad l = 0, 1, 2.
\end{align*}
\]

Here \( P_h u^{n,l} \) is the \( L^2 \) projection of \( u^{n,l} \) defined by (2.23). We can obtain the error equation
Recalling the fact that \((\zeta^n, v^n_h)_{K_j} = (\zeta^n, v^n_h)_{K_j} - \tau A(\zeta^n, v^n_h)_{K_j} + T_j^1(v^n_h),\)
\[(\zeta^{n+1}, v^n_h)_{K_j} = \frac{3}{4} (\zeta^n, v^n_h)_{K_j} + \frac{1}{4} (\zeta^{n+1}, \hat{v}^n_h)_{K_j} + \frac{\tau}{4} A(\zeta^{n+1}, \hat{v}^n_h)_{K_j} + T_j^2(v^n_h),\]
\[(\zeta^{n+1}, \hat{v}^n_h)_{K_j} = \frac{1}{3} (\zeta^n, v^n_h)_{K_j} + \frac{2}{3} (\zeta^{n+1}, \hat{v}^n_h)_{K_j} + \frac{2\tau}{3} A(\zeta^{n+1}, \hat{v}^n_h)_{K_j} + T_j^3(v^n_h),\]
where
\[T_j^1(v^n_h) = (\eta^{n+1}, \hat{v}^n_h)_{K_j} - (\eta^n, v^n_h)_{K_j} + \tau A(\eta^n, v^n_h)_{K_j},\]
\[T_j^2(v^n_h) = (\eta^n, v^n_h)_{K_j} + \left(\eta^{n+1}, \hat{v}^n_h\right)_{K_j} + \frac{2\tau}{3} A(\eta^{n+1}, \hat{v}^n_h)_{K_j} + (\zeta^n, v^n_h)_{K_j},\]
\[T_j^3(v^n_h) = (\eta^n, v^n_h)_{K_j} - \left(\eta^{n+1}, \hat{v}^n_h\right)_{K_j} + \frac{2\tau}{3} A(\eta^{n+1}, \hat{v}^n_h)_{K_j} + (\zeta^n, v^n_h)_{K_j}.\]

Similar to the stability analysis, we introduce some notations for simplicity,
\[D_1 = \zeta^{n+1} - \hat{\zeta}^n, \quad D_2 = 2\hat{\zeta}^{n+1} - \zeta^n - \hat{\zeta}^n, \quad D_3 = \zeta^{n+1} - 2\hat{\zeta}^{n+1} + \zeta^n.\]
Recalling the fact that
\[(\zeta^n, v^n_h)_{K_j} = (1 - s_2)(\hat{\zeta}^n, \hat{v}^n_h)_{K_j} + (\zeta^{n+1}, \hat{v}^n_h)_{K_j} = (1 - \frac{s_2}{2})(\hat{\zeta}^{n+1}, \hat{v}^n_h)_{K_j},\]
we follow the same lines as that in obtaining (4.34) to derive,
\[\begin{align*}
(D_1, \hat{v}^n_h)_{K_j} &= -s_2(\zeta^n, v^n_h)_{K_j} - \tau A(\zeta^n, v^n_h)_{K_j} + T_j^1(v^n_h), \\
(D_2, \hat{v}^n_h)_{K_j} &= -s_2(\zeta^n - \zeta^{n+1}, v^n_h)_{K_j} - \frac{\tau}{2} A(D_1, \hat{v}^n_h)_{K_j} + T_j^4(v^n_h), \\
(D_3, v^n_h)_{K_j} &= -s_2(\zeta^{n+1} - \zeta^n, v^n_h)_{K_j} - \frac{\tau}{3} A(D_2, \hat{v}^n_h)_{K_j} + T_j^5(v^n_h),
\end{align*}\]
where
\[\begin{align*}
T_j^4(v^n_h) &= 2(\eta^{n+1}, \hat{v}^n_h)_{K_j} + (\eta^n, v^n_h)_{K_j} - (\eta^{n+1}, \hat{v}^n_h)_{K_j} - \frac{\tau}{2} A(\eta^{n+1} - \eta^n, v^n_h)_{K_j}, \\
T_j^5(v^n_h) &= (\eta^{n+1}, \hat{v}^n_h)_{K_j} - 2(\eta^n, v^n_h)_{K_j} + \frac{\tau}{3} A(2\eta^{n+1} - \eta^n, \hat{v}^n_h)_{K_j} + (\zeta^n, v^n_h)_{K_j}.
\end{align*}\]
Similar to the first and second order case, some estimates for the projection error will be shown. The proof follows the same lines as that in proving Lemma 9, therefore we just list the results without the detailed proof.
Lemma 16. Suppose $u$ is sufficiently smooth with bounded derivatives, then there exists a positive constant $C$ independent of $h$, $\tau$ and $n$, such that for $\forall n \leq M$,

