Early Developments in Geometric Measure Theory

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Dedicated to the memory of William P. Ziemer

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1 Introduction.

The field of geometric measure theory (GMT) is at an interface of problems in mathematical analysis and geometry. It provides a framework for measure and integration over broadly defined "surfaces" in n-dimensional Euclidean $\mathbf{R}^n$ of any dimension $k < n$. The concept of surface should include oriented k-dimensional manifolds with lower dimensional singularities. However, to study geometric problems in the calculus of variations, much broader classes of surfaces are needed. A prototypical example is the higher dimensional Plateau problem: minimize k-dimensional area among all k-dimensional surfaces with given $(k-1)$-dimensional boundary. To prove that a minimum exists, closure and compactness theorems in a suitable topology are needed.

This article gives a concise overview of early developments in GMT, with emphasis on the decade 1960-1970. That decade was a time during which the field developed rapidly and was completely transformed in truly remarkable ways. I joined Federer's group at Brown University in 1958, and had the good fortune to participate in GMT research during the years immediately afterward. Federer and I collaborated on our joint paper *Normal and Integral Currents* [FF60] during my first year at Brown. Bill Ziemer also came to Brown in 1958 and finished the PhD under my supervision in 1961.

This article is organized as follows. Part I discusses mathematical results. We begin in Sections 2 to 5 with a concise review of pre-1960 background. Section 3 concerns De Giorgi's sets of finite perimeter. This is followed by results in Section 4 about BV functions of several variables, considered from a GMT perspective. Other work in the 1950s which directly influenced developments in GMT afterward included de Rham's theory of currents, Young's generalized surfaces and Whitney's geometric integration theory (Section 5). Section 6 concerns my paper with Federer [FF60] and subsequent results of Federer about integral currents. [FF60] is in a de Rham current setting, which involves orientations of k-dimensional "surfaces". Section 7 discusses alternative formulations which do not involve orientations of surfaces, including Ziemer's pioneering work in his thesis on integral currents mod 2.

The next Section 8 introduces various formulations of the higher dimensional Plateau (least area) problem, in both oriented and non-oriented versions. This is followed in Sections 9 and 10 with a brief summary of results about the notoriously difficult questions about regularity of solutions. Section 11 considers geometric problems in the calculus of variations, in the setting of Young's generalized surfaces. Such solutions exist without assumptions of convexity or ellipticity, needed for existence and regularity of "ordinary" solutions as integral currents. Finally, in Section 12 Almgren's varifolds, which are defined in a way formally similar to Young's generalized surfaces, are discussed.

The remainder of this article (Part II) makes shift from mathematical discussions to some remembrances of the mathematical milieu at Brown University in the 1960s, and of our graduate students and visitors then, (Section 13). Section 14 gives some personal remembrances about W.P. Ziemer, F.J. Almgren, E. De Giorgi, H. Federer, E.R. Reifenberg and L.C. Young, all of whom are no longer with us. In the list of References, I cite scientific obituary articles and also volumes of selected works of Almgren [AF99] and De Giorgi [DG06].

Federer's monumental book *Geometric Measure Theory* [Fe69] is a definitive treatment of results in that field up to its date of publication. Another thorough introduction to GMT is Simon's book [SL83]. Morgan's *Beginners Guide* [MF00] provides for non-experts a readable introduction to concepts and results in GMT, with many references. Almgren's survey paper [AF93] provides a good, concise overview of concepts and results in GMT, with emphasis on area minimizing surfaces. Ziemer's book [Z89] gives a readable introduction to the topics of our Sections 3 and 4. My historical survey [Fl15] gives a somewhat more detailed account of topics which are also included in this article.

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Part I – Mathematical Results

2 Background, pre 1950.

We begin with a brief overview of three topics of research before 1950, each of which influenced developments in GMT afterward. Throughout this paper, $\mathbf{R}^n$ denotes n-dimensional Euclidean space. Other notations are summarized in the Appendix.

a) Hausdorff measure. During the early 20th century several definitions of k-dimensional measure of a set $K \subset \mathbf{R}^n$ were given. Among them the Hausdorff definition is now most widely used. The Hausdorff measure of K is denoted by $H^k(K)$. A set K is called k-rectifiable if $0 < H^k(K) < \infty$ and K differs in arbitrarily small H^k measure from a finite union of closed sets $K_1, \ldots, K_m$. such that each K_i is the image of a set $D_i \subset \mathbf{R}^k$ under some Lipschitz function f_i. The k-rectifiable sets have an important role in the theory of rectifiable and integral currents (Section 6). A set K with $H^k(K)$ finite is called purely k-unrectifiable if K has the following property. Let ρ denote the orthogonal projection of $\mathbf{R}^n$ onto a k-dimensional plane π containing 0. Then $\rho(K)$ has k dimensional Lebesgue measure 0 for "almost all" such projections ρ.

Besicovitch showed for $k = 1, n = 2$, that any set $K \subset \mathbf{R}^2$ with $0 < H^1(K) < \infty$ is the disjoint union of 1-rectifiable and purely 1-unrectifiable subsets K_1, K_2. This result was extended by Federer to arbitrary dimensions n and $k < n$, in his fundamental paper [Fe47]. He also showed that, if K is

k-rectifiable, then all reasonable definitions of k-dimensional measure of K agree with $H^k(K)$. See also [Fe69, Sec. 3.3].

2.1 Remarks.

2.1 Every k-rectifiable set K has an approximate tangent k-plane at H^k-almost all points of K [Fe69, Sec. 3.2.16].

2.2 What we call k-rectifiable sets are called countably k-rectifiable sets in [Fe69, Sec. 3.2.14].

2.3 In the definition of k-rectifiable sets, the sets K_i can be chosen as subsets of C^1 manifolds [Fe69, Sec. 3.2.29].

b) The Plateau (least area) problem. The classical Plateau problem for two dimensional surfaces in $\mathbf{R}^3$ is as follows. Find a surface S^* of least area among all surfaces S with given boundary C. This is a geometric problem in the calculus of variations, which has been studied extensively. During the 1930s, J. Douglas and T. Rado independently gave solutions to a version of the Plateau problem. Their results were widely acclaimed. Douglas received a Fields Medal in 1936 for his work.

Douglas and Rado considered surfaces defined by "parametric representations," which were mappings f from a circular disk $D \subset \mathbf{R}^2$ into $\mathbf{R}^3$, such that the restriction of f to the boundary of D is a parametric representation of the boundary curve C. The Douglas-Rado result was later extended by Douglas and Courant to give a solution to the Plateau problem bounded by a finite number of curves and of prescribed Euler characteristic. However, all of these results depended on conformal parameterizations of surfaces, and the methods are inherently two dimensional. They also depend on prescribing in advance the topological type of the surfaces considered. For these reasons, it became clear by the late 1950s that entirely new formulations and methods were needed to study higher dimensional versions of the Plateau problem, for surfaces of any dimension and codimension $k < n$. This is the topic of

Sections 8-10. **References.** [Do31] [Ra30] [Co50] [Do39].

c) Surface area theory.

The issue of giving a suitable definition of area for surfaces, without traditional smoothness assumptions, goes back to Lebesgue's thesis at the beginning of the 20th century. As in part (b) consider surfaces defined parametrically. Let f be a continuous function from a region $D \subset \mathbf{R}^2$ into $\mathbf{R}^3$. The Lebesgue area $A_\ell(f)$ is defined as the lower limit of the elementary areas of approximating polyhedra. Lebesgue area theory flourished from the 1930s through the 1950s. T. Rado and L. Cesari were leaders in the field, and their books [Ra48] [Ce56] are important sources. In the years after WW2, they were joined by Federer who contributed many of the most significant advances during this period. A basic question in area theory is to find an integer valued multiplicity function $\Theta(x)$ which yields $A_\ell(f)$, when integrated over $f(D)$ with respect to Hausdorff measure H^2. If f is Lipschitz, then $A_\ell(f) = A(f)$, where $A(f)$ is the classical formula for area as an integral over D, and one can take $\Theta(x) = N(x)$ where $N(x)$ is the number of points (u_1, u_2) in D such that $f(u_1, u_2) = x$. However, the task of defining a suitable "essential" multiplicity $\Theta(x)$ for every f with $A_\ell(f)$ finite presented a major challenge. For this purpose, corresponding multiplicity functions were first defined for mappings from $\mathbf{R}^2$ into $\mathbf{R}^2$ in terms of topological indices.

An excellent overview of Federer's important contributions to area theory is given in Ziemer's part of the scientific obituary article [P12], and also in [FZ14]. Federer extended many results about Lebesgue area to k-dimensional surfaces, defined by mappings f from $D \subset \mathbf{R}^k$ into $\mathbf{R}^n$, with $2 < k \leq n$. In doing so, he used recent developments in algebraic topology. Topological indices for the case $k = 2$ were replaced by topological degrees, defined in terms of Čech cohomology groups. In [P12], Ziemer considered the paper [Fe55] as one of Federer's best efforts in area theory. In particular, it contains the basic idea that led to the Deformation Theorem of GMT, mentioned in

Section 6 below.

3 Sets of finite perimeter.

An important role of GMT is to provide theories of k-dimensional integration, without the usual smoothness assumptions. Included are versions of the classical theorems of Gauss-Green and Stokes. In this section we consider the case $k = n-1$ and outline De Giorgi's theory of sets of finite perimeter [DG54] [DG55]. These papers were influenced by related work of R. Caccoppoli.

