Hopf-Cole Transformation

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March 20, 2016
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Hopf-Cole Transformation

- **Hopf, Eberhard** The partial differential equation
  \[ u_t + uu_x = \mu u_{xx}. \]  


**Hopf-Cole transformation:**

\[ u_t + uu_x = \kappa u_{xx} \]  
Burgers equation,

\[ \Rightarrow B_t + \frac{(B_x)^2}{2} = \kappa B_{xx}, \quad B_x = u, \]  
Hamilton-Jacobi equation,

Introduce Hopf-Cole relation

\[ B(x, t) = -2\kappa \log[\phi(x, t)], \]

\[ \Rightarrow \phi_t = \kappa \phi_{xx}, \]  
Heat equation.
Solution formula for initial value problem for Burgers equation:

\[ u(x, t) = u(x, t, \kappa) = \frac{\int_{-\infty}^{\infty} \frac{x-y}{t} e^{-\frac{(x-y)^2}{4\kappa t}} - \frac{1}{2\kappa} \int_{0}^{y} u(z,0) dz}{\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\kappa t}} - \frac{1}{2\kappa} \int_{0}^{y} u(z,0) dz} \, dy. \]
Burgers equation:

- Cole derived the Burgers equation from gas dynamics:

\[
\frac{\partial w}{\partial t} + \beta \frac{\partial w}{\partial x} = \frac{4}{3} \nu^* \frac{\partial^2 w}{\partial x^2}.
\]

"for \( w = \) excess of flow velocity over a sonic velocity, where \( \beta = (\gamma + 1)/2 \), \( \nu^* = \) the kinematic viscosity at sonic condition"
Hopf-Cole Transformation:

- **Hopf**: "The reduction of (1) to the heat equation was known to me since the end of 1946. However, it was not until 1949 that I became sufficiently acquainted with the recent development of fluid dynamics to be convinced that a theory of (1) could serve as an instructive introduction into some of the mathematical problems involved."


- Hopf was inspired by the works of Burgers on turbulence and Friedrichs’ theory of N-waves.


- The Hopf-Cole transformation is embedded in this exercise in Forsyth’s book.
Hopf-Cole Transformation

Hopf:

- Solution formula for the inviscid Burgers equation
  \[ u_t + (u^2/2)_x = 0 \]
  in the zero dissipation limit \( \kappa \to 0^+ \):

  \[
  F(x, y, t) = \frac{(x-y)^2}{2t} + \int_{0-}^{y} u(z, 0) \, dz,
  \]
  \[
  \min_{y} F(x, y, t) = F(\xi, t),
  \]
  \[
  \lim_{\kappa \to 0^+} u(x, t, \kappa) = u(\xi, 0).
  \]

- Metastable states:

  \[
  \lim_{t \to \infty} \lim_{\kappa \to 0^+} u(x, t, \kappa) \neq \lim_{\kappa \to 0^+} \lim_{t \to \infty} u(x, t, \kappa).
  \]

- Modern theory of hyperbolic conservation laws.
Hopf-Cole Transformation

Hopf:

\[ b_t + \left( \frac{b^2}{2} \right)_x = \kappa b_{xx}, \quad b(x, 0) = A\delta(x), \text{ Burgers kernel} \]

By Hopf-Cole transformation,

\[
b(x, t; A) = \frac{\sqrt{\kappa}}{\sqrt{t}} (e^{\frac{A}{2\kappa}} - 1)e^{-\frac{x^2}{4\kappa t}} \]
\[
\sqrt{\pi} + \int_{\frac{x}{\sqrt{4\kappa t}}}^{\infty} (e^{\frac{A}{2\kappa}} - 1)e^{-y^2} dy.
\]

For initial data with finite mass, a solution of the Burgers equation tends to Burgers kernel:

\[
\int_{-\infty}^{\infty} |u(x, t) - b(x, t; A)| \, dx = O(1)t^{-\frac{1}{2}}, \text{ as } t \to \infty, \quad A = \int_{-\infty}^{\infty} u(x, 0) \, dx.
\]
Hopf-Cole Transformation