\[
\|\eta^{n,l}\|_{(t_{n+1})} + h^{1/2}\|\eta^{n,l}\|_{\Gamma_h(t_{n+1})} + h\|\partial_\tau \eta^{n,l}\|_{(t_{n+1})} \leq C h^{k+1}, \ l = 0, 1, \tag{5.54}
\]

\[
\|\eta^{n,2}\|_{(t_{n+3/2})} + h^{1/2}\|\eta^{n,2}\|_{\Gamma_h(t_{n+3/2})} + h\|\partial_\tau \eta^{n,2}\|_{(t_{n+3/2})} \leq C h^{k+1}. \tag{5.55}
\]

Moreover, for any $v^m_h \in H^1_h(t_n)$,

\[
(d_1\eta^{n+1} + d_2\eta^{n,1}, \tilde{v}^m_h)(t_{n+1}) + d_3(\eta^{n,2}, \bar{v}^m_h)(t_{n+3/2}) + d_4(\eta^n, v^m_h)(t_n) \leq C \tau h^{k+1}\|v^m_h\|_{(t_n)},
\]

with any four constants restricted by $d_1 + d_2 + d_3 + d_4 = 0$.

Denote $T^m(v^n_h) = \sum_{j=1}^N T^m_j(v^n_h)$ for $m = 1, \ldots, 5$. It is straightforward to derive the following results by a combination of the estimates in Lemmas 3, 15 and 16 as well as in (2.20)-(2.22).

Lemma 17. Suppose $u$ is sufficiently smooth with bounded derivatives, then we have the following estimates, for $m = 1, \ldots, 5$,

\[
T^m(v^n_h) \leq C(\tau h^{k+1} + \delta_{3m}\tau^4 + \delta_{5m}\tau^4)\|\tilde{v}^m_h\|_{(t_{n+1})} + C\tau h^{k+1}\|v^m_h\|_{(t_n)}, \tag{5.56}
\]

\[
T^m(v^n_h) \leq C(\tau h^{k+1} + \delta_{3m}\tau^4 + \delta_{5m}\tau^4 + \alpha\mu\tau h^k)\|\tilde{v}^m_h\|_{(t_{n+1})}, \tag{5.57}
\]

where $\delta_{3m}$ and $\delta_{5m}$ are the Kronecker symbols.

In particular, we also get the following results of the estimates for $T^4$ and $T^5$.

Lemma 18. Suppose $u$ is sufficiently smooth with bounded derivatives, then we have the following estimates, for $m = 4, 5$,

\[
T^m(v^n_h) \leq C(\tau h^{k+1} + \delta_{5m}\tau^4 + \alpha\mu\tau^2 h^k)\|\tilde{v}^m_h\|_{(t_{n+1})}. \tag{5.58}
\]