The classical Gauss-Green (divergence) Theorem says the following. Let $E \subset \mathbf{R}^n$ be a bounded open set with smooth boundary B, and let ζ be any smooth $\mathbf{R}^n$-valued function. Then

$$\int_E \mathrm{div}\zeta(x) dH^n(x) = \int_B \zeta(y) \cdot \nu(y) dH^{n-1}(y), \tag{3.1}$$

where $\nu(y)$ is the exterior unit normal at y and H^n is Lebesgue measure on $\mathbf{R}^n$. Choosing the exterior (rather than the interior) normal vector amounts to choosing an orientation for B. De Giorgi addressed the question of how to make sense of the right side of (3.1) without any smoothness assumptions on the topological boundary B of the set E. His program was as follows:

(a) Require only that E be a "set of finite perimeter" $P(E)$.

(b) In (3.1), replace B by a set $B_r \subset B$ called the "reduced boundary."

(c) Show that B_r is a $(n-1)$-rectifiable set and that there is an "approximate exterior unit normal" vector $\nu(y)$ at each $y \in B_r$.

To define sets of finite perimeter, let $E \subset \mathbf{R}^n$ be a bounded, Lebesgue measurable set. Let 1_E denote its indicator function. In the style of Schwartz distribution theory, think of ζ in (3.1) as any smooth $\mathbf{R}^n$-valued test function.

Let $\Phi = -\mathrm{grad}\, 1_E$ in the Schwartz distribution sense. De Giorgi called E a set of finite perimeter if Φ is a measure. This is equivalent to saying that 1_E is a bounded variation (BV) function of n variables.

An important part of his theory involves approximations of E by sequences E_j of sets with piecewise smooth boundaries B_j, and in particular by polygonal domains with piecewise flat boundaries. The convergence of E_j to E as $j \to \infty$ is in H^n-measure and the perimeter of E_j is $P(E_j) = H^{n-1}(B_j)$. If $P(E_j)$ is bounded, then the corresponding measures Φ_j converge weakly to Φ as $j \to \infty$. Another characterization of the perimeter $P(E)$ is as the lower limit of $P(E_j)$ as $j \to \infty$, among all such sequences E_j of polygonal domains.

In [DG55] De Giorgi defined in an elegant way the reduced boundary B_r and approximate exterior unit normal vectors $\nu(y)$. He then showed that (3.1) remains correct, with B replaced by B_r, see [DG55, Theorem III]. An English translation of this paper, written in Italian, is included in the selected papers of De Giorgi's book [DG06]. Another clear exposition of these results is given in [Z89, Sections 5.4-5.8]. The vectors $\nu(y)$ are called generalized exterior normal vectors there. A definition of $\nu(y)$ is also included in the Appendix to this article.

This work by De Giorgi significantly influenced a few years later the theory of integral currents of any dimension $k < n$ (Section 6). For example, the "slicing formula" which De Giorgi used to show that his definition of set of finite perimeter was equivalent to another definition of Caccoppoli anticipated the "coarea formula" in GMT, of which it is a particular case.

4 BV functions.

A real valued function g on an open set $G \subset \mathbf{R}^n$ is of bounded variation (BV) if $g \in L^1(G)$ and its Schwartz distribution theory gradient is a $\mathbf{R}^n$-valued measure with finite total variation. In this section, we summarize some results about the pointwise behavior of BV functions, and the convergence of smooth approximations obtained by local averaging.

For $n = 1$, a BV function g is continuous except for a countable set of points, at each of which g has right and left hand limits. In [Fe68], Federer announced a remarkable extension of this property to any BV function g of n variables. Following [Z89, Sec.5.9-5.14], an approximate lower limit $\lambda(x)$ and approximate upper limit $\mu(x)$ of $g(y)$ as $y \to x$ are defined. The set $D = \{x : \lambda(x) < \mu(x)\}$ corresponds (roughly speaking) to a set of jump discontinuities of g. It is shown that D is $(n-1)$-rectifiable. This result is also included in [Fe69, Theorem 4.5.9]. The statement of that theorem has 31 parts, which represent a nearly complete theory of BV functions as of 1969. Federer stated this theorem in terms of the normal current T_g of dimension n associated with the BV function g (see Section 5 below). Readers of this section of [Fe69] need some familiarity with concepts and notations about de Rham's currents.

Smooth approximations by local averaging. For any $g \in L^1(G)$, let $g_r(x)$ denote the average of $g(y)$ over the ball of radius r with center x. Then $g_r(x) \to g(x)$ as $r \to 0$, for almost all $x \in G$. For functions in the Sobolev space $W^{1,p}(G)$, with $p > 1$, more precise statements in terms of capacities are known. See [Z89, Secs. 2.6 and 3.1]. For $1 < p < n, g_r(x)$ tends to a limit $\tilde{g}(x)$ as $r \to 0$, for all $x \in G \backslash N$ where N is an "exceptional set" which has Bessel capacity 0. The function $\tilde{g}$ can be regarded as a "more precise" version of the function g. For $p > n$, $\tilde{g}$ is continuous on G.

A similar result was later proved for BV functions. In that result, the function

$$\tilde{g}(x) = \tfrac{1}{2}[\lambda(x) + \mu(x)]$$

is a "more precise" version of the BV function g. It is defined for $x \in G \backslash N$ where $H^{n-1}(N) = 0$. See [Fe69, Section 4.5.9] [Z89, Section 5.14].

The proof of this result makes use of a function $\gamma(A)$, defined for sets $A \subset \mathbf{R}^n$. It has the role of a capacity function, and is characterized by the following property. For any compact set K.

$$\gamma(K) = \inf\{\mathrm{P}(E) : \mathrm{K} \subset E, E \text{ open and bounded}\},$$

where $P(E)$ is the perimeter of E. The function γ was first introduced in [Fl60]. By using a "boxing inequality" due to Gustin [G60], it was shown [Fl60, Theorem 4.3] [Z89,Lemma 5.12.3] that $\gamma(A) = 0$ if and only if $H^{n-1}(A) = 0$. For functions $g \in W^{1,1}(G)$ it was then shown [Fl60, Theorem 5.2] that the exceptional set N satisfied $\gamma(N) = 0$, and hence $H^{n-1}(N) = 0$.

Boundary behavior. If G has smooth boundary B, then it is possible to assign values for a BV function (defined on G) at H^{n-1}-almost all points of B. See [MeZ77] [Z89, Sec. 5.10].

5 More background, 1950s.

In this section, we mention three publications, in addition to De Giorgi's papers on sets of finite perimeter, which significantly influenced directions that GMT followed in the years afterward.

a) de Rham's currents. The L. Schwartz theory of distributions appeared just at the end of WW2. Since then it has had a very profound influence on mathematical analysis. A Schwartz distribution T is defined as a linear functional on a space of smooth test functions on $\mathbf{R}^n$. Any such T has (by definition) partial derivatives of every order, which are also Schwartz distributions. Soon afterward, deRham's theory of currents on a smooth manifold V appeared [Rh55]. He was motivated primarily by questions in algebraic topology and differential geometry. However, deRham's currents turned out to provide a very convenient framework for studying questions in geometric measure theory. This connection was first made in [FF60].

For simplicity, we consider only $V = \mathbf{R}^n$. A current of dimension k is defined as a linear functional on a space of $\mathcal{D}_k$ of smooth differential forms ω of degree k, which have compact support. (deRham calls T a current of degree $n-k$.)

Example 5.1 Let $k = n$ and g an integrable function on $\mathbf{R}^n$. The corresponding current T_g of dimension n (degree 0) satisfies for every smooth test function ϕ on $\mathbf{R}^n$

$$T_g(\omega) = \int_{\mathbf{R}^n} g(x)\phi(x)dx \tag{5.1}$$

where $\omega = \phi(x)dx_1 \wedge \cdots \wedge dx_n$ is the corresponding differential form of degree n.

Example 5.2. Let $S \subset \mathcal{M}$, where $\mathcal{M}$ is a smooth k-dimensional submanifold of $\mathbf{R}^n$ and S has an orientation specified by a continuously varying unit tangent k-vector $\tau(x), x \in S$. The associated current T_S is defined by

$$T_S(\omega) = \int_S \omega = \int_S \omega(x) \cdot \tau(x) dH^k(x), \tag{5.2}$$

for all $\omega \in \mathcal{D}_k$. In (5.2), $\tau(x) = |\alpha(x)|^{-1}\alpha(x)$, where $\alpha(x) = v_1(x) \wedge \cdots \wedge v_k(x)$ and $v_1(x), \ldots, v_k(x)$ are linearly independent tangent vectors to $\mathcal{M}$ at x. See the Appendix for notations. The unit k-vector $\tau(x)$ is determined, up to sign, by the order of these basis vectors $v_1(x), \ldots, v_k(x)$ for the tangent space at x. Note that $-T_S$ has the opposite orientation from T_S.

For any current T of dimension $1 \leq k \leq n$, the boundary ∂T is defined as the current ∂T of dimension $k-1$ such that

$$\partial T(\omega) = T(d\omega) \tag{5.3}$$

for all $\omega \in \mathcal{D}_{k-1}$ where the k-form $d\omega$ is the exterior differential of ω. Formula (5.3) includes as a special case the classical theorem of Gauss-Green in (3.1). In this case $k = n-1$ and $S = B$ is the (smooth) boundary of E. The $(n-1)$-vector $\tau(x)$ in (5.2) is the adjoint of the unit exterior normal to E at the point $x \in B$. The $(n-1)$-form ω in (5.2) is adjoint to the 1-form determined by ζ in (3.1). See [Fl77, Section 8.7]. The classical Stokes formula for surfaces in $\mathbf{R}^3$ can also be rewritten in the form (5.3). Let S be a smooth surface in $\mathbf{R}^3$ with smooth boundary C, and with consistent orientations chosen for S and C. Let T_C and T_S denote the corresponding currents of dimensions 1,2 respectively. The classical Stokes' formula is equivalent to $T_C(\omega) = T_S(d\omega)$

for every 1 form $\omega \in \mathcal{D}_1$. In the classical statement of Stokes' formula, the adjoint of the 2-form $d\omega$ corresponds to the curl of ω. See [Fl77, Sec. 8.8].

b) Young's generalized parametric surfaces. L.C. Young is known for his work during the 1930s on generalized curves. This early work provided solutions to calculus of variations and optimal control problems with nonconvex integrands, which may have no solution in the traditional sense. A generalized curve solution involves an ordinary curve C, to which is attached a measure-valued function on the set of possible tangent vectors at each point of C. See [Y37].

