A Short Course on:

Spectral Methods for Fractional ODEs/PDEs

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Course Materials

We will present, in a tutorial level, new developments in spectral methods applied to fractional ODEs/PDEs. Specifically, this course provides a general framework for developing a new class of single- and multi-domain spectral methods for non-local mathematical models. The theory of Sturm-Liouville eigen-problems and Jacobi polynomials forms the backbone of spectral element methods for PDEs. Therefore, the present short course begins by introducing a theory on fractional Sturm-Liouville eigen-problems, which extends the classical Sturm-Liouville approximation theory to solutions of PDEs with fractional order. The proposed short course will be taught within five lectures, each for two hours, on August 15th, 17th, 19th, 22nd, and 24th, 2016. The breakdown of each lecture is given in the following:

Lecture 1: Jacobi Poly-fractonomials and Spectral Expansions

In the first part of this lecture, we generalize the well-known theory of Sturm-Liouville boundary-value problem and provide the corresponding explicit eigen-solutions in bounded and in unbounded domains. We first consider a regular fractional Sturm-Liouville problem of two kinds (RFSLP-I and RFSLP-II) of order $\nu \in (0, 2)$ in the standard domain $[-1, 1]$. The corresponding fractional differential operators in these problems are both of Riemann-Liouville and Caputo type, of the same fractional order $\mu = \nu/2 \in (0, 1)$. We obtain the analytical eigen-solutions to RFSLP-I &-II as non-polynomial functions, which we define as Jacobi poly-fractonomials. These eigen-functions are orthogonal with respect to the weight function associated with RFSLP-I &-II. Subsequently, we extend the fractional operators to a new family of singular fractional Sturm-Liouville problems of two kinds (SFSLP-I and SFSLP-II). We show that the primary regular boundary-value problems RFSLP-I&-II are indeed asymptotic cases for the singular counterparts SFSLP-I&-II. Furthermore, we prove that the eigenvalues of the singular problems are real-valued and the corresponding eigen-functions are orthogonal. In addition, we obtain the eigen-solutions to SFSLP-I &-II analytically, also as non-polynomial functions, hence completing the whole family of the Jacobi poly-fractonomials. We employ the new poly-fractonomial bases to demonstrate the exponential convergence of the approximation of a function, in agreement with the theoretical results. Following the same structure, we extend the aforementioned FSLPs from bounded to unbounded domains and present the exact eigen-functions.
In the second part of this lecture, we further generalize the theory of fractional Sturm-Liouville problems to a tempered class of boundary-value problems. Similarly, we introduce two classes of regular and singular tempered fractional Sturm-Liouville problems of two kinds (TFSLP-I and TFSLP-II) of order $\nu \in (0,2)$. In the regular case, the corresponding tempered differential operators are associated with tempering functions $p_I(x) = \exp(2\tau)$ and $p_{II}(x) = \exp(-2\tau)$, $\tau \geq 0$, respectively in the regular TFSLP-I and TFSLP-II, which do not vanish in $[-1,1]$. In contrast, the corresponding differential operators in the singular setting are associated with different forms of $p_I(x) = \exp(2\tau)(1-x)^{1+\alpha}(1+x)^{1+\beta}$ and $p_{II}(x) = \exp(-2\tau)(1-x)^{1+\alpha}(1+x)^{1+\beta}$, vanishing at $x = \pm 1$ in the singular TFSLP-I and TFSLP-II, respectively. We obtain the explicit eigen-solutions to the TFSLP-I & -II as non-polynomial functions, which we define as *tempered Jacobi poly-fractonomials*. Finally, we introduce these eigen-functions as new basis (and test) functions for spectrally-accurate approximation of functions and tempered-fractional differential operators. If there is no tempering, i.e. $\tau = 0$, we recover the eigen-solutions presented in the first part of the lecture.

**Lecture 2: Spectral Methods for FODEs**

In the first part of this lecture, we present efficient single-domain spectral methods for FODEs that lead to spectrally/exponentially fast decay of the error for smooth solutions. First, we develop a Petrov-Galerkin (PG) spectral method for Fractional Initial Value Problems (FIVPs) and Fractional Final Value Problems (FFVPs). These schemes are developed based on the theory of FSLPs, presented in the first lecture. Subsequently, we develop another single-domain spectral method, namely *discontinuous spectral method* of Galerkin sense for the aforementioned FIVPs and FFVPs, where the basis functions do not necessarily satisfy the initial/final conditions.

We also present a multi-domain algorithm for fractional differential equations. Since the kernel is singular or nearly singular, two main difficulties arise after the domain decomposition: how to properly account for the history/memory part and how to perform the integration accurately. To address these issues, we present a hybrid approach for the numerical integration based on the combination of three-term-recurrence relations of Jacobi polynomials and high-order Gauss quadrature. The different approximations used in the hybrid approach are justified theoretically and through numerical examples. Based on that, we present a multi-domain spectral method for high-order accurate time integrations and study its stability properties by identifying the method as a generalized linear method.
Lecture 3: Petrov-Galerkin Methods for FPDEs

In this lecture, we present a unified Petrov-Galerkin (PG) spectral method for a weak formulation of the general linear Fractional Partial Differential Equations (FPDEs) of the form $0_{D}^{2\tau}u + \sum_{j=1}^{d} c_{j}[a_{j}D_{x}^{2\mu_{j}}u] + \gamma u = f$, where $2\tau, \mu_{j} \in (0, 1)$, in a $(1+d)$-dimensional space-time domain subject to Dirichlet initial and boundary conditions. We perform the stability analysis (in 1-D) and the corresponding convergence study of the scheme (in multi-D). The unified PG spectral method applies to the entire family of linear hyperbolic-, parabolic- and elliptic-like equations. Besides the high-order spatial accuracy of the PG method, we demonstrate its efficiency and spectral accuracy in time-integration schemes for solving time-dependent FPDEs as well, rather than employing algebraically accurate traditional methods, especially when $2\tau = 1$. Finally, we formulate a general fast linear solver based on the eigen-pairs of the corresponding temporal and spatial mass matrices with respect to the stiffness matrices, which reduces the computational cost drastically. Moreover, in this lecture, we develop multi-domain spectral element methods for time- and space- fractional PDEs subject to Dirichlet initial/boundary conditions.

Lecture 4: Fractional Collocation Method

For an effective treatment of non-linear and multi-term fractional PDEs such as the fractional Burgers equation, we present a fractional spectral collocation method in this lecture, which introduces a new family of interpolants, called fractional Lagrange interpolants. Subsequently, we obtain the corresponding fractional differentiation matrices, and we present numerical solutions of a number of linear FODEs in addition to linear and non-linear FPDEs to investigate the numerical performance of the fractional collocation method. We finally, provide the extension of the present scheme for variable-order, distributed-order FPDEs, and some new methods with tunable accuracy.

Lecture 5: Applications of Fractional-Order Modelling

The final lecture includes several topics for the efficient implementation of spectral and spectral element methods to different applications.