Proof. By the scaling arguments and the quantities (2.15)-(2.17), we have,

\[
(\eta^n, v^n_h)_{K_j} = (1 - s_2)(\eta^n, \tilde{v}^n_h)_{K_j}, \ \ (\eta^{n,2}, \bar{v}^n_h)_{K_j} = (1 - \frac{s_2}{2})(\eta^{n,2}, \tilde{v}^n_h)_{K_j},
\]

which implies

\[
(\eta^{n,1} - \eta^n, \tilde{v}^n_h)_{K_j} = (\eta^{n,1}, \tilde{v}^n_h)_{K_j} - (\eta^n, \tilde{v}^n_h)_{K_j} = -s_2(\eta^{n,2}, \tilde{v}^n_h)_{K_j},
\]

\[
(2\eta^{n,2} - \eta^{n,1} - \eta^n, \tilde{v}^n_h)_{K_j} = 2(\eta^{n,2}, \tilde{v}^n_h)_{K_j} - (\eta^{n,1}, \tilde{v}^n_h)_{K_j} - (\eta^n, \tilde{v}^n_h)_{K_j} + s_2(\eta^{n,2}, \tilde{v}^n_h)_{K_j}.
\]

It follows from the estimates in Lemma 16 and $s_2 = \omega_{\tau}(t_{n+1})\tau$ that

\[
(\eta^{n,1} - \eta^n, \tilde{v}^n_h)(t_{n+1}) \leq C \tau h^{k+1}\|v^m_h\|_{(t_n)}, \tag{5.59}
\]

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Here we use the relationship (2.20)-(2.22) and \( \tau \leq 1 \). Take \( \hat{v}_h^n = \eta^{n_1} - \hat{\eta}^n \) in (5.59) and divide both sides by \( \|\eta^{n_1} - \hat{\eta}^n\|(t_{n+1}) \) to obtain,

\[
(2\eta^{n_2} - \eta^{n_1} - \hat{\eta}^n, \hat{v}_h^n) \leq C\tau h^{k+1}\|\hat{v}_h^n\|(t_{n+1}).
\]

By the inverse equality (2.26), we derive,

\[
\|\eta^{n_1} - \hat{\eta}^n\|_{\Gamma_h(t_{n+1})} \leq C\tau h^{k+\frac{1}{2}}.
\]

Similarly, we have

\[
\|2\eta^{n_2} - \eta^{n_1} - \hat{\eta}^n\|(t_{n+1}) \leq C\tau h^{k+1}, \quad \|2\eta^{n_2} - \eta^{n_1} - \hat{\eta}^n\|_{\Gamma_h(t_{n+1})} \leq C\tau h^{k+\frac{1}{2}}.
\]

Thus the boundedness (3.16) of \( A \) yields,

\[
A(\eta^{n_1} - \hat{\eta}^n, \hat{v}_h^n)(t_{n+1}) \leq \left( \mu C_{wx}\|\eta^{n_1} - \hat{\eta}^n\|(t_{n+1}) + 2\alpha \mu h^{-\frac{1}{2}}\|\eta^{n_1} - \hat{\eta}^n\|_{\Gamma_h(t_{n+1})} \right)\|\hat{v}_h^n\|(t_{n+1})
\]

\[
\leq \tau \left( Ch^{k+1} + 2\alpha \mu h^k \right)\|\hat{v}_h^n\|(t_{n+1}).
\]

Together with Lemma 16, we can easily get,

\[
\mathcal{T}^4(\hat{v}_h^n) \leq C(\tau h^{k+1} + \tau^2 h^{k+1} + \alpha \mu \tau^2 h^k)\|\hat{v}_h^n\|(t_{n+1})
\]

\[
\leq C(\tau h^{k+1} + \alpha \mu \tau^2 h^k)\|\hat{v}_h^n\|(t_{n+1}),
\]

Since \( \tau \leq 1 \). We follow the same lines and use Lemma 15 to obtain,

\[
\mathcal{T}^5(\hat{v}_h^n) \leq C(\tau h^{k+1} + \tau^4 + \alpha \mu \tau^2 h^k)\|\hat{v}_h^n\|(t_{n+1}).
\]

The proof is completed. \( \square \)