In 1948, Young extended the idea of generalized curve to nonparametric double integral problems in the calculus of variations [Y48a,b]. He obtained "generalized solutions" in the form of a pair of functions f, μ on a region D in $\mathbf{R}^2$. For $(u, v) \in D$, $f(u, v)$ is real valued and $\mu(u, v)$ is a probability measure on a space of possible gradient vectors. This led to the concept of Young measures, which later provided the basis for the study of minimum energy configurations in solid mechanics. The Tartar-Murat method of compensated compactness makes essential use of Young measures. Subsequently, they were also applied to problems in such diverse areas as hyperbolic PDEs, microstructures and phase transitions.

In the seminal paper [Y51], Young defined the notion of generalized parametric surface of dimension $k = 2$. One of his goals was to provide an alternative to the surface area theory formulations of geometric problems in the calculus of variations, which would also apply in any dimension $k \geq 2$. A generalized surface of any dimension k is defined as a nonnegative linear functional L on a space $\mathcal{E}_k$ of continuous functions $F(x, \alpha)$, with $x \in \mathbf{R}^n$ and α a k-vector. Such functions F are called geometric variational integrands (Section 11). His approach allowed the use of methods based on weak convergence and convex duality arguments. However, Young had a broader vision, including for example a possible Morse theory in terms of generalized surfaces. This was expressed, for example, in the introduction to his paper

[Y62]. In modified form, various parts of Young's vision were later achieved by Young himself, and also by others in the framework of integral currents, Whitney-type flat chains and Almgren's varifolds.

Young was my PhD thesis advisor, and he had a profound influence on my mathematical career. We wrote three joint papers on generalized surfaces [FY54, 56a,b], which are mainly of historical interest now. Some results in [FY56b] were precursors of later results about rectifiable and integral currents.

c) Whitney's geometric integration theory. H. Whitney's book [Wh57] was another influential source of ideas for developments in GMT soon afterward. Whitney began by asking what a theory of k-dimensional integration in $\mathbf{R}^n$ should look like. A central role is played by the spaces $\mathcal{P}_k(\mathbf{R}^1)$ of polyhedral chains P of dimension k with real coefficients, defined as follows.

A polyhedral convex cell σ is a bounded subset of some k dimensional plane $\pi \subset \mathbf{R}^n$, such that σ is the intersection of finitely many half k-planes of π. Each $P \in \mathcal{P}_k(\mathbf{R}^1)$ is a finite linear combination of oriented nonoverlapping polyhedral convex cells σ_i

$$P = \sum_i a_i \sigma_i \tag{5.4}$$

with real coefficients a_i. The mass $M(P)$ and boundary ∂P are defined by

$$M(P) = \sum_i |a_i| M(\sigma_i), \ \ \partial P = \sum_i a_i \partial \sigma_i.$$

Two possible norms on $\mathcal{P}_k(\mathbf{R}^1)$ were considered, called the flat and sharp norms. The flat $W(P)$ norm turned out to be particularly useful for subsequent developments in GMT. It is defined as

$$W(P) = \inf_{Q,R} \{M(Q) + M(R) : P = Q + \partial R\} \tag{5.5}$$

where Q, R are polyhedral chains with elementary k- and $(k+1)$-dimensional mass $M(Q)$ and $M(R)$. Whitney was particularly interested in characterizing the dual spaces to $\mathcal{P}_k(\mathbf{R}^1)$ with either flat or sharp norms. Elements of these dual spaces are called cochains, denoted by X in [Wh57]. Under either flat or sharp norm, the dual space contains all cochains which correspond to smooth differential forms of degree k. The cochain X_ω associated with such a k-form ω acts on P via

$$X_\omega(P) = \sum_i a_i T_{\sigma_i}(\omega)$$

with T_{σ_i} as in (5.2).

In [Wh57, Chap. 4] it is shown that, under the flat norm, a cochain corresponds to what Whitney called a flat differential form, which is defined pointwise in terms of directional derivatives. The perspectives of [Rh55] and [Wh57] are quite different. In [Rh55] deRham considered only "cochains" corresponding to smooth differential forms of degree k. These have the role of test functions in deRham's theory. The class of deRham's currents of dimension k is very large, including many currents which have no geometric properties at all. In contrast, Whitney's polyhedral chains correspond to test functions, and the large dual spaces include many cochains which do not correspond to differential forms in the usual sense.

For any abelian group G, the spaces $\mathcal{P}_k(G)$ and the Whitney flat norm can be defined as in (5.4)-(5.5), with coefficients $a_i \in G$. Of particular interest for the discussions in Sections 6 and 7 are the groups $\mathbf{Z}$ of integers and $\mathbf{Z}_2$ of integers mod 2.

6 Rectifiable and integral currents.

During the academic year 1958-59, Federer and I wrote the paper *Normal and integral currents* [FF60], for which we later received a Steele Prize from the American Mathematical Society. A brief overview of the principal motivations and results of this paper is given in this section, and in Section 8. As already mentioned, one of the goals of GMT is to provide a theory of integration over k-dimensional "surfaces" without traditional smoothness assumptions. In [FF60], this is addressed in a systematic way. The Introduction to [FF60] begins with the following paragraph (written by Federer):

"Long has been the search for a satisfactory analytic and topological formulation of the concept "k dimensional domain of integration in euclidean n-space." Such a notion must partake of the smoothness of differentiable manifolds and of the combinatorial structure of polyhedral chains with integer coefficients. In order to be useful for the calculus of variations, the class of all domains much have certain compactness properties. All these requirements are met by the *integral currents* studied in this paper."

In the discussion which follows, we refer to the Appendix for notation and definitions. We consider currents T with compact support spt$T$. $M(T)$ denotes the mass of a current T. Three types of convergence of sequences of currents are of interest: weak, strong and in the Whitney flat distance. As in [FF60], a current T is called *normal* if $N(T) = M(T) + M(\partial T)$ is finite. In particular, if g is a BV function on $\mathbf{R}^n$, then the current T_g defined by (5.1) is normal.

Any Lipschitz mapping f from $\mathbf{R}^m$ into $\mathbf{R}^n$ induces a corresponding mapping $f_\#$ of normal currents [FF60, Sec. 3]. If T is normal on $\mathbf{R}^m$, then $f_\#(T)$ is normal on $\mathbf{R}^n$.

Rectifiable currents. As in [FF60, Section 3], a current T is called *rectifiable* if, for every $\varepsilon > 0$, there exists an integral polyhedral chain P and a Lipschitz mapping f such that $M(T - f_\#(P)) < \varepsilon$. From [FF60, pp. 500-502], any rectifiable current T has the following representation. There exist a bounded k-rectifiable set K, and for H^k-almost all $x \in K$ there exist $\Theta(x)$ with positive integer values and an approximate tangent vector $\tau(x)$ with $|\tau(x)| = 1$, such that:

(6.1) $$T(\omega) = \int_K \omega(x) \cdot \tau(x)\Theta(x) dH^k(x), \forall \omega \in \mathcal{D}_k.$$

(6.2) $$M(T) = \int_K \Theta(x) dH^k(x).$$

In view of (6.2), $M(T)$ is also called the k-area of the rectifiable current T, and $\Theta(x)$ represents the number of times x is counted.

Integral currents. A current T is called *integral* if both T and its boundary ∂T are rectifiable currents.

Highlights of [FF60]. Among the main results are the following:

(a) **Deformation Theorem** [FF60, Thm. 5.5.]. This essential tool provides polyhedral chain approximations to integral currents in the Whitney flat metric (and hence also in the sense of weak convergence). The cells of the approximating integral polyhedral chains belong to the k dimensional skeleton of a cubical grid in $\mathbf{R}^n$. An immediate consequence

of the Deformation Theorem is a result [FF60, Thm. 5.11] which provides a way to characterize homology groups for subsets of $\mathbf{R}^n$. Such subsets need not be smooth manifolds, but are required to have a local Lipschitz neighborhood retract property.

(b) **Isoperimetric inequalities.** Other very useful tools are the isoperimetric inequalities for currents [FF60, Sec. 6]. The proofs rely on the Deformation Theorem. In [FF60, Remark 6.6] the best isoperimetric constant is obtained, by an argument due to Federer, through which the inequality [FF60, Corollary 6.5] was originally discovered.

(c) **Weak and flat convergence.** In [FF60,Sec. 7] it is shown that weak convergence of a sequence T_j to T as $j \to \infty$ is equivalent to convergence in the Whitney flat distance, provided that sptT_j is a subset of a fixed compact set and $N(T_j)$ is uniformly bounded. Again, the Deformation Theorem has an essential role in the proof.

(d) **Closure Theorem [FF60, Th. 8.12].** This result says that, if T_j is a sequence of integral currents such that $N(T_j)$ is uniformly bounded, sptT_j is contained in a compact subset of $\mathbf{R}^n$ and $T_j \to T$ weakly as

$j \to \infty$, then T is also an integral current. A corollary of the Closure Theorem is the following result [FF60, Cor. 8.13]: for any positive constants c, r, the set of integral currents T such that $N(T) \leq c$ and $\mathrm{spt}T \subset \mathcal{B}_r(0)$ is weakly compact.