Hopf:
On the other hand, for inviscid Burgers equation, the solution tends to *N*-waves:

\[
\int_{-\infty}^{\infty} |u(x, t) - N(x, t; p, q)| = O(1)t^{-\frac{1}{2}},
\]

\[
p = \min_x \int_{-\infty}^{x} u(x, 0) dx, \quad q = \max_x \int_{x}^{\infty} u(x, 0) dx,
\]

\[
N(x, t; p, q) = \begin{cases} \frac{x}{t}, & \text{for } -\sqrt{-2pt} < x < \sqrt{2qt}; \\ 0, & \text{otherwise.} \end{cases}
\]

\[
\lim_{t \to \infty} \lim_{\kappa \to 0^+} u(x, t, \kappa) = N - \text{waves, two time invariants;}
\]

\[
\neq \lim_{\kappa \to 0^+} \lim_{t \to \infty} u(x, t, \kappa) \text{ one time invariant.}
\]

*N*-waves represent metastable states for the Burgers solutions.
Outside of gas dynamics:


\[ V_t - 6VV_x + V_{xxx} = 0, \text{ KdV} \Rightarrow \phi_t - 6\phi^2\phi_x + \phi_{xxx} = 0, \text{ Modified KdV}, \]

\[ V = \phi^2 \pm \phi_x, \text{ Miura transformation}. \]

"It is rare and surprising to find a transformation between two simple nonlinear partial differential equations of independent interest. One is reminded of the Hopf-Cole transformation of quadratically nonlinear Burgers equation into the heat conduction (diffusion) equation. A number of investigators (including us) have attempted unsuccessfully to find a similar simple linearizing transformation for the KdV equation, but a complicated one will be given in VI."
Outside of gas dynamics:


Evolution of the profile of a growing interface: the Hamilton-Jacobi equation plus a noise $\eta$:

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(x, t).$$

Hopf-Cole transform $\Rightarrow$ linear equation with a source:

$$\begin{cases} \frac{\partial W}{\partial t} = \nu \nabla^2 W + \frac{\lambda}{2\nu} \eta(x, t) W, \\ W(x, t) = e^{\frac{\lambda}{2\nu} h(x, t)}. \end{cases}$$

New scaling, distinct from deterministic dissipation equations comes up due to the noise.

"We thus have an intriguing connection between evolutions of a hydrodynamic and a growth pattern!"
Hopf-Cole Transformation

- Scalar convex hyperbolic conservation law, \( f''(u) \neq 0 \),

\[ u_t + f(u)_x = 0 \Rightarrow \lambda_t + \lambda \lambda_x = 0, \text{ inviscid Burgers, } \lambda = f'(u). \]

- System of hyperbolic conservation laws, e.g. Euler equations in gas dynamics

\[ u_t + f(u)_x = 0, \; u \in \mathbb{R}^n, \; f'(u)r_j(u) = \lambda_j(u)r_j(u), \]
\[ l_j(u)f'(u) = \lambda_j(u)l_j(u), \; l_j(u) \cdot r_k(u) = \delta_{jk}, \; j, k = 1, 2, \ldots, n. \]

- "Convexity": \( \nabla_u \lambda_j(u) \cdot r_j(u) \neq 0 \) "genuine nonlinear" field, e.g. acoustic waves.

- \( j \)-simple waves: \( u(x, t) \) moves along integral curve of \( r_j(u) \).

\[ \lambda_t + \lambda \lambda_x = 0, \; \lambda(x, t) = \lambda_j(u(x, t)) \Rightarrow u_t + f(u)_x = 0. \]
• Viscous conservation laws, e.g. Compressible Navier-Stokes equations

\[ u_t + f(u)_x = (\mathbb{B}(u, \nu)u)_x. \]

• Dissipation parameters, e.g. \( \nu = (\mu, \kappa) \) viscosity and heat conductivity.

• The Burgers equation is used for construction of approximate \( j \)-simple waves for each genuinely nonlinear field

\[ \lambda_t + \lambda\lambda_x = \kappa\lambda_{xx}, \quad \lambda(x, t) = \lambda_j(u(x, t)). \]

• Burgers dissipation parameter \( \kappa \) is the diagonal element of the viscosity matrix \( \mathbb{B} \) in the characteristic coordinates of the hyperbolic part:

\[ \kappa = l_j(u)\mathbb{B}(u, \nu)r_j(u). \]
**Hopf-Cole Transformation**

- \( u_t + f(u)_x = 0 \) hyperbolic conservation laws
  
  Solutions of finite mass tends to \( N \)-waves at the rate of \( t^{-1/4} \) in \( L_1(x) \) as consequence of pointwise estimate.


- \( u_t + f(u)_x = (B(u, \nu)u_x)_x \), viscous conservation laws.
  