5.3.2 Error estimate for the third order scheme

**Theorem 19.** Let \( u_h^n \) be the numerical solution of the fully discrete scheme (4.31) with TVD-RK3 time-marching, and \( u \) be the exact solution of Eq.(1.1). Suppose \( u \) is sufficiently smooth with bounded derivatives, then we have the following error estimate,

\[
\max_{nT \leq t} \|u(x, t_n) - u_h^n\|(t_n) \leq C(h^{k+\frac{1}{2}} + \tau^3),
\]

under the CFL condition \( \tau h^{-1} \leq \rho \) with a fixed constant \( \rho > 0 \). Here the positive constant \( C \) is independent of \( h, \tau, n \) and \( u_h \).
Proof. Similar to the stability analysis, we take the test function $v^n_h = \zeta^n$, $4\zeta^{n,1}$ and $6\zeta^{n,2}$ in the error equation (5.50), respectively, and add them together to obtain the identity for $\zeta^n$,

$$3\|\zeta^{n+1}\|^2(t_{n+1}) - 3\|\hat{\zeta}^n\|^2(t_{n+1}) = \|D_2\|^2(t_{n+1}) + 3(\zeta^{n+1} - \hat{\zeta}^n, D_3)(t_{n+1}) + \sum_{j=1}^N s_2 G_1$$

$$- \sum_{j=1}^N s_2 \|\zeta^n\|^2_{K^j_{n+1}} - 2 \sum_{j=1}^N s_2 \|\hat{\zeta}^{n,2}\|^2_{K^j_{n+1}} - \tau \sum_{j=1}^N s_2 G_2 + \mathcal{T}^6,$$  

(5.60)

where

$$G_1 = 2(\hat{\zeta}^{n,2}, \zeta^{n,1} - \hat{\zeta}^n)_{K^j_{n+1}} - 3(\hat{\zeta}^n, \zeta^{n,1})_{K^j_{n+1}},$$

$$G_2 = A(\hat{\zeta}^n, \zeta^n)_{K^j_{n+1}} + A(\zeta^{n,1}, \zeta^{n,1})_{K^j_{n+1}} + 4A(\hat{\zeta}^{n,2}, \hat{\zeta}^{n,2})_{K^j_{n+1}},$$

$$\mathcal{T}^6 = T^1(\zeta^n) + 4T^2(\hat{\zeta}^n) + 6T^3(\zeta^{n,2}).$$  

(5.61)

Denote each line on the right hand of (5.60) by $\Phi_1$ and $\Phi_2$, respectively. In light of the definitions (5.51), the equalities (5.52) and the properties (3.17)-(3.18) of $A$, we follow the same lines as that in the stability analysis to derive,

$$\Phi_1 = \|D_2\|^2 + 3(D_3, D_1 + D_2 + D_3) + \sum_{j=1}^N s_2 G_1$$

$$= - \|D_2\|^2 + 2(D_2, D_2) + 3(D_3, D_1) + 3(D_3, D_2) + 3(D_3, D_3) + \sum_{j=1}^N s_2 G_1$$

$$= - \|D_2\|^2 + 3\|D_3\|^2 - \tau A(D_1, D_2) - \tau A(D_2, D_1) - \tau A(D_2, D_2)$$

$$\quad + \sum_{j=1}^N s_2 (\zeta^n - \hat{\zeta}^2, D_2 + 3D_1)_{K^j_{n+1}} + \sum_{j=1}^N s_2 G_1 + \mathcal{T}^7$$

$$= - \|D_2\|^2 + 3\|D_3\|^2 - \tau \alpha \sum_{j=1}^N \|D_1\|_{j+\frac{1}{2}} \|D_2\|_{j+\frac{1}{2}} - \frac{\alpha}{2} \tau \|D_2\|^2 + \sum_{j=1}^N s_2 \Phi_{11} + \mathcal{T}^7,$$

where we have dropped the symbol $t_{n+1}$ for simplicity, and

$$\Phi_{11} = (\hat{\zeta}^n - \hat{\zeta}^{n,2}, D_2 + 3D_1)_{K^j_{n+1}} + G_1 + (D_1, D_2)_{K^j_{n+1}} + \frac{1}{2} \|D_2\|^2_{K^j_{n+1}},$$

$$\mathcal{T}^7 = 2T^4(\hat{D}_2) + 3T^5(\hat{D}_1) + 3T^5(\hat{D}_2).$$  

(5.62)