(e) **Strong approximation Theorem.** [FF60, Thm. 8.22] provides the following results, which further justify the idea that (in a measure theoretic sense) any integral current T of dimension k nearly coincides with a finite sum of pieces of oriented smooth manifolds. In fact, there exist sequences of integral polyhedral chains T_j and diffeomorphisms f_j converging to the identity map, such that $N[f_{j\#}T_j - T]$ tends to 0 as $j \to \infty$. This remarkable result is due entirely to Federer. The convex cells of each polyhedral chain T_j are assigned orientations, consistent with the orientations of their boundaries. This imposes consistent orientations on the limit T. In the representation formula (6.1), the k-vector $\tau(x)$ prescribes the orientation of the approximate tangent space at x.

(f) **Minimal currents.** [FF60, Sec. 9] is concerned with integral currents which minimize k-area, subject to given boundary conditions. Some of these results are mentioned in Section 8.

Late in the autumn of 1958, I found a method which proved a result similar to the Closure Theorem in part (d) above, and mentioned it to Federer. My result was stated in terms of Young's generalized surfaces. Quite independently, Federer had developed other parts of a theory of integral currents. He soon convinced me of the advantages of the deRham current setting. We then began an intensive joint effort through the rest of the academic year 1958-59 and summer 1959. Federer undertook the task of organizing our results into the systematic and coherent form in which [FF60] appears.

Other results. During the 1960s, Federer wrote several important papers in addition to his monumental book [Fe69]. His last publication on surface area theory was [Fe61]. In it he used GMT methods to study surfaces of dimension $k \geq 2$, defined by parametric representations and with finite Lebesgue area. The influential paper [Fe65] has created linkages between Riemannian, complex and algebraic geometry. The technique of slicing for normal currents was introduced in this paper. Another important result in [Fe65] is his proof of mass minimality for complex subvarieties of Kähler manifolds. This led to the subject of calibration theory.

7 Integral currents mod 2.

With any integral current T is associated orientations of its approximate tangent spaces. If orientations are ignored, a different formulation is needed. One approach, pioneered in Ziemer's thesis [Z62], is to consider integral currents modulo 2. The integral current $-T$ has the opposite orientations to that of T. Since $-T + T = 0$, the current $2T$ should be identified with 0.

Ziemer began by considering what he called flat classes of dimension k. These are elements of the quotient group $W_k^2 = W_k/2W_k$, where W_k is the additive group of currents T of the form $T = R + \partial S$, where R and S are rectifiable currents. The boundary $\partial\mathcal{T}$ and mass $M(\mathcal{T})$ of any flat class $\mathcal{T}$ are well-defined. A flat class of finite mass is called a rectifiable class. Integral currents mod 2 are elements of $I_k^2 = I_k/2I_k$, where I_k is the additive group of integral currents of dimension k. If $\mathcal{T}$ is an integral current mod 2, then $N(\mathcal{T}) = M(\mathcal{T}) + M(\partial\mathcal{T})$ is finite.

In [Z62], Ziemer showed that counterparts of most of the main results about integral currents in [FF60] remain true for integral currents mod 2. In [FF60], weak convergence of sequences of integral currents had an important role. Weak convergence is not defined in the mod 2 case, but fortunately a version of the Whitney distance (mentioned in Section 6) is still available. During the 1960s, the theory of integral currents mod 2 developed further. See [Fe69, Sec. 4.2.26] for a concise presentation of these results.

In [Fl66] a different formulation was considered. It begins with the space $\mathcal{P}_k(G)$ of polyhedral chains with coefficients in a finite abelian group G (Section 5). When $G = \mathbf{Z}_2$, this amounts to ignoring orientations of polyhedral chains. The Whitney distance between polyhedral chains P_1 and P_2 is $W(P_1 - P_2)$. Let $C_k(G)$ be the W-completion of $\mathcal{P}_k(G)$. The elements of $C_k(G)$ are called *flat chains* over G. Rectifiable flat chains are defined in the same way as the definition of rectifiable currents in Section 6. The main result of [Fl66] states that every flat chain of finite mass is rectifiable. A consequence [Fl66, Cor. 7.5] is the important analogue of the Closure Theorem for integral currents, mentioned in Section 6(d).

8 Higher dimensional Plateau problem.

As mentioned at the end of Section 2(b), by the late 1950s it was clear that entirely new formulations and methods were needed to study the Plateau (least area) problem for surfaces of any dimension $k \geq 2$. The first major step in that direction was Reifenberg's paper [Re60]. In his formulation, a "surface" is a closed set $S \subset \mathbf{R}^n$ with $H^k(S) < \infty$. A closed set $B \subset S$ is called the boundary if an appropriate relationship in terms of Čech homology groups holds. Reifenberg proved that, given the boundary B, a set S^* which minimizes $H^k(S)$ exists. There were no earlier results to guide Reifenberg in this effort. His methods had to be invented "from scratch" and required remarkable ingenuity.

Oriented Plateau problem. Another formulation (often called the oriented Plateau problem) is in terms of integral currents. In this formulation, a rectifiable current B of dimension $k-1$ with $\partial B = 0$ is given. The problem is to find an integral current T^* which minimizes the mass (or k-area) $M(T)$ among all integral currents T with $\partial T = B$. Since $M(T)$ is weakly lower semicontinuous, the existence of a minimizing T^* is immediate from the weak compactness property mentioned at the end of Section 6(d). There remained the difficult task of describing regularity properties of T^*. This is the topic of Sections 9 and 10.

In [FF60, pp.518-9] the following monotonicity property was proved, which has had quite a useful role in later developments. Let T^* be mass minimizing, and $x \in$ spt $T^* -$ spt B. For $r > 0$, let T_r^* denote the part of T^* in the ball $\mathcal{B}_r(x)$ with center x and radius r. Then $r^{-k}M(T_r^*)$ is a nondecreasing function of r. The density $D(x)$ of T^* at x is defined as the limit as $r \to 0^+$ of $a(k)^{-1}r^{-k}M(T_r^*)$, where $a(k)$ is the k-area of the unit spherical ball in $\mathbf{R}^k$. Then $D(x) \geq 1$ for every $x \in$ spt $T^* -$ spt B and D is an upper semicontinuous function of x. Another useful concept introduced in [FF60, Sec. 9] is that of tangent cones to mass minimizing integral currents.

Nonoriented versions. In [Re60], orientations play no role. Other formulations of the Plateau problem which disregard orientations are in terms of integral currents mod 2 or Whitney's flat chains with coefficients in the group $\mathbf{Z}_2$ (Section 7). Since analogues of the Closure Theorem in Section 6 hold in either of these formulations, existence of a minimum follows in the same way as before. Yet another formulation is in terms of Almgren's varifolds (Section 12).

9 Regularity properties.

Once the existence of a solution to the higher dimensional Plateau problem is known, there remains the problem of proving regularity properties of its support. This problem has been extremely challenging, but remarkable progress was made during the 1960s and afterward. We give only a brief summary of these results, and refer the reader to other expository publications [AF93] MF00].

In [Re60], Reifenberg proved that his solution S^* (see Section 8) is topologically a k-dimensional disk in a neighborhood of H^k- almost every nonboundary point $x \in S^*$. The lower density equals 1 at such points. He then published a paper on his important "epiperimetric inequality" [Re64a] and a sequel [Re64b] in which he used the epiperimetric inequality to prove an "almost everywhere" regularity result. It states that the topological k-disks in [Re60] are smooth manifolds. In 1964 Reifenberg's promising career ended tragically in a fatal mountaineering accident.

Regularity for oriented Plateau problem. This problem is to prove smoothness of $\operatorname{spt} T^* - \operatorname{spt} \partial T^*$ for any integral current T^* which minimizes k-area, except at points of a singular set of lower Hausdorff dimension.

The result in [Fe65] about mass minimality of complex subvarieties, mentioned at the end of Section 6, provides a rich class of examples in which $\operatorname{spt} T^* - \operatorname{spt} \partial T^*$ can have a singular set of Hausdorff dimension $k-2$. In these examples, $k = 2\ell, n = 2m$ and the Kähler manifold is complex m dimensional space C^m, identified with $\mathbf{R}^{2m}$. The subvarieties corresponding to locally area minimizing integral currents are obtained by setting a finite number of homomorphic functions on C^m equal to 0. For instance, the equation $z_1 z_2 = 0$ in C^2 gives an example in which $T^* = T_1^* + T_2^*$, with $T_1^* = \pi_1 \cap \mathcal{B}, T_2^* = \pi_2 \cap \mathcal{B}$ where $\mathcal{B} = \mathcal{B}_1(0)$ is the unit ball in $\mathbf{R}^4$ and π_1, π_2 are mutually orthogonal planes which intersect at 0. At the singular point 0 the density is $D(0) = 2$ and $D(x) = 1$ at all other $x \in \operatorname{spt} T^*$.

The earliest "almost everywhere" regularity result was due to De Giorgi [DG61b, Thm. VII], for dimension $k = n-1$. It is stated in terms of reduced boundaries of sets of finite perimeter. Techniques used to prove this result appeared in a companion paper [DG61a]. In terms of integral currents, De Giorgi's regularity result implies that, for $k = n-1$, $\operatorname{spt} T^*$ is locally a smooth manifold near any point x with density $D(x) = 1$. By using an argument in [Fl62, Sec.3] De Giorgi's result then implies a corresponding almost everywhere regularity result about integral currents in $\mathbf{R}^n$ which minimize $(n-1)$-dimensional area. However, very few regularity results about the oriented Plateau problem were known during the 1960s for $k \leq n-2$.

Almgren's work on regularity. Beginning in the 1960s, Almgren was a leading contributor to results on regularity. His paper [AF68] represented a major advance. In it he obtained almost everywhere regularity results not only for the Plateau problem in all dimensions, but for a much broader class of geometric variational problems in which the integrand satisfies a suitable ellipticity condition. (Section 11.)