  Solutions of finite mass tends to Burgers and heat kernels also at the rate of \( t^{-1/4} \) in \( L_1(x) \) and as consequence of pointwise estimate.


- **Open problem: Metastability.**
Hopf-Cole transformation is used for finding exact expression of Burgers Nonlinear waves.

The Burgers nonlinear waves is used for construction of approximate nonlinear waves for system of conservation laws.

The linearized Hopf-Cole transformation is used for the explicit construction of Green’s function for Burgers equation linearized around a nonlinear wave.

The construction is Green’s function for systems is based on the Burgers Green’s function.

This is essential for the study of shock, initial layers for system of conservation laws.
Burgers shock formation, $\lambda_0 > 0$, using Hopf-Cole:

\[
\begin{cases}
(u_S)_t + u_S(u_S)_x = \kappa (b_S)_{xx}, \\
u_S(x, 0) = \begin{cases}
\lambda_0, & \text{for } x < 0, \\
-\lambda_0, & \text{for } x > 0,
\end{cases}
\end{cases}
\]

\[
u_S(x, t) = -\lambda_0 \frac{\text{Erfc}(\frac{-x - \lambda_0 t}{\sqrt{4\kappa t}}) - e^{-\frac{\lambda_0 x}{\kappa}} \text{Erfc}(\frac{x - \lambda_0 t}{\sqrt{4\kappa t}})}{\text{Erfc}(\frac{-x - \lambda_0 t}{\sqrt{4\kappa t}}) + e^{-\frac{\lambda_0 x}{\kappa}} \text{Erfc}(\frac{x - \lambda_0 t}{\sqrt{4\kappa t}})}.
\]

The thickness $T_0$ of the initial layer to form Burgers shock profile $b_S$, the time when the error function $\text{Erfc}$ approaches $\sqrt{\pi}$,

\[
b_S(x) = \lim_{t \to \infty} u_S(x, t) = -\lambda_0 \tanh(\frac{\lambda_0 x}{2\kappa}).
\]

\[
\frac{\lambda_0 T_0}{\sqrt{4\kappa T_0}} = O(1), \text{ or } T_0 = O(1) \frac{\kappa}{(\lambda_0)^2}.
\]
Burgers rarefaction wave

\[
\begin{aligned}
(h_R)_t + \left(\frac{(h_R)^2}{2}\right)_x &= 0, \\
h_R(x, 0) &= \begin{cases} 
-\lambda_0, & \text{for } x < 0, \\
\lambda_0, & \text{for } x > 0;
\end{cases}
\end{aligned}
\]

\[
b_R(x, t) = \lambda_0 \frac{e^{\frac{\lambda_0 x}{2\kappa}} \text{Erfc}(\frac{-x+\lambda_0 t}{\sqrt{4\kappa t}}) - e^{-\frac{\lambda_0 x}{2\kappa}} \text{Erfc}(\frac{x+\lambda_0 t}{\sqrt{4\kappa t}})}{\text{Erfc}(\frac{-x+\lambda_0 t}{\sqrt{4\kappa t}}) + e^{-\frac{\lambda_0 x}{2\kappa}} \text{Erfc}(\frac{x+\lambda_0 t}{\sqrt{4\kappa t}})}.
\]

Within the hyperbolic rarefaction wave region, 
\(x \in (-\lambda_0 t + M\sqrt{4\kappa t}, \lambda_0 t - M\sqrt{4\kappa t})\), and after initial layer time, 
the difference of the Burgers rarefaction wave \(b_R\) and the inviscid rarefaction wave \(x/t\):

\[
b_R(x, t) - \frac{x}{t} = O(1)\left[\frac{1}{|x - \lambda_0 t|} + \frac{1}{|x + \lambda_0 t|}\right], \ t > O(1)\frac{\kappa}{(\lambda_0)^2}.
\]
Hopf-Cole Transformation

Linear Hopf-Cole transformation
Burgers equation linearized around a given solution $\bar{u}(x, t)$:

$$\bar{u}_t + \left( \frac{\bar{u}^2}{2} \right)_x = \kappa \bar{u}_{xx}$$

$$\bar{U}_x = \bar{u}, \quad \bar{U}(x, t) = -2\kappa \log[\phi(x, t)],$$

$$\nu_t + (\bar{u}\nu)_x = \kappa \nu_{xx},$$  Burgers equation linearized around $\bar{u}(x, t)$.