A direct calculation of $\Phi_{11}$ gives that,

$$\Phi_{11} = -\frac{5}{2} \|\zeta^n\|^2_{K^j_{n+1}} - \frac{1}{2} \|\zeta^{n,1}\|^2_{K^j_{n+1}}.$$  

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Moreover, by the property (3.18) of $A$, we get,
\[
\Phi_2 = -\frac{\alpha}{2} \tau [\zeta^n]^2(t_n) - \frac{\alpha}{2} \tau [\zeta^{n,1}]^2(t_{n+1}) - 2\alpha \tau [\zeta^{n,2}]^2(t_{n+\frac{1}{2}}) - \sum_{j=1}^N \frac{s_j^2}{2} \|\hat{\zeta}^n\|^2_{K^{n+1}_j} + \sum_{j=1}^N \frac{s_j^2}{2} \|\zeta^{n,1}\|^2_{K^{n+1}_j} + T^6.
\]
Plug $\Phi_1$ and $\Phi_2$ in the identity (5.60) to obtain the energy equality for $\zeta^n$,
\[
3\|\zeta^{n+1}\|^2(t_{n+1}) - 3\|\zeta^n\|^2(t_n) = \Lambda_1 + \Lambda_2 + \Lambda_3,
\]
where
\[
\Lambda_1 = -\|D_2\|^2(t_{n+1}) + 3\|D_3\|^2(t_{n+1}) - \tau \alpha \sum_{j=1}^N [D_1]_{j+\frac{1}{2}} [D_2]_{j+\frac{1}{2}} - \frac{\alpha}{2} \tau [D_2]^2,
\]
\[
\Lambda_2 = -\frac{\alpha}{2} \tau [\zeta^n]^2(t_n) - \frac{\alpha}{2} \tau [\zeta^{n,1}]^2(t_{n+1}) - 2\alpha \tau [\zeta^{n,2}]^2(t_{n+\frac{1}{2}}),
\]
\[
\Lambda_3 = T^6 + T^7.
\]
Here $T^6$ and $T^7$ are defined by (5.61) and (5.62), respectively. The following proof is decomposed into five steps.

**Step 1.** Bound on $\|D_3\|(t_{n+1})$. Take the test function $\hat{v}_h^n = D_3$ in the third equality of (5.52), sum all over all $j$ and apply the boundedness (3.6) of $A$ to obtain,
\[
\|D_3\|(t_{n+1}) \leq C_{\text{wx}} \tau \left( \|\hat{\zeta}^{n,2}\|(t_{n+1}) + \|\hat{\zeta}_n\|(t_{n+1}) \right) + \alpha \mu \tau h^{-1}\|D_2\|(t_{n+1})
\]
\[
+ C(\tau h^{k+1} + \alpha \mu \tau h^k + \tau^4).
\]
Here we use the estimate (5.57). Denote $\lambda = \alpha \mu \tau h^{-1}$ as before. It is inferred that
\[
3\|D_3\|^2(t_{n+1}) \leq 6\lambda^2 \|D_2\|^2(t_{n+1}) + C \tau^2 \left( \|\hat{\zeta}^{n,2}\|^2(t_{n+1}) + \|\hat{\zeta}_n\|^2(t_{n+1}) \right)
\]
\[
+ C(\tau h^{k+1} + \alpha \mu \tau h^k + \tau^4)^2.
\]