The regularity issue for the oriented Plateau problem is especially challenging. Almgren wrestled with it for several years. After persistent, courageous efforts he produced a massive manuscript often called his "Big Regularity Paper." In it he showed that singular sets for the higher dimensional oriented Plateau problem indeed have Hausdorff dimension at most $k-2$. If $\sum$ denotes the singular set, this means that $H^{k-2+\epsilon}(\sum) = 0$ for any $\epsilon > 0$. The Big Regularity Paper has appeared in book form [AF00].

[Fe69, Sec.5.3-5.4] gives a systematic account of many results about regularity for the Plateau and other geometric variational problems as of the publication date. The regularity problem turned out to be less daunting for nonoriented versions of the Plateau problem. In [Fe70] Federer showed that the singular set has Hausdorff dimension at most $k-2$, for the Plateau problem formulated either using integral currents mod 2 or in Reifenberg's formulation.

Boundary regularity. Let T^* minimize $(n-1)$-area among all integral currents with $\partial T^* = B$, where $\operatorname{spt} B$ is a smooth oriented $(n-2)$-dimensional manifold. In Allard's 1968 PhD thesis, he considered the regularity of $\operatorname{spt} T^*$ near points of $\operatorname{spt} B$. The main results were announced in [AW69]. Regularity of $\operatorname{spt} T^*$ is proved near any boundary point x where the density of the total variation measure is 1/2. Sufficient geometric conditions for boundary regularity to hold at all points of $\operatorname{spt} B$ were also given. Later, other authors obtained regularity at all boundary points without these geometric conditions on $\operatorname{spt} B$.

10 Regularity results for $k = n-1$, Bernstein's Theorem.

It seemed at first that $(n-1)$-dimensional area minimizing integral currents might have no singular points. This was proved in [Fl62] for $n = 3$. Closely related to the regularity question in dimension $n-1$ is the question of whether the only cones in $\mathbf{R}^n$ which locally minimize $(n-1)$-area are hyperplanes. Using this connection, De Giorgi [DG65], Almgren [AF66] and Simons [SJ68] showed that there are no singular points for $n \leq 7$. However, Bombieri, De Giorgi and Giusti [BDG69] considered an example of a cone in $\mathbf{R}^8$ which provides a seven dimensional area minimizing integral current with a singularity at the vertex. This example (due to Simons) is as follows. Write $\mathbf{R}^8 = \mathbf{R}^4 \times \mathbf{R}^4$ and $x = (x', x'')$ with $x', x'' \in \mathbf{R}^4$. The cone satisfies $|x'| = |x''|$. Its intersection with any ball $\mathcal{B}^r(0)$ in $\mathbf{R}^8$ with center 0 defines a 7 dimensional integral current T. Simons conjectured and it was later proved in [BDG69] that T is area minimizing. The vertex 0 is a singular point. Federer [Fe70] showed that this example is generic in the following sense: if $k = n-1$, then the singular set for the oriented Plateau problem can have Hausdorff dimension at most $n-8$.

The classical Bernstein Theorem is as follows. Let f be a smooth, real valued function which satisfies the minimal surface PDE for all $x \in \mathbf{R}^2$. Then f is an affine function (equivalently, the graph of f is a plane.) A GMT proof of this well known result was given in [Fl62]. It used the monotonicity property (Section 8) and the result [Fl62], Lemma 2.2] that cones in $\mathbf{R}^3$ which locally minimize area must be planes. An interesting question was whether the corresponding result about smooth solutions f to the minimal surface PDE in all of $\mathbf{R}^m$ must be true when $m > 2$. This was proved by De Giorgi [DG65] for $m = 3$. In his proof, he showed that falsity of the Bernstein Theorem for functions on $\mathbf{R}^m$ would imply the existence of non-planar locally area minimizing cones in $\mathbf{R}^m$ of dimension $m-1$. For $m = 3$, this allowed

De Giorgi to use the same result as in [Fl62] about 2 dimensional locally area minimizing cones in $\mathbf{R}^3$. Making use of the same idea, Almgren and Simons then proved the Bernstein Theorem for $4 \leq m \leq 7$. However, the Bernstein Theorem is not correct for $m \geq 8$.

I was visiting Stanford in the Spring of 1969 when the startling news about this negative result arrived there. D. Gilbarg (an authority on nonlinear PDEs) was perplexed. It was unheard of that a result about PDEs should be true in 7 or fewer variables, but not in more variables. However, Gilbarg wisely observed that the Bernstein Theorem is really a geometric result, not a result about PDEs.

11 Geometric variational problems, generalized surfaces.

A real-valued function $F = F(x, \alpha)$ is often called a *geometric variational integrand* if F is continuous and satisfies, for every $x \in \mathbf{R}^n$ and simple k-vector α

$$F(x, c\alpha) = cF(x, \alpha), \text{ if } c \geq 0. \tag{11.1}$$

In the literature on surface area theory and its applications in the calculus of variations surfaces were defined using parametric representations. The homogeneity condition (11.1) was imposed to guarantee that the criterion J to be minimized does not depend on the particular parametric representation chosen for a surface. For example, in the classical Plateau problem, (Section 2b), J is the area $A(f)$ of the surface, where

$$A(f) = \iint\limits_D |\frac{\partial f}{\partial u_1} \wedge \frac{\partial f}{\partial u_2}| du_1 du_2.$$

In this case, $F(\alpha) = |\alpha|$.

Let G_k^n denote the oriented Grassmannian, with elements simple k-vectors α such that $|\alpha| = 1$. Since we consider in this section only such k-vectors,

F can equivalently be regarded as a function in $C(\mathbf{R}^n \times G_k^n)$. Given such an integrand F, we can consider the problem of minimizing

$$J(T,F) = \int_{\mathbf{R}^n} F(x,\tau(x))\Theta(x)dH^k(x) \tag{11.2}$$

among all integral currents T with given boundary $B = \partial T$. In (11.2), $\tau(x)$ and $\Theta(x)$ are as in (6.1) with $\Theta(x) = 0$ for $x \notin K$. In particular, if $F(x,\alpha) = |\alpha|$, then $J(T,F) = M(T)$ is the mass (or k-area) of T.

A solution T^* to this minimum problem exists under the following assumptions: there exist positive constants k_1, k_2 such that

$$k_1 \leq F(x,\alpha) \leq k_2 \text{ for all } (x,\alpha) \in \mathbf{R}^n \times G_k^n, \tag{11.3}$$

$$F \text{ is semi–elliptic.} \tag{11.4}$$

Condition (11.4) is defined in the Appendix, and a proof of the existence of T^* is sketched there. If F satisfies a stronger ellipticity condition, then any integral current T^* which minimizes $J(T,F)$ has an almost everywhere regularity property [Fe69,Sec.5.3]. This is very similar to a regularity result in [AF68], which is formulated in terms of varifolds.

Generalized surface solutions. When F is not semielliptic, then we can seek what Young called a "generalized solution" to the problem of minimizing $J(T,F)$ for given $F = F_1$ subject to a given boundary condition. As in Section 5, let $\mathcal{E}_k$ denote the space of all continuous geometric variational integrands F. Thus, $\mathcal{E}_k = C(\mathbf{R}^n \times G_k^n)$. In [Y51] Young defined a generalized surface of dimension k as any nonnegative linear functional L on $\mathcal{E}_k$, such that L has compact support. "Nonnegative" means that $L(F) \geq 0$ whenever $F(x,\alpha) \geq 0$ for all (x,α).

To any integral current T corresponds a generalized surface $L(T)$, such that $L(T)(F) = J(T,F)$ for all F as in (11.2). In the following discussion, for simplicity let us now admit only currents T with $\mathrm{spt}T \subset \mathcal{B}_r(0)$ for some fixed r. Let us fix an integrand $F = F_1$, and assume that F_1 satisfies (11.3) but not (11.4). Let

$$\Gamma_B = \{L(T) : T \text{ an integral current}, \ \partial T = B, \ \mathrm{spt} T \subset \mathcal{B}_r(0)\} \tag{11.5}$$

and $c\ell T_B$ the weak closure of Γ_B. Since the semi-ellipiticity condition (11.4) is not assumed, $L(T)(F_1) = J(T, F_1)$ may not have a minimum on Γ_B. We look instead for a generalized surface solution $L^* \in c\ell\Gamma_B$.

Since $L(F_1)$ is a weakly continuous function of L.

$$\inf_{\Gamma_B} J(T, F_1) = \inf_{\Gamma_B} L(T)(F_1) = \min_{c\ell\Gamma_B} L(F_1). \tag{11.6}$$

Denote the right side of (11.6) by ℓ_0, and choose L^* such that $L^*(F_1) = \ell_0$.

The restriction of any generalized surface L to the linear subspace $\mathcal{D}_k \subset \mathcal{E}_k$ is a current of dimension k, denoted by $T(L)$. It is shown in [Fl15, Appendix C] that L^* has the following representation. Let $T^* = T(L^*)$. Then T^* is integral, with corresponding rectifiable set K and integer valued multiplicities $\Theta(x)$ and approximate tangent k-vector $\tau(x)$ as in (6.1). Moreover,

$$L^*(F) = \int_K \Lambda_x(F)\Theta(x)dH^k(x), \text{ for all } F \in \mathcal{E}_k \tag{11.7}$$

$$\Lambda_x(F) = \int_{G_k^n} F(x, \alpha)d\lambda_x(\alpha) \tag{11.8}$$

and for H^k-almost all $x \in K$

$$\tau(x) = \int_{G_k^n} \alpha d\lambda_x(\alpha). \tag{11.9}$$

In addition, $1 \leq \lambda_x(G_k^n) \leq c$ for some constant c. In [Fl15], these measures λ_x are called *shape measures*. They are discussed in [Fl15,Appendix C].