Linearize the Hopf-Cole relation $V + \bar{U} = -2\kappa \log[\phi + \zeta]$:

$$V = -2\kappa \frac{\zeta}{\phi}, \text{ linearized Hopf-Cole relation, } \Rightarrow$$

$$\zeta_t = \kappa \zeta_{xx} \text{ and the solution representation to the solution of the linearized Burgers equation:}$$

$$\nu(x, t) = \frac{\partial}{\partial x} \left[ \frac{1}{\sqrt{4\pi \kappa t}} e^{-\frac{(x-y)^2}{4\kappa t}} \bar{\phi}(y, 0) V(y, 0) \right] dy \bar{\phi}(x, t).$$
Green’s function for shock profile

Green’s function $G_S(x, t; x_0, t - t_0)$ for the shock profile $b_S(x)$ using linearized Hopf-Cole:

$$(G_S)_t + b_S(G_S)_x = \kappa (G_S)_{xx}, \quad G_S(x, 0) = \delta(x - x_0);$$

$$G_S(x, t; x_0) = \frac{1}{\sqrt{4\pi \kappa t}} e^{-\frac{(x-x_0)^2}{4\kappa t}} e^{\frac{\lambda_0 x_0}{2\kappa}} + e^{-\frac{\lambda_0 x_0}{2\kappa}} e^{\frac{(\lambda_0)^2 t}{4\kappa}}.$$

The Green’s function as weighted combination of the heat kernel with speeds $\pm \lambda_0$:

$$G_S(x, t; x_0) = \begin{cases} 
H(x + \lambda_0 t, t), & \text{for } x > 0, \ x_0 > 0; \\
1 + e^{-\frac{\lambda_0 |x_0|}{\kappa}}, & \\
e^{-\frac{\lambda_0 |x|}{\kappa}} H(x + \lambda_0 t, t), & \text{for } x < 0, \ x_0 > 0; \\
H(x - \lambda_0 t, t), & \text{for } x < 0, \ x_0 < 0; \\
e^{-\frac{\lambda_0 |x|}{\kappa}} H(x - \lambda_0 t, t), & \text{for } x > 0, \ x_0 < 0. 
\end{cases}$$
Green’s function for rarefaction waves:

\[
G_R(x, t; x_0, t_0) = e^{-\frac{[x-x_0-(\lambda_0(t-t_0))]^2}{4\kappa(t-t_0)}} \frac{\text{Erfc}\left(\frac{-x_0+\lambda_0 t_0}{\sqrt{4\kappa t_0}}\right) + \text{Erfc}\left(\frac{x_0+\lambda_0 t_0}{\sqrt{4\kappa t_0}}\right)}{\text{Erfc}\left(\frac{-x+\lambda_0 t}{\sqrt{4\kappa t}}\right) + \text{Erfc}\left(\frac{x+\lambda_0 t}{\sqrt{4\kappa t}}\right)}.
\]

The propagation of waves is around the zero line of the exponential, along inviscid characteristics \( x = x_0 + \lambda_0(t - t_0) \). The essential support of the information is in the region given by

\[
\frac{t(x_0 - t_0 x/t)^2}{4\kappa t_0(t - t_0)} = O(1), \text{ or } |x - \frac{t}{t_0}x_0| = O(1)\sqrt{\kappa(t - t_0)\frac{t}{t_0}},
\]

varying from sub-linear, dissipative scale \( \sqrt{t - t_0} \) for \( t - t_0 \) small, to linear, hyperbolic scale \( t \) for \( t - t_0 \) large.
Open problem: Riemann problem

\[ u_t + f(u)_x = (B(u, \nu)u_x)_x, \]
\[ u(x, 0) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases} \]

\[ t \to \infty \Rightarrow \nu \to 0, \text{ zero dissipation limit,} \]
\[ u_t + f(u)_x = (B(u, \nu)u_x)_x \Rightarrow u_t + f(u)_x = 0. \]


Boltzmann equation

$$\partial_t f(x, t, \xi) + \xi \cdot \partial_x f(x, t, \xi) = \frac{1}{k} Q(f, f)(x, t, \xi)$$

Open problem: Riemann problem

$$f(x, t, \xi) = \begin{cases} M_l(\xi), & x < 0, \\ M_r(\xi), & x < 0. \end{cases}$$

$$t \to \infty \Rightarrow k \to 0, \text{ zero mean free path},$$

Boltzmann solutions \(\Rightarrow\) Euler solutions,