**Step 2.** Bound on $\Lambda_1$. By the Young’s inequality and the above estimate, we have,
\[
\Lambda_1 \leq \frac{\alpha}{8} \tau [\|D_1\|^2(t_{n+1}) + 3\alpha \tau [\|D_2\|^2(t_{n+1}) - \|D_2\|^2(t_{n+1}) + 3\|D_3\|^2(t_{n+1})
\]
\[
\leq \frac{\alpha}{4} \tau [\|\zeta^{n,1}\|^2(t_{n+1}) + \|\|\zeta^n\|^2(t_n) - (1 - 3\lambda - 6\lambda^2)\|D_2\|^2(t_{n+1})
\]
\[
+ C \tau^2 \left( \|\hat{\zeta}^{n,2}\|^2(t_{n+1}) + \|\hat{\zeta}_n\|^2(t_{n+1}) \right) + C(\tau h^{k+1} + \alpha \mu \tau h^k + \tau^4)^2.
\]
Thus we obtain the time step restriction
\[
(1 - 3\lambda - 6\lambda^2) > 0.
\]
It is sufficient to choose $\lambda \leq \frac{1}{5}$. Then
\[
A_1 \leq \frac{\alpha}{4} \tau \left[ \lVert \zeta^{n,1} \rVert^2(t_{n+1}) + \frac{\alpha}{4} \tau \lVert \zeta^n \rVert^2(t_n) + C(\tau h^{2k+1} + \tau^7)
\right]
+ C \tau \left( \lVert \hat{\zeta}^{n,2} \rVert^2(t_{n+1}) + \lVert \hat{\zeta}^n \rVert^2(t_{n+1}) \right).
\] (5.64)

Here we use $\tau \leq 1$.

**Step 3.** Bound on $T^6$. Recalling the definition (5.61) of $T^6$, we take $v^m_h = \zeta^n$, $\hat{\zeta}^{n,1}$ and $\hat{\zeta}^{n,2}$ for $m = 1, 2$ and 3, respectively, in the estimate (5.56) to yield,
\[
T^6 \leq C(\tau h^{k+1} + \tau^4) \left( \lVert \hat{\zeta}^n \rVert(t_{n+1}) + \lVert \hat{\zeta}^{n,1} \rVert(t_{n+1}) + \lVert \hat{\zeta}^{n,2} \rVert(t_{n+1}) \right)
+ C\alpha \tau h^{k+1} \left( \lVert \zeta^n \rVert(t_n) + \lVert \zeta^{n,1} \rVert(t_{n+1}) + \lVert \zeta^{n,2} \rVert(t_{n+1}) \right)
\leq \tau \left( \lVert \hat{\zeta}^n \rVert^2(t_{n+1}) + \lVert \zeta^{n,1} \rVert^2(t_{n+1}) + \lVert \zeta^{n,2} \rVert^2(t_{n+1}) \right)
+ \frac{\alpha}{4} \tau \left( \lVert \zeta^n \rVert^2(t_n) + \lVert \zeta^{n,1} \rVert^2(t_{n+1}) + \lVert \zeta^{n,2} \rVert^2(t_{n+1}) \right) + C\tau (h^{2k+1} + \tau^6). \quad (5.65)
\]

**Step 4.** Bound on $T^7$. Taking $v^m_h = D_2$ and $D_1 + D_2$ in the representations (5.53) of $T^4$ and $T^5$, respectively, we obtain the following estimate by Lemma 18,
\[
T^7 \leq C\tau (h^{k+1} + \tau^3 + \alpha \mu \tau h^k)(\lVert D_1 \rVert + \lVert D_2 \rVert)(t_{n+1})
\leq \tau \lVert D_1 \rVert^2(t_{n+1}) + \tau \lVert D_2 \rVert^2(t_{n+1}) + C\tau (h^{2k+2} + \tau^6 + \alpha^2 \mu^2 \tau^2 h^{2k})
\leq C \tau \left( \lVert \hat{\zeta}^{n,2} \rVert^2(t_{n+1}) + \lVert \zeta^n \rVert^2(t_{n+1}) + \lVert \zeta^{n,1} \rVert^2(t_{n+1}) \right) + C\tau (h^{2k+2} + \tau^6). \quad (5.66)
\]

Here we use the CFL condition $\tau \leq \rho h$ and the fact that
\[
\lVert D_1 \rVert^2(t_{n+1}) + \lVert D_2 \rVert^2(t_{n+1}) \leq C \left( \lVert \hat{\zeta}^{n,2} \rVert^2(t_{n+1}) + \lVert \zeta^n \rVert^2(t_{n+1}) + \lVert \zeta^{n,1} \rVert^2(t_{n+1}) \right).
\]