By the Riesz representation theorem, to each generalized surface L corresponds a measure m_L on $\mathbf{R}^n \times G_k^n$ with compact support. Instead of expressing $L(F)$ as an integral with respect to m_L, equations (11.7)-(11.8) express

$L(F)$ as a double integral, as shown in more detail in [Fl15, Appendix C]. Formula (11.9) is obtained by considering functions in $\mathcal{D}_k$, which have the form $F(x, \alpha) = \omega(x) \cdot \alpha$ where $\omega(x)$ can be chosen arbitrarily and using formula (6.1).

Singular generalized surfaces. The notion of singular generalized surface had an important role in [Y51] and also [FY56]. L is called singular if the current $T(L) = 0$.

An important class of singular generalized surfaces is obtained as follows. Let U_j, R_j be a sequence of integral currents such that $U_j = \partial R_j, M(U_j)$ tends to a positive limit a and $M(R_j)$ tends to 0 as $j \to \infty$. For a subsequence of j, the generalized surface $L(U_j)$ tends weakly to a nonzero limit L^s as $j \to \infty$, L^s is a singular generalized surface, with $L^s(F_0) = a$, where $F_0(\alpha) = |\alpha|$.

For $k = 2, n = 3$ one can think (for instance) of R_j as a thin tentacle-like body. Another possibility is that R_j is composed of thin platelets, or of many small bubbles lying close to some 2-dimensional surface.

Singular generalized surfaces had a prominent role in the early literature, including [Y51] and [FY56]. In [Fl15,Appendix C] the following is posed as an open question. Does every $L \in c\ell\Gamma_B$ have a decomposition of the form $L = L^s + L^1$, where L^s is singular and L^1 has a representation of the form (11.7)-(11.9) above? When $L = L^*$ minimizes $L(F_1)$ on $c\ell\Gamma_B$, then L has no singular part L^s.

Example (L.C. Young). Let $k = 1, n = 2$ and $e_1 = (1.0), e_2 = (0, 1)$ the standard basis for $\mathbf{R}^2$. We denote any 1-vector in $\mathbf{R}^2$ by v rather than α. Let σ be the line segment from 0 to e_1, with B corresponding to the initial endpoint 0 and final endpoint e_1. In (11.7)-(11.9), let

$$K = [0, 1] \times \{0\}, \Theta(x) = 1, \tau(x) = e_1 \text{ for } x \in K, \text{and}$$

$$\lambda_x = \lambda = 2^{-\frac{1}{2}}(\delta^+ + \delta^-)$$

$$v^\pm = 2^{-\frac{1}{2}}(e_1 \pm e_2).$$

where $\delta^{\pm}$ is the Dirac measure a $v^{\pm}$. This generalized curve L^* is weak limit as $j \to \infty$ of L_j, where $L_j = L(P_j)$ and P_j is a "sawtooth shaped" polygon with endpoints 0 and e_1 and with j teeth. Each tooth is an isocoles right triangle with hypotenuse on K. Moreover, $T(L^*) = T_\sigma$. Let

$$F_1(x, v) = g(x \cdot e_2)h(v),$$

where $g(0) = 0, g(u) > 0$ for $u \neq 0, h(v^{\pm}) = 0$, and $h(v) > 0$ for $v \neq v^+$ or $v^-(|v| = 1)$. Then $L^*(F_1) = 0$ and hence L^* is minimizing.

12 Varifolds.

In the mid 1960s, Almgren initiated the theory of varifolds, which has become an important tool in GMT and its applications. A varifold is defined as a measure on $\mathbf{R}^n \times \tilde{G}^n_k$, where $\tilde{G}^n_k$ is defined similar to G^n_k without considering orientations. [SL83] provides a good introduction to varifolds.

In [AF68] existence and almost everywhere regularity results were given for a varifold version of the problem of minimizing $J(T, F_1)$, with given boundary conditions. Boundaries were defined in terms of singular homology groups [AF68, pp 334-335]. Another goal was the study of varifolds which are minimal (not necessarily area minimizing) in the sense that first-order necessary conditions for minimum k-area are satisfied. Allard's paper [AW72] was an important contribution in that direction. He considered the first variation δV of an integral varifold V, which can be represented in terms of mean curvature and exterior normals at the boundary if V is a smooth manifold. In [AW72, Section 6] a compactness result about integral varifolds was proved. In this result, the bound on $M(\partial T)$ for the corresponding compactness result about integral currents (Section 6(d)) is replaced by a bound on the norm of δV.

Despite the formal similarity in the definitions of Young's generalized surfaces and Almgren's varifolds few connections between these two theories seem to have been made. As a step toward bridging this gap, the method outlined in Section 11 to describe generalized surface solutions to minimum problems could perhaps be adapted to similar nonoriented versions. Let $\tilde{\mathcal{E}}_k = C(\mathbf{R}^n \times \tilde{G}^n_k)$. Varifolds with compact support can be regarded as non-negative linear functionals on $\tilde{\mathcal{E}}_k$. It is natural to replace integral currents by rectifiable flat chains A over the group Z_2 with $N(A) = M(A) + M(\partial A)$ finite (Section 7). Associated with A should be a rectifiable set K and unoriented approximate tangent k-vectors $\tau(x)$. In analogy with (11.2) define the varifold $V(A)$ by

$$\begin{aligned} V(A)(F) &= J(A, F) \\ &= \int_{\mathbf{R}^n} F(x, \tau(x))\Theta(x) dH^k(x), \end{aligned} \tag{12.1}$$

for every $F \in \tilde{\mathcal{E}}_k$, where $\Theta(x) = 1$ for $x \in K, \Theta(x) = 0$ for $x \notin K$. As in (11.5) given a rectifiable flat chain B with $\partial B = 0$, let

$$\mathcal{F}_B = \{V(A) : A \text{ a flat chain over } Z_2, \partial A = B, \text{ spt } A \subset \mathcal{B}_r(0)\} \tag{12.2}$$

Given F_1 satisfying (11.3), the infimum of $V(A)(F_1)$ over $\mathcal{F}_B$ is attained at some varifold $V^* \in c\ell\mathcal{F}_B$. In Young's terminology, V^* is a generalized solution to the minimum problem.

It is conjectured that a representation corresponding to (11.7)-(11-8) holds, in which the generalized surface L^* is replaced by varifold V^*, and $\mathcal{E}_k$ is replaced by $\tilde{\mathcal{E}}_k$. Formula (11.9) is no longer available in the nonoriented formulation. However, the shape measures λ_x can be defined in the same way as for the oriented version, using polyhedral approximations. A possible alternative to this formulation could be in terms of integral currents mod 2 (Section 7) rather than in terms of rectifiable flat chains in the sense of [Fl66].

Part II – Remembrances

13 GMT at Brown in the 1960s.

Both Bill Ziemer and I came to the Brown University Mathematics Department in September 1958, he as a PhD student and I as an Assistant Professor. Nearly all of the Mathematics Department faculty members were young. There was an atmosphere of camaraderie and excitement about mathematics. Most important for me, Herbert Federer was there. The winter of 1958-59 was when we did most of the work which resulted in our Normal and Integral Currents paper, discussed in Section 6. Brown students, at both undergraduate and graduate levels, were good. Fred Almgren and Bill Ziemer were among the students in my real analysis course, which I taught that year.

Graduate Students in GMT. During the 1960's there were six PhD students in geometric measure theory at Brown. Their names and year of completion of the PhD are as follows:

- Federer's PhD students: Frederick Almgren (1962), John Brothers (1964), Lawrence Ernst (1970) and Robert Hardt (1971).
- Fleming's PhD students: William Ziemer (1961) and William Allard (1968).

In the 1960s, the students who chose to work in GMT were entering a field which was just being invented. There was a chance to contribute something really new, not just to add a few more bricks to a long standing mathematical edifice. The number of "mathematical descendents" of our small program in GMT is much larger. As of March 2018, the Mathematics Genealogy Project website listed 188 students and "grandstudents" of these six former Brown PhD students.

Both Federer and I encouraged regular discussions with students. A great deal of learning happened in one-on-one conversations with faculty, other students and former students in GMT. The task of mastering a difficult mathematical field is challenging, and at times discouraging. I told Bill Ziemer to start by reading both [FF60] and Whitney's book [Wh57]. An unnamed source told me (much later) that while struggling with this assignment, Bill occasionally wondered whether some other career (such as construction worker) might be better than life as a mathematician. However, he overcame the difficulties admirably. His thesis was published as [Z62]. As mentioned in Section 7, it was the beginning of an important GMT research direction, with applications to the nonoriented Plateau problem.

Bill Allard's thesis provided the first results on boundary regularity for the oriented Plateau problem [AW68] [AW69]. This topic seemed scary to me, with either total success or nothing as possible outcomes. I hinted that he might try something safer, but Bill didn't agree to this.

Federer set very high standards for his mathematical work and expected high-quality research from his students. In addition to Federer's own work, the contributions of his many mathematical descendents (PhD students, "grandstudents" and "greatgrandstudents") continue to have a major impact on GMT. While some students found Federer's courses daunting, he was very welcoming to anyone who was deeply committed to mathematics and who took the trouble to get to know him. The sections of [P12] written by Allard, Hardt and Ziemer, are eloquent testimonials to the great regard and esteem which former students have for Federer.

All of our graduate students had duties as teaching assistants. Several of them also gave Federer and me substantial help with book projects. During the writing of the first edition of my textbook [Fl77], several students read various chapters. John Brothers carefully read the entire manuscript and furnished many improvements to it. After the PhD, he joined the Mathematics Department at Indiana University and remained there for his entire career.

Our students also read parts of the manuscript for Federer's book [Fe69], and Allard read all of it. The introduction to [Fe69] says: "William K. Allard read the whole manuscript with great care and contributed significantly, by many valuable queries and comments, to the accuracy of the final version."

Some remembrances of Fred Almgren are included in Section 14.

Visitors. Among visitors to Brown in the field of GMT were E.R. (Peter) Reifenberg in the summer of 1963 and Ennio De Giorgi in the spring semester of 1964. I had met both of them in August 1962 at a workshop in Genoa, Italy. That workshop was unusually lively and productive. Someone described the language of the workshop as "lingua mista" – a mixture of English, Italian and bad French.

Other visitors included J. Marstrand, J. Michael and Bill Ziemer (summer 1963). Ubiritan D'Ambrosio came as a postdoc starting in January 1964, and later returned to Brazil for a distinguished career in math education. D'Ambrosio was fluent in Italian, and was quite helpful during De Giorgi's seminar talks at Brown. Harold Parks was a junior faculty member at Brown during the 1970s.

14 Personal remembrances.

I conclude this article with some personal remembrances about Bill Ziemer and five other deceased mathematicians (Almgren, De Giorgi, Federer, Reifenberg and Young), who had leading roles in the development of GMT. My list of References includes scientific obituary articles for each of these five.

Bill Ziemer was my first PhD student. After his time at Brown, he joined the Mathematics Department at Indiana University, where he remained until retirement. After the thesis, Bill Ziemer went on to a distinguished research career, making lasting contributions to a wide variety of topics. In addition to GMT, these topics include: surface area theory, capacity-like functionals, weakly differentiable functions, elliptic and parabolic PDEs and least gradient problems. Among the most satisfying aspects of my career has been the opportunity to follow the successes of my former students, and to form long-lasting friendships with them. I remember well a number of pleasant and productive visits to Indiana University, during which I enjoyed Bill and Suzannne Ziemer's warm hospitality. Particularly memorable was Bill's 60th birthday conference in 1994. In addition to the excellent conference program, I enjoyed an opera at IU's fine Music School and a visit to a Wordsworth exhibit at the museum.

During his retirement years, Ziemer remained remarkably productive, despite health issues. He continued to write books and research articles, and won an Outstanding Paper Prize from the Mathematical Society of Japan in 2016 [MaZ15]. Bill and I collaborated on a National Academy of Sciences Bibliographical Memoir about Herbert Federer [FZ14]. With Suzanne's help, he continued to write books and research articles, and lectured despite difficulties in speaking during his last years. I remember his lecture at the 2011 mini conference in Federer's memory, held at Brown University. Suzanne provided the audio while Bill showed the visual materials for the lecture.

Frederick Almgren was an Engineering undergraduate at Princeton, then served three years as a US Navy airplane pilot. It was clear upon his arrival at Brown as a new PhD student that Fred had excellent intuition and original ideas. He was not yet trained to think like a mathematician, but this was soon remedied. After the PhD, he joined the Princeton Mathematics Department, and remained there until his death in 1997. At Princeton, his enthusiasm for mathematics and ability to locate beautiful problems which were "ready to be solved" attracted many PhD students. Besides Almgren's profound contributions to GMT mentioned in earlier sections of this article, he did (with coauthors) important work on such other topics as curvature driven flows, liquid crystals, energy minimizing maps and rearrangements.

Fred's PhD thesis was a brilliant one. In a curious episode, the Brown Graduate School initially hesitated to accept it, on the grounds that the thesis had already been accepted by the journal Topology. A very firm stand by Herb Federer persuaded the Dean to withdraw his objection.

My thesis advisor L.C. Young had expressed the need for a kind of Morse theory in terms of multivariable calculus of variations. Soon after the thesis, Fred provided such a theory in terms of what he called varifolds. Varifolds are defined in a way very similar to Young's generalized surfaces, but the name varifold is much more appealing.

A lot was happening in geometric measure theory during the years 1958-62 when Fred was at Brown. He and I ate lunch regularly in the cafeteria. During these lunches, Fred found out more or less all I knew and of course I learned a great deal from him in return. It was clear even then that the regularity problem for the higher dimensional Plateau problem (and for other geometric problems in the calculus of variations) was going to be extremely difficult. I have the greatest admiration for Fred's determination and persistence in wrestling with these regularity problems through many years, culminating in his massive regularity proof, reported in [AF00].

After Fred left Brown we saw each other less often. One such occasion was in summer 1965 when we were both visiting De Giorgi at the Scuola Normale in Pisa. I still have very pleasant memories of excursions with Fred and my wife Flo to Lucca and Siena, which are interesting towns nearby. He knew how to enjoy life during the times when he was not immersed in mathematics.

We were very pleased to have Fred Almgren as an honored guest at the 1988 Brown Commencement, when he received a Distinguished Graduate School Alumnus Award. Each year this award is given to two or three of Brown's most distinguished former PhD graduates.

Ennio De Giorgi studied in Rome with M. Picone. After one year 1958-1959 at the Università di Messina, he moved to the Scuola Normale Superiore, (SNS) in Pisa, and remained there for the rest of his life. Besides his many contributions to GMT, De Giorgi is renowned for his work on PDEs (including the De Giorgi-Nash a priori estimates in the 1950s), gamma convergence and other topics. In Pisa De Giorgi lived in simple accommodations in a residence along the Arno River which belonged to the SNS. He had a wide circle of friends, and he enjoyed good food, conversation and mountain walks.

Beginning in 1988, De Giorgi spent long periods in Lecce, which was his boyhood home and still is the home of his extended family. He established mathematical ties in Lecce, and the Mathematics Department at Università di Lecce is now named after him. Besides his contributions to mathematics, De Giorgi was deeply involved with charitable and human rights issues. He was a devout Christian, with nuanced views about relationships between science and faith.

In the spring of 1964, De Giorgi visited Brown and Stanford universities. He came by ship (the Cristoforo Colombo), and I met him in New York. During the auto trip from New York to Providence, De Giorgi told me that he had just proved the Bernstein Theorem for minimal surfaces of dimension 3 in 4 dimensional space (already mentioned in Section 10). However, there was no mathematics library on the Cristoforo Colombo, and he wished to be certain about one technical point which he needed in the proof. I assured him that what he needed is OK.

After the 1960s De Giorgi's work and mine took different directions. However, we kept up a lifelong friendship and saw each other from time to time, both in Pisa and elsewhere. Communication became easier as De Giorgi's English improved and I learned a little Italian. Besides his mathematical work, De Giorgi told me about his trips to Eritrea and his work for Amnesty International. Our last meeting was in 1993 at the 75th birthday conference for Cecconi in Nervi.

Herbert Federer was born in Vienna, and immigrated to the United States in 1938. He received his PhD in 1944 at the University of California Berkeley under the supervision of A.P. Morse, and then served in the U.S.

Army at the Ballistics Research Laboratory in Aberdeen, MD. In the fall of 1945, he joined the Department of Mathematics at Brown University, where he remained until his retirement in 1985.

Federer is remembered for his many deep and original contributions to the fields of surface area and geometric measure theory (GMT). It is difficult to imagine that the rapid growth of GMT (beginning in the 1950s), as well as its subsequent influence on other areas of mathematics and applications, could have happened without his groundbreaking efforts.

He was fair-minded and very careful to give proper credit to the work of other people. He was also generous with his time when serious mathematical issues were at stake. He was the referee for John Nash's 1956 *Annals of Mathematics* paper "The imbedding problem for Riemannian manifolds," which involved a collaborative effort between author and referee over a period of several months. In the final published version, Nash stated, "I am profoundly indebted to H. Federer, to whom may be traced most of the improvements over the first chaotic formulation of this work." This paper provided the solution to one of the most daunting and longstanding mathematical challenges of its time.

I first met Herb Federer at the summer 1957 AMS Meeting at Penn State. Afterwards, he suggested to the Mathematics Department at Brown that I might be offered an assistant professorship. An offer was made, which I accepted. Upon our arrival in Providence in September 1958, my wife Flo and I were warmly welcomed by Herb and Leila Federer. The academic year 1958-1959 was the most satisfying time of my career. Most of our joint work on normal and integral currents was done then. This involved many blackboard sessions at Brown, as well as evening phone calls at home (no Skype in those days). Herb undertook the task of organizing our results into a systematic coherent form, which appeared as [FF60].

In later years, after our research paths had taken different directions, Federer and I didn't often discuss mathematics. However, we always brought each other up to date about family news.

Ernst (Peter) Reifenberg was a student of A.S. Besicovitch at Cambridge University. After a postdoctoral position at Berkeley, he joined the Mathematics Department at the University of Bristol. During the 1950s, he wrote a series of papers on surface area theory. He then produced the remarkable Acta Mathematica paper [Re60], and the subsequent papers [Re64a,b] on the epiperimetric inequality and its applications (Sections 8 and 9).

I first met Peter in Italy during August 1962. He and his wife Penny met my wife Flo and me at the Milan airport. We followed them in a rented car to Genoa to participate in a very productive workshop on GMT organized by J.P. Cecconi, already mentioned in Section 13. After the workshop, Flo and I met the Reifenbergs again at Zermatt, just after they had climbed the Matterhorn.

My friendship with Peter continued through mail correspondence and his visit to Brown during the summer of 1963. During the summer of 1964 we got news of his death while climbing with Penny in the Dolomites. It was caused by falling rocks, and not at all due to carelessness on Peter's part. A few months after his death, Penny Reifenberg sent to me his handwritten notes concerning regularity for the Plateau problem. I shared them with Fred Almgren. Those notes were too fragmentary to determine what new results Peter Reifenberg had obtained.