**Step 5.** Bound on $\lVert \zeta^{n,1} \rVert^2(t_{n+1})$ and $\lVert \hat{\zeta}^{n,2} \rVert^2(t_{n+1})$. Take the test function $\hat{v}_h^n = \zeta^{n,1}$ in the first equality of the error equation (5.50) to derive,
\[
\lVert \zeta^{n,1} \rVert^2(t_{n+1}) \leq \lVert \zeta^n \rVert(t_n) \lVert \hat{\zeta}^{n,1} \rVert(t_{n+1}) + 3\alpha \mu \tau h^{-1} \lVert \zeta^n \rVert(t_n) \lVert \hat{\zeta}^{n,1} \rVert(t_{n+1})
+ C(\tau h^{k+1} + \alpha \mu \tau h^k)\lVert \zeta^{n,1} \rVert(t_{n+1})
\leq \left( C \lVert \zeta^n \rVert(t_n) + Ch^{k+1} \right) \lVert \zeta^{n,1} \rVert(t_{n+1}).
\]

Here we use the Cauchy-Schwarz inequality, boundedness (3.6) of $A$ and the estimate (5.57) of $T^1$ for the first step. The CFL condition and (2.20) are used for the second step. The above estimate indicate that
\[
\lVert \zeta^{n,1} \rVert(t_{n+1}) \leq C \lVert \zeta^n \rVert(t_n) + Ch^{k+1}. \quad (5.67)
\]
Taking the test function \( \tilde{\nu}_h^n = \zeta^{n,2} \) in the second equality of the error equation (5.50) and by the similar analysis, we have,

\[
\| \zeta^{n,2}(t_{n+\frac{1}{2}}) \| \leq C \| \zeta^n \|(t_n) + C \| \zeta^{n,1}(t_{n+1}) \| + C h^{k+1},
\]

which implies

\[
\| \hat{\zeta}^{n,2}(t_{n+1}) \| \leq C \| \zeta^n \|(t_n) + C \| \zeta^{n,1}(t_{n+1}) \| + C h^{k+1}.
\]  

(5.68)

Here the properties (2.20)-(2.22) are used frequently. Finally, we combine the estimates (5.64)-(5.68) and the energy equality (5.63) together to yield,

\[
3 \| \zeta^{n+1} \|_2(t_{n+1}) - 3 \| \zeta^n \|_2(t_n) \leq C \tau \| \zeta^n \|_2(t_n) + C \tau(h^{2k+1} + \tau^6).
\]

Sum over all \( n \), use the Gronwall’s inequality and choose the initial condition \( u_h(x,0) = P_h u(x,0) \) to obtain,

\[
\| \zeta^n \|_2(t_n) \leq C(h^{2k+1} + \tau^6), \quad n \leq M.
\]

We finish the proof by applying the estimate (5.54),

\[
\| e^n \|_2(t_n) \leq C(h^{2k+1} + \tau^6), \quad n \leq M.
\]

Remark 20. We remark that it is not difficult to extend the error estimates to the upwind flux, which also starts from the energy identity and the analysis follows the same ways as the Lax-Friedrichs flux case, then we can obtain the optimal error estimate. The main difference lies in two places. One is the properties of the ALE-DG operator \( A \), which is also changed owing to the choice of the flux, and the other one is the Gauss-Radau projections (2.24) instead of \( L^2 \) projection.

6 Conclusion

In this paper, we have analyzed the stability and error estimates of the fully discrete ALE-DG schemes for the linear conservation laws, when the explicit TVD-RK time-marching methods up to the third order were adopted. The energy analysis and scaling arguments are the main techniques used in our work. We prove that the fully discrete schemes are stable under the appropriate CFL conditions and obtain the quasi-optimal error estimates in space and optimal in time for sufficiently smooth solutions.
References