Peter was fearless in his approach to mathematics. He could be undiplomatic, which sometimes led to worthwhile mathematical outcomes. On one hot, humid Friday afternoon during Peter's 1963 visit to Brown, I outlined at the blackboard a possible method to prove the main result of [Fl66]. Peter's skepticism was probably intended as friendly advice, but was expressed in a negative way. I worked intensively over the following weekend, and had verified the essential details of my argument by Monday. A version of this proof had a key role later in Almgren's proof of an existence theorem for geometric calculus of variations problems [AF68,p.347].

Peter Reifenberg was surely destined for a brilliant future in mathematics. His premature death was a great loss to GMT, as well as to his family and friends.

Laurence Young came from a mathematical family. Both parents, William H. and Grace C. Young, were distinguished English mathematicians, whose research was at the forefront of real analysis in the early 20th century. Laurence Young studied mathematics at Trinity College, Cambridge, and later became a Fellow there. He interspersed his studies at Cambridge with extended stays in Munich, where the great Greek mathematician Carathéodory became a mentor. From 1938 to 1948, Young was at the University of Capetown, South Africa. He then moved to the University of Wisconsin Mathematics Department where he remained until his retirement.

I first heard about Young's generalized surfaces in a seminar lecture which he gave in the spring of 1950. Later that year he became my PhD thesis advisor. He was not a "hands on" kind of thesis advisor, providing detailed suggestions in regular meetings with me. However, at a more fundamental level, Young's guidance was excellent. In the autumn of 1950, he mentioned to me an interesting research problem. Young intended to work on this problem himself, and later (jokingly) said that "I stole his problem." This problem was the basis for my PhD thesis, which later appeared in revised form as [FY54]. In the spring of 1951, I gave a first draft of my thesis to Young. He pronounced it "unreadable," which was certainly true. The process of revision extended into the hot, humid summer of 1951. My wife, Flo, typed the thesis, without the aid of modern text processing technology. Our departure to begin my new job at the RAND Corporation in Santa Monica, California was delayed until late August.

Young had a wide range of interests beyond mathematics. He was gifted in languages, music and chess, and kept remarkably fit physically. In Madison, the Young family home was on the shore of Lake Mendota. In winters, he often skated from home to the University several miles away.

In the years soon after my PhD, I visited the University of Wisconsin several times for extended periods. My wife Flo and I always remembered the warm hospitality which the Young family showed us. The kindness of his wife Elizabeth Young, when our oldest son was born in 1954, was particularly appreciated.

In 1995, the University of Wisconsin Mathematics Department hosted a celebration of Laurence Young's 90th birthday. It included a "mini-conference", in which Bill Ziemer was one of two distinguished speakers, followed by a gala dinner in the evening.

Appendix

Throughout this paper, $\mathbf{R}^n$ denotes euclidean n-dimensional space with elements denoted by x or y. $\mathcal{B}_r(x) = \{y \in \mathbf{R}^n : |x - y| \le r\}$ is the ball with center x and radius r. For $0 \le k \le n, H^k(K)$ is the Hausdorff measure of a set K.

Approximate exterior unit normal. Let E be a set of finite perimeter, $\Phi = -\mathrm{grad} 1_E$ in the Schwartz distribution sense and B_r the reduced boundary of E (Sec. 3). The total variation measure μ of the vector-valued measure Φ satisfies $\mu(K) = H^{n-1}(K)$ for any Borel set $K \subset B_r$. At any point $y \in B_r$, the approximate exterior unit normal vector $\nu(y)$ is the pointwise derivative of Φ with respect to μ in the sense of [Z89,Sec.5.5].

Exterior algebra and differential forms. The textbook [Fl77] gives an introduction to these topics. A more complete development is given in [Fe69,Chap.1 and Sec.4.1.6].

k-vectors

α denotes a k-vector, $k = 1, \cdots, n$.

$\alpha \wedge \beta$ is the exterior product of a k-vector α and ℓ-vector β. Note that $\beta \wedge \alpha = (-1)^{k\ell} \alpha \wedge \beta$.

α is a simple k-vector if $\alpha = v_1 \wedge \cdots \wedge v_k$ with $v_1, \cdots, v_k \in \mathbf{R}^n$. The norm $|\alpha|$ of a simple k-vector α is the k-area of the parallelopiped

$$P = \{x = c_1 v_1 + \cdots + c_k v_k, 0 \le c_j \le 1, \text{ for } j = 1, \cdots, k\}.$$

Orientations. Any k-plane π has the form

$$\pi = \{x = x_0 + c_1 v_1 + \cdots + c_k v_k\}$$

with $x_0, v_1, \cdots, v_k \in \mathbf{R}^n, c_1, \cdots, c_k \in \mathbf{R}^1$ and $v_1, \cdots, v_k$ linearly independent. If $\alpha = v_1 \wedge \cdots \wedge v_k$, then $\tau = |\alpha|^{-1} \alpha$ has norm $|\tau| = 1$. This k-vector τ assigns an orientation to π, with $-\tau$ the opposite orientation.

k-covectors and differential forms. A k-covector ω is defined similarly as for k-vectors, with the space $\mathbf{R}^n$ of 1-vectors replaced by its dual space of 1-covectors. The dot product of ω and α is denoted by $\omega \cdot \alpha$. A differential form ω of degree k is a k-covector-valued function on $\mathbf{R}^n$. The norm (or comass) of ω is

$$\|\omega\| = \sup\{\omega(x) \cdot \alpha, x \in \mathbf{R}^n, \alpha \,\text{simple},\ |\alpha| = 1\}.$$

The exterior differential of a smooth k-form ω is a $(k+1)$-form denoted by $d\omega$.

Currents. As in Section 5, a current T of dimension k is a linear functional on a space $\mathcal{D}_k$ of smooth k-forms. ∂T denotes the boundary of T. The mass (or k-area) of T is

$$M(T) = \sup\{T(\omega) : \|\omega\| \leq 1\}.$$

Let $N(T) = M(T) + M(\partial T)$. T is called *normal* if $N(T)$ is finite. The support spt T of a current T is the smallest closed set $\Gamma \subset \mathbf{R}^n$ such that $T(\omega) = 0$ whenever $\omega(x) = 0$ for all x in some open set containing Γ. A Lipschitz function f from $\mathbf{R}^m$ into $\mathbf{R}^n$ induces a mapping $f_\#$ of normal currents on $\mathbf{R}^m$ to normal currents on $\mathbf{R}^n$ [FF60,Defn.3.5].

Types of convergence for sequences of currents. A sequence of currents $T_1, T_2, \cdots$ converges to T weakly if

$$T(\omega) = \lim_{j \to \infty} T_j(\omega),\ \text{for all}\, \omega \in \mathcal{D}_k.$$

Strong convergence of T_j to T means that $M(T_j - T) \to 0$ as $j \to \infty$. Convergence of a sequence of integral currents of T_j to T in the Whitney flat distance means that $W(T_j - T) \to 0$ as $j \to \infty$, where

$$W(T) = \inf_{Q,R}\{M(Q) + M(R) : T = Q + \partial R,\ Q, R \,\text{integral}\}.$$

Semi-ellipticity (Section 11). As in [Fe69,Sec.5.1.2], F is called *semi-elliptic* if the following holds. For fixed x, let $\Phi_x(\alpha) = F(x, \alpha)$. For any oriented polyhedral convex cell σ, let T_σ be the corresponding integral current. Then $J(T_\sigma, \Phi_x) \leq J(T, \Phi_x)$ for any integral current T such that $\partial T = \partial T_\sigma$.

Existence Theorem. In Section 11, the following result was stated.

Theorem. *If F satisfies* (11.3) *and* (11.4), *then there exists an integral current T^* which minimizes $J(T, F)$ among all integral currents T with $\partial T = B$.*

Let us sketch a proof. We first impose the additional restriction spt$T \subset \mathcal{B}_r(0)$ for some r (sufficiently large). Condition (11.4) implies semicontinuity of $J(T, F)$ under weak convergence [Fe69,Sec.5.1.5]. The Closure Theorem in Section 6(d) can then be invoked to obtain the existence of a minimizing T^*. We will show that spt$T^* \subset \mathcal{B}_R(0)$ for some constant R, which does not depend on r.

Since the minimum of $J(T, F)$ is a nonincreasing function of r, the positive lower bound on F in (11.3) implies that $M(T^*) \leq C_1$ for some constant C_1. Choose r_0 such that spt$B \subset \mathcal{B}_{r_0}(0)$ and let r_1 denote the smallest $\rho \in [r_0, r]$ such that spt$T^* \subset \mathcal{B}_\rho(0)$. If $r_0 < r_1$, let

$$T_s = T^* \cap \mathcal{B}_s(0)^c, \;\; A(s) = M(T_s), \;\; \ell(s) = M(\partial T_s)$$

for $r_0 \leq s \leq r_1$. By the Eilenberg inequality [FF60,Cor.3.10]

$$(A.1) \qquad \int_s^{r_1} \ell(v) dv \leq A(s) \leq C_1.$$

Moreover, T_s minimizes $J(U, F)$ among all integral currents U with spt$U \subset \mathcal{B}_r(0)$ and $\partial U = \partial T_s$. We choose $U = U_s$ such that the isoperimetric inequality in [FF60,Remark6.2] holds. By using again (11.3), for some constants C_2, C_3

$$(A.2) \qquad A(s) \leq C_2 M(U_s) \leq C_3 |\ell(s)|^\lambda,$$

where $\lambda = k/(k-1)$. Let $t = r_1 - s$ and let $\psi(t)$ denote the left side of (A.1). By (A.2), for some constant C_4

$$(A.3) \qquad \psi(t)^{\frac{1}{\lambda}} \leq C_4 \psi^{'}(t), \;\; \psi(0) = 0.$$

An elementary calculation gives, with $\mu = k^{-1}$,

$$t \le C_5[\psi(t)]^\mu \le K \tag{A.4}$$

where the constant $K = C_5 C_1^\mu$ does not depend on r. We take $R = r_0 + K$.

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